

Quantifying stress states of theoretically modeled polarimetric measurements on dielectric media

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This work introduces and characterizes a theoretical model of a reflective polarimetric measurement technique determining the surface stress of a dielectric material, e.g., glass. We have developed a procedure to reconstruct the actual stress state from calculated Stokes vector components that would appear as polarization signals in a measurement. For the calculation, we consider a special geometry of the principal stress axis, where we chose the third principal axis corresponding to the z axis to be perpendicular to the surface and relaxed to a zero value. A created database represents the dependence of the reflected Stokes vector on the initial Stokes vector and the stress states included in the reflection matrix, which is iteratively calculated for several stress components and their orientations. The introduction of a model for the dependency of the reflected Stokes vector components on the stress states and the fitting of it to the database results in a system of equations of those dependencies that are solved for the stress state components and orientation. Finally, we found the theoretical determination accuracy of the model for surface stress magnitude to be on the order of a few megapascals and the orientation of the principal axes of stress can be determined with an accuracy of 20° .

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I. INTRODUCTION

Glass has developed throughout its more than 5000-year manufacturing history to be a versatile high-technology material that is used in a wide range of applications, such as smart device displays or safety glass in general [1–3]. Especially in the automotive sector an increasing number of sensors are nowadays included in windshields, drastically increasing the complexity in production [4], as well as their costs. Sufficient quality control to minimize the failure of expensive glass panes and to save resources and energy is of great interest. In the production of safety glass, residual stresses are purposely introduced during the tempering process to increase its stability and control its behavior at rupture [5]. An important part of quality control in the automotive industry is the prevention of breaking of glass panes because of unintended residual stress distributions. Therefore, quantitative, large-scale inline monitoring of the stress distribution is desirable.

The measurement of the stress distribution in glass panes is an indirect measurement and is based on the

change of light polarization when polarized light interacts with an anisotropic medium, where the anisotropy arises from stress. This is the well-known effect of photoelasticity, which was discovered by Brewster [6] in 1816. Currently, there are two commercially available devices that exploit this effect to estimate the surface stress of glass. The grazing angle surface polarimeter (GASP) measures the surface stress on the tin-doped side of uncoated thermally treated float glass [7]. The tin-doped side emerges from the nature of the float glass production process, which is based on liquid glass floating on liquid tin. That is why the application of this well-tried method is restricted to the tin-doped side of that special type of glass. Additionally, the exit and entrance prism of the GASP have to be in contact with the glass surface by means of an immersion fluid, which complicates the handling and the scan over the glass pane, especially if it is curved, which is typically the case for automotive glass. The scattered light polariscope (SCALP) measures the depth-dependent polarization changes of the backscattered light of a laser beam with an oblique incidence and extrapolates the fitted behavior to the surface [8]. Although this is also a well-established and typically used method, the following problems arise for operation on the production line. First, the SCALP also includes an entrance and exit prism is the same in contact with the glass surface by means of an immersion fluid, which causes the same problems as for the GASP. Second, the resolution of the stress measured with this technique and the applied fitting procedure are

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not completely sufficient for all types of glass, especially for those with low residual stresses, e.g., laminated glasses, as typically used for windshields. Third, the spatial orientation of the stress state inside the glass pane is not evaluated by this method.

In this work, we have developed a theoretical model for an experimental technique that uses the polarization state of the reflected beam to determine the surface stress conditions of an arbitrary dielectric material, on the example of glass.

This paper is organized as follows. First, we introduce the model and geometric considerations before explaining the creation of the database. We then quantify the model and consider the accuracy of the model before showing a first experimental approach and concluding the paper with a short outlook.

II. INTRODUCTION TO THE MODEL-BUILDING PROCEDURE

Comparison of the polarization state of the reflected light beam with that of the incoming light beam provides information on the stress-optical properties of the sample. To reconstruct the stress state from a measured polarization state, we developed a formalism that can describe the dependence of the polarization state on the stress conditions. The polarization state is represented by a Stokes vector in this theoretical work as well as in the experimental application. A phenomenological fitting procedure applied to that dependency enables us to create a system of equations that are solvable for the stress state, starting from the measured polarization states.

The procedure of determining the stress state can be tested by evaluating the fitting parameters from a theoretical database, which includes the originating stress states and the expected Stokes vector components. The database is generated by systematically calculating the expected polarization state of a measurement for several combinations of the stress state in the surface of the dielectric sample.

For each dataset, the Stokes vector components of a certain polarization state can be calculated with use of the Müller formalism [9] for the modeled setup of polarization optics and an optically anisotropic reflecting sample. There, the polarization states are represented by a Stokes vector, and the optical components, including the reflecting dielectric sample, are represented by a Müller matrix. The reflection matrix can be calculated first in the Jones formalism [10] for the special geometry of the stressed dielectric sample based on the (2×2) -matrix formalism for the general interface transition of light between two biaxial dielectric media, as shown by Abdulhalim [11]. Afterwards, the reflection matrix can be transferred to the Müller formalism with use of the transformation in Ref. [12]. The work of Abdulhalim [11] uses the

(2×2) -matrix formalism considering not only the transmission, as was the case for multilayer systems in former publications [13–15], but also the reflection of light at an interface. The (2×2) -matrix formalism is used instead of the original (4×4) -matrix formalism, as used, for example, in Refs. [16–18] to simplify the eigenvalue problems of those matrices, which especially improves the application to more complicated anisotropic geometries and for nonperpendicular incident angles.

In the following, we apply the formalism of Abdulhalim [11] to our geometry of stress-induced anisotropy. We introduce thereby a Müller-matrix representation of the reflecting sample. The resulting data are fitted and the dependency function is evaluated for its applicability to quantify optical polarization measurements with the originating stress state of the dielectric sample.

III. MÜLLER MATRIX OF THE REFLECTING MODEL SAMPLE

For the calculation of the polarization states after the reflecting sample, the Müller-Stokes formalism is used considering the Stokes vector of the incoming beam shaped by the Müller matrices of the optical components as well as the Müller matrix of the reflecting sample. The sample's reflection Müller matrix corresponds directly to

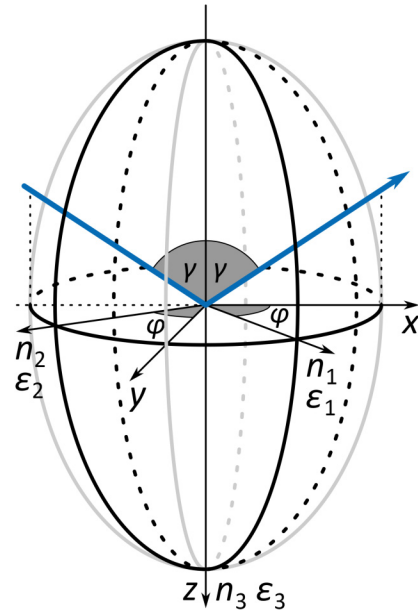


FIG. 1. Index ellipsoid of an optically anisotropic medium with a special geometry of the anisotropy axes corresponding to the surface. The third principal axis of the refractive index n_3 or the permittivity ϵ_3 , respectively, aligns with the z axis perpendicular to the x - y surface plane, while n_1 and n_2 are arbitrarily rotated by φ in the x - y surface plane. The blue lines indicate the incoming and reflected beams, defining the plane of incidence in the x - z plane with the incidence angle γ .

the Fresnel matrix in the Jones formalism. Our approach to find the matrix representation for our reflection geometry is thereby inspired by Abdulhalim [11].

A. Special geometry of a medium stressed along two directions in the x - y plane and relaxed in the z direction

The reflecting medium is a nonmagnetic, dielectric, and nonabsorbing material, which shows an optical anisotropy due to certain stresses. For the application of surface stress measurements, we simplified a general geometry considering a relaxed stress direction perpendicular to the surface. This defines the third principal axis of stress $\sigma_3 = 0$ perpendicular to the x - y surface plane. The resulting geometry is limited to scenarios of thermally tempered or float glass sufficiently far away from the edges, for which the absence of any stress component perpendicular to the surface is commonly assumed [19]. Therefore, the proposed geometry is in any case suitable for application of the method to many commonly used glasses sufficiently far away from the edges. However, the other generalities remain, resulting in the other two principal axes σ_1 and σ_2 being arbitrarily rotated by an azimuth angle φ in the x - y plane. Along those principal axes, the refractive indices n_1 , n_2 , and n_3 are present. The geometry of the corresponding index ellipsoid is sketched in Fig. 1. For the depicted geometry, the stress

optical law simplifies to [20]

$$n_1 = n_0 + C_1\sigma_1 + C_2\sigma_2, \quad (1a)$$

$$n_2 = n_0 + C_1\sigma_2 + C_2\sigma_1, \quad (1b)$$

$$n_3 = n_0 + C_2(\sigma_1 + \sigma_2), \quad (1c)$$

where n_0 is the refractive index of the material in a relaxed isotropic state, while C_1 and C_2 are the photoelastic constants. This defines the diagonalized refractive index tensor corresponding to the principal axes' coordinates. For our calculations, we use $n_0 = 1.52$, measured for the soda-lime glass used in the intended experiments, as well as the parameter set $C_1 = -0.65 \times 10^{-12} \text{Pa}^{-1}$ and $C_2 = -4.22 \times 10^{-12} \text{Pa}^{-1}$ for vitreous silica, taken from Ref. [21]. In Cartesian coordinates the refractive index tensor has to be rotated from the main axes' coordinates around the azimuth angle φ by using the rotation matrix $\mathbf{R}_z(\varphi)$ relative to the plane of incidence. Moreover, the plane of incidence can also be rotated from the x - z plane around the z axis by ϑ . That rotation is, in general, subtracted from the azimuth angle φ regarding the xyz coordinates, resulting in the rotation matrix $\mathbf{R}_z(\varphi - \vartheta)$. Figure 1 shows the case with $\vartheta = 0$. Thus, in the coordinates of the plane of incidence the permittivity tensor of a nonmagnetic, dielectric, and nonabsorbing material becomes [22]

$$\begin{aligned} \vec{\varepsilon} = \vec{n}^2 &= \mathbf{R}(\varphi - \vartheta) \vec{n}_{\text{diag}} \mathbf{R}(-\varphi + \vartheta) \mathbf{R}(\varphi - \vartheta) \vec{n}_{\text{diag}} \mathbf{R}(-\varphi + \vartheta) \\ &= \begin{pmatrix} \cos^2(\varphi - \vartheta)n_1^2 + \sin^2(\varphi - \vartheta)n_2^2 & \sin(\varphi - \vartheta)\cos(\varphi - \vartheta)(n_1^2 - n_2^2) & 0 \\ \sin(\varphi - \vartheta)\cos(\varphi - \vartheta)(n_1^2 - n_2^2) & \cos^2(\varphi - \vartheta)n_2^2 + \sin^2(\varphi - \vartheta)n_1^2 & 0 \\ 0 & 0 & n_3^2 \end{pmatrix}, \end{aligned} \quad (2)$$

transforming the diagonal matrix in the principal axes' coordinates by the rotation matrix around the coinciding z axis.

B. Reflection matrix of a biaxial medium with two principal axes arbitrarily rotated in the surface plane

The calculation of the reflection matrix for the special case of a biaxial medium with two principal axes

arbitrarily rotated in the surface plane is calculated in the same way as shown for other cases by Abdulhalim [11]. There, a certain condition for the electromagnetic fields of a light wave hitting an interface of, in general, two different optically anisotropic media is described. From Ref. [11] the relation between the electric field state $\psi_e = (\sqrt{\varepsilon_0}\mathbf{E}_x, \sqrt{\varepsilon_0}\mathbf{E}_y)^T \exp ik_z z$ and the 2×2 matrix $\mathbf{G}$ is given by

$$\mathbf{G}\psi_e = 0, \quad (3a)$$

$$\mathbf{G} = \begin{pmatrix} -(\varepsilon_{yx} + b_x\varepsilon_{yz} + b_xv_yv_z + v_xv_y) & v_x^2 + v_z^2 - b_yv_yv_z - b_y\varepsilon_{yz} - \varepsilon_{yy} \\ \varepsilon_{xx} + b_x\varepsilon_{xz} + b_xv_xv_z - v_y^2 - v_z^2 & \varepsilon_{xy} + b_y\varepsilon_{xz} + b_yv_xv_y + v_xv_y \end{pmatrix}, \quad (3b)$$

where v_p for $p \in \{x, y, z\}$ are the k -vector components normalized by k_0 , ε_{pq} for $p, q \in \{x, y, z\}$ are the components of the permittivity tensor $\hat{\varepsilon}$ in Cartesian coordinates, and

$$b_x = \frac{\varepsilon_{zx} + v_x v_z}{v_x^2 + v_y^2 - \varepsilon_{zz}} \quad \text{and} \quad (4a)$$

$$b_y = \frac{\varepsilon_{zy} + v_y v_z}{v_x^2 + v_y^2 - \varepsilon_{zz}}. \quad (4b)$$

The magnetic field state $\psi_h = (\sqrt{\mu_0} \mathbf{H}_x, \sqrt{\mu_0} \mathbf{H}_y)^T \exp ik_z z$ corresponds to

$$\psi_h = \mathbf{Q} \psi_e, \quad (5a)$$

$$\text{with } \mathbf{Q} = \begin{pmatrix} b_x v_y & b_y v_y - v_z \\ v_z - b_x v_x & -b_y v_x \end{pmatrix}, \quad (5b)$$

in relation to the electric field [11].

The solutions v_{zji} with $j \in \{1, 2, 3, 4\}$ and $i \in \{m, n\}$ of Eq. (3a) and the corresponding eigenvectors $V_{e/h,1/3,m/n}$ and $U_{e/h,2/4,m/n}$ for the particular medium and mode span the field vectors [11]

$$\begin{aligned} \psi_{e,m/n}(r, t) = & (V_{e1,m/n} \quad U_{e2,m/n}) \begin{pmatrix} A_{1x} \\ A_{2y} \end{pmatrix} \\ & + (V_{e3,m/n} \quad U_{e4,m/n}) \begin{pmatrix} A_{3x} \\ A_{4y} \end{pmatrix} \end{aligned} \quad (6a)$$

$$\begin{aligned} \text{and } \psi_{h,m/n}(r, t) = & (V_{h1,m/n} \quad U_{h2,m/n}) \begin{pmatrix} A_{1x} \\ A_{2y} \end{pmatrix} \\ & + (V_{h3,m/n} \quad U_{h4,m/n}) \begin{pmatrix} A_{3x} \\ A_{4y} \end{pmatrix}, \end{aligned} \quad (6b)$$

where

$$V_{h,1/3} = \mathbf{Q} V_{e,1/3}, \quad U_{h,2/4} = \mathbf{Q} U_{e,2/4} \quad (7)$$

and

$$A_{jl} = \mathbf{E}_{jl} \exp(ik_0(v_{jx}x + v_{jy}y + v_{jz}z) - i\omega t) \quad (8)$$

are the amplitudes of the eigenmodes with $j \in \{1, 2, 3, 4\}$ and $l \in \{x, y\}$. With use of the continuity conditions for the electromagnetic field, the Fresnel reflectivity matrix results in

$$\mathbf{R}_{mn} = (\mathbf{E}_{12m}^{-1} \mathbf{E}_{34m} - \mathbf{H}_{12m}^{-1} \mathbf{H}_{34m}) (\mathbf{H}_{12n}^{-1} \mathbf{H}_{12m} - \mathbf{E}_{12n}^{-1} \mathbf{E}_{12m}), \quad (9)$$

with

$$\mathbf{E}_{ij,m/n} = (V_{ei,m/n} \quad U_{ej,m/n}), \quad (10a)$$

$$\mathbf{H}_{ij,m/n} = (V_{hi,m/n} \quad U_{hj,m/n}). \quad (10b)$$

Considering the transition of an isotropic medium m (e.g., air or a vacuum) to an anisotropic and biaxial dielectric material n , different modes have to be considered for

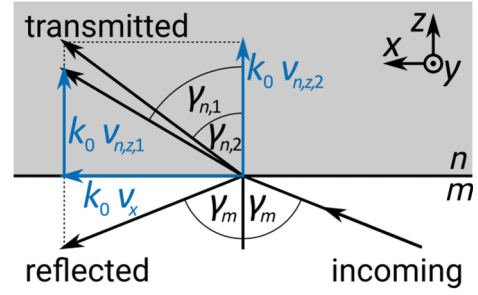


FIG. 2. For the transition from an isotropic medium m such as air or a vacuum into an anisotropic n such as stressed glass, the reflection of a light beam consists of one mode, while for the transmitted beam an additional mode appears due to birefringence. For both modes of the transmitted beam, the x component of the k vector stays the same as for the incoming beam, but the z component differs for both modes. The sketch of the transition is a special case for m isotropic and the plane of incidence coinciding with the x - z plane. The continuity and Maxwell's equations define the reflection and transmission coefficients in the Fresnel matrices. The reflection matrix is transformed to the Müller-matrix formalism in order to calculate the expected reflected Stokes vectors for a certain stress state.

the transmitted light wave in the anisotropic medium n due to birefringence [11]. Figure 2 schematically shows the situation for refraction. In the isotropic medium m , the incoming and reflected waves are in the same mode for both directions of polarization. The incoming wave mode can be described by its k vector $\vec{k}_1 m = \vec{k}_2 m$ and the reflected wave mode can be described by $\vec{k}_3 m = \vec{k}_4 m$. Both waves are directed along the same angle γ_m to the interface normal due to the law of reflection. The transmitted wave is divided into two modes $\vec{k}_{1n}$ and $\vec{k}_{2n}$ with corresponding angles $\gamma_{1,n}$ and $\gamma_{2,n}$ to the interface normal. In that special case of materials and with the geometry described in Sec. III A, a few simplifications directly arise. Since the plane of incidence is in the x - z plane, the y component of the wave vector $\vec{k}/k_0$ becomes $v_y = 0$. Even if the plane of incidence is rotated by ϑ around the z axis in Fig. 1, it can be considered as an additional rotation to the azimuth angle $\varphi - \vartheta$ instead of an additional y component. Equation (2), describing the geometry of the permittivity, defines $\varepsilon_{xz} = \varepsilon_{zx} = \varepsilon_{yz} = \varepsilon_{zy} = 0$ and $\varepsilon_{xy} = \varepsilon_{yx}$. With those simplifications, in Eq. (4b), it directly follows that $b_y = 0$. Hence, Eqs. (3b) and (4a) result in

$$\mathbf{G}_n = \begin{pmatrix} -\varepsilon_{xy} & v_{xn}^2 + v_{zn}^2 - \varepsilon_{yy} \\ \varepsilon_{xx} + b_x v_{xn} v_{zn} - v_{zn}^2 & \varepsilon_{xy} \end{pmatrix}, \quad (11a)$$

$$\text{with } b_{xn} = \frac{v_{xn} v_{zn}}{v_{xn}^2 - \varepsilon_{zz}}, \quad (11b)$$

and Eq. (5b) becomes

$$\psi_h = \mathbf{Q}\psi_e, \quad (12a)$$

$$\text{with } \mathbf{Q} = \begin{pmatrix} 0 & -v_{zn} \\ v_{zn} - b_{xn}v_{xn} & 0 \end{pmatrix}. \quad (12b)$$

The eigenvalues v_{zn} as solutions of $\det(\mathbf{G}_n) = 0$ are presented in Ref. [11] and simplify to

$$v_{z1n} = \sqrt{\frac{-b_n - \sqrt{b_n^2 - 4d_n}}{2}}, \quad (13a)$$

$$v_{z2n} = \sqrt{\frac{-b_n + \sqrt{b_n^2 - 4d_n}}{2}}, \quad (13b)$$

$$\text{with } b_n = -\varepsilon_{xx} - \varepsilon_{yy} + v_{x,n}^2 \left(1 + \frac{\varepsilon_{xx}}{\varepsilon_{zz}}\right), \quad (13c)$$

$$d_n = \varepsilon_{xx}\varepsilon_{yy} - \varepsilon_{xy}\varepsilon_{yx} + v_{x,n}^2 \left(\frac{\varepsilon_{xy}\varepsilon_{yx}}{\varepsilon_{zz}} - \varepsilon_{xx}\right) + \frac{v_{x,n}^2}{\varepsilon_{zz}}(v_{x,n}^2\varepsilon_{xx} - \varepsilon_{xx}\varepsilon_{yy}), \quad (13d)$$

$$v_{x,n} = v_{x,m} = \sin \gamma_m. \quad (13e)$$

The eigenvectors should be determined under the consideration that for the special case where the first principal axis is parallel to the x direction and the second principal axis is parallel to the y direction, i.e., $\varepsilon_{xy} = \varepsilon_{yx} = 0$, the components g_{11} and g_{22} are not the denominator. Furthermore, the convention ensures that the odd modes become p -polarized and the even modes becomes s -polarized in that special case. Considering those conditions, the relevant solutions of Eq. (11) are

$$V_{e1,n} = N^{-1} \begin{pmatrix} 1 \\ -\frac{g_{11}}{g_{12}} \end{pmatrix}, \text{ with } N = \sqrt{1 + \left(\frac{g_{11}}{g_{12}}\right)^2}, \quad (14a)$$

$$U_{e2,n} = M^{-1} \begin{pmatrix} -\frac{g_{22}}{g_{21}} \\ 1 \end{pmatrix}, \text{ with } M = \sqrt{1 + \left(\frac{g_{22}}{g_{21}}\right)^2}, \quad (14b)$$

in the biaxial medium n with the third principal axis in the z direction. For the isotropic medium m (e.g., air or a vacuum), the $\mathbf{G}$ matrix in Eq. (3) becomes

$$\mathbf{G}_m = \begin{pmatrix} 0 & v_{xm}^2 v_{zm}^2 - n_m^2 \\ n_m^2 + v_{zm}^2 \frac{v_{xm}^2 - 1}{v_{xm}^2 - n_m^2} & 0 \end{pmatrix} \quad (15)$$

and the eigenvectors become

$$V_{e,1/3,m} = (1 \ 0)^T \quad (16a)$$

$$\text{and } U_{e,2/4,m} = (0 \ 1)^T \quad (16b)$$

for the eigenvalues $v_{z,1/2,m} = n_m \cos \gamma_m$ and $v_{z,3/4,m} = -n_m \cos \gamma_m$, with $n_m = 1$, which result from the (2×2) -matrix formalism as expected from the sketch in Fig. 2. Including these solutions in Eqs. (6)–(10), one gets the reflection Fresnel matrix R_{mn} in the basis of the modes. It can be transferred to the sp basis by the transformation shown in Eq. (22) in Ref. [11] and then translated to the reflection Jones matrix in the Müller formalism [9] as conventionally done as, for example, in Ref. [12].

The resulting Müller matrix $R_{\text{Müller}}(\sigma_1, \sigma_2, \varphi)$ is used to perform step-by-step numerical calculations to predict measured data from the reflection matrix. Those calculations start from Eq. (1) to determine the stress dependence of the reflection matrix $\mathbf{R}_{mn}(\sigma_1, \sigma_2, \varphi)$ in Eq. (9). The reflection Müller matrix $R_{\text{Müller}}(\sigma_1, \sigma_2, \varphi)$, given in Eq. (A8) in the Appendix, directly results from the stress-dependent $\mathbf{R}_{mn}(\sigma_1, \sigma_2, \varphi)$ when one considers the solutions of the eigenvectors $V_{e/h,1/3,m/n}$ and $U_{e/h,2/4,m/n}$. Their solutions for air in Eq. (16) lead to the electric and magnetic field matrices $\mathbf{E}_{ij,m}$ and $\mathbf{H}_{ij,m}$ for air in Eq. (A1). Also, for anisotropic material n , the eigenvector solutions in Eq. (14), given in Eqs. (A2) and (A3) with all the information inserted, deliver the matrices $\mathbf{E}_{ij,n}$ and $\mathbf{H}_{ij,n}$ for the anisotropic material n in Eq. (A4). From there, it is explained in the Appendix how the reflection Müller matrix $R_{\text{Müller}}$ in Eq. (A8) is finally calculated.

IV. CREATING A DATABASE AND FITTING THE THEORETICAL MODEL

With the Müller-Stokes formalism, the expected reflected polarization states represented by a Stokes vector can be determined by multiplying the Müller matrix of the reflecting sample $R_{\text{Müller}}$ by the Stokes vector before the sample [23]:

$$\vec{S} = R_{\text{Müller}}(\sigma_1, \sigma_2, \varphi) \vec{S}_{\text{init}}. \quad (17)$$

The initial Stokes vector represents a -45° -polarized light beam and is multiplied by the Müller matrix of a compensator with its slow axis oriented at 0° . Thus, the initial polarization state becomes [23]

$$\vec{S}_{\text{init}} = (1 \ 0 \ -\cos \delta \ \sin \delta)^T, \quad (18)$$

where δ is the retardation of the optional compensator, resulting in $\cos(\delta = \pi/2) = 0$ and $\sin(\delta = \pi/2) = 1$ for a quarter-wave plate compensator and $\cos(\delta = 0) = 1$ and $\sin(\delta = 0) = 0$ for no compensator. The latter is the case in this work.

The resulting Stokes vector $\vec{S}$ is calculated for each reflection Müller matrix $\vec{R}(\sigma_1, \sigma_2, \varphi)$ of each stress state $(\sigma_1, \sigma_2, \varphi)$. By that a database is created for the sweeps

of σ_1 and σ_2 between -15 and 15 MPa in 1 MPa steps and between -5 and 5 MPa in 0.2 MPa steps, while φ is iterated between 0° and 90° in 1° steps. Additionally, the database is generated for the geometry shown in Fig. 1 as well as for the plane of incidence turned by $\vartheta = -45^\circ$ or the sample axes turned by 45° , respectively. The database is then separated into different signal types for the first two Stokes vector components $\iota \in \{0, 1\}$ and the orientation of the plane of incidence $\vartheta \in \{0^\circ, -45^\circ\}$. For each of the four signal types $S_{0/1,0^\circ/-45^\circ}(\sigma_1, \sigma_2, \varphi)$, those Stokes vector components show a nearly linear relation with σ_1 and σ_2 for each fixed φ . This behavior can be described by

$$S(\sigma_1, \sigma_2, \varphi) = A\sigma_1 + B\sigma_2 + c. \quad (19)$$

For different azimuth angles φ , the slopes $A(\varphi)$ and $B(\varphi)$ follow opposing third-order polynomials of φ . Hence, Eq. (19) is extended to a third-order parametric family of φ :

$$S(\sigma_1, \sigma_2, \varphi) = (a + d\varphi^3 + e\varphi^2 + f\varphi)\sigma_1 + (b - d\varphi^3 - e\varphi^2 - f\varphi)\sigma_2 + c. \quad (20)$$

For each of the four signals, this dependency on σ_1 , σ_2 , and φ is fitted. The fit procedure uses a series of one-dimensional fits along the dimensions σ_1 , σ_2 , and φ instead of one three-dimensional fit in the following way: First, one figures out the signal axis intersecting c by linear fitting the profile line, where $\sigma_1 = \sigma_2$, for each azimuth angle $\varphi \in [5^\circ, 85^\circ]$ considered in the database. Since for $\sigma_1 = \sigma_2$ the azimuthal orientation is irrelevant, all surfaces for each azimuth intersect in this profile line, which allows a global fit of this profile line including the data points for each azimuth angle in order to achieve a more accurate results for the intersect. The surface from the database and the fitted linear profile along the diagonal $\sigma_1 = \sigma_2$ are plotted in Fig. 3(a).

Second, the slopes of the surface in the σ_1 direction and the σ_2 direction are determined by linear fitting the profiles for each fixed value of σ_2 and σ_1 , respectively. This is performed by the use of Eq. (19) for a fixed σ_2 ,

$$S(\sigma_1, \sigma_2 = \text{const}, \varphi = \text{const}) = A\sigma_1 + B\sigma_2 + c = A\sigma_1 + c_B, \quad (21)$$

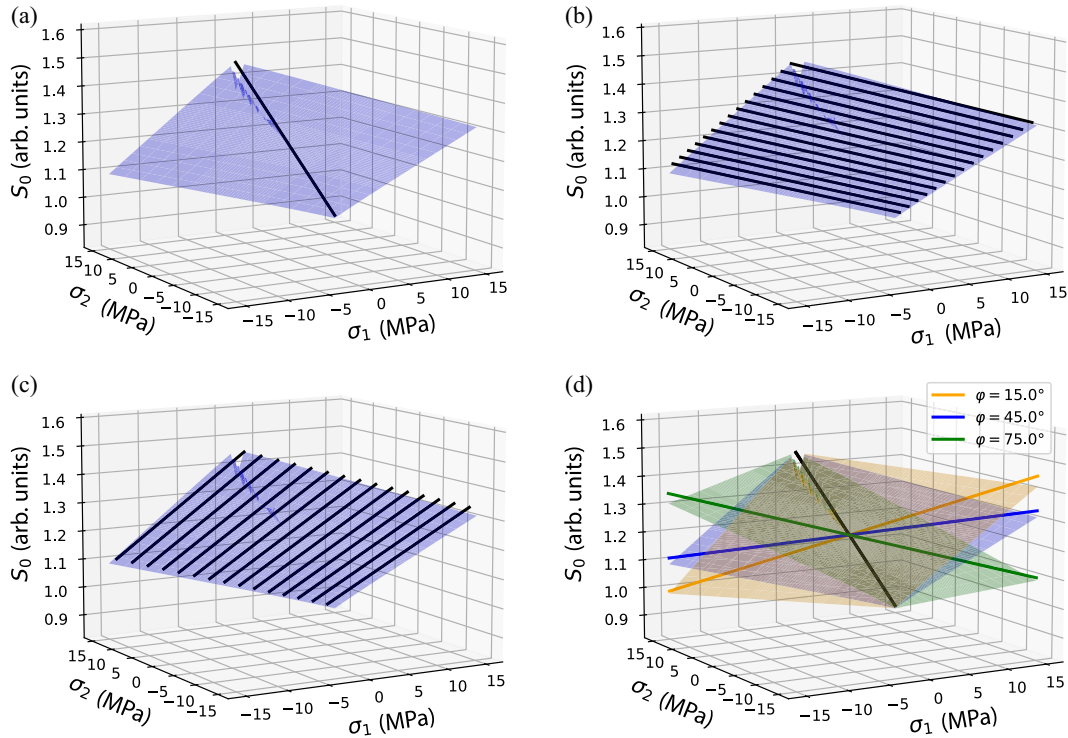


FIG. 3. For each azimuth angle $\varphi \in [5^\circ, 85^\circ]$, the database includes a nearly flat surface representing the calculated Stokes vector component S_0 plotted against the stress component values σ_1 and σ_2 . In (a)–(c), the zeroth Stokes vector component S_0 is plotted against σ_1 and σ_2 for the azimuth angle $\varphi = 45^\circ$. Also, the results of the fits for the one-dimensional profiles (a) along the diagonal to fit the signal axis intersect, (b) along the x direction, and (c) along the y direction to fit the corresponding slopes are shown. In (d) S_0 is plotted against σ_1 and σ_2 for $\varphi \in \{15^\circ, 45^\circ, 75^\circ\}$ together with the coinciding diagonal profiles and the diagonals perpendicular to the coinciding diagonal as an indication of the fit results for each angle φ . All plots refer to the data of the zeroth Stokes vector component S_0 and the plane of incidence in the x - z plane with a 0° rotation of the sample orientation around the z axis.

and for a fixed σ_1 ,

$$\begin{aligned} S(\sigma_1 = \text{const}, \sigma_2, \varphi = \text{const}) \\ = A\sigma_1 + B\sigma_2 + c = B\sigma_2 + c_A, \end{aligned} \quad (22)$$

as fit functions of the profiles in order to obtain the slopes $A(\varphi)$ and $B(\varphi)$ for each φ . Those profiles in the σ_1 direction for several fixed σ_2 values and in the σ_2 direction for several fixed σ_1 values are plotted on the surface plot for the azimuth angle $\varphi = 45^\circ$ in Figs. 3(b) and 3(c), respectively. The slopes for each profile are averaged to one slope value along the σ_1 direction and one slope value along the σ_2 direction. That is done for all azimuth angles $\varphi \in [5^\circ, 85^\circ]$, since for the angles near 0° and 90° the surface is no longer a differentiable linear function for the data calculated by the formalism.

For all fits in the steps described, the data for $\sigma_1 = \sigma_2 < 0$ are ignored due to the singularities of the calculation results, visible in Fig. 3.

Third, considering the resulting slopes for each φ , a dependency on φ appears in the form of a third-order polynomial function. Therefore, the φ dependency of Eq. (20) is determined with use of the results for $A(\varphi)$ and $B(\varphi)$ and the fit functions

$$A(\varphi) = d\varphi^3 + e\varphi^2 + f\varphi + a \quad (23)$$

and

$$B(\varphi) = -d\varphi^3 - e\varphi^2 - f\varphi + b, \quad (24)$$

respectively. The φ dependency of the surfaces in the database is depicted in Fig. 3(d) for $\varphi \in \{15^\circ, 45^\circ, 75^\circ\}$. Furthermore, diagonal profiles are used as in Fig. 3(a), and the profiles perpendicular to the coinciding diagonals are shown as lines to compare the fit results following Eq. (20) with the database. The profiles perpendicular to the diagonals refer to the results for the slopes $A(\varphi)$ and $B(\varphi)$ fitted with Eqs. (23) and (24), respectively, indicating their φ dependence. That fit completes the parametrization of the family of surface functions $S_0(\sigma_1, \sigma_2, \varphi)$ in Eq. (20). This procedure is performed for the first two Stokes vector components in the detector's plane of incidence orientations of $\vartheta = 0^\circ$ and $\vartheta = -45^\circ$. Hence, these parametrized surface functions are used to determine the stress state $(\sigma_1, \sigma_2, \varphi)$ from the measured or calculated Stokes vector components.

V. QUANTIFICATION PROCEDURE

The fitting procedure using Eq. (20) to describe the stress state $(\sigma_1, \sigma_2, \varphi)$ dependencies of any first or second Stokes vector component signal results, for the combination of all eight signals described, in the possibility to directly quantify the stress state.

This quantification is performed with a system of three equations of the form of Eq. (20) for the three signals $S_\kappa(\sigma_1, \sigma_2, \varphi)$, $S_\lambda(\sigma_1, \sigma_2, \varphi)$, and $S_\mu(\sigma_1, \sigma_2, \varphi)$, which are chosen from four combinations of Stokes vector components $\{S_{\iota, \vartheta}$, where $\vartheta \in \{0^\circ, -45^\circ\}$, $\iota \in \{0, 1\}\}$ calculated for the database with the plane of incidence orientation ϑ to the x - z plane around the z axis and the Stokes vector component ι . For each of these four combinations of three Stokes vector components, the system of the three equations, including the fitted parameters for each signal determined by the fitting procedure described above, is solved for $(\sigma_1, \sigma_2, \varphi)$ in the following manner: First, Eq. (20) for signal μ can be written as

$$\begin{aligned} 0 = (a_\mu + d_\mu\varphi^3 + e_\mu\varphi^2 + f_\mu\varphi)\sigma_1 \\ + (b_\mu - d_\mu\varphi^3 - e_\mu\varphi^2 - f_\mu\varphi)\sigma_2 + c_\mu - S_\mu, \end{aligned} \quad (25)$$

and its right-hand side is calculated for each azimuth angle φ considering the φ -dependent values for σ_1 and σ_2 . These arise by one solving the Eq. (20) for the signals κ and λ for σ_1 and σ_2 considering each originating azimuth angle φ from the database. Since the absolute value of right-hand side of Eq. (25) is calculated for each originating azimuth angle φ and the φ -dependent σ_1 and σ_2 , the minimum of those is evaluated and the corresponding φ is considered as the azimuth angle corresponding to the searched for stress state causing the measured signal.

Second, Eq. (20) for the signals κ and λ is solved for σ_1 and σ_2 once again, but this time only for the already determined φ . Hence, one has determined the complete stress state $(\sigma_1, \sigma_2, \varphi)$, which has caused the Stokes vector components S_κ , S_λ , and S_μ representing the measurable signals.

The results for σ_1 , σ_2 , and φ for each of the four combinations are averaged afterwards to increase the accuracy.

VI. ACCURACY OF THE QUANTIFICATION BY THE FITTED MODEL

To evaluate the accuracy of the fitting and quantification procedure, self-verification is performed, where all the datasets in the database are quantified and the stress states determined are compared with the originating ones. As explained in Sec. IV, the database has been created and our model has been fitted to it, which is used for the quantification of each dataset, as described in Sec. V. With use of the resulting parametrized fits, the stress state $(\sigma_1, \sigma_2, \varphi)$ is individually reconstructed from all four combinations of the Stokes vector components S_κ , S_λ , and S_μ , where κ, λ, μ corresponds to the index combinations ι, ϑ with $\vartheta \in \{0^\circ, -45^\circ\}$ and $\iota \in \{0, 1\}$ of each dataset. The four resulting values of σ_1 , σ_2 , and φ are averaged to obtain the final reconstructed stress state result for each dataset. The thereby determined angles φ are compared with the corresponding angles that were originally

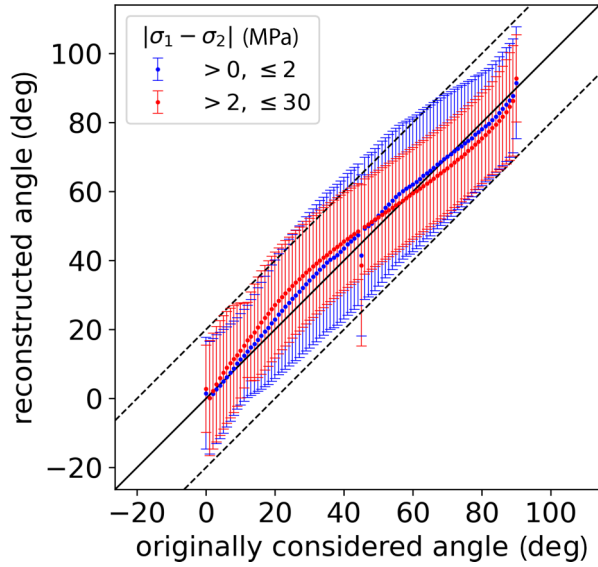


FIG. 4. Azimuth angles φ determined by the quantification of the single datasets plotted against the originating angles φ of the corresponding dataset. The determined values are averaged and plotted with the standard deviation as error bars for datasets with the same originating angles. The different colors represent the averaged angles separated in ranges of $|\sigma_1 - \sigma_2|$. The identity line is plotted as a solid black line and the $\pm 20^\circ$ deviation lines are plotted as dashed black lines.

considered to calculate each dataset of Stokes vectors. That comparison is depicted in Fig. 4. The reconstructed angles are averaged for all datasets with the same originating angles, but separately for certain ranges of certain values of $|\sigma_1 - \sigma_2|$, e.g., lower stress component differences $0 \text{ MPa} < |\sigma_1 - \sigma_2| \leq 2 \text{ MPa}$ and higher differences $2 \text{ MPa} < |\sigma_1 - \sigma_2| \leq 30 \text{ MPa}$. The error bars of the reconstructed angles are the standard deviations of their averages for each originating angle. Additionally, for the angles in the regions where the angle φ is expected to be 0° or 90° , respectively, the reconstructed angles have been corrected to values modulo 90° to reasonably average the values in those regions. Admittedly, this step artificially creates better results for the averaged angles in those regions, since this procedure would not be possible for actually measured or calculated datasets without the reference to the originating stress state.

On the one hand, the angles determined by the quantification of datasets with stress differences $|\sigma_1 - \sigma_2| \leq 2$ show higher standard deviations than those with $|\sigma_1 - \sigma_2| > 2$, as visible in Fig. 4. That is actually caused by the small differences $|\sigma_1 - \sigma_2| \leq 2$ since the angles for datasets with $|\sigma_1 - \sigma_2| = 0$ are not definitely determinable, as the stress or index ellipses are circles. Consequently, the datasets with stress component differences close to zero are more error-prone.

On the other hand, the standard deviations of the other range $|\sigma_1 - \sigma_2| > 2$ are around 15° , while all means deviate

by less than half the standard deviation from the identity line, indicating a decisively greater accuracy degraded only by outliers. The outliers originate from those datasets where the fitted linear dependencies differ most from the slightly parabolic surfaces in Fig. 3; that is, those data points far from the intersecting diagonal.

For both difference ranges in Fig. 4, a conspicuous deviation of the averaged reconstructed angle from the identity line and the overall trend reconstructed angles appears at $\varphi = 45^\circ$ as well as, but less distinctly, at $\varphi = 0^\circ$ and $\varphi = 90^\circ$. That is presumably because at $\varphi = 0^\circ$ and $\varphi = 90^\circ$ the surfaces in the S_i - σ_1 - σ_2 spaces are not linear for the plane of incidence orientation of $\vartheta = 0^\circ$ but show a sharp bend. The same is the case for the orientation of the principal axis $\varphi = 45^\circ$ and the orientation plane of incidence of $\vartheta = -45^\circ$.

The values of σ_1 and σ_2 determined are calculated with the previously determined angles φ of the corresponding datasets. In Fig. 5 the averages of those determined stress values with their standard deviations as error bars are plotted against the originally considered values in the corresponding datasets. Decreasing standard deviations for smaller absolute stress values are seen in Fig. 5. Furthermore, the determined averages deviate more from the identity line for positive stresses than for negative stresses. Both effects are caused by the accuracy of the surface fit functions, which differ less from the calculated data surfaces in Fig. 3 for small stress values and, also, typically

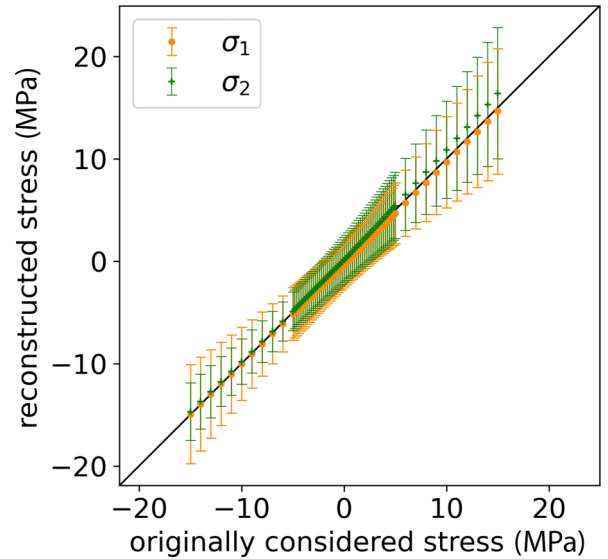


FIG. 5. Stress values σ_1 and σ_2 determined by the quantification of the single datasets plotted against the originating stress value of the corresponding dataset. The determined values are averaged and plotted with the standard deviation as error bars for datasets with the same originating angles. The point density of the data in the region between -5 and 5 MPa is too high to visualize the data points separated from each other. The identity line is plotted in black.

follow rather the parabolic arm of the surface's negative stress region than that of the positive stress region caused by the fitting procedure.

The overall accuracy $\Delta\sigma$ of the quantification procedure, including the fitting procedure, is 5 MPa or less for $-5 \text{ MPa} \leq \sigma_1, \sigma_2 \leq 5 \text{ MPa}$ and 8 MPa or less for $-15 \text{ MPa} \leq \sigma_1, \sigma_2 \leq 15 \text{ MPa}$, while the angle φ is determined with accuracy $\Delta\varphi \leq 20^\circ$. For differences of stress values $|\sigma_1 - \sigma_2| > 2 \text{ MPa}$, the corresponding angle φ is determined more accurately than $\Delta\varphi \leq 15^\circ$.

VII. EXPERIMENTAL APPROACHES

Surface stress measurements using the SCALP are often performed simultaneously with four-point bending tests, for example, by Aben *et al.* [8] or more recently by Efferz *et al.* [24]. In contrast, we developed a forcing apparatus applying stress along one of the surface plane directions in order to recreate parts of the simulated data experimentally, e.g., the profiles of the surface plots in Fig. 3, where either $\sigma_1 = 0$ or $\sigma_2 = 0$. Measurements with the reflection polarimeter, whose results are theoretical calculated by the Müller-Stokes formalism here, are in a developmental stage. However, SCALP measurements have been performed as a reference to validate the forcing apparatus and the linearity of those measurements our model depends on. For a compressive forcing along the y direction, we have taken SCALP measurements along the x and y directions (corresponding to 0° and 90°), linearly fit them, and calculated the principal axes of stress $\sigma_1 = \sigma_x$ and $\sigma_2 = \sigma_y$, according to Ref. [5], depending on the applied compressive force. The linear dependencies are shown in Fig. 6. The first principal axis σ_1 , corresponding to the x direction, is nearly independent of the forcing compared with the stress along the second principal axis σ_2 , corresponding to the y direction, which is the direction of the applied

force. The measured surface stress σ_1 corresponds with an offset of approximately 1 MPa quite well to the applied pressure. This indicates the validity of our forcing apparatus. Hence, we will be able to experimentally recreate the theoretical database used by our reflection polarimeter measurements in the next step.

VIII. CONCLUSION

In conclusion, we showed that a direct quantification of the stress state $(\sigma_1, \sigma_2, \varphi)$ is possible for our simplified fit model of the theoretical data, limited only by the accuracy defined as the standard deviation of $\Delta\sigma \leq 5 \text{ MPa}$ for $-5 \text{ MPa} \leq \sigma_1, \sigma_2 \leq 5 \text{ MPa}$. In comparison, this determination error is higher than the error of the SCALP, which is a widely used state-of-the-art instrument for surface stress measurements. In the manufacturer's data on the GlasStress website [25], the SCALP measurement errors are specified as $\Delta\sigma < 1 \text{ MPa}$ for $\sigma < 4 \text{ MPa}$ and 5% for $\sigma \geq 4 \text{ MPa}$. However, our method allows a complete determination of the stress state under the geometric conditions described in Sec. III A and in Fig. 1 with an error for the azimuth angle of at maximum $\Delta\varphi \leq 20^\circ$. Also, for higher absolute stress values, the error, with which the stress values $-15 \text{ MPa} \leq \sigma_1, \sigma_2 \leq 15 \text{ MPa}$ are determined, increases to $\Delta\sigma \leq 8 \text{ MPa}$. That is not too far away from the state of the art, where one has to consider, for example, in case of the SCALP, that one probes the stress in only one direction and not the complete stress state.

Our work is considering the self-verification of the quantitative stress state reconstruction of theoretically calculated Stokes vector components. That means that there are two differences from the practical case. The practical case would use a measured database instead of a calculated one to quantitatively reconstruct an unknown sample of the same material used for the database recording. The first difference would be the precision of the measurement and the experimental errors, which will have an influence on the overall accuracy of the database fits and thus the reconstruction. The second difference is that an unknown sample is probed. Reconstructing the azimuth for azimuths near 0° and 90° can end up in a situation where the principal stresses σ_1 and σ_2 could be confused corresponding to a false assignment of φ to 0° instead of 90° or the other way around. That would not be recognized by the corresponding actual azimuth and therefore would not be prevented by use of the modulo 90° correction as described for the self-verification in Sec. VI. Ultimately, this could cause an interchange of the σ_1 and σ_2 values that could not be identified.

Nevertheless, our method paves the way for quantitative reconstruction of the surface stress state by a polarimetric measurement on dielectric materials using a reference database of the same material as probed. We showed for our model a theoretical precision of at least the same order

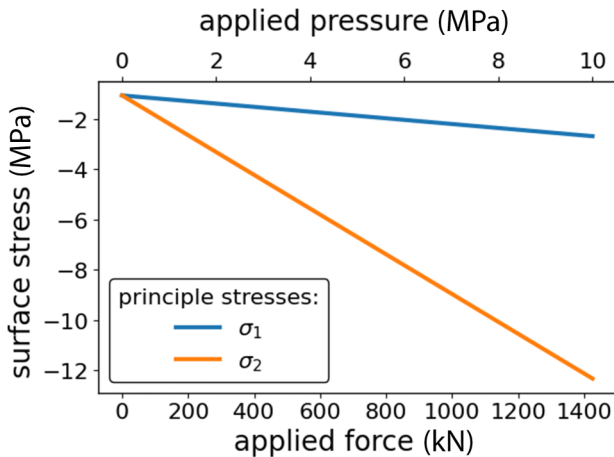


FIG. 6. Principal axes of stress calculated from linear fits of SCALP measurements in the x and y directions depending on an applied compressive force along the y direction.

of magnitude as the state-of-the-art instrument for surface stress measurements, with the extension of being able to quantify the complete stress state, including both stress values of the principal axes in the surface plane as well as their azimuthal orientation in the plane.

Of course, the model has its limitations to dielectric material and the positions of it, where the proposed geometry is present. Furthermore, the model is designed for an idealized case of no impurities in the dielectric medium or the atmosphere causing scattering effects. In most cases of float glass or tempered glass applications, the presence of impurities and their disturbing effects should be negligibly small.

While the considered surface stress range of -15 to 15 MPa rather covers the laminated glasses used as automotive windshields, thermally tempered glass with residual surface stresses in the range of -80 to -120 MPa [5] is typically used for automotive side windows and is closer to the extreme case of glass rupture. One can imagine by extrapolating the trend of the rising error bars in Fig. 5 a decreasing precision for measurements of such high absolute surface stress values. This is again intrinsically caused by the actual parabolic behavior of the surfaces in Fig. 3 and cannot be solved by a linear description of the total stress range. However, one could think about a local linear description in the regions of interest. For thermally tempered glass this would be, for example, a range of -80 to -120 MPa. Since this is a range of 40 MPa, one would expect similar accuracies as for the range of 30 MPa evaluated here.

IX. OUTLOOK

For the theoretical calculations and the stress determination procedure, efforts are being made to extend the formalism to a generally absorbing material by including the imaginary part in the complex refractive index representing the absorption in the equations. On the experimental side, measurements are currently being prepared to test the same fitting procedure for a measured database. In that case, the database is measured by external application of a mechanical stress on glass panes in a single direction for several angles $\tilde{\varphi} \in [0^\circ, 180^\circ]$, thus imitating a stressed glass state $(\sigma_1, \sigma_2 = 0, \varphi)$ and $(\sigma_1 = 0, \sigma_2, \varphi)$ for $\varphi \in [0^\circ, 90^\circ]$. These data can be extrapolated to complete surfaces $S(\sigma_1, \sigma_2, \varphi = \text{const})$ in the linear regions of the relations $S(\sigma_1, \sigma_2 = 0, \varphi = \text{const})$ and $S(\sigma_1 = 0, \sigma_2, \varphi = \text{const})$. With the same procedure as described in this work, self-verification can be performed to compare the theoretically possible and experimentally achievable precision of the quantification method.

Moreover, the determination of certain parameters, for example, the stress state $(\sigma_1, \sigma_2, \varphi)$, using a database containing datasets with other parameters, such as Stokes vector components, is a typical application for machine

learning. We plan to feed machine learning algorithms with the databases of measured or calculated signal datasets in order to evaluate the stress states, replacing the manual numerical fitting and solving procedure. Probably, the accuracy of the quantitative determination of the stress states can be increased by avoiding the restriction of fitting a linear model and by using a “semianalytical” solution of the resulting system of equations.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX

We provide the following calculations in order to retrace the intermediate steps between the equations given in the main text.

For air or a vacuum as the medium m , the matrices in Eq. (10) become

$$\mathbf{E}_{12m} = \mathbf{E}_{34m} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\text{A1a})$$

$$\begin{aligned} \mathbf{H}_{12m} &= \begin{pmatrix} \mathbf{Q}_{1m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \mathbf{Q}_{2m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -v_{z2m} \\ v_{z1m} - b_{x1m}v_{x1m} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\cos \gamma_m \\ \left(\cos \gamma_m - \frac{\sin^2 \gamma_m}{\cos \gamma_m} \right) & 0 \end{pmatrix}, \end{aligned} \quad (\text{A1b})$$

$$\begin{aligned} \text{and } \mathbf{H}_{34m} &= \begin{pmatrix} \mathbf{Q}_{3m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \mathbf{Q}_{4m} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -v_{z4m} \\ v_{z3m} - b_{x3m}v_{x3m} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \cos \gamma_m \\ -\left(\cos \gamma_m - \frac{\sin^2 \gamma_m}{\cos \gamma_m} \right) & 0 \end{pmatrix} \end{aligned} \quad (\text{A1c})$$

by insertion of the results of Eqs. (16a) and (16b) together with the results of Eqs. (7) and (12).

From Eqs. (14a) and (14b) with all inserted information calculated in Sec. III A, we obtain

$$\begin{aligned} V_{e1n} &= N^{-1} \begin{pmatrix} 1 \\ -\frac{g_{11}}{g_{12}} \end{pmatrix} = N^{-1} \begin{pmatrix} 1 \\ \frac{\varepsilon_{xy}}{v_{xn}^2 + v_{zn}^2 - \varepsilon_{yy}} \end{pmatrix} \\ &= N^{-1} \begin{pmatrix} 1 \\ \frac{\sin(\varphi - \vartheta) \cos(\varphi - \vartheta) (n_1^2 - n_2^2)}{v_{xn}^2 + v_{zn}^2 - \cos^2(\varphi - \vartheta) n_2^2 - \sin^2(\varphi - \vartheta) n_1^2} \end{pmatrix}, \end{aligned} \quad (\text{A2a})$$

$$\begin{aligned} \text{with } N &= \sqrt{1 + \left(\frac{g_{11}}{g_{12}}\right)^2} = \sqrt{1 + \left(\frac{-\varepsilon_{xy}}{v_{xn}^2 + v_{zn}^2 - \varepsilon_{yy}}\right)^2} \\ &= \sqrt{1 + \left(\frac{\sin(\varphi - \vartheta) \cos(\varphi - \vartheta) (n_1^2 - n_2^2)}{v_{xn}^2 + v_{zn}^2 - \cos^2(\varphi - \vartheta) n_2^2 + \sin^2(\varphi - \vartheta) n_1^2}\right)^2}, \end{aligned} \quad (\text{A2b})$$

and

$$\begin{aligned} U_{e2n} &= M^{-1} \begin{pmatrix} -\frac{g_{22}}{g_{21}} \\ 1 \end{pmatrix} = M^{-1} \begin{pmatrix} \frac{-\varepsilon_{xy}}{\varepsilon_{xx} + b_{x2n} v_{xn} v_{zn} - v_{zn}^2} \\ 1 \end{pmatrix} \\ &= M^{-1} \begin{pmatrix} \frac{\sin(\varphi - \vartheta) \cos(\varphi - \vartheta) (n_2^2 - n_1^2)}{\cos^2(\varphi - \vartheta) n_2^2 + \sin^2(\varphi - \vartheta) n_1^2 + b_x v_{xn} v_{zn} - v_{zn}^2} \\ 1 \end{pmatrix}, \end{aligned} \quad (\text{A3a})$$

$$\begin{aligned} \text{with } M &= \sqrt{1 + \left(\frac{g_{22}}{g_{21}}\right)^2} = \sqrt{1 + \left(\frac{\varepsilon_{xy}}{\varepsilon_{xx} + b_{x2n} v_{xn} v_{zn} - v_{zn}^2}\right)^2} \\ &= \sqrt{1 + \left(\frac{\sin(\varphi - \vartheta) \cos(\varphi - \vartheta) (n_2^2 - n_1^2)}{\cos^2(\varphi - \vartheta) n_2^2 + \sin^2(\varphi - \vartheta) n_1^2 + b_x v_{xn} v_{zn} - v_{zn}^2}\right)^2}. \end{aligned} \quad (\text{A3b})$$

Thus, Eq. (10) for the anisotropic medium n becomes

$$\mathbf{E}_{12n} = (V_{e1n} \quad U_{e2n}) = \begin{pmatrix} N^{-1} & -M^{-1} \frac{g_{2n,22}}{g_{2n,21}} \\ -N^{-1} \frac{g_{1n,11}}{g_{1n,12}} & M^{-1} \end{pmatrix} \quad (\text{A4a})$$

$$\begin{aligned} \text{and } \mathbf{H}_{12n} &= (V_{h1n} \quad U_{h2n}) = \begin{pmatrix} \mathbf{Q}_{1n} \begin{pmatrix} N^{-1} \\ -N^{-1} \frac{g_{1n,11}}{g_{1n,12}} \end{pmatrix} & \mathbf{Q}_{2n} \begin{pmatrix} -M^{-1} \frac{g_{2n,22}}{g_{2n,21}} \\ M^{-1} \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} v_{z1n} N^{-1} \frac{g_{1n,11}}{g_{1n,12}} & -v_{z2n} M^{-1} \\ (v_{z1n} - b_{x1n} v_{xn}) N^{-1} & - (v_{z2n} - b_{x2n} v_{xn}) M^{-1} \frac{g_{2n,22}}{g_{2n,21}} \end{pmatrix}. \end{aligned} \quad (\text{A4b})$$

With the results in Eqs. (A1) and (A4), the reflection matrix (9) becomes

$$\mathbf{R}_{mn} = (\mathbf{E}_{12m}^{-1} \mathbf{E}_{34m} - \mathbf{H}_{12m}^{-1} \mathbf{H}_{34m}) (\mathbf{H}_{12n}^{-1} \mathbf{H}_{12m} - \mathbf{E}_{12n}^{-1} \mathbf{E}_{12m}) = (\mathbb{I}_2 - \mathbf{H}_{12m}^{-1} \mathbf{H}_{34m}) (\mathbf{H}_{12n}^{-1} \mathbf{H}_{12m} - \mathbf{E}_{12n}^{-1}) \quad (\text{A5})$$

for the anisotropic medium n and the medium m being air or a vacuum. The reflection matrix $\mathbf{R}_{mn}$ is related to the mode basis, i.e., the matrix elements represent the transitions between the corresponding modes [11]. For the geometry used with

the plane of incidence in the x - z plane, the transformation from the xy field component basis to the ps basis in Ref. [11] simplifies to

$$\begin{pmatrix} \mathbf{E}_x \\ \mathbf{E}_y \end{pmatrix}_{j,m/n} = \mathbf{T}_{j,m/n} \begin{pmatrix} \mathbf{E}_p \\ \mathbf{E}_s \end{pmatrix}_{m/n} = \begin{pmatrix} \cos \gamma_{j,m/n} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{E}_p \\ \mathbf{E}_s \end{pmatrix}_{m/n} \quad (\text{A6})$$

for mode type $j \in \{1, 2\}$, which corresponds to $j \in \{3, 4\}$ in the case of the reflected beam. Thus, the reflection matrix transforms to the sp basis with

$$\begin{aligned} \tilde{R} &= \mathbf{T}_{jm}^{-1} \mathbf{R}_{mn} \mathbf{T}_{jm} = \mathbf{T}_m^{-1} \mathbf{R}_{mn} \mathbf{T}_m \\ &= \begin{pmatrix} \frac{1}{\cos \gamma_m} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \begin{pmatrix} \cos \gamma_m & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} r_{11} & \frac{r_{12}}{\cos \gamma_m} \\ r_{21} \cos \gamma_m & r_{22} \end{pmatrix}, \end{aligned} \quad (\text{A7})$$

since in the isotropic medium all modes' j angles are $\gamma_{jm} = \gamma_m$, as shown in Fig. 2. To transfer the reflection matrix from the Jones formalism to the Müller-Stokes formalism [9], the relation

$$\begin{aligned} R_{\text{Müller}} &= A (\tilde{R} \otimes \tilde{R}^*) A^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{pmatrix} \begin{pmatrix} r_{pp} \tilde{R}^* & r_{ps} \tilde{R}^* \\ r_{sp} \tilde{R}^* & r_{ss} \tilde{R}^* \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & 0 & \frac{1}{2} & \frac{i}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{pmatrix} \begin{pmatrix} r_{pp} \tilde{r}_{pp} & r_{pp} \tilde{r}_{sp} & r_{ps} \tilde{r}_{pp} & r_{ps} \tilde{r}_{sp} \\ r_{pp} \tilde{r}_{ps} & r_{pp} \tilde{r}_{ss} & r_{ps} \tilde{r}_{ps} & r_{ps} \tilde{r}_{ss} \\ r_{sp} \tilde{r}_{pp} & r_{sp} \tilde{r}_{sp} & r_{ss} \tilde{r}_{pp} & r_{ss} \tilde{r}_{sp} \\ r_{sp} \tilde{r}_{ps} & r_{sp} \tilde{r}_{ss} & r_{ss} \tilde{r}_{ps} & r_{ss} \tilde{r}_{ss} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i \\ 1 & -1 & 0 & 0 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{pmatrix} \begin{pmatrix} r_{pp} \tilde{r}_{pp} + r_{ps} \tilde{r}_{sp} & r_{pp} \tilde{r}_{pp} - r_{ps} \tilde{r}_{sp} & r_{pp} \tilde{r}_{sp} + r_{ps} \tilde{r}_{pp} & i(-r_{pp} \tilde{r}_{sp} + r_{ps} \tilde{r}_{pp}) \\ r_{pp} \tilde{r}_{ps} + r_{ps} \tilde{r}_{ss} & r_{pp} \tilde{r}_{ps} - r_{ps} \tilde{r}_{ss} & r_{pp} \tilde{r}_{ss} + r_{ps} \tilde{r}_{ps} & i(-r_{pp} \tilde{r}_{ss} + r_{ps} \tilde{r}_{ps}) \\ r_{sp} \tilde{r}_{pp} + r_{ss} \tilde{r}_{sp} & r_{sp} \tilde{r}_{pp} - r_{ss} \tilde{r}_{sp} & r_{sp} \tilde{r}_{sp} + r_{ss} \tilde{r}_{pp} & i(-r_{sp} \tilde{r}_{sp} + r_{ss} \tilde{r}_{pp}) \\ r_{sp} \tilde{r}_{ps} + r_{ss} \tilde{r}_{ss} & r_{sp} \tilde{r}_{ps} - r_{ss} \tilde{r}_{ss} & r_{sp} \tilde{r}_{ss} + r_{ss} \tilde{r}_{ps} & i(-r_{sp} \tilde{r}_{ss} + r_{ss} \tilde{r}_{ps}) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} r_{pp} \tilde{r}_{pp} + r_{ps} \tilde{r}_{sp} + r_{sp} \tilde{r}_{ps} + r_{ss} \tilde{r}_{ss} & r_{pp} \tilde{r}_{pp} - r_{ps} \tilde{r}_{sp} + r_{sp} \tilde{r}_{ps} - r_{ss} \tilde{r}_{ss} & r_{pp} \tilde{r}_{pp} - r_{ps} \tilde{r}_{sp} - r_{sp} \tilde{r}_{ps} + r_{ss} \tilde{r}_{ss} & i(-r_{pp} \tilde{r}_{sp} + r_{ps} \tilde{r}_{pp} - r_{sp} \tilde{r}_{ss} + r_{ss} \tilde{r}_{ps}) \\ r_{pp} \tilde{r}_{pp} + r_{ps} \tilde{r}_{sp} - r_{sp} \tilde{r}_{ps} - r_{ss} \tilde{r}_{ss} & r_{pp} \tilde{r}_{pp} - r_{ps} \tilde{r}_{sp} - r_{sp} \tilde{r}_{ps} + r_{ss} \tilde{r}_{ss} & r_{pp} \tilde{r}_{pp} - r_{ps} \tilde{r}_{sp} - r_{sp} \tilde{r}_{ps} + r_{ss} \tilde{r}_{ss} & i(-r_{pp} \tilde{r}_{sp} + r_{ps} \tilde{r}_{pp} - r_{sp} \tilde{r}_{ss} + r_{ss} \tilde{r}_{ps}) \\ r_{pp} \tilde{r}_{ps} + r_{ps} \tilde{r}_{ss} + r_{sp} \tilde{r}_{pp} + r_{ss} \tilde{r}_{sp} & r_{pp} \tilde{r}_{ps} - r_{ps} \tilde{r}_{ss} + r_{sp} \tilde{r}_{pp} - r_{ss} \tilde{r}_{sp} & r_{pp} \tilde{r}_{ps} - r_{ps} \tilde{r}_{ss} + r_{sp} \tilde{r}_{pp} - r_{ss} \tilde{r}_{sp} & i(-r_{pp} \tilde{r}_{ss} + r_{ps} \tilde{r}_{ps} - r_{sp} \tilde{r}_{sp} + r_{ss} \tilde{r}_{pp}) \\ i(r_{pp} \tilde{r}_{ps} + r_{ps} \tilde{r}_{ss} - r_{sp} \tilde{r}_{pp} - r_{ss} \tilde{r}_{sp}) & i(r_{pp} \tilde{r}_{ps} - r_{ps} \tilde{r}_{ss} - r_{sp} \tilde{r}_{pp} + r_{ss} \tilde{r}_{sp}) & i(r_{pp} \tilde{r}_{ps} - r_{ps} \tilde{r}_{ss} - r_{sp} \tilde{r}_{pp} + r_{ss} \tilde{r}_{sp}) & r_{pp} \tilde{r}_{ss} - r_{ps} \tilde{r}_{ps} - r_{sp} \tilde{r}_{sp} + r_{ss} \tilde{r}_{pp} \end{pmatrix} \end{aligned} \quad (\text{A8})$$

is used, where $\tilde{R}^*$ is the conjugate transpose of $\tilde{R}$ [9].

- [1] M. Ballauff, J. M. Brader, S. U. Egelhaaf, M. Fuchs, J. Horbach, N. Koumakis, M. Krüger, M. Laurati, K. J. Mutch, G. Petekidis, M. Siebenbürger, T. Voigtmann, and J. Zausch, Residual stresses in glasses, *Phys. Rev. Lett.* **110**, 215701 (2013).
- [2] G. Whittleston and Armin Berenjian, History and manufacturing of glass, *Am. J. Mater. Sci.* **7**, 18 (2017).
- [3] P. Richet, *Encyclopedia of Glass Science, Technology, History, and Culture* (John Wiley & Sons, Inc., Hoboken, New Jersey, 2021), 1st ed., Vol. 1.
- [4] M. Jacoby, Automotive glass presents unique challenges for manufacturing and recycling, *Chem. Eng. News* **100**, 20 (2022), last viewed on 02.12.2024.
- [5] Y. Chen, D. Lochegnies, R. Defontaine, J. Anton, H. Aben, and R. Langlais, Measuring the 2D residual surface stress mapping in tempered glass under the cooling jets: The influence of process parameters on the stress homogeneity and isotropy, *Strain* **49**, 60 (2013).
- [6] D. Brewster, X. On the communication of the structure of doubly refracting crystals to glass, muriate of soda, fluor spar, and other substances, by mechanical compression and dilatation. by David Brewster, LL. D. F. R. S. Lond. and Edin. in a letter addressed to the Right Hon. Sir Joseph Banks, Bart. G.C.B. P.R.S., *Philos. Trans. R. Soc. Lond.* **106**, 156 (1816).
- [7] Strainoptics Inc., GASP Bulletin GSP-0920, available online at: <http://www.strainoptics.com>, last viewed on 20.08.2024 at: <https://www.strainoptics.com/wp-content/uploads/2021/09/GASP-Bulletin-2020-Now-with-4-Models.pdf>.
- [8] H. Aben, J. Anton, A. Errapart, S. Hödemann, J. Kikas, H. Klaassen, and M. Lamp, On non-destructive residual stress measurement in glass panels, *Est. J. Eng.* **16**, 150 (2010).
- [9] Proceedings of the Winter Meeting of the Optical Society of America, *J. Opt. Soc. Am.* **38**, 657 (1948).
- [10] R. C. Jones, A new calculus for the treatment of optical systems I. Description and discussion of the calculus, *J. Opt. Soc. Am.* **31**, 488 (1941).
- [11] I. Abdulhalim, Exact 2×2 matrix method for the transmission and reflection at the interface between two arbitrarily oriented biaxial crystals, *J. Opt. A: Pure Appl. Opt.* **1**, 655 (1999).
- [12] N. Podraza and G. Jellison, in *Encyclopedia of Spectroscopy and Spectrometry (Third Edition)*, edited by J. C. Lindon, G. E. Tranter, and D. W. Koppenaal (Academic Press, Oxford, 2017) 3rd ed., p. 482.
- [13] P. Yeh, Extended Jones matrix method, *J. Opt. Soc. Am.* **72**, 507 (1982).
- [14] A. Lien, Extended Jones matrix representation for the twisted nematic liquid-crystal display at oblique incidence, *Appl. Phys. Lett.* **57**, 2767 (1990).
- [15] C. Gu and P. Yeh, Extended Jones matrix method. ii, *J. Opt. Soc. Am. A* **10**, 966 (1993).
- [16] S. Teitler and B. W. Henvis, Refraction in stratified, anisotropic media, *J. Opt. Soc. Am.* **60**, 830 (1970).
- [17] D. W. Berreman, Optics in stratified and anisotropic media: 4×4 -matrix formulation, *J. Opt. Soc. Am.* **62**, 502 (1972).
- [18] P. Yeh, Electromagnetic propagation in birefringent layered media, *J. Opt. Soc. Am.* **69**, 742 (1979).
- [19] K. Thiele, S. Müller-Braun, and J. Schneider, Evaluation methods for surface compression stress measurements with unknown principle stress directions, *Glass Struct. Eng.* **7**, 121 (2022).
- [20] H. Aben and C. Guilleminot, *Photoelasticity of Glass* (Springer-Verlag Berlin Heidelberg, Berlin and Heidelberg, 1993), 1st ed.
- [21] W. Primak and D. Post, Photoelastic constants of vitreous silica and its elastic coefficient of refractive index, *J. Appl. Phys.* **30**, 779 (1959).
- [22] H. Uranus, H. Hoekstra, and E. van Groesen, Finite difference scheme for planar waveguides with arbitrary index profiles and its implementation for anisotropic waveguides with a diagonal permittivity tensor, *Opt. Quantum Electron.* **35**, 407 (2003).
- [23] H. G. Tompkins and E. A. Irene, eds. *Handbook of Ellipsometry* (William Andrew, Inc., Norwich, NY, 2005).
- [24] L. Efferz, K. Thiele, M. Schuster, C. Schuler, G. Siebert, and J. Schneider, Relation between edge stress, bending strength, surface stress and fracture pattern of thermally toughened glass, *Glass Struct. Eng.* **9**, 307 (2014).
- [25] GlasStress OÜ, SCALP-05 specifications, last viewed online on 10.12.2024 at: <https://www.glasstress.com/web/scalp-05/>.