

Periodic transient beam loading effect in triple radio-frequency systems

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For fourth-generation synchrotron light sources, the triple rf system has been proposed to achieve further bunch lengthening compared to the commonly used double rf system, as well as to meet specific requirements for longitudinal injection. In a previous study for the double rf system [T. He *et al.*, Periodic transient beam loading effect with passive harmonic cavities in electron storage rings, *Phys. Rev. Accel. Beams* **25**, 024401 (2022)], it was shown that the periodic transient beam loading (PTBL) effect, also referred to as mode 1 instability, could limit the maximum bunch lengthening. Intuitively, in the triple rf system, the harmonic cavity (HC) operating at a higher harmonic of the fundamental rf frequency is expected to help mitigate the mode 1 instability driven by the lower-order HC, as the two HCs are tuned in opposite directions. However, contrary to this intuition, both tracking simulations and semianalytical calculations presented in this paper show that the two HCs of the triple rf system together actually strengthen the PTBL effect. To better understand this unexpected behavior, we introduce a novel concept of steady-state response matrix of cavity voltage, which simplifies the semianalytical algorithm and directly links cavity voltage perturbations to the complex bunch form factor perturbations. Additionally, we propose a modified semianalytical method that efficiently and accurately determines the PTBL threshold for the triple rf system. Using the parameters of the Hefei Advanced Light Facility storage ring as a case study, we apply this modified method to explore the relationship between the threshold R/Q values of the two HCs under different bunch lengthening scenarios. All results are well validated through tracking simulations. Our findings show that the R/Q values of the two HCs must be sufficiently low to avoid the PTBL effect under the desired optimum bunch lengthening condition. This result provides valuable guidance for the design of the triple rf system.

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I. INTRODUCTION

The fourth-generation synchrotron light sources (4GLS), which aim to achieve ultralow emittance, are inevitably troubled by Touschek scattering and intrabeam scattering effects [1]. The former effect limits the beam lifetime, and the latter causes significant beam emittance growth, especially in low energy rings. To effectively mitigate the aforementioned two effects, the double rf system, consisting of main cavities and harmonic cavities (HCs), is commonly used to stretch the bunch length [2]. In addition, bunch lengthening with HCs can also offer significant benefits in the reduction of parasitic energy loss that leads to heating of vacuum components, and the enhancement of Landau damping that fights against beam instabilities [3].

In general, the more the bunch is lengthened, the more significant these benefits become.

Recently, the triple rf system, which incorporates an additional frequency HC compared to the double rf system, has been proposed to achieve further bunch lengthening. Theoretically, under the optimum bunch lengthening condition [4], the triple rf system can stretch the bunch length by a factor of close to 9, effectively doubling the bunch lengthening capability of the double rf system. A detailed theoretical comparison for the case of NSLS-II-U can be found in Ref. [5]. Experimentally, MAX-IV has integrated a fifth HC into the existing double rf system and plans to test the bunch lengthening capability of the triple rf system in the beginning of 2025 [6].

In addition to improving bunch lengthening, the triple rf system can also better serve on axis longitudinal injection with a small dynamic aperture [7–11]. Compared with earlier longitudinal injection schemes based on a single [7] or double rf system [8], the triple rf system ensures a larger temporal separation between the injected beam and the circulating beam, greatly reducing the technical challenge for the short-pulse kicker. This makes it a promising injection solution for HEPS [9] and SAPS [10,11].

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However, it is important to note that the previous studies on the triple rf system were typically conducted using ideal cavity voltages and analytical formulas, ignoring beam loading and potential instabilities—issues that could significantly impact the practical implementation of the triple rf system.

Actually, for the double rf system, it has been found that beam instabilities driven by the HC's fundamental impedance are the primary limiting factors for maximum bunch lengthening, e.g., dipole-quadrupole coupling Robinson instability [12,13], D-mode Robinson instability related to passive superconducting HCs [14,15], and mode 1 instability [16]. This paper focuses on the periodic transient beam loading (PTBL) effect, which is also referred to as a special type of mode 1 instability with a very low oscillation frequency. Venturini first demonstrated using ALS-U parameters that the imaginary part of the HC's fundamental impedance drives a strong mode 1 instability with a low oscillation frequency [16]. Subsequently, one of the current authors comprehensively studied the PTBL effect for the Hefei Advanced Light Facility (HALF) storage ring employed a passive superconducting HC, using both semianalytical calculations and tracking simulations [17]. It was found that when this instability occurs at levels somewhat higher than the threshold, the beam often exhibits a large transient distribution similar to that observed in the case of long-gap filling patterns. This transient distribution undergoes extremely slow periodic variations, which can be regarded as a quasisteady state. As such, referring to this behavior as mode 1 instability is inappropriate, and we will continue to use the term PTBL effect to describe this beam dynamics. Following this, the PTBL effect was observed in simulations for Diamond-II [18] and SOLEIL-II [19], as well as in experiments at MAX-IV [20,21].

Based on experiences with the double rf system, it is anticipated that the PTBL effect could pose a critical limitation for the triple rf system. Intuitively for the triple rf system, it is in general believed that the higher-order HC with a detuning direction opposite to that of the lower-order HC would help suppress the PTBL effect. However, we demonstrate in this paper that this intuition is incorrect: both HCs work together to enhance the PTBL effect. A novel concept of steady-state response matrix of cavity voltage is proposed for the first time to simplify the semianalytical algorithm and helps us better understand this abnormal PTBL effect. Additionally, we present a modified semianalytical method that allows for efficient and accurate determination of the PTBL threshold. Using HALF as a case study, we conduct a detailed investigation of the PTBL effect through both tracking and modified semianalytical methods. Our results show that sufficiently low R/Q values of the two HCs are necessary to achieve

the optimum lengthening condition at nominal beam current without being adversely affected by the PTBL effect.

The rest of the paper is organized as follows: In Sec. II, we present analytical formulas to calculate the rf parameters for different symmetric bunch lengthening scenarios in the triple rf system and describe in detail the self-consistent semianalytical and tracking methods used in this study. Section III presents the PTBL behavior in the triple rf system observed using the aforementioned two methods, and provides a corresponding theoretical explanation focused on the effects of bunch centroid and length perturbations on the real-part voltage perturbations of the two HCs. In Sec. IV, we introduce the modified semianalytical method, which enables rapid determination of the PTBL threshold. Section V explores the relationship between the threshold R/Q values of the two HCs in different bunch lengthening scenarios. Finally, conclusions and outlooks are presented in Sec. VI.

II. THEORY AND METHODS

A. Bunch lengthening conditions for the triple rf system

1. Optimum bunch lengthening

For a triple rf system, the total cavity voltage can be expressed as

$$V_T = V_{mc} \sin(\omega_{rf}t + \phi_s) + k_1 V_{mc} \sin(n_1 \omega_{rf}t + \phi_{hc1}) + k_2 V_{mc} \sin(n_2 \omega_{rf}t + \phi_{hc2}), \quad (1)$$

where V_{mc} and ϕ_s are the voltage amplitude and phase of the main cavity (MC), ω_{rf} is the angular rf frequency, k_1 and k_2 are the voltage ratios between the two HCs and the MC, ϕ_{hc1} and ϕ_{hc2} are the voltage phases of the two HCs, n_1 and n_2 denote the order of the two HCs, and the subscripts “1” and “2” represent the HCs with a lower and higher order, respectively.

In order to achieve the so-called optimum bunch lengthening condition, the first to fourth derivatives of the total cavity voltage at the synchronous phase should be set to 0 [5]:

$$\begin{aligned} \sin \phi_s + k_1 \sin \phi_{hc1} + k_2 \sin \phi_{hc2} &= U_0 / eV_{mc} \\ \cos \phi_s + k_1 n_1 \cos \phi_{hc1} + k_2 n_2 \cos \phi_{hc2} &= 0 \\ \sin \phi_s + k_1 n_1^2 \sin \phi_{hc1} + k_2 n_2^2 \sin \phi_{hc2} &= 0 \\ \cos \phi_s + k_1 n_1^3 \cos \phi_{hc1} + k_2 n_2^3 \cos \phi_{hc2} &= 0 \\ \sin \phi_s + k_1 n_1^4 \sin \phi_{hc1} + k_2 n_2^4 \sin \phi_{hc2} &= 0 \end{aligned} \quad , \quad (2)$$

where the first-row subequation represents the energy gained from the rf system should fully compensate for the radiation loss of the synchronous particles, and U_0 is the radiation energy loss per turn.

Solving for ϕ_s , k_1 , k_2 , ϕ_{hc1} , and ϕ_{hc2} , we obtain [5]:

$$\begin{aligned}\sin \phi_s &= \frac{n_1^2 n_2^2}{a_{n_1 n_2}} \sin \phi_{s0} \\ k_1 &= -\frac{1}{n_1} \sqrt{\frac{a_{n_1 n_2} (n_2^2 - 1) - n_1^2 n_2^4 \sin^2 \phi_{s0}}{(n_1^2 - 1)(n_1^2 - n_2^2)^2}} \\ k_2 &= -\frac{1}{n_2} \sqrt{\frac{a_{n_1 n_2} (n_1^2 - 1) - n_2^2 n_1^4 \sin^2 \phi_{s0}}{(n_2^2 - 1)(n_2^2 - n_1^2)^2}}, \quad (3) \\ \tan \phi_{hc1} &= \frac{\text{sgn}(n_1 - n_2) n_1 n_2^2 \sin \phi_{s0}}{\sqrt{a_{n_1 n_2}^2 - n_1^4 n_2^4 \sin^2 \phi_{s0}}} \\ \tan \phi_{hc2} &= \frac{\text{sgn}(n_1 - n_2) n_1^2 n_2 \sin \phi_{s0}}{\sqrt{a_{n_1 n_2}^2 - n_1^4 n_2^4 \sin^2 \phi_{s0}}}\end{aligned}$$

where $a_{n_1 n_2} = (n_1^2 - 1)(n_2^2 - 1)$ and $\sin \phi_{s0} = U_0 / eV_{mc}$.

Once we obtain the cavity voltages and phases of the triple rf system, the total longitudinal rf potential is also obtained as

$$\begin{aligned}\Phi(\tau) &= \frac{eV_{mc}}{T_0 E_0 \omega_{rf}} \left\{ \cos \phi_s - \cos(\omega_{rf} \tau + \phi_s) \right. \\ &\quad + \frac{k_1}{n_1} [\cos \phi_{hc1} - \cos(n_1 \omega_{rf} \tau + \phi_{hc1})] \\ &\quad + \frac{k_2}{n_2} [\cos \phi_{hc2} - \cos(n_2 \omega_{rf} \tau + \phi_{hc2})] \\ &\quad \left. - \omega_{rf} \tau \sin \phi_{s0} \right\}. \quad (4)\end{aligned}$$

Then the longitudinal bunch density distribution can be written as [22]:

$$\rho(\tau) = \rho_0 \exp \left[\frac{\Phi(\tau)}{\alpha_c \sigma_\delta^2} \right], \quad (5)$$

where ρ_0 is the normalized constant, α_c is the momentum compaction factor, and σ_δ is the natural energy spread.

Through Eqs. (3)–(5), we can obtain the optimum bunch lengthening distribution for the triple rf system.

2. Symmetric over- and less-optimum bunch lengthening

Besides the aforementioned optimum bunch lengthening, the over- and less-optimum lengthening are also frequently discussed. Sometimes, the overoptimum lengthening is desired to further improve the Touschek beam lifetime. However, due to the limitations imposed by beam instabilities, either the overoptimum or even the optimum lengthening may not be achievable, in which case the less-optimum lengthening must be considered. For the convenience of discussion, we will confine ourselves to the symmetric bunch lengthening distribution.

Actually, we can achieve a symmetric over- or less-optimum lengthening by slightly regulating the cavity voltage and phase of the lower-order HC from the optimum lengthening condition. To do this, the value of $V_{hc1} \sin \phi_{hc1}$ should be kept constant so that the synchronous phase remains unaffected. That is, for a minor variation of voltage ΔV_{hc1} and phase $\Delta \phi_{hc1}$, to maintain the symmetric bunch distribution, we should have:

$$\frac{\Delta V_{hc1}}{V_{hc1}^{\text{opt}}} + \frac{\Delta \phi_{hc1}}{\tan \phi_{hc1}^{\text{opt}}} = 0, \quad (6)$$

where V_{hc1}^{opt} and ϕ_{hc1}^{opt} denote the voltage amplitude and phase of the lower-order HC under the optimum bunch lengthening condition. To facilitate subsequent discussions, we introduce two relative coefficients: $k_V = 1 + \Delta V_{hc1} / V_{hc1}^{\text{opt}}$ and $k_\phi = 1 + \Delta \phi_{hc1} / \phi_{hc1}^{\text{opt}}$. Notice that $k_V > 1$ means the HC voltage is increased with respect to the optimum lengthening condition. Substituting these coefficients into Eq. (6), we obtain:

$$(1 - k_V) + (1 - k_\phi) \frac{\phi_{hc1}^{\text{opt}}}{\tan \phi_{hc1}^{\text{opt}}} = 0. \quad (7)$$

It is generally true that $|\phi_{hc1}^{\text{opt}}| < 10^\circ$, which implies $\tan \phi_{hc1}^{\text{opt}} \approx \phi_{hc1}^{\text{opt}}$. Therefore, we can approximate $k_V + k_\phi \approx 2$. Keeping this relationship, it is trivial to achieve a symmetric over- or less-optimum lengthening distribution. In the following, we will demonstrate this using the parameters of the HALF storage ring.

3. Example of HALF

HALF, a fourth-generation 2.2 GeV synchrotron light source, is being built in Hefei, China. Table I shows the main parameters of the HALF storage ring relevant to this

TABLE I. Main parameters of HALF.

Parameter	Symbol	Value
Ring circumference	C	480 m
Beam energy	E_0	2.2 GeV
Nominal beam current	I_0	350 mA
Longitudinal damping time	τ_z	14 ms
Momentum compaction	α_c	9.4×10^{-5}
Natural energy spread	σ_δ	7.4×10^{-4}
Harmonic number	h	800
Energy loss per turn	U_0	400 keV
Voltage of MC	V_{mc}	1.2 MeV
Natural rms bunch length	σ_τ	7 ps
Harmonic number of 3HC	n_1	3
Harmonic number of 5HC	n_2	5
Quality factor of 3HC	Q_1	1.68×10^5
Quality factor of 5HC	Q_2	1×10^5
R/Q of 3HC	R_1/Q_1	40 Ω
R/Q of 5HC	R_2/Q_2	40 Ω

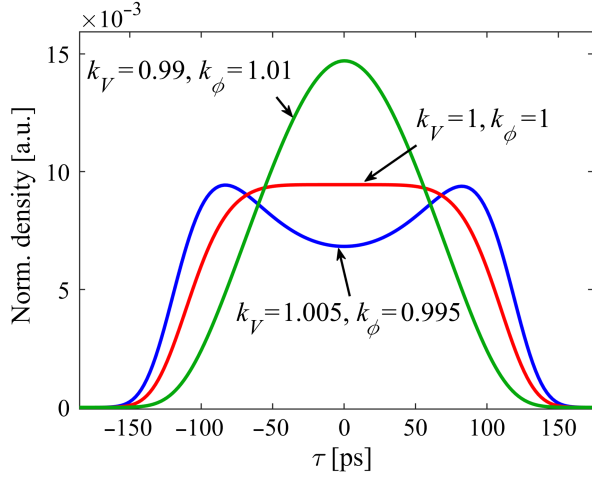


FIG. 1. Three types of symmetrical bunch distributions obtained by slightly adjusting the values of k_V and k_ϕ while keeping $k_V + k_\phi = 2$.

study. In addition to the basic parameters generated from the lattice design [23], the characteristic parameters of the two HCs of order 3 and 5, based on an assumed triple rf system, are also provided. It should be noted that the two HCs are assumed to work in active mode, with their voltages induced by both the generator and beam current. For the active triple rf system, ideally, the specific bunch lengthening condition can be satisfied by easily regulating the cavity voltages.

It should also be noted that the present rf system adopted by the HALF storage ring is a superconducting double rf system, including one HC of order 3 [24,25]. As a result, this study mainly focuses on the superconducting triple rf system, which includes the two HCs of order 3 and 5 (referred to as 3HC and 5HC in the following). As shown in Table I, both the R/Q values of the two HCs are set to $40\ \Omega$, and the loaded quality factors are set at the level of 1×10^5 . These settings are consistent with the actual characteristics held by active superconducting cavities.

Figure 1 shows the resulting bunch density distributions, which include three types. The red curve corresponds to the optimum bunch lengthening, with an rms length of 64.5 ps. The blue curve represents a symmetric, overoptimum bunch lengthening, with an rms length of 74 ps, obtained by setting “ $k_V = 1.005$ and $k_\phi = 0.995$.” The green curve denotes a symmetric, less-optimum lengthening, with an rms length of 50 ps, obtained by setting “ $k_V = 0.99$ and $k_\phi = 1.01$.” Note that to achieve the symmetrical distribution, we maintain “ $k_V + k_\phi = 2$.”

It should be noted that the above discussions only consider the impact of rf cavity voltages on bunch lengthening. While at optimum lengthening, short-range wakefields can cause significant potential-well distortion, resulting in asymmetric distributions. Nonetheless, in that

case, it is still possible to restore the bunch distribution to a symmetric state by finely adjusting the voltage amplitude and phase of the lower-order HC.

B. Self-consistent iterative algorithm based on steady-state response matrix of rf cavity voltage

For the triple rf system operating at currents on the order of hundreds of mA, the cavity voltage cannot be simply expressed as in Eq. (1), as the impact of beam loading must be considered. In particular, for nonuniform filling, the beam loading transients lead to different cavity voltages seen by different bunches [26]. Therefore, the beam equilibrium distribution must be solved in a self-consistent manner. Several self-consistent algorithms are currently available [27–32]. These algorithms account for the interplay between the bunch distribution and cavity voltage, typically requiring iterative computations to converge on the bunch distribution.

In this study, we follow the self-consistent algorithm framework proposed in Ref. [28], which was initially developed for the double rf system. We have extended this framework by adding a second HC voltage calculation module, enabling it to be used for the triple rf system. We also handle the beam-cavity interaction by distinguishing between short-range and long-range effects. The short-range effects are integrated into the lumped short-range wakefields, while the long-range effects are addressed in matrix form.

A significant improvement over Ref. [28] in this paper is that we find the matrix representing long-range effects can be expressed as the product of a steady-state response matrix (SSRM) of cavity voltage (excited by unit point charges) and an array of factors containing information about the bunch form factors and charges, i.e.,

$$\tilde{V}_b = \tilde{F} R_V, \quad (8)$$

where $\tilde{V}_b$ is an array with h elements arranged in a row, representing the steady-state beam loading voltages seen by each bunch. h is the harmonic number (or the number of buckets). $\tilde{F} = (q_1 \tilde{F}_1, q_2 \tilde{F}_2, \dots, q_h \tilde{F}_h)$, with q_i being the bunch charge, $\tilde{F}_i = \int_{-\infty}^{\infty} \rho_i(\tau) e^{-j\omega_r \tau} d\tau$ being the complex bunch form factor. For the empty buckets, the corresponding charge values are treated as 0. R_V represents the SSRM, a matrix composed of $h \times h$ elements:

$$R_V = \begin{bmatrix} V_0 e^{h\phi_b} & V_0 e^{\phi_b} & \dots & V_0 e^{(h-1)\phi_b} \\ V_0 e^{(h-1)\phi_b} & V_0 e^{h\phi_b} & \dots & V_0 e^{(h-2)\phi_b} \\ \vdots & \vdots & \ddots & \vdots \\ V_0 e^{\phi_b} & V_0 e^{2\phi_b} & \dots & V_0 e^{h\phi_b} \end{bmatrix}, \quad (9)$$

where $V_0 = \frac{\omega_r R_L / Q_L}{1 - \exp[(j - \frac{1}{2Q_L})\omega_r T_0]}$ is the steady-state cavity voltage induced by a unit point charge, with R_L , Q_L ,

and ω_r being the cavity loaded impedance, loaded quality factor, and resonant angular frequency, respectively, and T_0 being the revolution time. In addition, $\phi_b = (j - \frac{1}{2Q_L})\omega_r T_b$, with $T_b = T_0/h$ being the duration between two adjacent buckets.

Looking at Eq. (9), we see that R_V is a circulant matrix, where each row is a cyclic shift of the row above it. The first row of R_V represents the steady-state voltage induced by a point bunch with unit charge, as seen sequentially by each bunch. This row can be analytically determined by utilizing the attenuation and rotation characteristics of voltage phasors [33]. In addition to the analytical expression, the first row can also be obtained from a simple, discrete equation for cavity voltage that excludes the generator current [34]. In cases such as the complex ARES system, composed of three types of coupled cavities [34], while no analytical formula exists for the steady-state cavity voltage, an accurate, discrete equation can still be used [34]. Once the first row of R_V is known, the entire matrix can be easily created.

This raises the question: why can we construct an R_V to solve for the rf cavity steady-state voltage? The answer lies in the fact that the rf cavity can be treated as a linear, time-invariant system [35], and the filled bunches can be modeled as periodic impulses in a fixed time sequence. Due to the system's linearity, the steady-state voltage response follows the impulse of the filled bunches, with consistent relative changes in amplitude and phase, and the responses from different impulses can be superimposed.

Utilizing Eq. (8), we optimize the self-consistent algorithm. Figure 2 shows the schematic of the iterative calculation for bunch equilibrium distributions. During the iterative process, R_V , which depends solely on the rf cavity parameters, remains constant. As a result, the calculation for updating the cavity voltage matrix can be eliminated, significantly improving algorithm efficiency.

It is well known that the cavity voltage will deviate from its initial set value due to changes in beam loading during

the iteration process. To address this, we have incorporated a module to regulate the generator-induced voltage, ensuring the average cavity voltage remains at the set value. Similar to Ref. [28], we use a small coefficient to weight the bunch distribution for the next round of calculations:

$$\rho'_{\text{new}} = r\rho_{\text{new}} + (1-r)\rho_{\text{old}}, \quad (10)$$

where ρ_{new} denotes the bunch distribution derived directly from the total cavity voltages, r is a small coefficient generally less than 1, and ρ_{old} represents the input of bunch distribution at this current round.

When applying this algorithm to the triple rf system, we found that a relatively small value of r , e.g., $r \leq 0.02$, is necessary to ensure good convergence of the bunch distributions. In the calculations presented in this paper, r is set to 0.01 unless otherwise specified. For comparison, in the case of the double rf system, r is typically set as a random number between 0 and 0.5, which is sufficient to obtain good convergence.

C. Macroparticle tracking

We use the STABLE code to perform tracking simulations for the triple rf system. STABLE is a multibunch, multi-particle tracking code specifically developed for studying the longitudinal beam dynamics in the double rf system. One of its key advantages is its high computational efficiency, which is achieved through GPU-accelerated algorithms [36]. Recently, a proportional-integral (PI) feedback module has been incorporated into the code to model the low-level rf feedback in active cavities [15]. To adapt STABLE for the triple rf system, a second HC calculation module was implemented. Figure 3 shows the schematic of the triple rf system with PI feedback as modeled in the STABLE tracking code. During tracking, the rf cavity can be set either as an ideal cavity, neglecting the beam loading effect, or as an actual cavity with PI feedback to stabilize the beam-loaded cavity voltage.

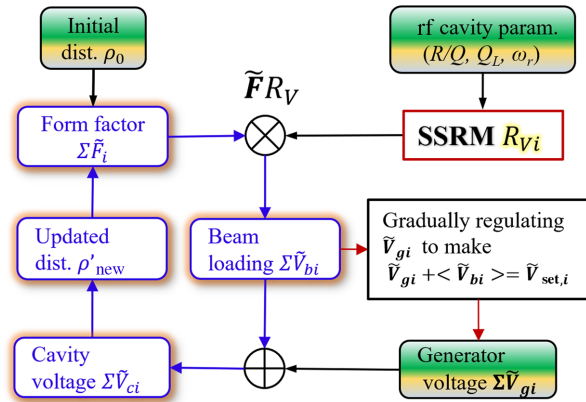


FIG. 2. Schematic of the iterative calculation for steady-state bunch distributions for the triple rf system. $\tilde{V}_{gi}$ represents the generator-induced voltage, $\tilde{V}_{bi}$ is the beam loading voltage, $\tilde{V}_{ci}$ is the total cavity voltage, and $\tilde{V}_{\text{set},i}$ means the set cavity voltage.

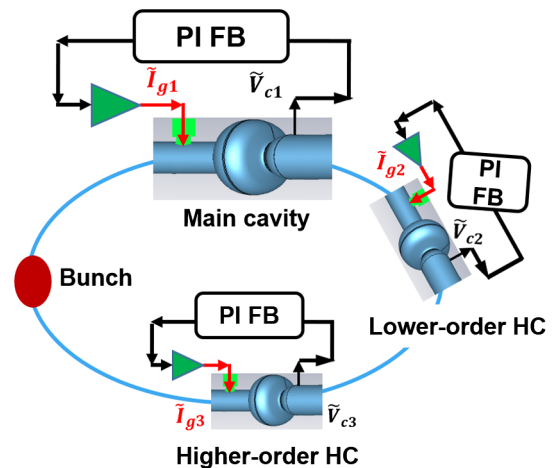


FIG. 3. Schematic of the triple rf system with PI feedback.

In the simulations presented in this paper, we focus on a uniform filling pattern. Each bunch is represented by 10,000 macroparticles, and the simulation is typically run for 300,000 turns. To minimize generator power consumption during routine operation, we set the detuning of the lower-order HC to make the theoretical beam loading voltage as close as possible to the target cavity voltage. To ensure the beam loading voltage aligns with the target under a given beam current, the corresponding loaded quality factor is adjusted appropriately, typically ranging from 1×10^5 to 5×10^6 . This range is feasible for superconducting rf cavities. For the higher-order HC, we primarily adjust its detuning by setting the corresponding loading angle to 0. Generally, the MC has little effect on the PTBL effect studied in this paper, so it is treated as ideal in the simulations to highlight the influence of the two HCs. More details on configuring rf cavity parameters are presented in the Appendix.

Regarding the PI feedback, improper settings may lead to undesired beam instabilities. To avoid this, we set the proportional gain to 1, while the integral gain is adjusted according to the R/Q value, typically in the range of 0.01 to 0.1. Additionally, the loop delay is set to the time interval of 800 buckets, approximately 1.6 μ s.

III. PERIODIC TRANSIENT BEAM LOADING EFFECT

Similarly to Ref. [17], we perform the aforementioned self-consistent semianalytical calculations and tracking simulations using the parameters given in Table I to investigate the PTBL effect in the triple rf system.

A. Unexpected PTBL behavior

To demonstrate the PTBL effect driven by the two HCs with harmonic orders 3 and 5, we first consider the optimum bunch lengthening condition for the following three cases: (i) actual 3HC and ideal 5HC, (ii) ideal 3HC and actual 5HC, and (iii) both actual 3HC and 5HC. Here, “actual” refers to the inclusion of beam loading, while “ideal” denotes the neglect of beam loading.

Figure 4 shows the resulting distributions of bunch length and centroid obtained at the end of the self-consistent iterative calculations. We set the number of iterations to 10,000 to ensure full convergence of the resulting distributions. For the sake of comparison, the distributions are manually shifted so that the maximum bunch length occurs at bunch number 400.

As shown in Fig. 4, the resulting distributions in all three cases exhibit significant transient behavior, similar to the PTBL effect observed in the double rf system in Ref. [17]. Specifically, each bunch near bunch number 400 is stretched into two far-separated sub-bunches, while most other bunches experience less stretching. In the first two cases, we find that considering either the beam loading of 3HC or 5HC individually leads to a pronounced PTBL effect. Moreover, when the beam loading of the two HCs is

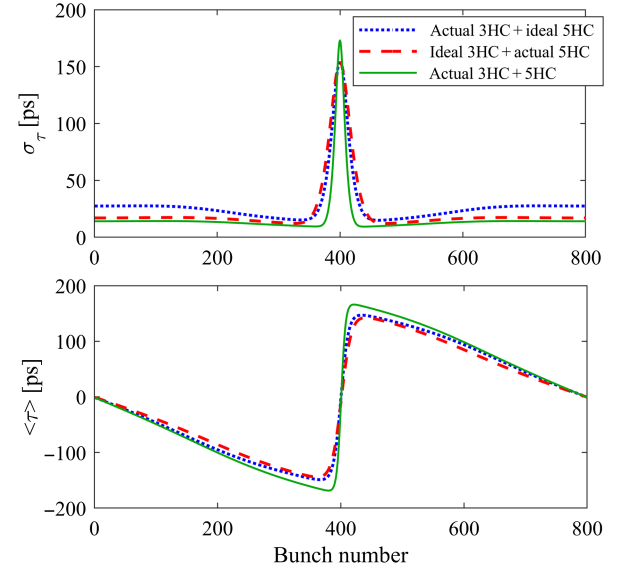


FIG. 4. Bunch-by-bunch centroids (bottom) and rms lengths (top) for 800 bunches at a beam current of 350 mA, obtained at the end of self-consistent iterative calculations, including the three cases of “actual 3HC + ideal 5HC,” “ideal 3HC + actual 5HC,” and “actual 3HC + 5HC.” During the iteration process, the generator-induced voltage is gradually regulated to maintain the cavity voltage at the initial set value.

considered simultaneously, the PTBL effect becomes stronger, with a more severe transient distribution. This observation suggests that the two HCs drive the PTBL in the same direction, despite being detuned in opposite directions, which contrasts with the intuitive expectation that the 5HC should help suppress the PTBL caused by the 3HC.

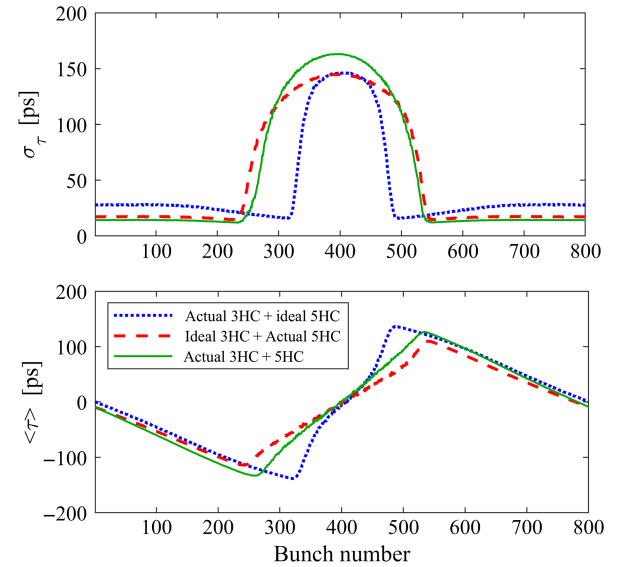


FIG. 5. Bunch-by-bunch centroids (bottom) and rms lengths (top) for 800 bunches at a beam current of 350 mA, obtained at the end of tracking simulation. The legends show the three cases concerned.

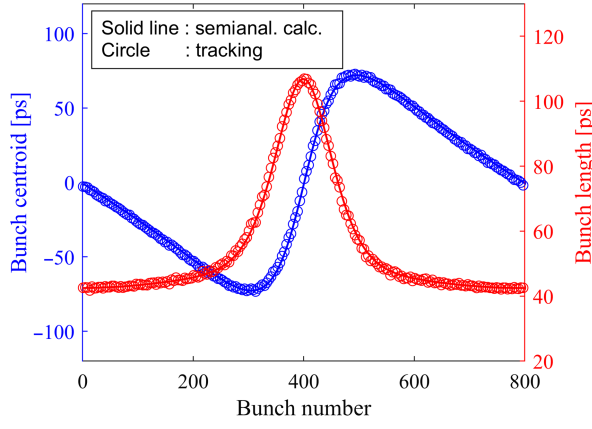


FIG. 6. Bunch-by-bunch centroids (blue) and rms lengths (red) for 800 bunches at a beam current of 55 mA, obtained at the end of tracking simulation (circle mark) and self-consistent iterative calculations (solid line). Beam loading effects are considered for the two HCs.

To validate the results shown in Fig. 4, we also performed tracking simulations for the three cases. Figure 5 shows the resulting distributions of bunch length and centroid. A comparison of these distributions confirms that the two HCs together enhance the PTBL effect, supporting the findings from the semianalytical calculations.

It is worth noting that there are noticeable discrepancies between the distributions obtained from the semianalytical calculations and the tracking simulations for all three cases. We suspect that the tracking results are not fully convergent, as even after 200,000 turns, the centroid and length of some bunches continue to change slowly. However, increasing the tracking turns to 5×10^6 shows that the resulting distributions still change slightly. When comparing the bunch density distributions at the bunch numbers with significant discrepancies, we observe that these bunches are typically overstretched and split into two far-separated sub-bunches. It is reasonable to infer that when a bunch splits rapidly into two sub-bunches due to the strong PTBL effect, the particles within these sub-bunches may not easily merge in subsequent tracking turns. To confirm this, we reduced the beam current to 55 mA to alleviate the PTBL effect, and observed that the results from both methods agree well, as shown in Fig. 6. This confirms that the discrepancies are due to the excessively strong PTBL effect at higher beam currents.

In addition to comparing the saturation transients, which suggest that considering the two HCs simultaneously strengthens the PTBL effect, we also analyze the growth rates of mode $l = 1$ in the three cases. Given the rapid growth of PTBL, it is challenging to obtain accurate growth rates directly from tracking data of the bunch centroids. To address this, we first transform the bunch coordinates into modal coordinates [37], and then compute the corresponding modulus, as shown in Fig. 7. Fitting the modulus data during the exponential growth stage yields growth rates of

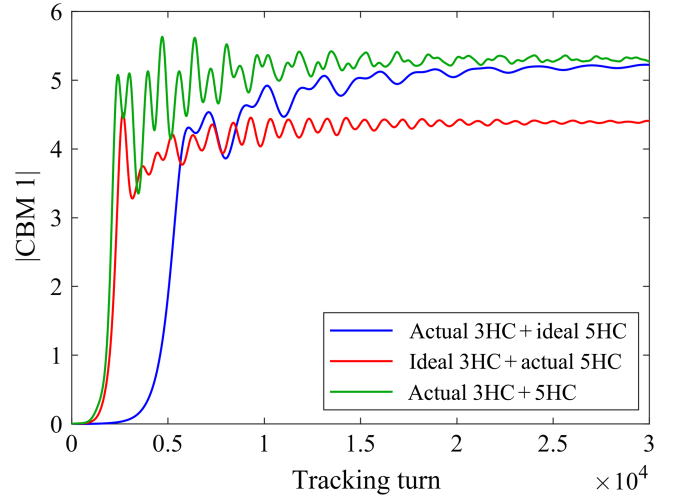


FIG. 7. Modal coordinates (absolute value) of coupled-bunch mode $l = 1$ as a function of the tracking turns. The legends show the three cases concerned. The longitudinal coordinates are computed by $|1/h \sum_{i=0}^{h-1} \langle \tau \rangle_i e^{-j2\pi i/h}|$, where $\langle \tau \rangle_i$ denotes bunch centroid.

1164, 2238, and 2419 s^{-1} for the cases of “actual 3HC + ideal 5HC,” “ideal 3HC + actual 5HC,” and “actual 3HC + 5HC,” respectively.

Overall, we have demonstrated that, in the triple rf system with identical parameter settings, the PTBL effect becomes more severe when beam loading from both the 3HC and 5HC is considered, compared to cases where either of them is taken into account. This finding contradicts conventional expectations. In the following, we will attempt to explain this observation.

B. Theoretical analysis

1. The real-part cavity voltage distribution

During beam tracking, the evolution of bunch distributions is primarily driven by the real-part cavity voltages. Therefore, it is natural to examine the real-part voltage distributions of the two HCs at the initial stage of PTBL onset. If the two distributions are in phase, it indicates that the PTBL effect is enhanced by the two HCs; otherwise, the effect is weakened. Due to the strong PTBL effect at a beam current of 350 mA and R/Q values of 40Ω , significant transients occur within only a few tracking turns, making it difficult to capture the voltage distributions at the initial stage of PTBL. To circumvent this, we investigated the case of 55 mA beam current, as mentioned previously. In this case, we recorded the real-part cavity voltages on a turn-by-turn basis during the tracking simulation.

Figure 8 shows the resulting perturbation distributions of cavity voltages obtained by subtracting the mean voltage. It is evident that the two distributions are indeed in phase, as expected. This confirms that the two HCs contribute to driving the PTBL effect in the same direction, thereby enhancing the PTBL effect.

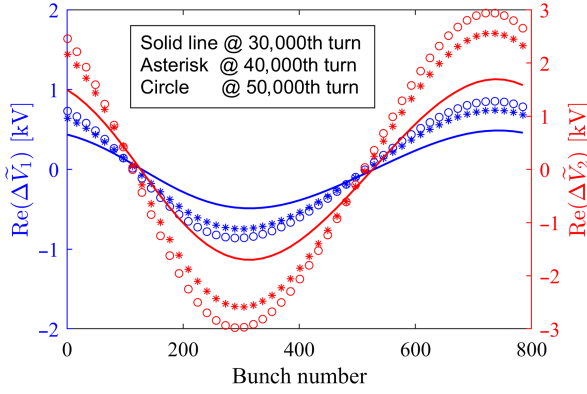


FIG. 8. Real-part cavity voltage perturbation distributions of 3HC (blue) and 5HC (red) at the initial stage of PTBL, obtained by tracking simulations. The legend shows the corresponding number of tracking turns.

A natural question arises: why are the real-part voltage distributions of the two HCs in phase during the onset of PTBL? To address this, we introduce a novel perturbation model to compute the cavity voltage distributions, as described below.

2. Perturbation model of complex bunch form factors

In order to estimate the PTBL threshold in the double rf system, He proposed a steady-state time-domain perturbation model that accounts only for the bunch centroid perturbation [38]. Based on this model, an analytical formula for calculating the threshold detuning of the HC was derived, which showed good agreement with tracking simulations for relatively large R/Q values of superconducting HCs. However, the model does not perform as well for smaller R/Q values, which can reasonably be attributed to the omission of bunch length perturbation. In such cases, where the bunch length is close to the optimum lengthening value, it becomes highly sensitive to centroid perturbations.

In this study, we initially applied this perturbation model and obtained the real-part cavity voltage distributions of the two HCs, which showed an antiphase relationship. These results prompted us to re-evaluate the perturbation model for analyzing the PTBL effect in the triple rf system. We also noted that other researchers have pointed out the shortcomings of considering only the bunch centroid perturbations [21] and highlighted the necessity of including bunch length perturbations when accurately determining the PTBL threshold for the double rf system [39–42]. By incorporating the impact of the azimuthal mode $m = 2$, the PTBL threshold results calculated using the analytical method based on the Vlasov equation are more consistent with tracking simulations [39] and match well with experimental measurements [40]. Ultimately, we concluded that a more accurate model must account for both bunch centroid and length perturbations simultaneously.

The perturbation in the cavity voltage distribution arises from the perturbation in the bunch distributions.

According to Eq. (8), the beam loading voltage is completely determined by two terms: SSRM and the bunch form factors. SSRM is treated as a constant matrix, while the bunch form factors are derived from the bunch distributions. For a Gaussian bunch, the bunch form factor can be expressed as

$$\tilde{F} = e^{-j\omega_r \langle \tau \rangle} \int_{-\infty}^{\infty} \rho(t + \langle \tau \rangle) e^{-j\omega_r t} dt = e^{-j\omega_r \langle \tau \rangle} e^{-\frac{\omega_r^2 \sigma_\tau^2}{2}}, \quad (11)$$

where $\langle \tau \rangle$ is the bunch centroid, and σ_τ is the bunch rms length. For a symmetric bunch, $\tilde{F}$ is a real number as $\langle \tau \rangle = 0$. However, under perturbation, the bunch deviates from symmetry, and while $\langle \tau \rangle$ is no longer exactly 0, it remains close to 0. This implies that $\omega_r \langle \tau \rangle$ can be treated as a small value, much less than 1. Therefore, Eq. (11) can be approximated as

$$\begin{aligned} \tilde{F} &\approx (1 - j\omega_r \langle \tau \rangle) e^{-\frac{\omega_r^2 \sigma_\tau^2}{2}} \\ \frac{\partial \text{Re}(\tilde{F})}{\partial \sigma_\tau} &\approx -\omega_r^2 \sigma_\tau e^{-\frac{\omega_r^2 \sigma_\tau^2}{2}}. \end{aligned} \quad (12)$$

This approximation holds well for a symmetric Gaussian distribution. For the optimum bunch lengthening distribution, as shown in Fig. 1, we used both Eq. (12) and the exact formula to compute the $\tilde{F}$ values. The maximum relative error was found to be less than 6%. From Eq. (12), we observe that the real part of $\tilde{F}$ is influenced by bunch length, while the imaginary part is affected by both bunch centroid and length. For the distribution of $\tilde{F}$ across all bunches, the real-part perturbations are primarily driven by bunch length perturbations, while the imaginary-part perturbations are mainly driven by centroid perturbations.

To directly observe the influence of bunch distributions on the $\tilde{F}$ distribution, we performed self-consistent calculations for the final case described in Sec. III A. At the initial stage of PTBL onset, we recorded the bunch density distributions, which were subsequently used to compute the perturbations of bunch length and centroid, as well as the real- and imaginary-part perturbations of the bunch form factor. These results are shown in Fig. 9. We observe a phase difference of π between the perturbation distribution of the real part (and imaginary part) of $\tilde{F}$ and the perturbations of bunch rms length (and centroid). Additionally, there is a phase advance of $\pi/2$ between the perturbations of bunch length and centroid, and between the real- and imaginary-part perturbations of $\tilde{F}$. Notably, the perturbation amplitudes are of similar magnitude. Under the same bunch perturbations, the $\tilde{F}$ of 5HC shows a larger perturbation amplitude than that of 3HC, but both distributions are in phase.

Based on the observations and analyses above, we will now investigate further the impact of the real- and imaginary-part $\tilde{F}$ perturbations on the real-part cavity voltage distribution.

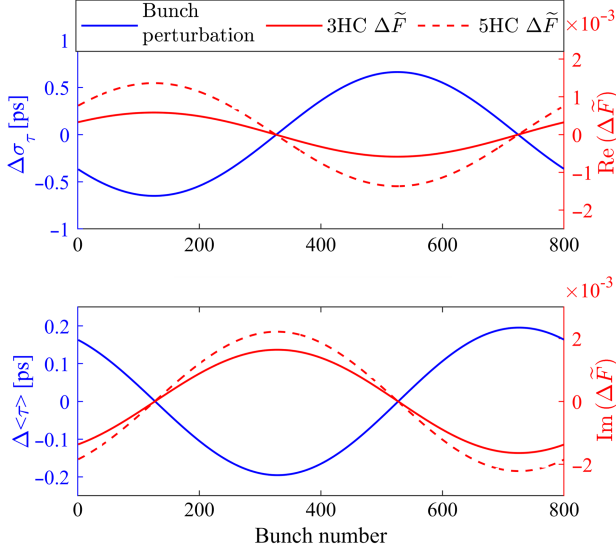


FIG. 9. Perturbation distributions of bunch moments including rms length (top blue) and centroid (bottom blue), and perturbation distributions of bunch form factor including the real part (top red) and the imaginary part (bottom red). $\Delta\tilde{F}$ is obtained by subtracting the average with $\tilde{F}$, and so are $\Delta\sigma_\tau$ and $\Delta\langle\tau\rangle$. These results are obtained based on the setting of $R_1/Q_1 = 40\ \Omega$ and $R_2/Q_2 = 40\ \Omega$ under optimum lengthening condition.

3. The effect of $\tilde{F}$ perturbation on the real-part cavity voltage perturbation

Given the complex form factor perturbation $\Delta\tilde{F}$ in the two HCs, we can calculate the corresponding beam loading voltage perturbation $\Delta\tilde{V}_b$ using Eq. (8):

$$\Delta\tilde{V}_b = \Delta\tilde{F}R_V. \quad (13)$$

Note that $\Delta\tilde{V}_b$ is equivalent to the cavity voltage perturbation, as the generator-induced voltage is constant across all bunches. Using Eq. (13), we can easily distinguish the influence of the real and imaginary parts of $\Delta\tilde{F}$ on the cavity voltage perturbation.

Figure 10 presents the resulting cavity voltage perturbations in the two HCs, obtained by directly substituting the $\Delta\tilde{F}$ data (shown in Fig. 9) into Eq. (13). When considering only the $\text{Re}(\Delta\tilde{F})$, as shown in Fig. 10(a), the real-part cavity voltage perturbations in the two HCs are in phase. The perturbation amplitude in 3HC is approximately one-fifth of that in 5HC. Conversely, when only considering the $\text{Im}(\Delta\tilde{F})$, as shown in Fig. 10(b), the cavity voltage perturbations in the two HCs are in antiphase. The perturbation amplitude in 3HC is about one-thirtieth of that in 5HC. By comparing the results in Figs. 10(a) and 10(b), we observe the following: For 3HC, the cavity voltage perturbations induced by the real- and imaginary-part $\Delta\tilde{F}$ are in phase. For 5HC, the perturbations are in antiphase. The $\text{Re}(\Delta\tilde{F})$ induces a stronger cavity voltage perturbation amplitude, which is a common feature in the two HCs. When

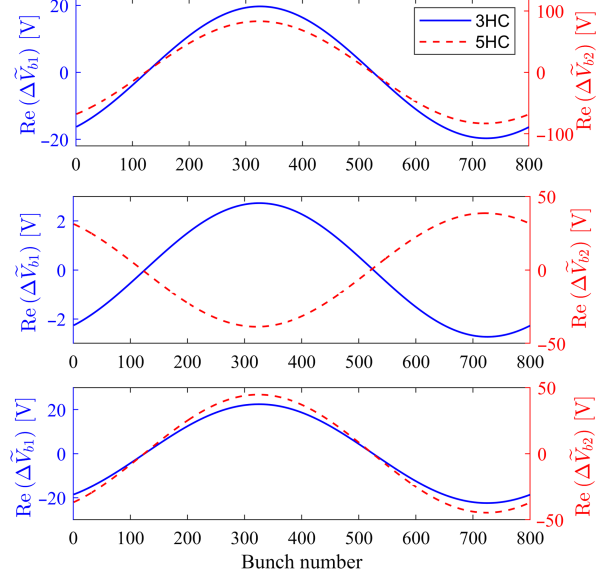


FIG. 10. Real-part perturbation distributions of beam loading voltage in 3HC (blue solid line) and 5HC (red dashed line), obtained with only the real-part perturbation (a), only the imaginary-part perturbation (b), and the full complex perturbation (c) of $\tilde{F}$.

considering the full $\Delta\tilde{F}$, as shown in Fig. 10(c), the perturbations in the two HCs are in phase, with comparable amplitudes. These results clearly indicate that the two HCs jointly strengthen the PTBL effect, as a result of the dominant contribution from the real part of $\Delta\tilde{F}$.

We also found that changing only the detuning direction of the HC, without altering its absolute value, causes only a phase shift of π in the perturbation of $\text{Re}(\Delta\tilde{V}_b)$ induced by the $\text{Im}(\Delta\tilde{F})$. However, this does not affect the perturbation of $\text{Re}(\Delta\tilde{V}_b)$ induced by the $\text{Re}(\Delta\tilde{F})$.

IV. A MODIFIED SEMIANALYTICAL METHOD TO EFFICIENTLY DETERMINE THE PTBL THRESHOLD

As discussed in Sec. III A, the self-consistent iterative calculation can predict the PTBL effect. However, near the PTBL threshold, thousands of iterations are often required to clearly observe the PTBL phenomena. Inspired by the work in Sec. III B, we propose a modified semianalytical method that significantly reduces the number of iterations needed to determine the PTBL threshold and reach PTBL saturation. The core idea is to initially impose a coupled-bunch mode-1 perturbation on the ideal equilibrium bunch distribution and observe how the perturbation evolves in subsequent iterations. If the perturbation amplitude increases with iterations, it indicates that the system is above the PTBL threshold; if the perturbation amplitude decays, it indicates that the system is below the threshold. This modified method is implemented in the following three steps: Step 1: Compute the ideal equilibrium bunch

distribution based on the ideal cavity voltage settings. Step 2: Apply a sinusoidal perturbation (in mode 1 form) to the bunch distribution using the SSRM-based self-consistent algorithm. This perturbation can be added to either the real part or the imaginary part of the complex bunch form factor corresponding to the lower-order HC. In this study, we generally use the real part, and the perturbation is implemented as $\tilde{F}_i = \tilde{F} + \epsilon_0 \sin(2\pi i/h)$, where $\tilde{F}$ here is the complex form factor calculated from the ideal equilibrium distribution obtained in the first step, $i = 1, 2, \dots, h$ is the bunch index, and ϵ_0 is the perturbation amplitude (typically set to 1×10^{-6}). The second step usually takes fewer than 500 iterations. Step 3: Continue the iterations starting from the perturbed distribution. By tracking the changes in perturbation amplitude (either in the complex form factor or the bunch distribution), we can determine whether the PTBL effect is present. If the perturbation amplitude grows, it indicates the system is above the PTBL threshold; if it decays, the system is below the threshold. Typically, 500 iterations are sufficient for this determination.

To demonstrate the effectiveness of the modified semi-analytical method, we first perform tracking simulations to determine the PTBL threshold current, which is found to be 50 mA by scanning the beam current. Two test cases are then selected, with beam currents of 51 mA (slightly above the PTBL threshold) and 49 mA (slightly below the PTBL threshold). We apply the modified semi-analytical method to these cases, following the three steps outlined above. To clearly show the trend in perturbation amplitude changes, we set the third step to 1000 iterations. The resulting normalized perturbation amplitudes as a function of the iteration number are shown in Fig. 11. In the first case (51 mA), the centroid perturbation amplitude continues to increase after 400 iterations, indicating that the system is above the PTBL threshold. Conversely, in the second case

(49 mA), the perturbation amplitude decreases with iterations, indicating that the system is below the PTBL threshold. These results strongly validate the effectiveness of the modified semi-analytical method.

It should be noted that this modified semi-analytical method overcomes the limitations of the analytical approach proposed in Ref. [38] by considering not only the bunch centroid perturbation but also the bunch length perturbation. This improvement enables us to accurately determine the PTBL threshold. Furthermore, in the example case of HALF near the PTBL threshold, a maximum of 1000 iterations (approximately 25 s on our desktop computer) are sufficient to evaluate the PTBL. In comparison, the original method requires the PTBL to grow naturally from the equilibrium distribution, a process that typically demands over 5000 iterations (around 125 s on our desktop computer).

V. THRESHOLD R/Q VALUES OF THE TWO HCS

Section III A shows that the triple rf system, using the parameters listed in Table I, cannot achieve optimum bunch lengthening due to the serious PTBL effect. Based on the lessons learned from studies on the double rf system [17], we know that the PTBL effect is closely related to the R/Q values of the HCs. In order to stretch the bunch length as much as possible at the target of 350 mA, it is necessary to know the threshold R/Q values of the two HCs in different bunch lengthening scenarios, which can provide valuable information to guide the cavity designs. In the following, both the modified semi-analytical method and the tracking method are employed to calculate the threshold R/Q values. This will be achieved by scanning the lower-order HC R/Q value corresponding to each given R/Q value of the higher-order HC. Several impact factors on the threshold R/Q values will be investigated.

A. Impact of bunch lengthening

By adjusting the lower-order HC voltage amplitude and phase, as described in Sec. II A 2, we can achieve different symmetric bunch lengthening scenarios, either less optimum or overoptimum. In this section, we consider three specific bunch lengthening scenarios, corresponding to rms lengths of 64.5, 40, and 30 ps, which are associated with the following parameter settings: “ $k_V = 1, k_\phi = 1$ ” (optimum lengthening), “ $k_V = 0.98, k_\phi = 1.02$ ” (less-optimum lengthening), and “ $k_V = 0.96, k_\phi = 1.04$ ” (less-optimum lengthening). The overoptimum lengthening is not considered here due to the observed strong PTBL effect in the case of optimum lengthening.

Figure 12 shows the resulting threshold R/Q values, obtained from both tracking simulations and the modified semi-analytical method. The results from the two methods are in very good agreement, with an error margin of less than 1 Ω . The threshold curves reveal a negative correlation between the threshold R/Q values of the two HCs: as one

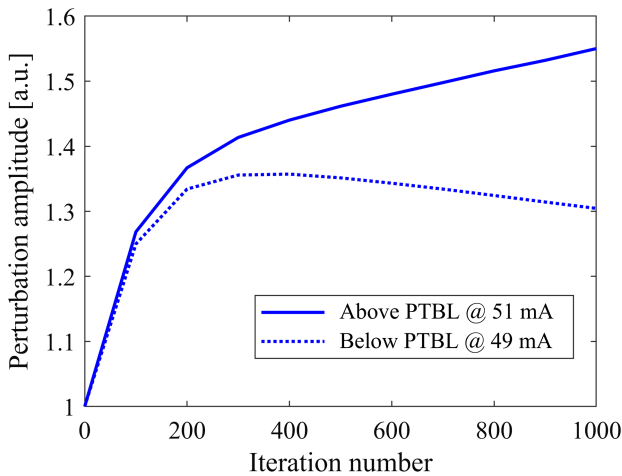


FIG. 11. Evolution of bunch centroid perturbation amplitude during the third step iterations. Two cases are shown: one is above the PTBL threshold at a beam current of 51 mA and the other is below the PTBL threshold at 49 mA.

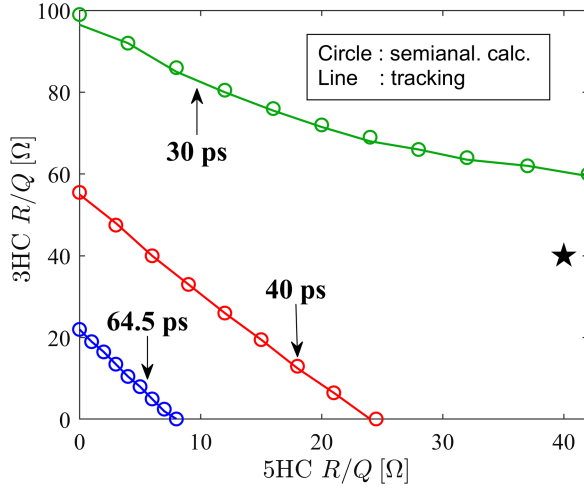


FIG. 12. Threshold R/Q values of the two HCs obtained by the modified semianalytical calculation and the tracking simulation. The black star stands for the designed R/Q values listed in Table I.

R/Q value increases, the other must decrease. This negative correlation supports the observation that the PTBL effect is enhanced when the beam loading effects of the two HCs are considered simultaneously.

Moreover, the threshold curve for the optimum bunch lengthening can be well fitted by the linear equation “ $R_1/Q_1 + 2.76R_2/Q_2 = 21.8$.” This implies that to avoid PTBL at a beam current of 350 mA for HALF, the condition “ $R_1/Q_1 + 2.76R_2/Q_2 < 21.8$ ” must be satisfied. This imposes a strict requirement on the R/Q values of the two HCs. The threshold R/Q values can be increased by sacrificing bunch lengthening, for instance, achieving higher threshold values than the current R/Q values of 40Ω for the case of 30 ps bunch length. However, such a sacrifice is undesirable, as the goal of using the triple rf system would not be fulfilled. In the following, we will explore other methods to increase the threshold R/Q values.

B. Impact of beam current

Reference [17] states that decreasing the beam current helps to increase the threshold R/Q values. It is worth investigating whether the triple rf system leads to the same conclusion. Keeping the same bunch lengthening scenarios as in the case of a beam current of 350 mA, we calculate the threshold R/Q values at a lower beam current of 100 mA. The threshold results are shown in Fig. 13. For the sake of comparison, the threshold R/Q values of the two HCs multiplied by the beam current are displayed. Tracking results are presented only for the case of $I_0 = 100$ mA, as the accuracy of the semianalytical results at $I_0 = 350$ mA has already been established in Fig. 12.

From Fig. 13, we observe that the threshold R/Q values are also inversely proportional to the beam current in the

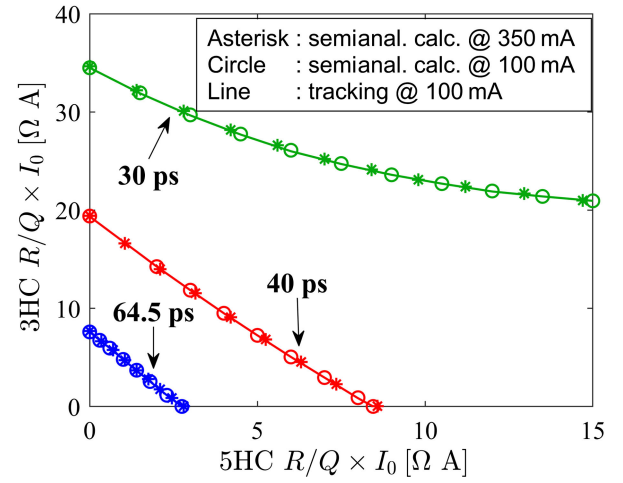


FIG. 13. Threshold R/Q values of the two HCs multiplied by beam currents, obtained by the modified semianalytical method at $I_0 = 350$ mA, $I_0 = 100$ mA, and the tracking method at $I_0 = 100$ mA.

triple rf system. This means that a lower beam current allows for larger threshold R/Q values for the two HCs.

C. Impact of the order of the two HCs

For the triple rf system assumed for HALF, in addition to the two HCs of order 3 and 5, it is also possible to use the two HCs of order 2 and 3. Compared to the former, the latter can theoretically achieve a longer optimum bunch length of 92 ps [5]. Similarly to the analysis in Sec. VA, two other less-optimum symmetric bunch distributions with rms lengths of 50 and 40 ps can be achieved with the settings “ $k_V = 0.987$, $k_\phi = 1.013$ ” and “ $k_V = 0.980$, $k_\phi = 1.020$,” respectively. For these three cases, we continue using two methods to determine the threshold R/Q values, and the results are shown in Fig. 14.

We also find that the PTBL effect cannot be avoided unless the R/Q values of the two HCs are below the fitted linear curve “ $R_1/Q_1 + 2.44R_2/Q_2 = 25.3$.” This indicates that new constraints must be imposed on the cavity designs if optimum bunch lengthening is required. While reducing the bunch length can significantly increase the threshold R/Q values, for the case of 40Ω R/Q values designed for HALF, lowering the bunch length to 50 ps is still insufficient to avoid PTBL. Further reducing the bunch length to 40 ps can completely avoid PTBL, but this is far from the original intention of using the triple rf system.

D. Impact of higher MC voltage

It is known that raising the MC voltage is beneficial for increasing the threshold R/Q value of the HC in the double rf system [17]. It is worth studying whether this conclusion applies to the triple rf system, and to what extent increasing the MC voltage can improve the threshold R/Q values of

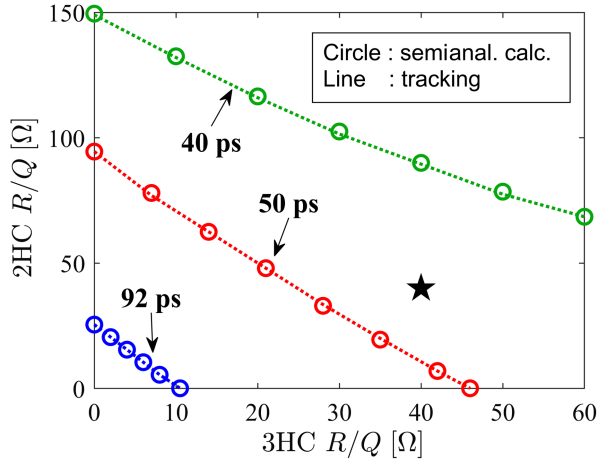


FIG. 14. Threshold R/Q values of the two HCs of order 2 and 3, obtained by the modified semianalytical method and the tracking method. The black star stands for the designed R/Q values of 40Ω .

the two HCs. In this subsection, we only consider the case of optimum bunch lengthening. Figure 15 shows the threshold R/Q values for MC voltage amplitudes of 1.2, 1.5, and 1.8 MV, computed using two methods. It can be seen that increasing the MC voltage can indeed improve the threshold R/Q values, but this improvement is not particularly effective for the case of optimum bunch lengthening.

E. Impact of lower HC voltages

When reducing the two HC voltages, it is expected that the PTBL threshold R/Q values will increase, but at the expense of reducing bunch elongation. It should be noted

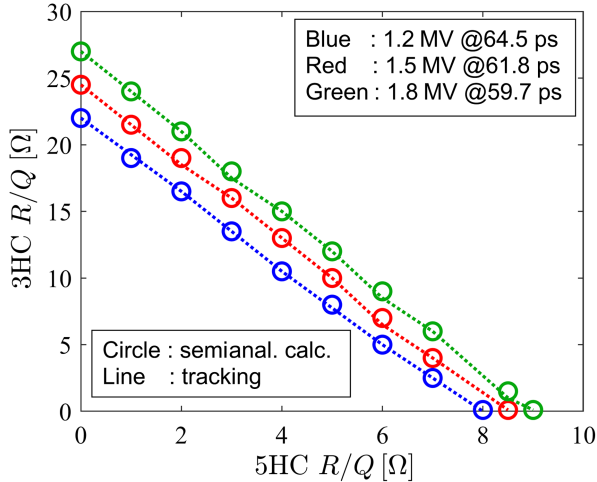


FIG. 15. Threshold R/Q values of the two HCs in the cases of optimum bunch lengthening under different MC voltages, obtained by the modified semianalytical method and the tracking method. The legend shows the MC voltage amplitude and the bunch rms length.

TABLE II. rf cavity parameter setup.

Cases (ps)	ϕ_s (rad)	V_{hc1} (kV)	ϕ_{hc1} (rad)	V_{hc2} (kV)	ϕ_{hc2} (rad)
50 ^a	2.740	552.2	-0.142	110.9	3.057
50 ^b	2.690	517.2	-0.314	87.2	2.708
45 ^a	2.740	549.9	-0.143	110.9	3.057
45 ^b	2.734	480.6	-0.187	64.8	2.938

^aObtained by adjusting the 3HC voltage amplitude and phase.

^bObtained by adjusting the two HCs voltage amplitude and phase, and the MC synchronous phase.

that the less-optimum bunch lengthening scenarios discussed above are mainly achieved by fine-tuning the lower-order HC voltage. However, if the triple rf cavities are tuned simultaneously, there are other settings for the two HCs that can yield the same rms bunch length while using lower voltage amplitudes. To achieve this, it is key to maintain the first three equality relationships in Eq. (2) when adjusting the two HC voltages. This approach can produce a flat-top but less-optimum bunch distribution.

Table II shows two types of settings for two bunch lengths of 50 and 45 ps, respectively. Superscripts “1” and “2” are used to distinguish between the two methods (see the bottom explanation in Table II for details). The MC voltage (not shown in Table II) is kept at 1.2 MV. It can be seen that method 2 allows for a further reduction in the two HC voltages compared to method 1.

Figure 16 shows the resulting threshold R/Q values obtained by two methods for the four setting cases listed in Table II. Clearly, the threshold R/Q values increase as the voltages decrease, with the increment being modest at 50 ps but relatively pronounced at 45 ps. It should be noted that, in these cases, the PTBL effect still cannot be avoided at a beam current of 350 mA with the current R/Q values of 40Ω .

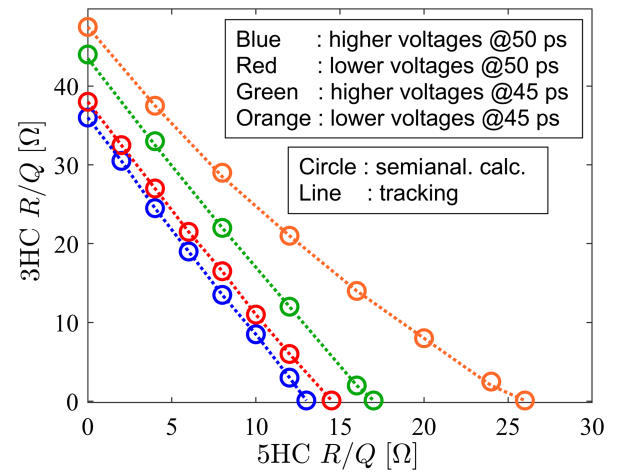


FIG. 16. Threshold R/Q values of the two HCs in the cases of relatively higher and relatively lower HC voltages, calculated via the modified semianalytical method and the tracking method. Two bunch lengths of 50 and 45 ps are considered.

In this subsection, we will not discuss the case of a bunch length lower than 45 ps. On one hand, for shorter bunch lengths, the 5HC voltage based on method 2 would become very low. Although the PTBL threshold could be significantly increased further, the detuning of the 5HC would exceed the revolution frequency, causing other types of coupled bunch instability. On the other hand, and most importantly, the case of a shorter bunch length has lost its research significance for the triple rf system.

VI. CONCLUSION AND OUTLOOK

This paper systematically investigates the PTBL effect in a triple rf system, using the HALF storage ring as a case study. A key finding is that the two HCs of the triple rf system jointly enhance the PTBL effect, contrary to the intuitive expectation that the higher-order HC would suppress this effect driven by the lower-order HC. This unexpected result is well supported by both tracking simulations and self-consistent semianalytical calculations, and can be well understood when considering bunch length perturbations rather than solely focusing on bunch centroid perturbations.

This paper first introduces a novel concept, the steady-state response matrix (SSRM) of rf cavity voltage, which simplifies the semianalytical algorithm and thus can accelerate the iterative calculations. SSRM also enables direct computation of the beam loading voltage perturbations from the bunch form factor perturbations. This approach allows a comparison of the real-part cavity voltage perturbations caused by both the real and imaginary parts of the bunch form factor perturbations, leading to the conclusion that the real-part bunch form factor perturbations dominate the abnormal PTBL behavior. Based on this understanding, we propose a modified semianalytical method in three steps, which introduces a large PTBL perturbation initially, rather than allowing the PTBL effect to grow from an equilibrium state. This modified method is efficient in determining the PTBL threshold, with results in good agreement with tracking simulations.

For the HALF case study, both the tracking method and this modified method are used to explore the relationship between the threshold R/Q values of the two HCs in various bunch lengthening scenarios. The results reveal that the threshold R/Q values of the two HCs are negatively correlated, reaffirming that the two HCs work together to enhance the PTBL effect. Additionally, the threshold R/Q values are significantly lower than the designed R/Q values at the nominal beam current, particularly under the optimal bunch lengthening scenario. Reducing the beam current or bunch lengthening (by lowering the HC voltage) can effectively increase the threshold R/Q values. However, this is far from the original objective of using the triple rf system. These findings provide valuable guidance for the design of HCs in the triple rf system, emphasizing that

lower R/Q values are necessary to achieve optimum bunch lengthening without triggering the PTBL effect.

In this study, PI feedback was integrated into the tracking simulations to stabilize the HC voltages. It was observed that the PI feedback did not affect the PTBL effect. The potential for other feedforward or feedback techniques to address PTBL warrants further investigation. It is important to note that this paper primarily focuses on the PTBL effect in the triple rf system using superconducting cavities. In future studies, we will investigate beam dynamics in room-temperature cavity schemes, which are expected to be more complex due to their relatively low quality factors. Additionally, a more comprehensive study of the triple rf system will include the impact of filling patterns, higher order modes, rf noise, and other factors.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available [43], embargo periods may apply.

APPENDIX: TO SET RF CAVITY PARAMETERS IN TRACKING SIMULATION AND SEMIANALYTICAL CALCULATION

In this paper, the complex rf cavity voltage is expressed in phasor notation. Once the ideal voltage phasors for the triple rf system are obtained using the analytical equations outlined in Sec. II A, we can adjust the rf parameters to match the actual operational conditions. These parameters include the loaded shunt impedances, loaded quality factors, and detuning frequencies. The phasor diagram is shown in Fig. 17.

For the lower-order HC, both the ideal cavity voltage phasor and the beam loading voltage phasor lie within the same quadrant. Thus, it is possible to align these two phasors in order to minimize both the generator current and the generator power. The beam loading voltage phasor is generally expressed as

$$\tilde{V}_{b1} = 2I_0 R_1 \tilde{F}_1 \cos(\psi_1) e^{i\psi_1} = 2I_0 R_1 F_1 \cos(\psi_1) e^{i(\psi_1 + \phi_{F_1})}, \quad (\text{A1})$$

where I_0 is the beam current, R_1 is the loaded shunt impedance, $\tilde{F}_1 = F_1 e^{i\phi_{F_1}}$ is the bunch form factor phasor, obtained based on the ideal bunch distribution. F_1 is the amplitude of the bunch form factor, and ϕ_{F_1} is its phase.

To align the beam loading voltage phasor with the ideal cavity voltage phasor, we have the condition:

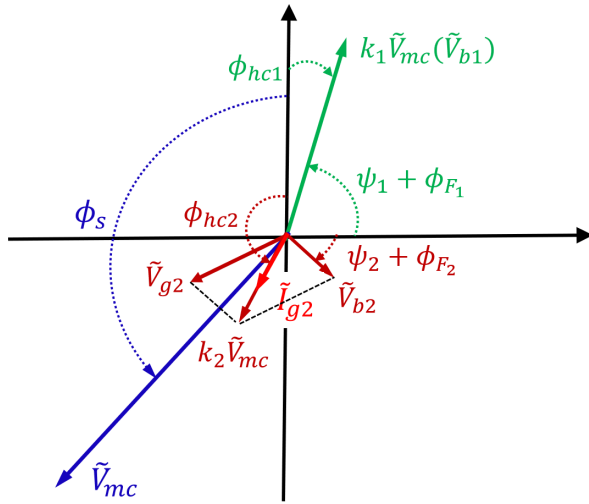


FIG. 17. Phasor diagram of the triple rf system. $\tilde{V}_{mc}$ and ϕ_s are the MC voltage and phase, respectively. Subscripts “1” and “2” correspond to the lower- and higher-order HCs, respectively. k_i is the voltage amplitude ratio, ϕ_{hci} is the voltage phase, ψ_i represents the detuning angle, $\tilde{V}_{bi}$ is the beam loading voltage phasors, $\tilde{I}_{g2}$ and $\tilde{V}_{g2}$ represent the generator current and voltage phasors of the higher-order HC.

$$\psi_1 = \phi_{\text{hc1}} - \phi_{F_1} + \frac{\pi}{2}, \quad (\text{A2})$$

$$R_1 = \frac{k_1 V_{\text{mc}}}{2I_0 F_1 \cos(\psi_1)}. \quad (\text{A3})$$

Once R_1 is determined using Eqs. (A2) and (A3), the quality factor Q_1 can then be calculated. Next, the detuning frequency Δf_1 can be determined by

$$\Delta f_1 = \frac{n_1 f_{\text{rf}} \tan(\psi_1)}{2Q_1}, \quad (\text{A4})$$

where n_1 is the order of lower-order HC and f_{rf} is the rf fundamental frequency.

For the higher-order HC, the generator current is indispensable. To minimize the reflected rf power, the loading angle should be set close to 0. In this case, the detuning frequency can be determined using the equation [44]:

$$\Delta f_2 = \frac{n_2 f_{\text{rf}} I_0 F_2 \cos(\phi_{\text{hc}2})}{k_2 V_{\text{mc}}} \frac{R_2}{Q_2}. \quad (\text{A5})$$

For the MC, we assume it operates ideally, meaning that V_{mc} and ϕ_s are treated as constants. If the beam loading effects for the MC need to be considered, its parameters are set similarly to those of the higher-order HC.

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