

Intrabeam scattering study for longitudinal strong focusing storage rings and lattice optimization

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In low and medium energy electron storage rings, intrabeam scattering (IBS) is a significant collective effect that can lead to an increase in equilibrium emittance, thereby diminishing the light source's flux. Typically, the bunch length of the electron beam in storage rings is considered constant around the ring circumference. However, in the concept of steady-state microbunching storage rings, the bunch length can vary significantly around the ring in an effort to achieve high coherence in the longitudinal dimension. In this paper, we have extended the Bjorken and Mtingwa (B-M) model to accommodate this requirement, making it versatile enough to be used for arbitrary planar storage rings. Additionally, this paper addresses the simplification of equilibrium emittance calculations involving IBS through certain approximations, making it possible to directly calculate equilibrium emittance by a formula rather than conventional iteration method. This method reveals the physical relationship between equilibrium emittance and the IBS growth rates, and also allows for the IBS effect optimization of the ring lattice by directly using equilibrium emittance as the optimization target rather than the IBS growth rate.

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I. INTRODUCTION

Electron storage rings, which store and circulate electrons continuously, can generate extremely bright short-wavelength radiation, which is valuable for cutting-edge scientific research in fields like biology, chemistry, and materials science. These light sources, known as synchrotron radiation sources (SR), face several challenges that can restrict their brightness, with one of the key factors being the emittance of the electron beam. For instance, in the most advanced types of storage rings, referred to as diffraction-limited storage rings [1–7], the emittance of the electron beam is reduced to the diffraction limit, which is defined as $\epsilon \leq \frac{\lambda}{4\pi}$, where ϵ is the transverse emittance of electron bunch, and λ is the radiation wavelength targeted for the application. This enhances the coherence of the radiation process, resulting in greater radiation brightness while maintaining a constant beam current. However, several factors will hinder any further decrease in

emittance, with intrabeam scattering (IBS) being a significant one. IBS is a scattering phenomenon caused by the Coulomb force among electrons within a single bunch, which causes an increase in beam emittance [8]. The IBS is dominant in low energy and medium energy SR, as many studies can be found in the literature [6,7,9–11].

IBS was first analytically calculated by Piwinski [12] and Bjorken and Mtingwa [13] independently. In this paper, we will refer to the two models as the Piwinski model and the B-M model. Over the years, to simplify the computation, many authors have derived high energy approximations to the full theory [14–17]. In particular, Bane showed the equivalence between his high energy modification and the theory described in Ref. [17]. Additionally, there are several research studies that expanded upon the Piwinski model and the B-M model to coupled storage rings [18–21]. It is important to highlight that Boaz Nash's research extended the B-M model to a fully coupled storage ring, and likewise, the work of Kubo and Oide [20] was implemented into the SAD software [22]. Specifically, there are several research efforts focused on applying the IBS theory to lattice optimization [23].

When examining the history of electron storage rings, the longitudinal dynamics is conventionally considered as being quasistatic. In particular, we assume that the length of the electron bunches remains constant along the ring. The

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concept of steady-state microbunching (SSMB), first proposed by Ratner and Chao [24], aims to maintain very short length of electron bunch in storage rings in order to increase the longitudinal coherence and consequently producing very high power radiation. This indicates the electron bunch will be strongly manipulated in longitudinal dimension, and the consensus of a constant bunch length along the ring is broken in an SSMB storage ring.

In general, Boaz Nash's research adequately fulfills the needs for calculating the IBS effects in the SSMB storage ring. However, to carry out its application still requires a well-defined specific lattice. In this paper, we will implement a straightforward modification to the B-M model to incorporate more complex longitudinal dynamics, while it is also very flexible to be used in arbitrary planar nonperiodical lattice.

Furthermore, in the study of lattice optimization with IBS [23], the author takes the IBS growth rate as optimization objective. However, equilibrium emittance is where optimization effort should focus on directly, not the growth rates and then applying a questionable iterative process after the growth rates. Equilibrium emittance with IBS involved is not utilized as a optimization target in lattice design because its calculation relies on an iterative method that is quite time-consuming. Furthermore, this iterative process does not accurately represent the physical relationship between equilibrium emittance and the growth rate of IBS. In this paper, we propose a different optimization approach. We will first simplify the IBS growth rate formula in a planar storage ring and give the approximate equilibrium emittance directly based on the IBS growth rate. We then employ this equilibrium emittance as the optimization objective for lattice design.

The structure of this paper is delineated as follows. Section II provides a concise introduction to the fundamental principles of IBS and the B-M model, along with the modifications to the B-M model pertaining to a storage ring with nonconstant bunch length. Section III presents a comparative analysis of various methodologies applied to the SSMB storage ring in contrast to a conventional storage ring characterized by a planar configuration and a constant electron bunch length. In Sec. IV, we examine the approximation techniques for IBS calculations specifically to a planar storage ring, offering a direct formula for determining the equilibrium emittance, as opposed to the iterative methods typically employed. Section V illustrates the application of this formula in the context of IBS optimization for a hybrid-MBA lattice design. Finally, Sec. VI concludes with a brief summary.

II. BASIC THEORY OF IBS

A. Review of B-M model

The fundamental principle underlying the IBS phenomenon is that the momentum of a particle is altered while its

position remains constant during the collision of two arbitrary beam particles, which occurs as a result of the Coulomb force. This process can be likened to thermal effects that contribute to the increase in beam emittance. The rate of this growth can be determined through the integration of the probability of collisions and the distribution of the beam. For the purposes of this discussion, we will assume that the beam distribution follows a Gaussian profile. Consequently, the beam distribution can be represented as [13]

$$\rho(\vec{r}, \vec{r}') = \frac{N}{\Gamma} e^{-S(\vec{r}, \vec{r}')}, \quad (1)$$

where N is the electron number in the bunch, $\vec{r}$ and $\vec{r}'$ represent the phase space coordinate $(x, x', y, y', z, \delta)$, and

$$S(\vec{r}, \vec{r}') = \sum_{i,j=1}^3 \left(\frac{1}{2} A_{ij} r'_i r'_j + B_{ij} r'_i r_j + \frac{1}{2} C_{ij} r_i r_j \right) \quad (2)$$

$$\Gamma = \int d^3\vec{r} d^3\vec{r}' e^{-S(\vec{r}, \vec{r}')},$$

while A_{ij}, B_{ij}, C_{ij} should be the elements of Σ^{-1} , and Σ is the distribution's second moment matrix in any position of the storage rings. As the distribution is given by [25]

$$\psi(X) = \frac{1}{(2\pi)^3 \sqrt{\det \Sigma}} \exp \left(-\frac{1}{2} \tilde{X} \Sigma^{-1} X \right), \quad (3)$$

while X here is the phase space coordinate, $\det \Sigma$ represents the determinant of matrix Σ .

In the foundational works of Bjorken and Mtingwa [13], coupling effects are not considered, and the longitudinal bunch length is assumed to remain constant along the ring. Consequently, the growth rate can be expressed as [17]

$$\frac{1}{T_i} = 4\pi A(\log) \left\langle \int_0^\infty \frac{d\lambda \lambda^{1/2}}{[\det(L + \lambda I)]^{1/2}} \left\{ Tr(L^{(i)}) Tr\left(\frac{1}{L + \lambda I}\right) - 3 Tr L^{(i)} \frac{1}{L + \lambda I} \right\} \right\rangle, \quad (4)$$

where i represents three different dimensions x, y, z or h, v, s , and h or x represents horizontal dimension, v or y represents vertical dimension, s or z represents longitudinal dimension, Tr represents the matrix trace, $(\log)$ represents Coulomb log factor, $\langle \cdot \rangle$ represents averaging over the entire ring.

$$A = \frac{r_0^2 c N}{64\pi^2 \beta^3 \gamma^4 \epsilon_x \epsilon_y \epsilon_s \sigma_p}, \quad (5)$$

r_0 is the classical electron radius, c is speed of light in vacuum, β represents the electron velocity over c , γ is the

Lorentz factor, L is the matrix constructed by A_{ij} (and γ comes from the frame transformation). Assuming the absence of coupling and the presence of horizontal dispersion, while considering the bunch length to be constant, the matrix L can be represented as

$$\begin{aligned} L^{(h)} &= \frac{\beta_x}{\epsilon_x} \begin{bmatrix} 1 & 0 & -\gamma\phi_x \\ 0 & 0 & 0 \\ -\gamma\phi_x & 0 & \gamma^2\mathcal{H}_x/\beta_x \end{bmatrix} \\ L^{(v)} &= \frac{\beta_y}{\epsilon_y} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ L^{(s)} &= \frac{\gamma^2}{\sigma_p^2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ L &= L^{(h)} + L^{(v)} + L^{(s)}, \end{aligned} \quad (6)$$

with $\phi_x = \eta'_x + \frac{\alpha_x\eta_x}{\beta_x}$, $\mathcal{H}_x = \gamma_x\eta_x^2 + 2\alpha_x\eta_x\eta'_x + \beta_x\eta_x'^2$, $\beta_x, \alpha_x, \gamma_x$ and $\beta_y, \alpha_y, \gamma_y$ are the horizontal and vertical Twiss functions, respectively, η_x and η'_x are horizontal dispersion and dispersion angle, respectively. Equation (6) can be obtained from the distribution

$$\begin{aligned} S^{(h)} &= \frac{\beta_x}{2\epsilon_x} x_\beta^2 + \frac{\alpha_x}{\epsilon_x} x_\beta x'_\beta + \frac{\gamma_x}{2\epsilon_x} x_\beta'^2 \\ S^{(v)} &= \frac{\beta_y}{2\epsilon_y} y_\beta^2 + \frac{\alpha_y}{\epsilon_y} y_\beta y'_\beta + \frac{\gamma_y}{2\epsilon_y} y_\beta'^2 \\ S^{(s)} &= \frac{\delta^2}{\sigma_p^2} + \frac{(z - z_0)^2}{2\sigma_s^2}, \end{aligned} \quad (7)$$

where ϵ_x and ϵ_y are the horizontal and vertical emittances, respectively, σ_p and σ_s are the relative energy spread and bunch length, respectively, z_0 is the beam position centroid, which normally is zero, x_β, x'_β and y_β, y'_β are the horizontal and vertical betatron motion, respectively, and there is the relation

$$x_\beta = x - \eta_x\delta, \quad x'_\beta = x' - \eta'_x\delta. \quad (8)$$

B. The modification for a longitudinal strong focusing planar storage ring

In a conventional storage ring, the length of the electron bunch is regarded as constant along the circumference. So the longitudinal phase space distribution is represented as S^s in Eq. (7), which yields the matrix of L^s in Eq. (6).

In an SSMB storage ring (the longitudinal strong focusing can be achieved), the motion of longitudinal phase space is no longer static. Under the assumption of a planar storage ring, we can modify the longitudinal distribution by first defining the longitudinal Twiss parameters, namely, $\alpha_s, \beta_s, \gamma_s$. Subsequently, the term S^s in Eq. (7) may be substituted with

$$S^{(s)} = \frac{\beta_s}{2\epsilon_s} \delta^2 + \frac{\alpha_s}{\epsilon_s} s_0\delta + \frac{\gamma_s}{2\epsilon_s} s_0^2, \quad (9)$$

and the longitudinal position of an arbitrary particle is determined by

$$z = s_0 + s_\beta. \quad (10)$$

The term s_β is derived from the dispersion effect, which is a consequence of the symplectic condition and can be expressed as follows

$$s_\beta = \eta'_x x_\beta - \eta_x x'_\beta. \quad (11)$$

By substituting Eqs. (10) and (11) into Eq. (9), one can derive the matrix L^s for a longitudinal strong focusing storage ring as follows

$$L^{(s)} = \frac{1}{\epsilon_s} \begin{bmatrix} \gamma_s\eta_x^2 & 0 & -\gamma\alpha_s\eta_x \\ 0 & 0 & 0 \\ -\gamma\alpha_s\eta_x & 0 & \gamma^2\beta_s \end{bmatrix}. \quad (12)$$

III. COMPARISON OF DIFFERENT METHODS OF IBS CALCULATION FOR SSMB STORAGE RINGS AND CONVENTIONAL STORAGE RINGS

There are two methods that we will use for comparison.

- (a) The initial B-M model, where bunch length is regarded as constant, implemented in ELEGANT [26] and
- (b) Modified B-M model, by replacing the matrix L^s in Eq. (6) with Eq. (12).

A. Comparison for an SSMB storage ring

The primary characteristic of an SSMB storage ring is the variation in bunch length throughout the ring. It is evident that Eqs. (4) and (6) are no longer capable of accurately representing the IBS growth rate under these conditions. To elucidate this phenomenon more effectively, we present an example that employs two different methods above for calculating the IBS growth rate in an SSMB ring.

To achieve a minimal equilibrium bunch length in the straight section, the design of the ring incorporates a low global momentum compaction factor as well as a low local momentum compaction factor [27,28], and the bunching system will utilize a laser modulator characterized by a very short wavelength. The parameters of the ring are detailed in

TABLE I. Parameters of an SSMB ring.

Parameters	Units	Value
Nominal energy	MeV	400
Circumference	m	165.62
Natural emittance	nm rad	1.06
Horizontal/vertical tune	...	15.211/13.188
Phase slippage factor	...	9.3×10^{-7}
Natural energy spread	...	3.16×10^{-4}
Horizontal/vertical damping time	ms	429.3/429.3
Longitudinal damping time	ms	214.7
Coupling	...	0.0012
Bunch length at straight	nm	106.45
Electron number per bunch	...	20,000
Bunching system harmonic	...	162,227,080
Bunching system voltage	kV	100

Table I, with the wavelength of the bunching system approximately equal to $1 \mu\text{m}$.

The Twiss parameters for a single super-period are illustrated in Fig. 1, whereas the entire ring comprises a total of four super-periods.

As previously indicated, the bunch length of the electron beam will exhibit variability throughout the ring. In fact, it can be expressed as follows

$$\sigma_s(s) = \sqrt{\beta_s(s)\epsilon_s + \mathcal{H}_x(s)\epsilon_x}. \quad (13)$$

In Fig. 2, we present a graphical representation of the various components of the σ_s for the SSMB storage ring. It is evident that the contribution from horizontal emittance is predominant in the arc region.

We have implemented the modified B-M method into a calculation program to determine the local growth rate in three dimensions for the SSMB ring. The results are presented in Fig. 3.

As illustrated in Fig. 3, variation of bunch length within the arc region yields significantly different results for IBS growth rate between the ELEGANT simulations and the modified B-M method. In regions where the $\mathcal{H}_x$ function approaches zero, the growth rates of the IBS calculated using various methodologies exhibit a high degree of

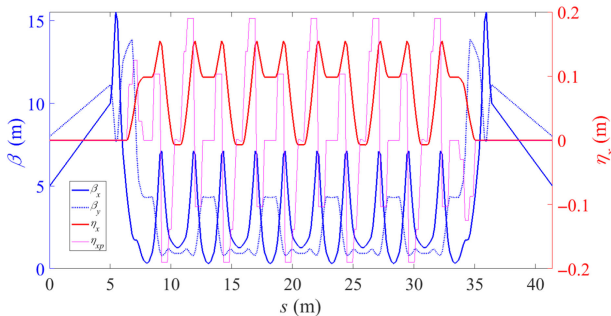


FIG. 1. Twiss parameters of a single super-period.

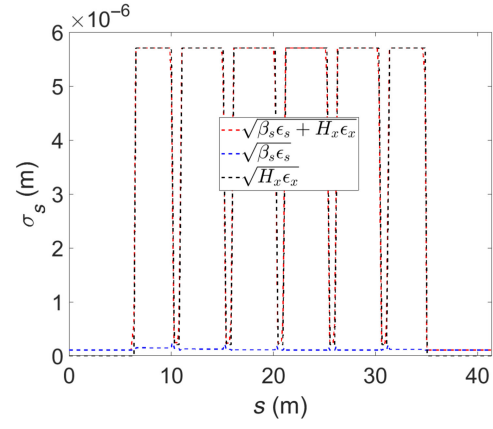


FIG. 2. Bunch length variation in a single super-period.

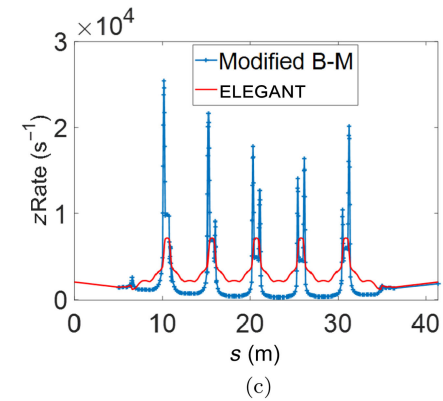
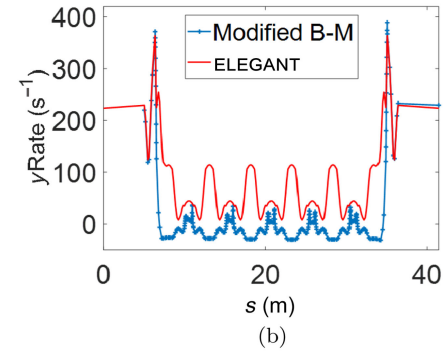
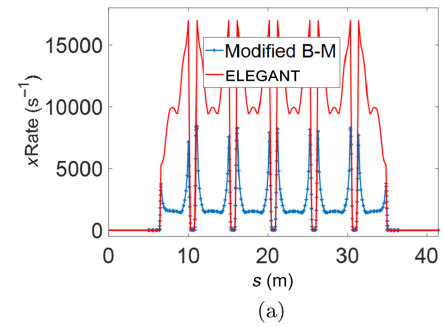


FIG. 3. The IBS growth rate calculated by different methods for an SSMB storage ring. (a) Horizontal IBS growth rate, (b) vertical IBS growth rate, and (c) longitudinal IBS growth rate.

similarity. In large dispersion regions, while ELEGANT assumes a constant bunch length, the growth rates in both horizontal and longitudinal dimensions computed by ELEGANT exhibit significant discrepancies compared to our modified B-M model. This outcome aligns with expectations.

Furthermore, we compute the equilibrium emittance incorporating the effects of IBS through an iterative application of the formulas presented below, which are analogous to those utilized in the ELEGANT simulation framework.

$$\begin{aligned}\frac{d\epsilon_x}{dt} &= \frac{\epsilon_x}{T_x} + \left(\frac{\epsilon_{x0}}{1 + \kappa} - \epsilon_x \right) \frac{2}{\tau_x} \\ \frac{d\epsilon_y}{dt} &= \frac{\epsilon_y}{T_y} + \left(\frac{\epsilon_{y0}\kappa}{1 + \kappa} - \epsilon_y \right) \frac{2}{\tau_y} \\ \frac{d\epsilon_s}{dt} &= \frac{\epsilon_s}{T_s} + (\epsilon_{s0} - \epsilon_s) \frac{2}{\tau_s},\end{aligned}\quad (14)$$

where κ denotes the coupling between the horizontal and vertical dimensions. The parameters τ_x , τ_y , and τ_s correspond to the damping times in the horizontal, vertical, and longitudinal directions, respectively. Similarly, T_x , T_y , and T_s represent the growth times of the IBS in the horizontal, vertical, and longitudinal dimensions, respectively. Additionally, ϵ_{x0} signifies the zero current emittance, while ϵ_{s0} indicates the longitudinal emittance at zero current. The condition for equilibrium is $\frac{d\epsilon_x}{dt} = 0$, $\frac{d\epsilon_y}{dt} = 0$, $\frac{d\epsilon_s}{dt} = 0$.

We need to emphasize more on the iteration equation used for equilibrium emittance calculation. This formula is valid when we only consider the core part of a bunch for IBS calculation but let the tail of the bunch for Touschek study. And when we say the core of the bunch, we typically talk about the particles inside the 3 times of the rms beam size. The diffusion induced by intrabeam scattering is not only relevant to the first order of emittance but also to high orders of emittance, which will result in the distortion of the beam distribution away from Gaussian. So here, we do an approximation, by studying the core of beam only and assuming the distribution of core will not be significantly distorted from Gaussian. In principle, the beam distribution could be solved by first calculating the high order IBS diffusion term exactly, and inserting the diffusion term into

TABLE II. Equilibrium emittance in an SSMB ring with IBS. Here, we use two bunch lengths for IBS calculation in ELEGANT. One is 106 nm at straight. Another is the average bunch length of 3.45 μm .

Method used	ϵ_x (nm)	ϵ_y (pm)	σ_s (nm)	σ_δ
ELEGANT at 106 nm	27.36	32.83	298	8.85×10^{-4}
ELEGANT at 3.45 μm	8.27	9.92	5871	5.37×10^{-4}
Modified B-M	11.93	14.31	296	8.80×10^{-4}

TABLE III. Bunching system parameters used for a conventional storage ring.

Parameter	Units	Value
Natural energy spread	...	2.31×10^{-4}
Bunch length at straight	μm	80.70
Electron number per bunch	...	2×10^9
Bunching system harmonic	...	165
Bunching system voltage	kV	100

the Fokker-Planck equation to determine the final distribution. But it is above the scope of this paper.

The equilibrium emittance, with IBS considered, is presented in Table II. As anticipated, the equilibrium emittance calculated from the ELEGANT simulation exhibits a notable difference from the results obtained through the modified B-M method. The parameters utilized for the IBS calculations are detailed in Table I.

B. Comparison for a conventional storage ring

In order to conduct a comparison for a conventional storage ring, it is feasible to change the settings of the bunching system while maintaining the lattice configuration depicted in Fig. 1 as previously utilized. The parameters that have been altered from those presented in Table I are detailed in Table III.

The variation in bunch length is illustrated in Fig. 4, which indicates that there is no substantial alteration in bunch length, in contrast to the behavior observed in SSMB storage rings.

We also compute the local IBS growth rate for this ring, with the results presented in Fig. 5. It is evident that, with a few exceptions, the growth rates obtained from various methods are largely consistent. In the conventional ring configuration, the modified B-M model demonstrates a strong agreement with the results from ELEGANT. Additionally, we calculated the equilibrium emittance considering IBS for this ring, as detailed in Table IV, which corroborates the findings illustrated in Fig. 5.

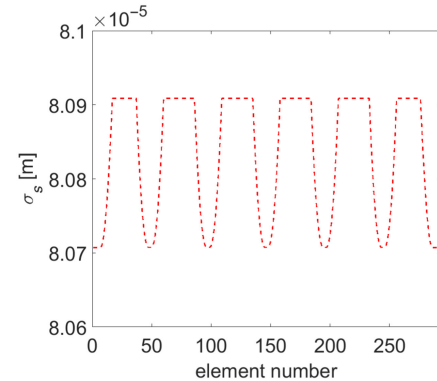


FIG. 4. Bunch length variation along a single super-period in a conventional storage ring.

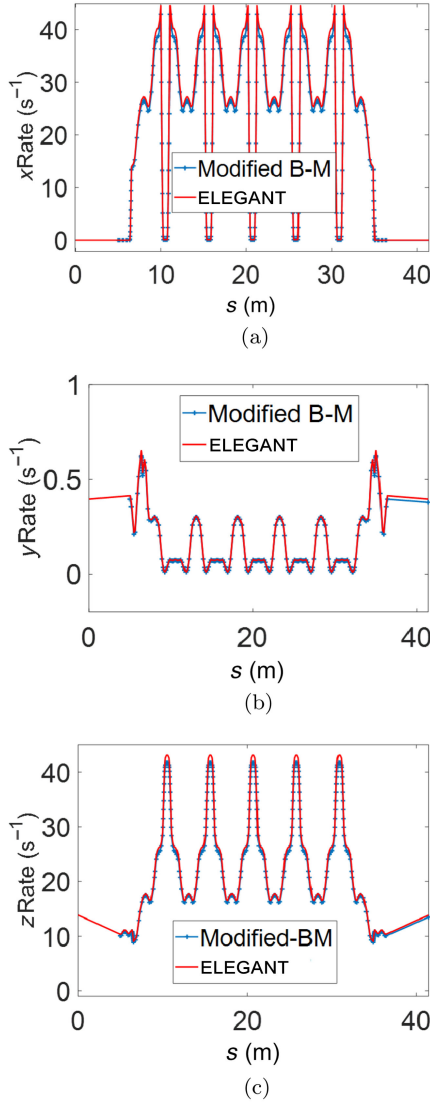


FIG. 5. The IBS growth rate calculated by different methods for a conventional storage ring. (a) Horizontal IBS growth rate, (b) vertical IBS growth rate, and (c) longitudinal IBS growth rate.

TABLE IV. Equilibrium emittance of a conventional storage ring with IBS.

Method used	ϵ_x (nm)	ϵ_y (pm)	σ_s (μm)	σ_δ
ELEGANT	129.12	154.94	641.67	1.83×10^{-3}
Modified B-M	134.68	161.62	652.43	1.86×10^{-3}
Relative deviation	4.31%	4.31%	1.68%	1.61%

IV. APPROXIMATION FOR A PLANAR STORAGE RING

The IBS growth rate shown in Eq. (4) is widely used in equilibrium emittance calculation in storage rings, as we also used in Sec. III. However, the growth rate is expressed by a complex integral, with emittance coupled in the integral. So, the normal method used (at least in electron storage rings) to predict the equilibrium emittance with IBS is the iterative method, as shown in Eq. (14). In this section, we will derive a more simple expression for IBS growth rate without integrals based on some approximation for a planar storage ring. Then, we predict the equilibrium emittance with IBS involved by a simple equation but not iteration process.

Kubo and Oide [20] and Nagaitsev [29] both simplified the integral in Eq. (4) to the type of symmetric Elliptic integrals, making it more efficient for numerical calculation. We will start our derivation based on Kubo's research and rewrite Eq. (4) as

$$\frac{1}{T_i} = 4\pi A(\log) \langle \text{Tr}(L^{(i)} * D_p) \rangle, \quad (15)$$

where $i = 1, 2, 3$ represents the three dimensions x, y, z or h, v, s . And the IBS momentum diffusion rate D_p can be expressed with Elliptic integrals as below (the Elliptic integrals are contained in dw_k)

$$D_p = \begin{bmatrix} \sum_{k=1}^3 R_{1k} R_{1k} dw_k & \sum_{k=1}^3 R_{1k} R_{2k} dw_k & \sum_{k=1}^3 R_{1k} R_{3k} dw_k \\ \sum_{k=1}^3 R_{2k} R_{1k} dw_k & \sum_{k=1}^3 R_{2k} R_{2k} dw_k & \sum_{k=1}^3 R_{2k} R_{3k} dw_k \\ \sum_{k=1}^3 R_{3k} R_{1k} dw_k & \sum_{k=1}^3 R_{3k} R_{2k} dw_k & \sum_{k=1}^3 R_{3k} R_{3k} dw_k \end{bmatrix}, \quad (16)$$

where R_{ij} is the element of matrix R . While matrix R can be defined by matrix L in Eq. (6) as

$$L = R^T \begin{bmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{bmatrix} R. \quad (17)$$

R is an orthogonal matrix, which satisfies

$$R^T = R^{-1}, \quad (18)$$

u_1, u_2, u_3 are eigenvalues of matrix L .

Then R can be calculated as

$$R = \begin{bmatrix} \sqrt{\frac{a\beta_x/\epsilon_x - u_3}{u_1 - u_3}} & 0 & -\sqrt{\frac{b\beta_x/\epsilon_x - u_3}{u_1 - u_3}} \\ 0 & 1 & 0 \\ \sqrt{\frac{u_1 - a\beta_x/\epsilon_x}{u_1 - u_3}} & 0 & \sqrt{\frac{u_1 - b\beta_x/\epsilon_x}{u_1 - u_3}} \end{bmatrix}, \quad (19)$$

with a, b, d, u_1, u_2, u_3 given by

$$\begin{aligned} a &= 1 + \frac{\gamma_s \eta_x^2 \epsilon_x}{\beta_x \epsilon_s} \\ b &= \frac{\gamma^2 \mathcal{H}_x}{\beta_x} + \frac{\gamma^2 \beta_s \epsilon_x}{\beta_x \epsilon_s} \\ d &= -\gamma \phi_x - \frac{\alpha_s \eta_x \gamma \epsilon_x}{\beta_x \epsilon_s}, \end{aligned} \quad (20)$$

$$\begin{aligned} u_1 &= \frac{\beta_x}{2\epsilon_x} (a + b + \sqrt{(a - b)^2 + 4d^2}) \\ u_2 &= \frac{\beta_y}{\epsilon_y} \\ u_3 &= \frac{\beta_x}{2\epsilon_x} (a + b - \sqrt{(a - b)^2 + 4d^2}). \end{aligned} \quad (21)$$

Defining $t_1 = \frac{1}{u_1}$, $t_2 = \frac{1}{u_2}$, $t_3 = \frac{1}{u_3}$, the Elliptic integrals and dw_k are defined as

$$\begin{aligned} g_1 &= \int_0^{\pi/2} \frac{2t_1 \sin^2 s \cos s \, ds}{\sqrt{\left(\sin^2 s + \frac{t_1}{t_2} \cos^2 s\right) \left(\sin^2 s + \frac{t_1}{t_3} \cos^2 s\right)}} \\ g_2 &= \int_0^{\pi/2} \frac{2t_2 \sin^2 s \cos s \, ds}{\sqrt{\left(\sin^2 s + \frac{t_2}{t_1} \cos^2 s\right) \left(\sin^2 s + \frac{t_2}{t_3} \cos^2 s\right)}} \\ g_3 &= \int_0^{\pi/2} \frac{2t_3 \sin^2 s \cos s \, ds}{\sqrt{\left(\sin^2 s + \frac{t_3}{t_1} \cos^2 s\right) \left(\sin^2 s + \frac{t_3}{t_2} \cos^2 s\right)}} \end{aligned}$$

$$dw_1 = g_2 + g_3 - 2g_1$$

$$dw_2 = g_1 + g_3 - 2g_2$$

$$dw_3 = g_1 + g_2 - 2g_3. \quad (22)$$

In a planar storage ring, we assume that $\epsilon_x \gg \epsilon_y$. Typically, in a dispersion location, the condition $b \gg 1$ can be satisfied. In a dispersion-free location, the parameter $b = \frac{\gamma^2 \beta_s \epsilon_x}{\beta_x \epsilon_s} \sim \frac{\gamma^2 \epsilon_x}{\beta_x \sigma_p^2} > 1$, which can also be achieved in fourth-generation storage rings and in an SSMB storage rings. Additionally, the condition $b \gg a$ is generally met in most scenarios. Consequently, it can be asserted that $u_1 \gg u_3$ is

consistently valid in majority cases, and $u_2 \gg u_3$ can also be maintained in a planar storage ring. This is equivalent to stating that

$$\frac{t_3}{t_2} \gg 1, \quad \frac{t_3}{t_1} \gg 1. \quad (23)$$

Under the conditions of Eq. (23), we are able to simplify Eq. (22) to

$$\begin{aligned} g_1 &= \frac{2}{\sqrt{u_1 u_2} \left(\sqrt{\frac{u_1}{u_2}} + 1 \right)} \\ g_2 &= \frac{2}{\sqrt{u_1 u_2} \left(\sqrt{\frac{u_2}{u_1}} + 1 \right)} \\ g_3 &= \begin{cases} \frac{2}{\sqrt{u_1 u_2}} \ln \sqrt{\frac{u_1}{u_3}} & u_2 > u_1 \\ \frac{2}{\sqrt{u_1 u_2}} \ln \sqrt{\frac{u_2}{u_3}} & u_2 \leq u_1. \end{cases} \end{aligned} \quad (24)$$

Now, we get an exact expression for the defined Elliptic integrals in Eq. (22), which make it possible to get the exact expression for IBS growth rate as

$$\begin{aligned} \frac{1}{T_1} &= 4\pi A (\log) \frac{2}{\epsilon_x} \left\langle \frac{1}{\sqrt{u_1 u_2}} P_m \right\rangle \\ \frac{1}{T_2} &= 4\pi A (\log) \frac{2}{\epsilon_y} \left\langle \frac{1}{\sqrt{u_1 u_2}} T_m \right\rangle \\ \frac{1}{T_3} &= 4\pi A (\log) \frac{2}{\epsilon_z} \left\langle \frac{1}{\sqrt{u_1 u_2}} G_m \right\rangle, \end{aligned} \quad (25)$$

while P_m , T_m , and G_m can be regarded as emittance independent but only related to Twiss functions. The expression for P_m , T_m , and G_m can refer to the Appendix.

In dispersive regions, we posit that the condition $\gamma \phi_x > 1$ is satisfied in the majority of scenarios. Furthermore, in a typical ring configuration, the condition $\frac{\gamma_s \eta_x^2 \epsilon_x}{\beta_x \epsilon_s} \ll 1$ also holds, thereby ensuring that the relationship $b \gg d$ is maintained.

Then we can make more approximation for u_1 as

$$\begin{aligned} u_1 &= \frac{\beta_x}{2\epsilon_x} (a + b + \sqrt{(a - b)^2 + 4d^2}) \\ &\approx \frac{\beta_x}{2\epsilon_x} 2b \\ &\approx \frac{\beta_s \gamma^2}{2\epsilon_s} \left(1 + 2 \frac{\mathcal{H}_x \sigma_p^2}{\epsilon_x} \right). \end{aligned} \quad (26)$$

It can also be written as

$$u_1 \approx \begin{cases} \frac{1}{\epsilon_x} \gamma^2 \mathcal{H}_x & \mathcal{H}_x \sigma_p^2 \gg \epsilon_x \\ \frac{1}{\epsilon_s} \frac{\beta_s \gamma^2}{2} & \mathcal{H}_x \sigma_p^2 \ll \epsilon_x, \\ \frac{\beta_s \gamma^2}{2 \epsilon_s \epsilon_x} (\epsilon_x + \Gamma_t \epsilon_s) & \mathcal{H}_x \sigma_p^2 \sim \epsilon_x \end{cases} \quad (27)$$

while $\Gamma_t = 2\gamma_s \mathcal{H}_x$.

To date, we have effectively decomposed the growth rates into two distinct components: one associated with beam emittance and the other pertaining to beam optics, as delineated below

$$\begin{bmatrix} \frac{1}{T_1} \\ \frac{1}{T_2} \\ \frac{1}{T_3} \end{bmatrix} = \begin{cases} \begin{bmatrix} \frac{1}{\epsilon_x \epsilon_s \sqrt{\epsilon_y \epsilon_x}} \left\langle \frac{A_n P_m}{\sqrt{\beta_y \mathcal{H}_x}} \right\rangle \\ \frac{1}{\epsilon_y \epsilon_s \sqrt{\epsilon_y \epsilon_x}} \left\langle \frac{A_n T_m}{\sqrt{\beta_y \mathcal{H}_x}} \right\rangle \\ \frac{1}{\epsilon_s^2 \sqrt{\epsilon_y \epsilon_x}} \left\langle \frac{A_n G_m}{\sqrt{\beta_y \mathcal{H}_x}} \right\rangle \end{bmatrix} & \mathcal{H}_x \sigma_p^2 \gg \epsilon_x \\ \begin{bmatrix} \frac{1}{\epsilon_x \epsilon_y \sqrt{\epsilon_y \epsilon_s}} \left\langle \frac{\sqrt{2} A_n P_m}{\sqrt{\beta_y \beta_s}} \right\rangle \\ \frac{1}{\epsilon_x \epsilon_y \sqrt{\epsilon_y \epsilon_s}} \left\langle \frac{\sqrt{2} A_n T_m}{\sqrt{\beta_y \beta_s}} \right\rangle \\ \frac{1}{\epsilon_x \epsilon_s \sqrt{\epsilon_y \epsilon_s}} \left\langle \frac{\sqrt{2} A_n G_m}{\sqrt{\beta_y \beta_s}} \right\rangle \end{bmatrix} & \mathcal{H}_x \sigma_p^2 \ll \epsilon_x \\ \begin{bmatrix} \frac{1}{\epsilon_x \sqrt{\epsilon_x \epsilon_y \epsilon_s}} \frac{1}{\sqrt{\epsilon_x + (\Gamma_t) \epsilon_s}} \left\langle \frac{\sqrt{2} A_n P_m}{\sqrt{\beta_y \beta_s}} \right\rangle \\ \frac{1}{\epsilon_y \sqrt{\epsilon_x \epsilon_y \epsilon_s}} \frac{1}{\sqrt{\epsilon_x + (\Gamma_t) \epsilon_s}} \left\langle \frac{\sqrt{2} A_n T_m}{\sqrt{\beta_y \beta_s}} \right\rangle \\ \frac{1}{\epsilon_s \sqrt{\epsilon_x \epsilon_y \epsilon_s}} \frac{1}{\sqrt{\epsilon_x + (\Gamma_t) \epsilon_s}} \left\langle \frac{\sqrt{2} A_n G_m}{\sqrt{\beta_y \beta_s}} \right\rangle \end{bmatrix} & \mathcal{H}_x \sigma_p^2 \sim \epsilon_x, \end{cases} \quad (28)$$

while $A_n = \frac{(\log) r_0^2 c N}{8\pi \beta^3 \gamma^5}$.

The reason for deriving the growth rates in the form presented in Eq. (28) is that, given the zero current emittance, we can subsequently predict the equilibrium emittance. In this section, we will demonstrate the methodology for predicting the equilibrium emittance utilizing Eq. (28). We will focus on the scenario $\langle \mathcal{H}_x \sigma_p^2 \rangle \gg \epsilon_x$ first [we also give the results for other two conditions in Eqs. (34) and (35)]. The zero current emittance of a storage ring is denoted as $\epsilon_{x0}, \epsilon_{y0}, \epsilon_{s0}$. Consequently, the growth rates at the zero current emittance can be articulated as follows

$$\begin{bmatrix} \frac{1}{T_{10}} \\ \frac{1}{T_{20}} \\ \frac{1}{T_{30}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\epsilon_{x0} \epsilon_{s0} \sqrt{\epsilon_{y0} \epsilon_{x0}}} \left\langle \frac{A_n P_m}{\sqrt{\beta_y \mathcal{H}_x}} \right\rangle \\ \frac{1}{\epsilon_{y0} \epsilon_{s0} \sqrt{\epsilon_{y0} \epsilon_{x0}}} \left\langle \frac{A_n T_m}{\sqrt{\beta_y \mathcal{H}_x}} \right\rangle \\ \frac{1}{\epsilon_{s0}^2 \sqrt{\epsilon_{y0} \epsilon_{x0}}} \left\langle \frac{A_n G_m}{\sqrt{\beta_y \mathcal{H}_x}} \right\rangle \end{bmatrix}. \quad (29)$$

If we denote the equilibrium emittance as $\epsilon_x, \epsilon_y, \epsilon_s$, it follows that the equation presented below must be satisfied,

$$\begin{aligned} \frac{\epsilon_x}{T_1} - \frac{2\epsilon_x}{\tau_1} + \frac{2\epsilon_{x0}}{\tau_1} &= 0 \\ \frac{\epsilon_y}{T_2} - \frac{2\epsilon_y}{\tau_2} + \frac{2\epsilon_{y0}}{\tau_2} &= 0 \\ \frac{\epsilon_s}{T_3} - \frac{2\epsilon_s}{\tau_3} + \frac{2\epsilon_{s0}}{\tau_3} &= 0, \end{aligned} \quad (30)$$

where τ_1, τ_2 , and τ_3 are the damping time of horizontal, vertical, and longitudinal direction, respectively.

Equation (30) illustrates that the equilibrium condition is maintained through the interplay of IBS growth, radiation damping, and radiation quantum excitation.

Defining

$$\begin{aligned} a_0 &= \frac{\tau_1}{2T_{10}} \\ b_0 &= \frac{\tau_2}{2T_{20}} \\ c_0 &= \frac{\tau_3}{2T_{30}} \end{aligned} \quad (31)$$

and

$$\begin{aligned} \epsilon_x &= x_t \epsilon_{x0} \\ \epsilon_y &= y_t \epsilon_{y0} \\ \epsilon_s &= z_t \epsilon_{s0}, \end{aligned} \quad (32)$$

x_t, y_t, z_t represent the ratio of equilibrium emittance to zero current emittance in horizontal, vertical, and longitudinal direction, respectively.

Then, substituting Eqs. (28), (29), (31), and (32), into Eq. (30), we can get

$$\begin{aligned} (x_t - 1) \sqrt{x_t} \sqrt{\left(\frac{b_0}{a_0} (x_t - 1) + 1 \right)} \left(\frac{c_0}{a_0} (x_t - 1) + 1 \right) &= a_0 \\ y_t &= \frac{b_0}{a_0} (x_t - 1) + 1 \\ z_t &= \frac{c_0}{a_0} (x_t - 1) + 1. \end{aligned} \quad (33)$$

In Eq. (33), x_t can be resolved expeditiously, allowing for the prediction of the equilibrium emittance without the need for iterative processes.

In a similar manner, if the condition $\langle \mathcal{H}_x \sigma_p^2 \rangle \ll \epsilon_x$ holds, then Eq. (33) will be modified to

$$(x_t - 1)x_t \sqrt{\left(\frac{b_0}{a_0}(x_t - 1) + 1\right) \left(\frac{c_0}{a_0}(x_t - 1) + 1\right)} = a_0$$

$$y_t = \frac{b_0}{a_0}(x_t - 1) + 1$$

$$z_t = \frac{c_0}{a_0}(x_t - 1) + 1.$$
(34)

Otherwise, if the condition $\langle \mathcal{H}_x \sigma_p^2 \rangle \sim \epsilon_x$ holds, then Eq. (33) will be modified to

$$(x_t - 1) \sqrt{x_t} \sqrt{\left(\frac{b_0}{a_0}(x_t - 1) + 1\right) \left(\frac{c_0}{a_0}(x_t - 1) + 1\right)}$$

$$\times \sqrt{\frac{x_t \epsilon_{x0} + \left(\frac{c_0}{a_0}(x_t - 1) + 1\right) \langle \Gamma_t \rangle \epsilon_{s0}}{\epsilon_{x0} + \langle \Gamma_t \rangle \epsilon_{s0}}} = a_0$$
(35)

$$y_t = \frac{b_0}{a_0}(x_t - 1) + 1$$

$$z_t = \frac{c_0}{a_0}(x_t - 1) + 1,$$

with

$$\frac{b_0}{a_0} = \frac{\epsilon_{x0} \tau_2}{\epsilon_{y0} \tau_1} \left\langle \frac{T_m}{P_m} \right\rangle$$

$$\frac{c_0}{a_0} = \frac{\epsilon_{x0} \tau_3}{\epsilon_{s0} \tau_1} \left\langle \frac{G_m}{P_m} \right\rangle.$$
(36)

Indeed, since $\epsilon_{x0} \sim \Gamma \epsilon_{s0}$, Eq. (35) has the same physical significance as Eqs. (33) and (34). This can be easily verified by selecting a value $0.1 < k_0 < 10$ with $\Gamma \epsilon_{s0} = k_0 \epsilon_{x0}$.

Equations (33)–(35) indicate the physical connection between the equilibrium emittance with IBS and the lattice information represented by the ratio of radiation damping time to the IBS growth time under the condition of zero current emittances. In detail, the solution of x_t in Eqs. (33)–(35) is highly affected by the parameter a_0 , which represents the ratio of damping time to IBS growth time at zero current emittance. The increase of a_0 will lead to the growth of x_t , and the increase of a_0 means that the growth rate of IBS at zero current emittance increases faster than the damping rate. Also, the solution of x_t will be affected by the parameters $\frac{b_0}{a_0}$ and $\frac{c_0}{a_0}$, while those two parameters in Eq. (36) represent the ratio of vertical beam parameters to horizontal

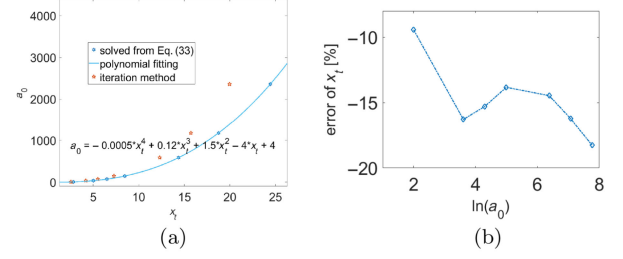


FIG. 6. Solution of x_t for different a_0 with Eq. (33) and the iteration method. (a) The scaling law solved from Eq. (33) for x_t and a_0 and (b) the relative error for x_t by iteration method and Eq. (33) for different a_0 .

beam parameters at zero current emittance and the ratio of longitudinal beam parameters to horizontal beam parameters at zero current emittance ($\frac{b_0}{a_0}$ can also be regarded as the ratio of horizontal IBS growth time to vertical growth time at zero current emittance, and $\frac{c_0}{a_0}$ the ratio of horizontal IBS growth time to longitudinal IBS growth time at zero current emittance. Here, we assume the ratio of damping time is constant).

We can take a very simple example to make the scaling law more clear for equilibrium emittance with the IBS effect. We take $\frac{b_0}{a_0} \sim 0.01$ and $\frac{c_0}{a_0} \sim 0.5$ for simplification, which is calculated from the lattice we used in Table III and set coupling to 0.01. $\frac{b_0}{a_0} \sim 0.01$ means the vertical IBS growth rate is much smaller than horizontal IBS growth rate at zero current emittance, and $\frac{c_0}{a_0} \sim 0.5$ means the longitudinal IBS growth rate is very close to horizontal IBS growth rate at zero current emittance. Under this condition, we can solve the Eq. (33) for different a_0 but keeping $\frac{b_0}{a_0}$ and $\frac{c_0}{a_0}$ constant, which corresponds to different beam current for the same lattice. The results are shown in Fig. 6. We solve the x_t in Eq. (33) for different a_0 and fitting it, meanwhile the x_t is also calculated by iteration method and plotted in Fig. 6(a). Figure 6(b) shows the relative error of x_t between two different methods, iteration method and solved from Eq. (33) at different a_0 . The error is between 10% and 20%, which is of course acceptable.

In summary, this approach then provides a comprehensive and rigorous theory to calculate equilibrium emittance with IBS, in contrast to the iterative procedure, which is only a phenomenological way to look for some semi-convincing result. The importance of this method is to reveal the physical connection between equilibrium emittance with IBS growth time at zero current emittance, but not the accuracy of equilibrium emittance calculation. Another potential application of our method will be used to optimize equilibrium emittance at high current at lattice design stage, as the quickly predicted equilibrium emittance can be used as objective in optimization algorithm. And we give an optimization example in next section.

V. IBS OPTIMIZATION FOR A HYBRID MBA LATTICE

In this study, we present an optimization example for a hybrid multibend achromat (MBA) lattice. The lattice configuration selected is analogous to that described in this paper [30], and the input parameters for the IBS calculation are comparable to those utilized in the research presented in [7]. We propose two optimization scenarios employing a multiobjective genetic algorithm (MOGA) for comparative analysis. The first scenario focuses on optimizing two objectives: the natural emittance, denoted as ϵ_0 , and the phase advance between the two dispersion bumps, which is pertinent to nonlinear optimization within the hybrid MBA lattice. The second scenario aims to optimize the IBS equilibrium emittance predicted by Eq. (33), alongside the phase advance between the two dispersion bumps in the hybrid MBA lattice. The definition of the phase advance objective is as follows

$$\begin{aligned} \text{obj}_2 = & |\phi_x(b_2) - \phi_x(b_1) - n_1\pi| \\ & + |\phi_y(b_2) - \phi_y(b_1) - n_2\pi|, \end{aligned} \quad (37)$$

where $\phi_x(b_2)$ and $\phi_x(b_1)$ represent the horizontal phase at the second and first dispersion bumps, respectively, while $\phi_y(b_2)$ and $\phi_y(b_1)$ denote the vertical phase at the second and first dispersion bumps, respectively. The variables n_1 and n_2 are defined as arbitrary odd integers. It is advisable to minimize the value of obj_2 toward zero.

A. One objective is the natural emittance

The MOGA optimization was conducted over approximately 10,000 generations with a population size of 2000. The results are presented in Fig. 7. We have selected a solution characterized by the objectives of (73.30 pm, 0.60 rad). The ring comprises 20 super-periods, and the Twiss functions for a single period are illustrated in Fig. 8.

The parameters of the ring and the equilibrium emittance considering IBS are presented in Table V. In this table, the equilibrium emittance is determined through an iterative process. A comparison is made between the equilibrium

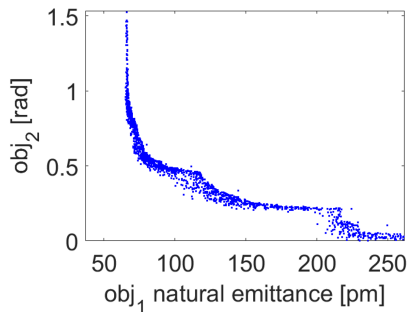


FIG. 7. MOGA results when the first objective is natural emittance.

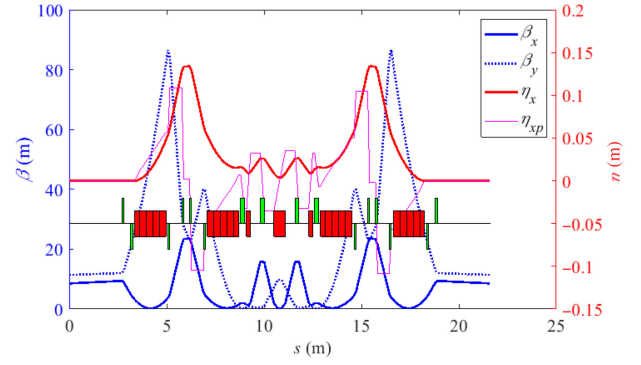


FIG. 8. Twiss functions of one solution when the first objective is the natural emittance.

emittance derived from the iterative calculations and that predicted by Eq. (33). The findings of this comparison are detailed in Table VI. It is evident from Table VI that the discrepancy between the predictions obtained from Eq. (33) and the iterative results is within an acceptable range. Furthermore, the predictive results facilitate the optimization of equilibrium emittance with IBS during the lattice design phase.

B. One objective is equilibrium emittance with IBS

We then substitute the objective of the natural emittance with the predicted emittance incorporating IBS. Two scenarios with values of κ set at 0.1 and 1, respectively, are established, where κ denotes the coupling factor between horizontal and vertical emittance. The results

TABLE V. Parameters of hybrid MBA ring. The optimization objective is natural emittance.

Parameter	Units	Value
Nominal energy	GeV	2.2
Circumference	m	432
Natural emittance	pm rad	73.30
Energy loss per turn	keV	104.8
Horizontal/vertical tune	...	48.199/32.198
Phase slippage factor	...	2.49×10^{-4}
Natural energy spread	...	6.56×10^{-4}
Horizontal/vertical damping time	ms	31.3/60.5
Longitudinal damping time	ms	57.9
Coupling for IBS	...	1
Bunch length at straight	mm	5.87
Electron number per bunch	...	6.25×10^9
Average beam current	mA	500
rf harmonic	...	720
rf voltage	kV	300
Horizontal equilibrium emittance with IBS	pm	97.10
Equilibrium bunch length with IBS	mm	7.96
Equilibrium energy spread with IBS	...	8.87×10^{-4}

TABLE VI. Comparison of IBS equilibrium emittance. Iteration and formula prediction are used. The coupling is $\kappa = 1$ in the upper table, $\kappa = 0.01$ in the bottom table.

Method	ϵ_x (pm)	ϵ_y (pm)	σ_s (mm)	σ_δ
Iteration	97.10	97.10	7.96	8.87×10^{-4}
Predicted by Eq. (33)	101.50	101.50	8.15	9.09×10^{-4}
Relative error	4.62%	4.62%	2.39%	2.48%

Method	ϵ_x (pm)	ϵ_y (pm)	σ_s (mm)	σ_δ
Iteration	265.84	2.66	11.20	12.48×10^{-4}
Predicted by Eq. (33)	284.02	2.84	11.51	12.84×10^{-4}
Relative error	6.84%	6.84%	2.77%	2.88%

show that the scenario with $\kappa = 0.1$ yields superior outcomes.

In this optimization example, we select a solution with a relatively lower equilibrium emittance to demonstrate its effectiveness by using the IBS equilibrium emittance as the objective, while the other objective value is $\text{obj}_2 = 4.21$ rad. This solution is referred to as Solu2. In an actual lattice design, both objectives should be balanced. However, we do not address this here because our goal is to show the effectiveness of substituting the natural emittance objective with the IBS equilibrium emittance.

In the case of Solu2, we show the detailed parameters of the ring and IBS equilibrium emittance in Table VII, and plot the Twiss functions in Fig. 9.

The parameters of beam current and bunching system used for the IBS calculation are the same as in Table VII.

TABLE VII. Parameters of hybrid MBA ring from Solu2. The optimization objective is equilibrium emittance with IBS.

Parameter	Units	Value
Nominal energy	GeV	2.2
Circumference	m	432
Natural emittance	pm rad	100.46
Energy loss per turn	keV	125.9
Horizontal/vertical tune	...	47.200/28.201
Phase slippage factor	...	2.34×10^{-4}
Natural energy spread	...	7.10×10^{-4}
Horizontal/vertical damping time	ms	25.79/50.38
Longitudinal damping time	ms	50.97
Coupling for IBS	...	1
Bunch length at straight	mm	6.27
Electron number per bunch	...	6.25×10^9
Average beam current	mA	500
rf harmonic	...	720
rf voltage	kV	300
Horizontal equilibrium emittance with IBS	pm	75.99
Equilibrium bunch length with IBS	mm	8.25
Equilibrium energy spread with IBS	...	9.34×10^{-4}

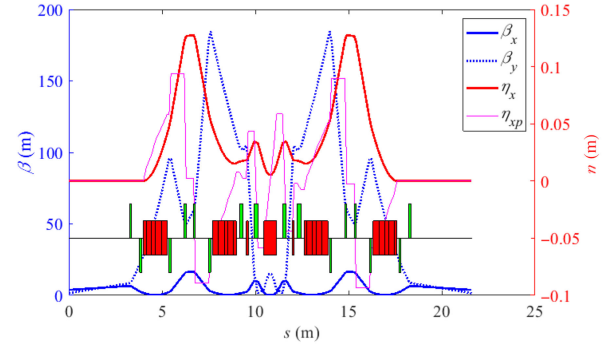


FIG. 9. Twiss functions of Solu2 when the first objective is equilibrium emittance with IBS.

To enhance the reliability of the comparison, we have recalculated the parameters in Table V by raising the energy loss per turn to match the level presented in Table VII. In an actual lattice design, a damping wiggler can be employed to increase the energy loss per turn. The required length of the damping wiggler to emit an additional 20 keV per turn is approximately 0.4 m, with a peak magnetic field of 4 T. We will disregard the horizontal emittance contribution from the wiggler since only a short length is necessary, and it can be placed in a dispersion-free region. Therefore, we have recalculated all the parameters from Table V within the same lattice by numerically increasing the damping, as detailed in Table VIII.

The comparison of the results presented in Tables VII and VIII indicates that the optimization is effective, resulting in a reduction of the equilibrium emittance with IBS by approximately 10%. However, it is important to

TABLE VIII. Parameters of hybrid MBA ring. The optimization objective is natural emittance.

Parameter	Units	Value
Nominal energy	GeV	2.2
Circumference	m	432
Natural emittance	pm rad	61.01
Energy loss per turn	keV	125.9
Horizontal/vertical tune	...	48.199/32.198
Phase slippage factor	...	2.49×10^{-4}
Natural energy spread	...	6.56×10^{-4}
Horizontal/vertical damping time	ms	26.05/50.36
Longitudinal damping time	ms	48.20
Coupling for IBS	...	1
Bunch length at straight	mm	5.87
Electron number per bunch	...	6.25×10^9
Average beam current	mA	500
rf harmonic	...	720
rf voltage	kV	300
Horizontal equilibrium emittance with IBS	pm	87.08
Equilibrium bunch length with IBS	mm	7.90
Equilibrium energy spread with IBS	...	8.77×10^{-4}

note that the nonlinear performance may be adversely affected. Furthermore, when seeking the optimal solution for equilibrium emittance in the context of IBS, it is not necessarily the case that a smaller natural emittance yields better results. While a reduced natural emittance leads to a decrease in the parameter $\mathcal{H}_x$ in the dipole and subsequently along the ring, it also results in an increase in the IBS growth rates associated with a smaller natural emittance. The optimization process aims to identify the optimal balance between $\mathcal{H}_x$ and natural emittance.

An additional aspect that warrants discussion is the relationship between the growth rate of IBS in the horizontal dimension and the average value of $\mathcal{H}_x$ along the ring, which exhibits a roughly proportional correlation. Meanwhile, the natural emittance is directly proportional to the average value of $\mathcal{H}_x$ within the dipole magnets. Therefore, to achieve further minimization of the equilibrium emittance associated with IBS, one could focus on reducing the maximum value of $\mathcal{H}_x$ in the dipoles, while simultaneously ensuring that the natural emittance does not decrease excessively by appropriately adjusting the lattice configuration.

A more pragmatic approach to optimization may involve considering two objectives: equilibrium emittance and a nonlinear indicator, while imposing a constraint on the phase advance between the two dispersion bumps. Although this topic extends beyond the current scope of this paper, it represents a potential avenue for future research stemming from our study.

VI. SUMMARY

In this study, we examined the IBS calculations within an SSMB storage ring, where the bunch length exhibits variation throughout the ring. Our findings indicate that this issue can be addressed by adapting and extending the B-M model through appropriate modifications to the matrix L^s . It is more straightforward and easily to be used in arbitrary planar SSMB storage rings characterized by longitudinal strong focusing. Additionally, we derived an approximate formula for IBS growth rates specifically to planar storage rings, which provides a comprehensive and rigorous theory to calculate equilibrium emittance with IBS. This formula facilitates the prediction of equilibrium emittance in relation to IBS through direct calculation rather than iterative methods, thereby enabling the use of equilibrium emittance as a target within optimization algorithms. We provide an example demonstrating the optimization of equilibrium emittance in the context of IBS, utilizing the predicted equilibrium emittance, and validate its effectiveness.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX: EXPRESSION FOR P_m , T_m , AND G_m

P_m , T_m , and G_m can be expressed as

$$\begin{aligned} P_m = & \beta_x \frac{a-b+m_a}{2m_a} \left(\frac{1}{\sqrt{\frac{u_2}{u_1}+1}} + f_{\min} - 2 \frac{1}{\sqrt{\frac{u_1}{u_2}+1}} \right) \\ & + \beta_x \frac{b-a+m_a}{2m_a} \left(\frac{1}{\sqrt{\frac{u_2}{u_1}+1}} + \frac{1}{\sqrt{\frac{u_1}{u_2}+1}} - 2f_{\min} \right) \\ & + \gamma^2 \mathcal{H}_x \frac{b-a+m_a}{2m_a} \left(\frac{1}{\sqrt{\frac{u_2}{u_1}+1}} + f_{\min} - 2 \frac{1}{\sqrt{\frac{u_1}{u_2}+1}} \right) \\ & + \gamma^2 \mathcal{H}_x \frac{a-b+m_a}{2m_a} \left(\frac{1}{\sqrt{\frac{u_2}{u_1}+1}} + \frac{1}{\sqrt{\frac{u_1}{u_2}+1}} - 2f_{\min} \right) \\ & - 2\beta_x \gamma \phi_x \frac{d}{m_a} \left(3 \frac{1}{\sqrt{\frac{u_1}{u_2}+1}} - 3f_{\min} \right) \end{aligned} \quad (A1)$$

$$\begin{aligned} T_m = & \beta_y \left(\frac{1-2\sqrt{\frac{u_1}{u_2}}}{1+\sqrt{\frac{u_1}{u_2}}} + f_{\min} \right) \\ G_m = & \eta_x^2 \gamma_s \frac{a-b+m_a}{2m_a} \left(\frac{1}{\sqrt{\frac{u_2}{u_1}+1}} + f_{\min} - 2 \frac{1}{\sqrt{\frac{u_1}{u_2}+1}} \right) \\ & + \eta_x^2 \gamma_s \frac{b-a+m_a}{2m_a} \left(\frac{1}{\sqrt{\frac{u_2}{u_1}+1}} + \frac{1}{\sqrt{\frac{u_1}{u_2}+1}} - 2f_{\min} \right) \\ & + \gamma^2 \beta_s \frac{a-b+m_a}{2m_a} \left(\frac{1}{\sqrt{\frac{u_2}{u_1}+1}} + f_{\min} - 2 \frac{1}{\sqrt{\frac{u_1}{u_2}+1}} \right) \\ & + \gamma^2 \beta_s \frac{b-a+m_a}{2m_a} \left(\frac{1}{\sqrt{\frac{u_2}{u_1}+1}} + \frac{1}{\sqrt{\frac{u_1}{u_2}+1}} - 2f_{\min} \right), \end{aligned}$$

where $m_a = \sqrt{(a-b)^2 + 4d^2}$ and $f_{\min} = \ln \sqrt{\frac{\min(u_1, u_2)}{u_3}}$. In dispersive locations, we know that $b \gg a$ and $b \gg d$ (this is satisfied when $\gamma\phi_x \gg 1$). Then we can get the approximation below

$$\begin{aligned} \frac{a-b+m_a}{2m_a} &\sim \frac{d^2}{4(b-a)^2} \ll 1 \\ \frac{b-a+m_a}{2m_a} &\sim 1 \\ \frac{b-a+m_a}{2m_a} &\gg \frac{a-b+m_a}{2m_a}. \end{aligned} \quad (\text{A2})$$

Then Eq. (A1) can be rewritten as

$$\begin{aligned} P_m &= \beta_x \left(\frac{1}{\sqrt{\frac{u_2}{u_1} + 1}} + \frac{1}{\sqrt{\frac{u_1}{u_2} + 1}} - 2f_{\min} \right) \\ &\quad + \gamma^2 \mathcal{H}_x \left(\frac{1}{\sqrt{\frac{u_2}{u_1} + 1}} + f_{\min} - 2 \frac{1}{\sqrt{\frac{u_1}{u_2} + 1}} \right) \\ &\quad - 2\beta_x \gamma \phi_x \frac{d}{m_a} \left(2 \frac{1}{\sqrt{\frac{u_2}{u_1} + 1}} - \frac{1}{\sqrt{\frac{u_1}{u_2} + 1}} - f_{\min} \right) \\ T_m &= \beta_y \left(\frac{1 - 2\sqrt{\frac{u_1}{u_2}}}{1 + \sqrt{\frac{u_1}{u_2}}} + f_{\min} \right) \\ G_m &= \eta_x^2 \gamma_s \left(\frac{1}{\sqrt{\frac{u_2}{u_1} + 1}} + \frac{1}{\sqrt{\frac{u_1}{u_2} + 1}} - 2f_{\min} \right) \\ &\quad + \gamma^2 \beta_s \left(\frac{1}{\sqrt{\frac{u_2}{u_1} + 1}} + \frac{1}{\sqrt{\frac{u_1}{u_2} + 1}} - 2f_{\min} \right). \end{aligned} \quad (\text{A3})$$

We know that $\frac{1}{\sqrt{\frac{u_2}{u_1} + 1}} < 1$ and $\frac{1}{\sqrt{\frac{u_1}{u_2} + 1}} < 1$. Utilizing the approximation $u_1 \gg u_3$ and $u_2 \gg u_3$ again, we can obtain the condition below

$$f_{\min} = \ln \sqrt{\frac{\min(u_1, u_2)}{u_3}} \gg 1. \quad (\text{A4})$$

So

$$\begin{aligned} f_{\min} &\gg \frac{1}{\sqrt{\frac{u_2}{u_1} + 1}} \\ f_{\min} &\gg \frac{1}{\sqrt{\frac{u_1}{u_2} + 1}}. \end{aligned} \quad (\text{A5})$$

The parameters P_m, T_m, G_m , as well as the IBS growth rates, are predominantly influenced by $f_{\min}$. During the

iteration process to calculate equilibrium emittance with IBS, when the emittance of the beam is modified, the ratio $\frac{\min(u_1, u_2)}{u_3}$ may experience slight variations; however, the corresponding change in $\ln \sqrt{\frac{\min(u_1, u_2)}{u_3}}$ is negligible. Consequently, it is reasonable to consider P_m, T_m , and G_m as invariant throughout the iteration process of calculating IBS equilibrium emittance. Alternatively, one could assert that P_m, T_m , and G_m exhibit minimal dependence on beam emittance, being primarily associated with lattice optics.

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