

Analysis of the impact of β -beat induced asymmetry on nonlinear dynamics of storage rings

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In a storage ring, focusing perturbations mainly from quadrupole strength errors and sextupole alignment errors induce β -beat, breaking the periodic symmetry of lattice and deteriorating the performance of nonlinear dynamics. There is a general negative correlation between β -beat and nonlinear performance. In this paper, the impact of β -beat-induced asymmetry on nonlinear dynamics of storage ring is studied with multi-bend achromat (MBA) lattices. For on-momentum dynamic aperture (DA), the asymmetry parameters are introduced to quantitatively characterize the breaking of lattice symmetry, which are calculated from the Fourier harmonics of the β -functions and resonance driving terms (RDTs). Compared to the commonly used β -beat, the asymmetry parameters exhibit stronger correlations with reduced DA, which is further enhanced when considering the frequency diffusion rate. In addition, the commonly used β -beat serving as a reasonable indicator of DA degradation is physically explained based on RDT variations. For off-momentum nonlinear dynamics, the half-integer resonance crossing of chromatic tunes is found to have a small effect on six-dimensional (6D) local momentum aperture for the hybrid MBA lattice with β -beat, when the tune shifts with momentum are reasonably controlled. Moreover, the reduced off-momentum DAs with β -beat have stronger correlations with the off-momentum asymmetry parameters than the on-momentum ones.

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I. INTRODUCTION

In storage rings, quadrupole strength errors, sextupole alignment errors, and other effects introduce focusing perturbations, affecting linear optics and nonlinear dynamics [1,2]. For linear optics, these perturbations result in tune shifts and distortion of β -functions, commonly known as β -beat [3], which can be usually corrected to about 1% rms level or even lower values in horizontal and vertical planes using orbit-response-matrix-based methods [4–8]. For nonlinear dynamics, the impact of β -beat is more complex. The distortions in β -functions and phase advances deviate the nonlinear driving terms from their optimized values. Furthermore, the random nature of the errors, as well as asymmetric optics with super-bends, double mini- β straight sections, or cell replacement [9–11], breaks the periodic symmetry of the lattice, exciting additional structural resonances [1,12]. These degradations in nonlinear dynamics can cause a significant

reduction in the dynamic aperture (DA) and momentum aperture (MA), affecting the beam injection and lifetime of a storage ring.

In characterizing the impact of such errors, the magnitude of β -beat is a commonly used metric [5–8], which is typically large for modern light-source storage rings before optics corrections. There is a general negative correlation between β -beat and nonlinear performance, including DA and beam lifetime [1,7,13–15]. To enhance the correlation between the deviation of β -functions and nonlinear dynamics, the asymmetry parameter is introduced in this paper, which better characterizes the periodicity breaking. Furthermore, the correlation between β -beat and DA can be understood through the formalism of resonance driving terms (RDTs), since the geometric RDTs are strongly related to β -functions [16]. Recently, B. Wei *et al.* [17] found that minimizing the variation of RDTs along the longitudinal position of the storage ring is much more effective in increasing DA than minimizing the commonly used one-turn RDTs. In this paper, this longitudinal variation of RDTs is used to study the β -beat-induced DA reduction.

In addition to DA, β -beat also has a significant impact on off-momentum nonlinear dynamics. Hybrid multi-bend achromat (HMBA) lattices are used in the design of some fourth-generation synchrotron light sources [18–23], in which the chromatic tunes can cross the half-integer

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resonance due to large higher-order chromaticities. The impact of half-integer resonance on the local momentum aperture (LMA) in the dispersion-free straight section of the HMBA lattice was studied [24,25]. LMAs are typically smaller in the dispersion bumps, where the impact of β -beat on half-integer resonance crossing is more significant. However, it is found in this paper that the half-integer resonance is not the actual limit of six-dimensional (6D) LMA provided that the chromatic tunes are reasonably controlled. Moreover, instead of on-momentum optics, it is more direct to use off-momentum optics to study the off-momentum nonlinear dynamics, as in Ref. [26]. In this paper, the off-momentum DA reduction with β -beat is analyzed using off-momentum β -functions and off-momentum RDT variations.

In this paper, the impact of β -beat-induced asymmetry on both on- and off-momentum nonlinear dynamics is analyzed, with two typical types of multi-bend achromat (MBA) lattices, HMBA and conventional MBA lattices, as examples. To isolate optics distortion, which has an important effect on nonlinear dynamics once the closed orbit is corrected, from other error effects, this paper focuses on the relationship between β -beat and nonlinear performance. This paper is organized as follows. Section II analyzes the impact of β -beat on the DA, where asymmetry parameters are introduced and RDT variations are used. Section III studies the impact of β -beat on off-momentum nonlinear dynamics, including half-integer resonance crossing in an HMBA lattice and off-momentum DA reduction. Finally, conclusions are given in Sec. IV.

II. IMPACT ON ON-MOMENTUM DYNAMIC APERTURE

MBA lattices can be typically categorized into two types: HMBA lattice with concentrated sextupoles in dispersion bumps for chromaticity correction [18] and conventional MBA lattice with distributed sextupoles [27]. The degradation of nonlinear dynamics due to β -beat will be investigated by analyzing these two types of MBA lattices.

A. The HALF 6BA lattice

The Hefei Advanced Light Facility (HALF) storage ring lattice is taken as an example of an HMBA lattice. HALF is a fourth-generation synchrotron light source with a beam energy of 2.2 GeV and a natural emittance of 86 pm. rad. The storage ring, consisting of 20 identical cells, employs an H6BA lattice with long and short straight sections [22,28]. The horizontal and vertical tunes are 48.15 and 17.15, respectively. The magnet layout and optical functions of a lattice cell are shown in Fig. 1.

Random normal quadrupole strength errors ($\Delta K/K$) of all 320 quadrupole magnets are introduced to generate

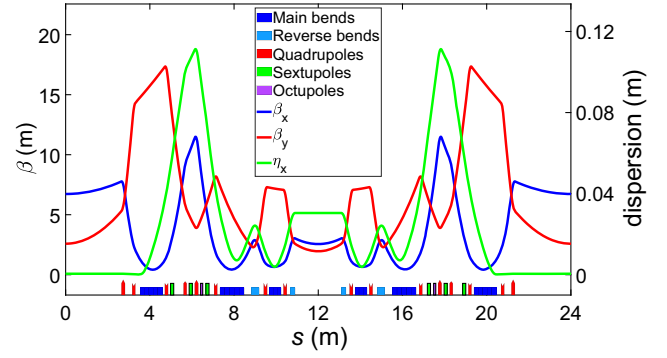


FIG. 1. Magnet layout and optical functions of an H6BA lattice cell of the HALF storage ring.

β -beat, and the betatron tunes are fitted to the nominal values (48.15, 17.15) by jointly scaling the strengths of the focusing and defocusing quadrupole families. The feed-down effect from sextupoles effectively acts as quadrupole errors, and our simulations found that this feed-down effect produces the same result as quadrupole strength errors. Therefore, no offset is introduced to the sextupoles. To simplify the analysis, multiple errors and coupling errors are disregarded in this paper. Totally 1000 error seeds are generated with a maximum RMS $\Delta K/K$ of about 1×10^{-3} and a maximum β -beat (sum of RMS $\Delta\beta_x/\beta_x$ and RMS $\Delta\beta_y/\beta_y$) of about 15%. The distribution of $\Delta K/K$ and β -beat values for all the seeds is presented in Fig. 2. Note that the error levels cover a range of β -beat below 2%, which is consistent with the practical β -beat level after correction.

For all the 1000 error seeds, the DA areas were tracked at the center of the long straight section. The tracking was performed using the ELEGANT code [29]. Since the β -function values in the straight section vary for different error seeds and the DA is proportional to the square root of β -functions, the DA area is normalized for a better comparison:

$$\bar{S}_{DA} = \frac{S_{DA}}{\sqrt{\beta_{x0} \cdot \beta_{y0}}}, \quad (1)$$

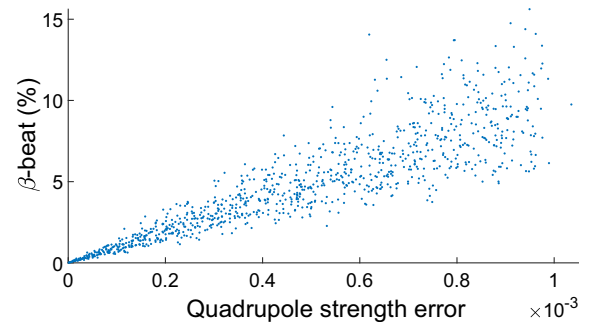


FIG. 2. Distribution of quadrupole strength error and β -beat values for the HALF lattice.

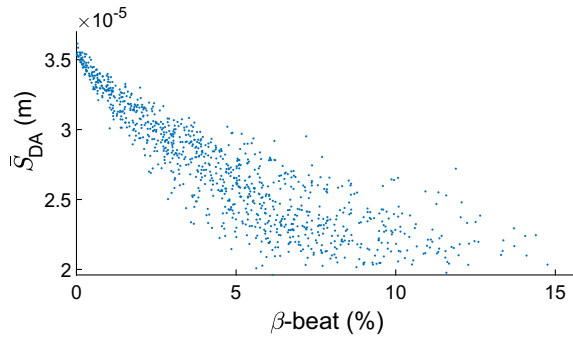


FIG. 3. Correlation between β -beat and the normalized DA area for the HALF lattice. The Spearman coefficient of β -beat and the normalized DA area is 0.904.

where β_{x0} and β_{y0} are the β -function values at the center of the long straight section, S_{DA} is the DA area tracked in x - y plane, and $\bar{S}_{DA}$ is the normalized DA area. The correlation between β -beat and $\bar{S}_{DA}$ for all error seeds is shown in Fig. 3. It is seen that $\bar{S}_{DA}$ generally decreases as β -beat grows, with a Spearman correlation coefficient of 0.904. In this paper, the Spearman correlation coefficients are considered only as absolute values, which represent both positive and negative correlations. Therefore, the value of β -beat is conventionally used to describe the correction performance of a storage ring. However, it is also seen that the monotonicity is not strong enough, and there are error seeds with the same β -beat but different normalized DA areas.

Strictly speaking, with the errors induced, the ideal symmetric lattice with 20 periods degrades into an asymmetric lattice with only one super period. Due to the breaking of symmetry, the nominal non-structural resonance lines can transform into structural ones, which are more deleterious to nonlinear dynamics. The breaking of symmetry can be more explicitly expressed by performing the Fourier transformation. Figure 4 shows the harmonics of the fast Fourier transform (FFT) of β_x for the ideal lattice and an error lattice from Fig. 2. It is seen that, for the ideal lattice, there are only harmonics of the number of periods, i.e., $20n$ ($n = 1, 2, \dots$), whereas for the error lattice, the harmonics other than $20n$ appear due to the symmetry breaking.

To quantify the breaking of symmetry, the asymmetry parameter of the β -function, α_{β_u} ($u = x, y$), is defined based on the proportion of harmonics, which is calculated as

$$\alpha_{\beta_u} = 1 - \frac{\sum A_{\beta_u, 20n}}{\sum A_{\beta_u, i}}, \quad (2)$$

where $A_{\beta_u, 20n}$ is the amplitude of the $20n$ th harmonic and $A_{\beta_u, i}$ is the amplitude of the i th harmonics ($i = 1, 2, \dots$). The harmonics are calculated by the FFT of β -function.

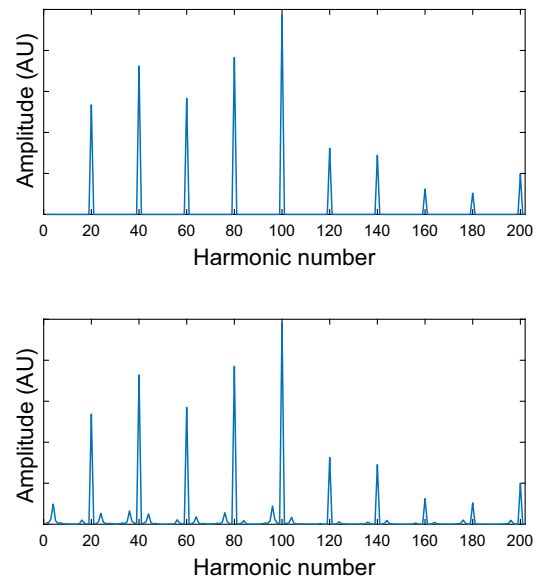


FIG. 4. Harmonics of the FFT of β_x for the ideal lattice (upper) and an error lattice from Fig. 2 (lower). The vertical axes of the two plots share the same scale. There are no harmonics other than $20n$ ($n = 1, 2, \dots$) in the ideally 20-fold symmetric lattice.

Considering both β_x and β_y , the total asymmetry parameter, α_β , is defined as the RMS value of α_{β_x} and α_{β_y} . The value of α_β is in the range of $0 \sim 1$, with 0 representing ideal symmetry and larger α_β representing greater asymmetry. The definition of α_β is reasonable, as demonstrated in the following simulations.

The correlation between β -beat, α_β and the normalized DA area for the error seeds is shown in Fig. 5. There is a strong positive correlation between β -beat and α_β . It can also be seen that for a given α_β , the normalized DA area is restricted to a narrower range compared to the case with a given β -beat. The correlation between α_β and the normalized DA area is stronger, with a Spearman coefficient of 0.935. As β -beat calculates the RMS value of the β deviations, it contains both symmetric and asymmetric

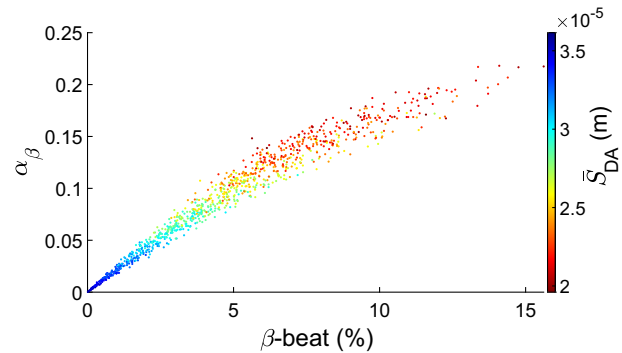


FIG. 5. Correlation between β -beat, α_β , and the normalized DA area for the HALF lattice. The Spearman coefficient of the correlation between α_β and the normalized DA area is 0.935.

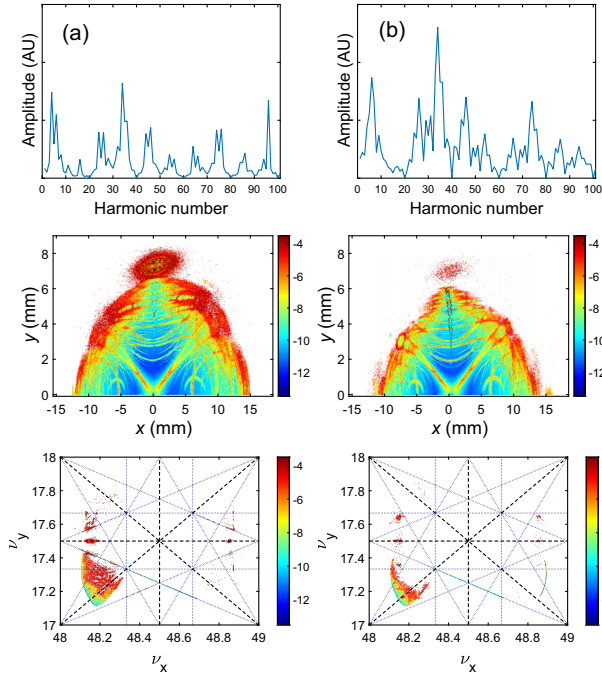


FIG. 6. Two error seeds for the HALF lattice with the same β -beat but different α_β and normalized DA areas. The left three plots are for seed A, and the right three for seed B. Upper: the harmonics of the FFT of β -functions, where the harmonics of $20n$ are excluded for better comparison. Middle: the FMA of DAs in x - y space. Bottom: the FMA of DAs in tune space. The color bars denote frequency diffusion rates.

components. In contrast, α_β only calculates the components that break the lattice periodicity and excite more deleterious resonances. This suggests that the asymmetry of the β -function is a more effective parameter than β -beat in denoting the reduction of DA. Figure 6 shows the harmonics of the FFT of β -functions and frequency map analysis (FMA) of DAs for two error seeds from Fig. 5 with the same β -beat but different α_β . In this figure, the harmonics are the sum of Fourier harmonics of β_x and β_y . Table I lists their β -beat, α_β , $\bar{S}_{DA}$, and β_{x0}/β_{y0} at the center of the long straight section. It is seen that despite having the same β -beat, the seed with lower α_β exhibits a larger normalized DA area, while the other seed with the higher α_β exhibits a smaller normalized DA area.

TABLE I. The values of β -beat, α_β , $\bar{S}_{DA}$, and β_{x0}/β_{y0} for the two error seeds in Fig. 6, where β_{x0}/β_{y0} are the horizontal and vertical β -functions at the center of the long straight section.

	Seed A	Seed B
β -beat	5.6%	5.6%
α_β	0.09	0.15
$\bar{S}_{DA}(\text{m}^{-1})$	2.8×10^{-5}	2.2×10^{-5}
β_{x0}/β_{y0} (m)	6.8/2.6	6.8/2.4

In the normal form analysis, the nonlinear part of the Hamiltonian for particle motion in a storage ring can be expanded into a series of nonlinear driving terms [16]. Generally, minimizing RDTs can improve DA. Compared to the commonly used one-turn RDTs, the variation of RDTs along the longitudinal position of the ring is a much more effective indicator of nonlinear dynamics [17]. The longitudinal variation of n th-order geometric RDTs, denoted as $f_{n,\text{avg}}$, is calculated as

$$f_{n,\text{avg}} = \sqrt{\sum_{j+k+l+m=n} (f_{jklm0,\text{avg}})^2}, \quad (3)$$

where $f_{jklm0,\text{avg}}$ is the average of a geometric RDT over the positions of nonlinear magnets. Considering both third- and fourth-order RDT variations, the overall RDT variation, f_{avg} , is defined as the weighted sum, $f_{3,\text{avg}} + w \cdot f_{4,\text{avg}}$, where the weight coefficient w is set to $0.01 \text{ m}^{1/2}$ in this paper. In the HALF lattice with the working point on the difference resonance $2\nu_x - 2\nu_y$, the corresponding fourth-order RDT h_{20020} grows significantly large with β -beat, dominating the contributions of all RDTs. Therefore, the variation of the RDT h_{20020} is assigned an additional weighting factor of 0.001 in the calculation of $f_{n,\text{avg}}$ for the HALF lattice.

The geometric RDTs caused by sextupoles are determined by three parameters: the integrated strengths of sextupoles, the β -functions at sextupole locations, and the betatron phase advances between the sextupoles. For all error seeds, the sextupole strengths are fixed. Since the phase advance is calculated with the β -function as $\Delta\phi_u = \int_{s_1}^{s_2} ds/\beta_u(s)$, the deviation of the β -function, characterized by β -beat and α_β , is strongly correlated with an increase in RDT variations, as shown in Fig. 7. It is shown in Ref. [17] that solutions with larger RDT variations generally have smaller DAs. Therefore, the commonly used β -beat provides a reasonable indication of DA degradation, as previously shown in Fig. 5.

Accordingly, given that the reduction in DA shows a stronger correlation with the asymmetry of the β -function

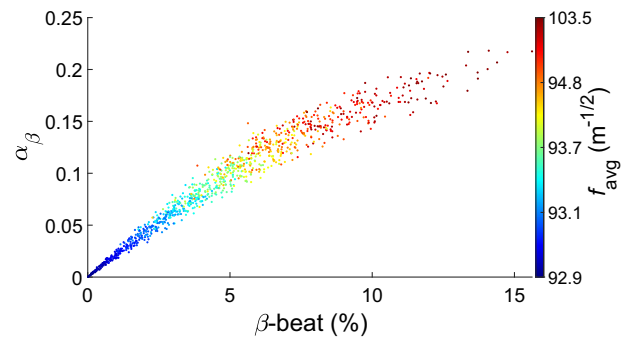


FIG. 7. Correlation between β -beat, α_β , and RDT variations for the HALF lattice.

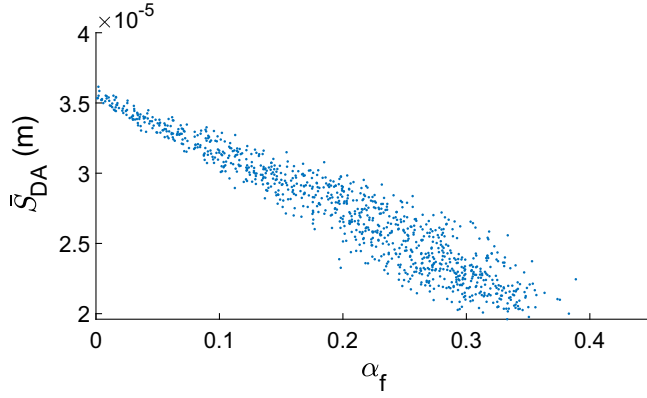


FIG. 8. Correlation between α_f and the normalized DA area for the HALF lattice. The Spearman coefficient is 0.945.

than with β -beat, the asymmetry of RDTs along the longitudinal position can also characterize the degradation of nonlinear dynamics [30]. The asymmetry of n th-order geometric RDT variations, denoted as $\alpha_{f,n}$, is defined in a similar way to α_β :

$$\alpha_{f,jklm0} = 1 - \frac{\sum A_{f,jklm0,20n}}{\sum A_{f,jklm0,i}}, \quad (4)$$

$$\alpha_{f,n} = \text{RMS}(\alpha_{f,jklm0}), \text{ with } j+k+l+m=n, \quad (5)$$

where $A_{f,jklm0,20n}$ and $A_{f,jklm0,i}$ are the amplitudes of harmonics calculated by the FFT of f_{jklm0} . Considering the asymmetry of both third- and fourth-order RDT variations, the overall asymmetry of the RDT variations, α_f , is defined as the average of $\alpha_{f,3}$ and $\alpha_{f,4}$. The value of α_f is also in the range of $0 \sim 1$, which represents the degree of asymmetry of the RDT variations. Figure 8 shows the correlation between α_f and the normalized DA area for the error seeds, with a higher Spearman correlation coefficient of 0.945. This indicates that the asymmetry of RDT variations is a better indicator for characterizing the reduction in DA, since it considers both the distortion of RDTs and the breaking of symmetry.

For the DA, in addition to the area of DA, the stability of particle motion inside the DA is also important, which can be quantified by the frequency diffusion rate [31,32]. Figure 9 shows the FMA of the DAs for two error seeds with almost the same β -beat and normalized DA area but different average diffusion rates. Considering both the DA area and particle motion stability, the seed in the upper plot has a better DA. To consider both the DA area and particle motion stability, the average diffusion rate of all tracked particles is regarded as a more comprehensive representation of the DA. Further considering the normalization in Eq. (1), the normalized average diffusion rate, denoted as $\bar{D}$, is calculated as

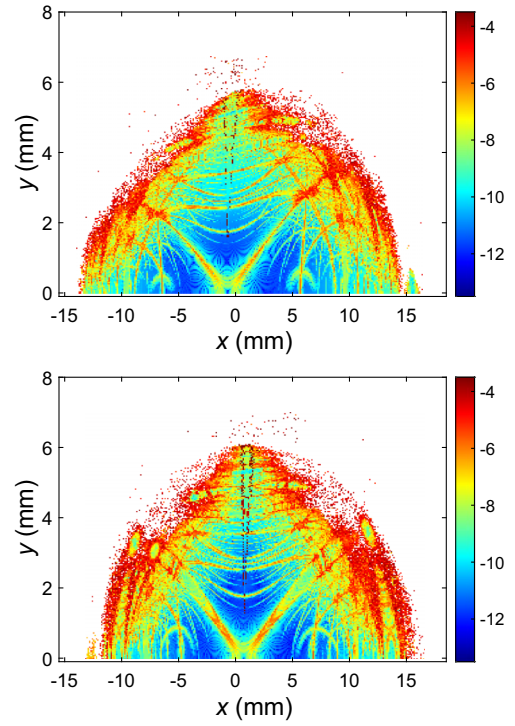


FIG. 9. The FMA of DAs in x - y space for two error seeds of the HALF lattice, which have the similar β -beat and normalized DA area but different average frequency diffusion rates. The color bars denote diffusion rates. The seed in the upper plot has lower average diffusion rate (i.e., more stable particle motion) than the seed in the lower plot.

$$\bar{D} = \frac{\sum D_i}{N_{\text{particle}} \sqrt{\beta_{x0} \cdot \beta_{y0}}}, \quad (6)$$

where D_i is the diffusion rate of each tracked particle, with the value of 0 assigned to lost particles; N_{particle} is the number of all tracked particles; and $\sqrt{\beta_{x0} \cdot \beta_{y0}}$ is the normalization factor. In the following, $\bar{D}$ instead of $\bar{S}_{\text{DA}}$ is used to better represent the DA.

Figure 10 shows the correlation between α_f and $\bar{D}$ for the error seeds, which is quite strong with a Spearman coefficient of 0.963. Table II compares the Spearman correlation coefficients of each parameter used above with $\bar{S}_{\text{DA}}$ and $\bar{D}$ for the error seeds of the HALF lattice. It is seen that the Spearman coefficients with $\bar{D}$ are stronger than those with $\bar{S}_{\text{DA}}$. We also see that, compared with β -beat and f_{avg} , the asymmetry parameters, α_β and α_f , have stronger correlations with the DA. This reveals the significance of β -beat-induced asymmetry in nonlinear dynamics.

In realistic operation, systematic symmetry breaking introduced by insertion devices (IDs) can be more severe than random lattice errors. To verify the robustness of the proposed asymmetry parameters, simulations were performed for the HALF lattice by introducing planar IDs

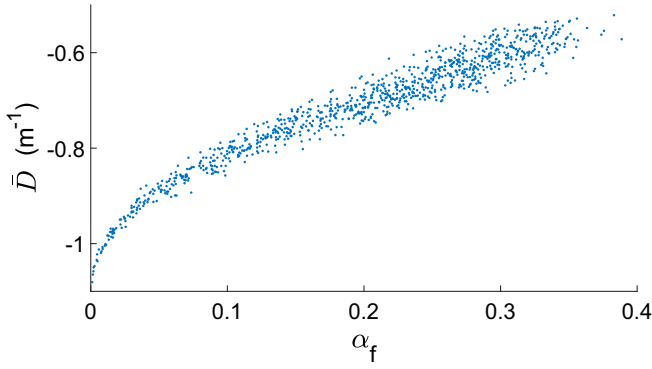


FIG. 10. Correlation between α_f and $\bar{D}$ for the HALF lattice. The Spearman coefficient is 0.963.

with simplified models. To simplify the analysis of the RDTs, a pseudo-quadrupole—acting as a focusing quadrupole in the vertical plane and as a drift in the horizontal plane, was introduced to model the planar IDs. In this study, 10 planar IDs were evenly inserted into 10 long straight sections of HALF. We generated 100 configurations; for each configuration, the field strengths of the IDs were drawn from a normal distribution, with the mean and variance of the distribution increasing across the 100 configurations. The five upstream and five downstream quadrupoles were used to compensate for the optics distortion locally, followed by a global tune correction. The results after the correction show that the asymmetry parameters still have stronger correlations with the DA compared with β -beat. Figure 11 shows the correlations of α_β and β -beat with $\bar{D}$.

As for f_{avg} , it does not have a good correlation with $\bar{D}$ in the case of IDs. The main reason is that, despite the relatively high β -beat in the ID straight sections, the variations of the β -functions at the sextupole and octupole locations are very small due to the local compensation, which differs from the case of random quadrupole errors studied before. When the relative change in f_{avg} is subtle, f_{avg} is unable to reveal the small difference in the DA. In contrast, the asymmetry parameters are calculated from Fourier analysis and indicate the relative perturbation to the reference lattice, which can resolve and better reveal small differences in nonlinear performance. In this regard, the asymmetry parameters are more effective than f_{avg} .

TABLE II. Spearman correlation coefficients of the normalized DA area and normalized average diffusion rate with β -beat, f_{avg} , α_β , and α_f for the error seeds of the HALF lattice.

	$\bar{S}_{\text{DA}}$	$\bar{D}$
β -beat	0.904	0.917
f_{avg}	0.905	0.923
α_β	0.935	0.948
α_f	0.945	0.963

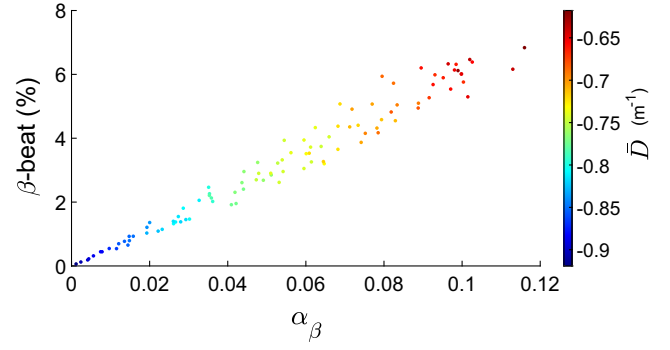


FIG. 11. Correlations of α_β and β -beat with $\bar{D}$ for 100 configurations of the HALF lattice with planar IDs. The Spearman coefficients of α_β and β -beat with $\bar{D}$ are 0.990 and 0.980, respectively. The lowest value of $\bar{D}$ is different from that in Fig. 10 due to different grids used in the DA tracking.

B. An HOA 6BA lattice

To enhance generality, a conventional 6BA lattice with distributed chromatic correction [33] is taken as another example, which is based on the nonlinear cancelation of a higher-order achromat (HOA). The storage ring has an energy of 2.2 GeV and a circumference of 297.6 m, consisting of 14 HOA 6BA lattice cells. In the lattice cells, the horizontal and vertical tunes of the bend unit cell are (0.4, 0.1) to make an HOA over five identical unit cells. The natural emittance is 132.8 pmrad, and the horizontal and vertical tunes are 36.30 and 12.14, respectively. The magnet layout and optical functions of a lattice cell are shown in Fig. 12, and the FMA of the DA is shown in Fig. 13.

Similarly, 1000 error seeds were generated with RMS values in the same range as those of the HALF H6BA lattice, and the same tracking and calculation were performed. Since the working point of the HOA 6BA lattice is not on the difference resonance, all fourth-order RDTs are equally weighted in the calculation of $f_{4,\text{avg}}$. The correlation between α_f and $\bar{D}$ for all the error seeds is shown in Fig. 14, with a strong Spearman coefficient of

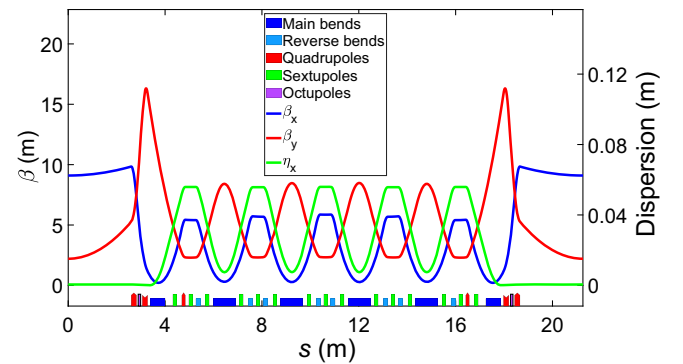


FIG. 12. Magnet layout and optical functions of an HOA 6BA lattice cell.

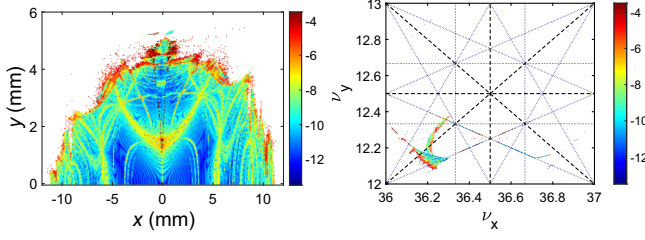


FIG. 13. FMA of the DA of the HOA 6BA lattice, tracked at the center of the straight section.

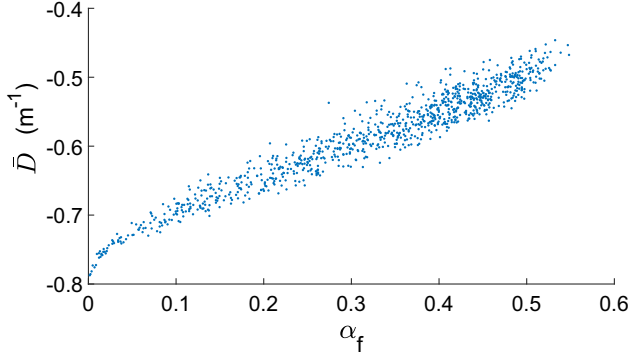


FIG. 14. Correlation between α_f and $\bar{D}$ for the error seeds of the HOA 6BA lattice. The Spearman coefficients is 0.957.

0.957. Table III lists the Spearman correlation coefficients of each parameter used before with $\bar{D}$. It is seen that the asymmetry parameters have stronger correlations with the DA.

The results of these two lattices reveal that the asymmetry parameters, α_β and α_f , are effective indicators of DA performance, offering a potential advantage for the correction and optimization of storage rings with errors. Although α_f shows the strongest correlation, the experimental measurement of RDTs [34] is considerably more complicated than that of the β -functions. Therefore, the β -function-based parameters can be more practical indicators.

III. IMPACT ON OFF-MOMENTUM NONLINEAR DYNAMICS

Momentum-dependent tune shifts and off-momentum DAs can affect the LMA. With errors included and

TABLE III. Spearman correlation coefficients of normalized average diffusion rate with β -beat, f_{avg} , α_β and α_f for the error seeds of the HOA 6BA lattice.

	$\bar{D}$
β -beat	0.928
f_{avg}	0.919
α_β	0.950
α_f	0.957

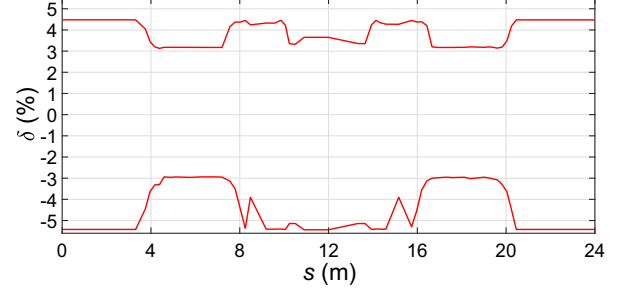


FIG. 15. The 6D LMA along a cell of the HALF ideal lattice.

subsequent symmetry breaking, the half-integer resonance crossing of chromatic tunes [35], which is seen in some HMBA lattices with relatively large higher-order chromaticities [19,22,25,36], can cause a significant reduction in the LMA. As in Sec. II, the off-momentum DA reduction due to β -beat can be similarly studied, where the parameters based on off-momentum optics rather than on-momentum optics may be more effective.

A. The HALF 6BA lattice

The HALF lattice is used as an example to study the reduction of LMA due to resonance crossing in the presence of β -beat. The 6D LMA of the HALF ideal lattice is shown in Fig. 15. Figure 16 shows the

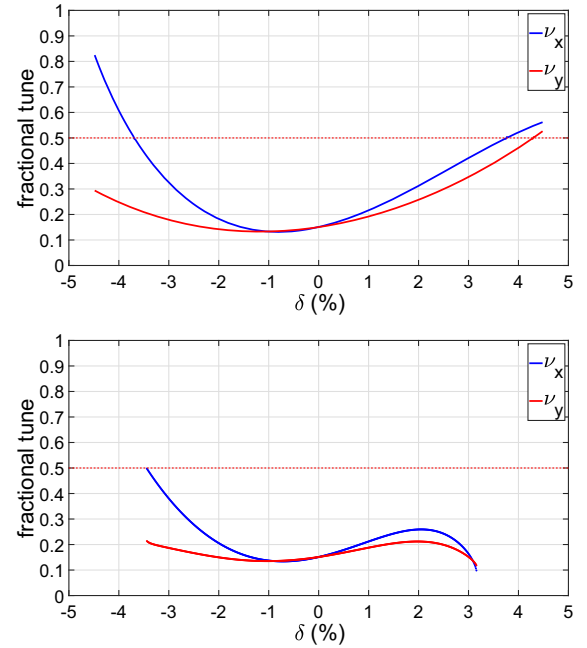


FIG. 16. Momentum-dependent tune shifts tracked at two different locations on the reference orbit of the HALF ideal lattice. Upper: tracked at the center of the dispersion-free long straight section; lower: tracked at the location of SF in dispersion bump. In the lower plot, the particle starts not from the off-momentum orbit (dispersive orbit) but at the origin of the x -axis, so the betatron oscillation is induced.

momentum-dependent tune shifts tracked at two different locations on the reference orbit, which is at the origin of transverse coordinates. The upper plot of Fig. 16 shows the tune shifts tracked at the center of the dispersion-free long straight section, from which it is seen that the horizontal tune crosses the half-integer resonance line $\nu_x = 48.5$ at relative momentum deviations $\delta = -3.7\%$ and 3.8% . The lower plot shows the tune shifts tracked at the focusing sextupole (SF) in dispersion bump, which are different from the upper plot due to the inclusion of off-momentum amplitude-dependent tune shifts (ADTSs). The SF location is chosen for its maximum dispersion and minimal LMA, where the effect of induced betatron oscillation is most visible. Detailed tracking found that these tune shifts remain largely unchanged when errors are included.

To better reveal the resonance effect with β -beat, the 4D dynamic LMA without an RF cavity was first studied, where the previously generated error seeds were used. Figure 17 shows the 4D LMAs for the error seeds tracked at the center of the long straight section. The 4D LMAs in the positive direction are limited by the structural resonance line $\nu_y = 20$ to about 11%, and they decrease as β -beat grows, affected by the integer resonance lines $\nu_y = 19, 18$, the half-integer resonance lines $\nu_y = 17.5$ and $\nu_x = 48.5$, as well as the off-momentum DA reduction. Nevertheless, the LMAs in the positive direction are large enough for most of the error seeds. The 4D LMAs in the negative direction are limited to about -5.4% with no further reduction, corresponding to the integer resonance

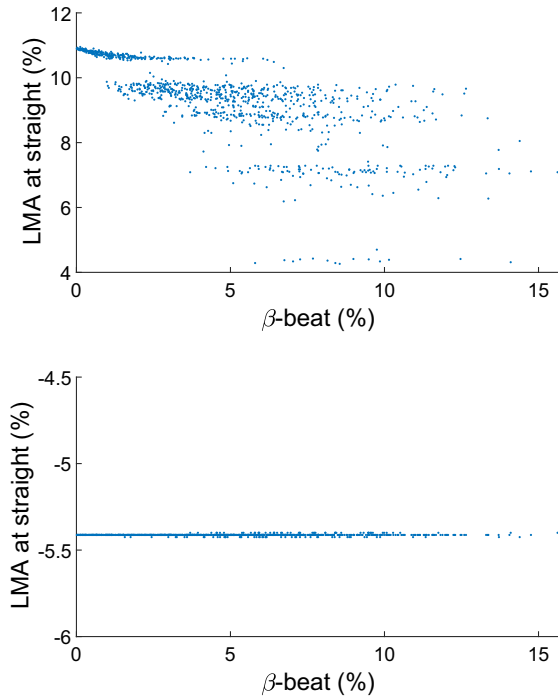


FIG. 17. The 4D LMAs for the error seeds of the HALF lattice tracked at the center of the long straight section. Upper: LMAs in the positive direction; lower: LMAs in the negative direction.

line $\nu_x = 50$, which is also a structural half-integer resonance line ($50/20 = 2.5$). Since the RF momentum acceptance of HALF is about 5%, the resonance lines $\nu_x = 49$ and $\nu_x = 48.5$ are not the actual limits of the 6D LMA at the long straight section even with large β -beat values. A similar result was also found in Ref. [24], where the half-integer resonance does not limit the LMA in the straight section.

The smallest 6D LMA in the dispersion bumps, shown in Fig. 15, is the primary limit of Touschek lifetime. In the LMA tracking at the SF location, the tunes of an off-momentum particle include an additional contribution from the horizontal betatron amplitude due to dispersion. The off-momentum tunes with horizontal amplitudes tracked at the SF location are seen in Fig. 18, where the tunes increase with horizontal amplitude in the negative- δ direction and decrease in the positive- δ direction. Consequently, as shown in the lower plot of Fig. 16, the tunes at the SF location cross the half-integer resonance line $\nu_x = 48.5$ at about $\delta = -3.4\%$ in the negative direction, and the tunes drop rapidly to the integer resonance line $\nu_x = 48$ at about $\delta = 3.2\%$ in the positive direction.

Figure 19 shows the 4D LMAs for the error seeds tracked at the SF location. In the negative direction, the LMA without errors is about -5.2% . With small β -beat, it collapses to about -3.4% , which corresponds to the half-integer resonance line $\nu_x = 48.5$. As β -beat grows, it does not further decrease. There are still some error seeds with larger LMAs near -5% , which is very likely due to the insufficient tracking resolution rather than physical

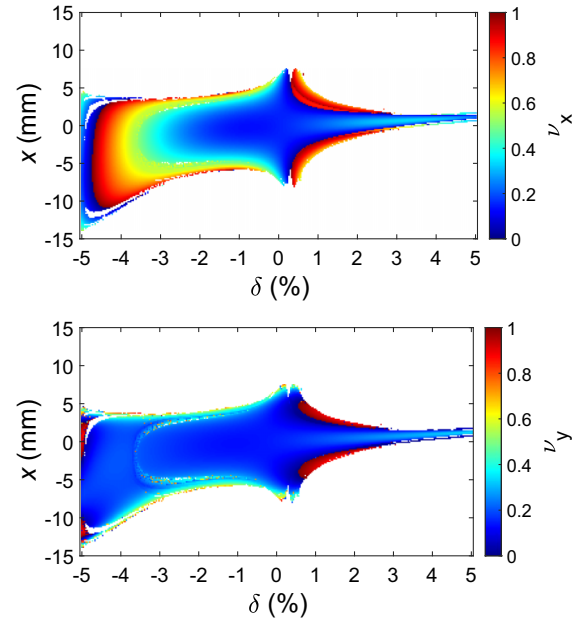


FIG. 18. Horizontal DAs and tune footprints of off-momentum particles for the HALF lattice tracked at the SF location. The color bars of the left and right plots denote the fractional parts of horizontal and vertical tunes, respectively.

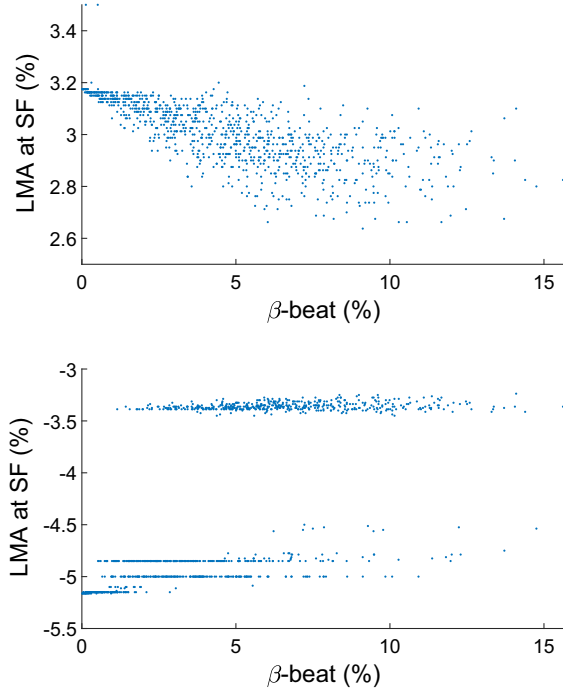


FIG. 19. The 4D LMAs for the error seeds of the HALF lattice tracked at the SF location. Upper: LMAs in the positive direction; lower: LMAs in the negative direction.

existence [37]. In the positive direction, the LMA without errors is about 3.2%. It continuously decreases as β -beat grows, which is due to the reduction in the off-momentum horizontal DAs.

It is seen from Fig. 15 that the 6D LMA at the dispersion-free long straight section is -5.4% to 4.5% , the asymmetry of which is due to RF bucket distortion caused by higher-order momentum compaction. The combination of Figs. 15 and 17 indicates that β -beat has a small effect on the 6D LMA in the straight section. For the 6D LMA at the SF location with high dispersion, it is -3.0% to 3.2% , as seen from Fig. 15. Since the 4D LMA in the negative direction is larger than -3% even with high β -beat, as seen from Fig. 19, the 6D LMA is not affected by the half-integer resonance. In the positive direction, β -beat has a limited impact on the 6D LMA. Therefore, with β -beat considered, the half-integer resonance line $\nu_x = 48.5$ has little impact on the overall 6D LMA of the HALF lattice. Figure 20 shows the 6D LMAs for the error seeds of the HALF lattice with β -beat of 3% and 6%. It is seen that, compared to the ideal lattice, the 6D LMAs of the error seeds exhibit a small reduction in the straight section and essentially no reduction in the dispersion bumps.

For other HMBA lattices, the 6D LMAs are similar to that of HALF, with the lowest part in the dispersion [21,24,38]. In hybrid lattice design, it is generally hard to control tune shifts with momentum due to relatively large higher-order chromaticities. The analysis here indicates

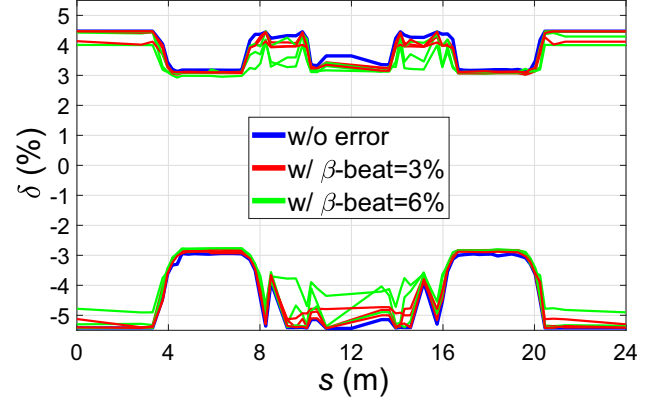


FIG. 20. The 6D LMAs for the error seeds of the HALF lattice with β -beat of 3% and 6%. Three seeds are presented for each β -beat.

that when tune shifts with momentum are reasonably controlled, for example, without crossing half-integer resonance within $|\delta| < 3.5\%$, the impact of half-integer resonance is quite small on the 6D LMA even with relatively high β -beat. Therefore, in the design and optimization of an HMBA lattice, excessive control of tune shifts with momentum is not necessary, allowing greater flexibility for the optimization of the DA.

B. The HOA 6BA lattice

For the HOA 6BA lattice in Sec. II B, the tune shifts with momentum are quite smaller, controlled away from integer and half-integer resonance lines within $|\delta| < 5\%$, as shown in Fig. 21. Therefore, the off-momentum horizontal DAs are relatively important for the LMA. For the off-momentum DAs, the correlations can be analyzed similarly to Sec. II but based on off-momentum optics. According to Ref. [26], the variation of n th-order off-momentum RDTs at an energy deviation δ , denoted as $f_{n,\text{avg}}(\delta)$, is calculated as

$$f_{n,\text{avg}}(\delta) = \sqrt{\sum_{j+k+l+m=n} (f_{jklm,\text{avg}}(\delta))^2}, \quad (7)$$

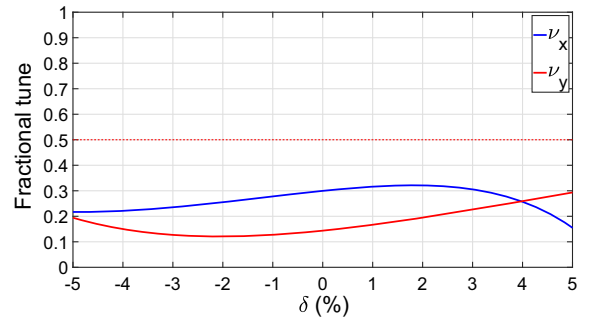


FIG. 21. Tune shifts with momentum for the HOA 6BA lattice.

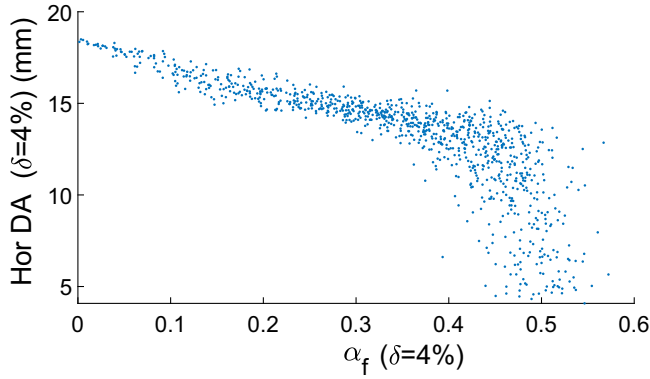


FIG. 22. Correlation between the off-momentum asymmetry parameter $\alpha_f(\delta = 4\%)$ and off-momentum horizontal DA ($\delta = 4\%$) tracked at the center of straight section for the error seeds of the HOA 6BA lattice. The Spearman coefficient is 0.918.

where $f_{jklm,avg}(\delta)$ is the variation of an off-momentum RDT for the energy deviation δ . The asymmetry parameters based on off-momentum β -functions and off-momentum RDTs are also introduced, denoted as $\alpha_\beta(\delta)$ and $\alpha_f(\delta)$, respectively, which are calculated in the same way as the on-momentum ones in Sec. II.

Figure 22 shows the correlation between $\alpha_f(\delta)$ and the off-momentum horizontal DAs tracked at the center of the straight section, where the previously generated error seeds are used and the momentum deviation $\delta = 4\%$. In the figure, the off-momentum horizontal DA is calculated as the absolute sum of the positive and negative horizontal DAs. There is a sharp decrease in the off-momentum horizontal DAs with large β -beat, due to that the off-momentum ADTSs cross the half-integer resonance line $\nu_x = 36.5$. Table IV lists the Spearman correlation coefficients of the off-momentum ($\delta = 4\%$) horizontal DA with the on- and off-momentum asymmetry parameters. In the calculation of off-momentum RDT variations, the fourth-order term $f_{2020}(\delta = 4\%)$ is assigned an additional weighting factor of 0.001, because the chromatic tunes are near the resonance $2\nu_x + 2\nu_y$. From Table IV, it is seen that the off-momentum horizontal DA has stronger correlations with the off-momentum asymmetry parameters than with the on-momentum ones.

TABLE IV. Spearman correlation coefficients of the off-momentum horizontal DA with on- and off-momentum α_β and α_f for the error seeds of the HOA 6BA lattice.

Horizontal DA ($\delta = 4\%$)	
$\alpha_\beta(\delta = 0)$	0.889
$\alpha_f(\delta = 0)$	0.863
$\alpha_\beta(\delta = 4\%)$	0.912
$\alpha_f(\delta = 4\%)$	0.918

IV. CONCLUSION

To isolate optics distortion from other error effects and elucidate its impact, the effect of β -beat induced asymmetry on both the on- and off-momentum nonlinear dynamics of the storage ring was studied, with the HALF H6BA lattice and an HOA 6BA lattice as examples. To better analyze the impact, error seeds with relatively high β -beat were generated, and their betatron tunes were fitted to the nominal values. In the study of on-momentum DA, the breaking of the periodic symmetry of the lattice was quantitatively characterized by introducing the asymmetry parameters, α_β and α_f , which are calculated from the FFT of the β -functions and RDT variations. Compared to the commonly used β -beat, α_β and α_f have stronger correlations with the normalized DA area, and the correlations are further enhanced when the DA is represented by the normalized average diffusion rate, which represents both the DA area and particle motion stability. In the case of optics distortion caused by IDs, the asymmetry parameters still maintain a strong correlation with the DA. In addition, the commonly used β -beat serving as a reasonable indicator in characterizing the nonlinear dynamics degradation was physically explained based on RDT variations.

In the study of off-momentum nonlinear dynamics, the impact of half-integer resonance crossing on the LMA was studied for the HALF H6BA lattice with β -beat. The half-integer resonance was demonstrated to have a small effect on the 6D LMA in both the dispersion-free straight section and dispersion bumps, even though the 4D LMA in the dispersion bumps collapses from -5.4% to -3.4% due to β -beat. This indicates that for an HMBA lattice, when tune shifts with momentum are reasonably controlled, the reduction in the 6D LMA can be kept small, even with relatively high β -beat. Consequently, more flexibility in the HMBA lattice design can be focused on the optimization of the DA. In addition, the reduction in off-momentum DAs with β -beat for the HOA 6BA lattice was studied. It was found that the asymmetry parameters based on off-momentum optics, rather than on-momentum optics, are more effective in characterizing the reduction in off-momentum horizontal DAs with β -beat.

Therefore, the asymmetry parameters show potential as useful indicators for the analysis and optimization of lattice correction. They can also be used to guide the design of lattices in which optical symmetry is broken by superbends, double mini- β ID straight sections, high- β injection straight sections, cell replacement, etc.

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DATA AVAILABILITY

The data are not publicly available. The data are available from the authors upon reasonable request.

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