

Vlasov solver for transverse mode coupling instability of double-rf system under arbitrary settings

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The Vlasov solver provides an efficient method to analyze transverse mode coupling instability (TMCI). However, many traditional Vlasov solvers are based on the small amplitude approximation, and are applicable only to single rf (radio frequency) systems, whereas harmonic cavities are widely utilized in colliders and synchrotron light sources to improve beam stability. In this paper, we have developed a new Vlasov solver suitable for any double rf system to study TMCI. The results of the Vlasov solver are in good agreement with the simulation results obtained using ELEGANT.

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I. INTRODUCTION

The advent of the fourth generation diffraction-limited storage ring increases the use of the double rf system [1–6]. A double rf system consists of two rf cavities that operate at two different frequencies. The primary cavity functions at a harmonic number h_1 , while the higher harmonic cavity (HHC) operates at h_2 , an integer multiple h of h_1 (i.e., $h_2 = h \cdot h_1$). The superposition of these frequencies enables manipulation of the longitudinal density distribution of electrons in the storage ring [7–9]. Utilizing a double rf system to stretch the bunch and improve the longitudinal emittance can decrease intrabeam scattering and the peak current, which can decrease the effects of wakefields [10–12]. Additionally, HHCs also provide Landau damping and increase the instability threshold [13–15]. A double rf system is particularly important for the new generation of ultralow emittance storage rings [16]. Therefore, it is essential to analyze instabilities, especially those resulting in rapid particle loss, such as transverse mode coupling instability (TMCI), for a double rf system.

The Vlasov solver is a traditional tool used for instability analysis. However, most Vlasov solvers rely on the small amplitude approximation based on single rf systems and, as a result, are unsuitable for double rf scenarios. They assume linear synchrotron motion, a decaying exponential radial density distribution, and a constant synchrotron tune [17,18].

Subsequently, extensive research was proposed to address these limitations. In 2014, Burov developed the nested head-tail Vlasov solver (NHTVS), which is capable of handling small synchrotron tune spread but is restricted to Gaussian bunches [19]. In 2018, Venturini introduced the radial discretization method, applicable for large synchrotron tune spread and flat-top distributions under optimal lengthening conditions [20]. These methods may result in significant inaccuracies if the longitudinal distribution or synchrotron tune spread deviates from Gaussian and flat-top cases. At present, there is no general Vlasov solver available for suboptimal lengthening conditions and overstretching conditions with two stable fixed points (SFPs) to analyze TMCI.

Formulating the eigenvalue problem from the Vlasov equation of the double rf system presents challenges. The settings of a double rf system lead to asymmetry of the longitudinal potential well and can even result in two SFPs in the longitudinal plane, complicating the determination of the matrix elements.

The paper is organized as follows: In Sec. II, we introduce the definitions and derive the eigenvalue problem using perturbation formalism near the distribution determined by a double rf system. In Sec. III, we examine single rf and optimal lengthening conditions as benchmarks. We then apply our solver to suboptimal conditions. In Sec. IV, we introduce the Vlasov solver for overstretching conditions.

II. NOTATION, LINEARIZED VLASOV THEORY

A. Equations of motion

We consider an ultrarelativistic electron executing synchrotron oscillation in a bucket generated by a double rf system, in the absence of longitudinal wakefields. The equations of synchrotron motion are

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$$\frac{dz}{ds} = -\eta\delta, \quad (1)$$

$$\frac{d\delta}{ds} = \frac{eV_1}{E_0 C} \left[\sin\left(\phi_s - \frac{2\pi h_1}{C}z\right) + r \sin\left(\phi_{2s} - \frac{2\pi h_2}{C}z\right) - \sin\phi_s - r \sin\phi_{2s} \right]. \quad (2)$$

Here z represents the longitudinal position relative to the synchronous particle of energy E_0 that has traveled a distance of $s = ct$, with $z > 0$ meaning the head of the bunch, and $\delta = (E - E_0)/E_0$ is the relative energy deviation with respect to the synchronous particle. Additionally, ϕ_s and ϕ_{2s} are the synchronous phase angles of the main cavity and HHC, respectively. In this manuscript, we adopt the convention that $\pi/2 < \phi_s < \pi$. $r = V_2/V_1$ is the ratio of the voltage in HHC to that in the main cavity, η is the slippage factor, e is the elementary charge, and C is the circumference of the ring along which a synchronous particle completes one revolution in a period of T_0 . For single rf, $\delta(s) \propto z$, the synchrotron motion is linear under the small amplitude approximation. Conversely, for most double rf conditions, the synchrotron oscillation is nonlinear, and the Hamiltonian may exhibit one SFP or two SFPs, as depicted in Fig. 1.

The Hamiltonian of synchrotron motion is expressed as

$$\begin{aligned} H(z, \delta; s) = & -\frac{\eta}{2}\delta^2 - \frac{eV_1}{2\pi h_1 E_0} \left[\cos\left(\phi_s - \frac{2\pi h_1}{C}z\right) \right. \\ & + \frac{r}{h} \cos\left(\phi_{2s} - \frac{2\pi h_2}{C}z\right) \\ & - \frac{2\pi h_1}{C}z(\sin\phi_s + r \sin\phi_{2s}) \\ & \left. - \cos\phi_s - \frac{r}{h} \cos\phi_{2s} \right]. \end{aligned} \quad (3)$$

The equations of betatron motion are

$$\frac{dy}{ds} = p_y, \quad (4)$$

$$\frac{dp_y}{ds} = -\left(\frac{\omega_\beta}{c}\right)^2 y + \frac{1}{E_0} F_y(z, s), \quad (5)$$

where ω_β is the betatron oscillation frequency, $F_y(z, s)$ is the transverse wake force generated by the longitudinal distribution $\rho(z)$ at time s/c .

B. Sacherer integral equation

Following a similar derivation to that outlined in Chao's book [21], we can derive the integral equation of TMCI.

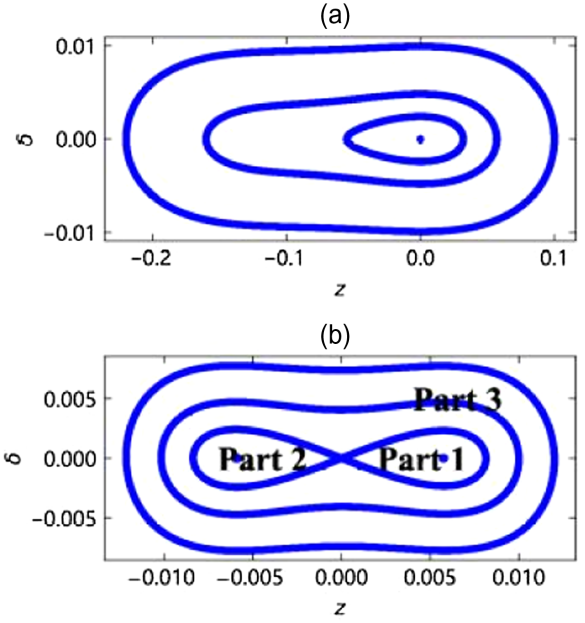


FIG. 1. Hamiltonian of suboptimal lengthening (a) and overstretching bunches (b).

The phase-space distribution $\psi(y, p_y, z, \delta; s)$ evolution is described by the Vlasov equation

$$\frac{\partial \psi}{\partial s} + \frac{\partial \psi}{\partial y} \frac{dy}{ds} + \frac{\partial \psi}{\partial p_y} \frac{dp_y}{ds} + \frac{\partial \psi}{\partial z} \frac{dz}{ds} + \frac{\partial \psi}{\partial \delta} \frac{d\delta}{ds} = 0. \quad (6)$$

Employing action-angle variables (J, ϕ) in the synchrotron phase space, and polar coordinates (q, θ) in the transverse phase space (see Appendix A), the Vlasov equation is transformed into

$$\frac{\partial \psi}{\partial s} + \frac{\omega_\beta}{c} \frac{\partial \psi}{\partial \theta} + \frac{F_y}{E_0} \frac{\partial \psi}{\partial p_y} + \frac{\omega_s}{c} \frac{\partial \psi}{\partial \phi} = 0. \quad (7)$$

Assuming a small perturbation on ψ , we can expand the phase-space density function as

$$\psi(q, \theta, J, \phi; s) = \psi_0(q, J) + \psi_1(q, \theta, J, \phi) e^{-i\Omega s/c}, \quad (8)$$

where Ω is the mode frequency. Due to the decoupling between the transverse and longitudinal planes, the phase-space density distribution factorizes into two independent terms

$$\psi_0(q, J) = f_0(q)g_0(J) \quad (9)$$

$$\psi_1(q, \theta, J, \phi) = f_1(q, \theta)g_1(J, \phi). \quad (10)$$

Here f_0 and g_0 represent the unperturbed transverse and longitudinal distributions, respectively. In this configuration, the manipulation of the potential well via the double-rf system invalidates classical phase-space distribution

models. Both the water-bag and air-bag models fundamentally assume an undistorted quadratic potential well [21], a condition that no longer holds in the present case. To include the effects of the double-rf system, the unperturbed phase-space distribution of the electron bunch g_0 takes the following form:

$$g_0(z, \delta) = \frac{1}{\sqrt{2\pi}\sigma_\delta} \exp\left[-\frac{\delta^2}{2\sigma_\delta^2}\right] \rho(z), \quad (11)$$

where $\rho(z)$ is the longitudinal line density [22]. In this paper, we consider no potential well distortion that

$$\rho(z) = A \exp\left[-\frac{V_{pw}(z)}{\eta\sigma_\delta^2}\right], \quad (12)$$

where $V_{pw}(z)$ is given by Eq. (A1), and the normalization constant A comes from

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dz d\delta g_0(z, \delta) = 1. \quad (13)$$

The terms f_1 and g_1 are transverse and longitudinal phase-space perturbations, respectively. Considering betatron oscillation dipole mode only, the transverse phase-space perturbation is expected to be

$$f_1(q, \theta) = -D \frac{df_0(q)}{dq} e^{i\theta}. \quad (14)$$

where D is the dipole displacement of the transverse distribution. And the perturbed wake force is (see Appendix B) [21]

$$F_{y,m=1}(z, s) = -\frac{De^2}{cT_0} e^{-i\Omega s/c} \int_{-\infty}^{\infty} dz' \rho_1(z') W_{m=1}(z - z'). \quad (15)$$

where $W_{m=1}$ is the wakefield of moment $m = 1$, $\rho_1(z) = \int \delta g_1(z, \delta)$. Substituting the phase-space distribution [Eq. (8)], the dipole transverse perturbation [Eq. (14)], and the perturbed wake force [Eq. (15)] into the Vlasov equation, and replacing the convolution of perturbation and wakefields with the product of their counterparts in the frequency domain, we can write the equation with vanishing chromaticity ξ , keeping only the first-order term of the phase-space distribution (see Appendix C).

$$\begin{aligned} & i(\Omega - \omega_\beta) g_1(J, \phi) - \omega_s \frac{\partial g_1(J, \phi)}{\partial \phi} \\ & - \frac{Nce^2}{4\pi E_0 \omega_\beta T_0} g_0(J) \int_{-\infty}^{\infty} d\omega e^{i\omega z/c} \tilde{\rho}_1(\omega) Z_{m=1}^\perp(\omega) = 0. \end{aligned} \quad (16)$$

where N is the number of particles per bunch, $\tilde{\rho}_1$ denotes the Fourier transform of $\rho_1(z)$ with respect to z , and $Z_{m=1}^\perp$ represents impedance of $m = 1$, which is the Fourier transform of $W_{m=1}^\perp$.

To analyze azimuthal modes, we make a Fourier expansion of g_1 in the angular variable with the head-tail phase factor following Sacherer's method [17]

$$g_1(J, \phi) = \sum_{l=-\infty}^{\infty} R_l(J) e^{il\phi} e^{i\xi\omega_\beta z(J, \phi)/c\eta}. \quad (17)$$

Substituting Eq. (17) into Eq. (16), subsequently multiplying both sides by $e^{-il\phi}$ and integrating over ϕ from 0 to 2π , we find the Sacherer integral equation

$$\begin{aligned} & (\Omega^{(l)} - \omega_\beta) R_l(J) \\ & = \sum_{l'} \left[-\frac{iNce^2}{2E\omega_\beta T_0} g_0(J) \right. \\ & \quad \left. \int_0^\infty dJ' R_{l'}(J') \mathcal{F}_{l,l'}(J, J') + l' \delta_{l,l'} R_{l'}(J) \omega_s(J) \right], \end{aligned} \quad (18)$$

where the kernel and function of beam distribution are

$$\mathcal{F}_{l,l'}(J, J') = \int_{-\infty}^{\infty} d\omega' Z_{m=1}^\perp(\omega') S_l^*(\omega', J) S_{l'}(\omega', J'), \quad (19)$$

$$S_{l'}(\omega', J) = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{il'\phi - i\frac{z(J, \phi)}{c}(\omega' - \frac{\xi\omega_\beta}{\eta})}. \quad (20)$$

Here, c is the speed of light, $g_0(J)$ is the normalized longitudinal phase-space density, $R_l(J)$ is the radial density perturbation of the l mode, $\omega_s(J)$ is the angular synchrotron frequency, $Z_{m=1}^\perp(\omega)$ is the transverse impedance with $m = 1$, and ξ is the chromaticity.

In the Gaussian model, where $z(r, \phi) = r \cos \phi$ and $z(J, \phi) = r(J) \cos \phi$, Eq. (20) becomes

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi e^{il'\phi - ix \cos \phi} = i^{-l} J_l(x), \quad (21)$$

which has appeared in many Vlasov solvers like NHTVS [19]. For a distorted potential well resulting from a double-rf system or wakefields, others have derived spectrum functions that are analogous to Eq. (20) [23,24].

C. Discretization method

To find the radial modes described by Eqs. (18)–(20), Oide proposed using a set of piecewise step functions as the orthogonal basis instead of polynomial orthogonal basis [25]. Then Venturini introduced an equivalent yet more straightforward method, known as radial discretization, based on Oide's approach. In the polynomial basis method, calculating the orthogonal integral numerically is

challenging because the Gaussian approximation proves inadequate, even at high Hamiltonian values of a potential well influenced by a double-rf system.

We opt for Venturini's radial discretization for radial modes, that we alter the integral over action by a summation. Then, we transform the integral equation into an eigenvalue problem

$$(\Omega^{(l)} - \omega_\beta)R_l(J_i) = \sum_{i'l'} [A_{ii'l'} + B_{ii'l'}]R_{l'}(J_{i'}), \quad (22)$$

and the two matrices are

$$A_{ii'l'} = -\frac{iNce^2}{2E_0\omega_\beta T_0} g_0(J_i) \Delta J_{i'} \mathcal{F}_{l,l'}(J_i, J_{i'}), \quad (23)$$

$$B_{ii'l'} = l\delta_{l,l'}\delta_{i,i'}\omega_s(J_i), \quad (24)$$

where $\Delta J_i = J_i - J_{i-1}$ is the step length of discretized action, while incoherent shifted synchrotron tune and phase-space density can be obtained, as shown in Appendixes A and B, respectively. The matrix Eq. (23) describes the interaction between different modes through wakefields, while the diagonal one [Eq. (24)] gives the incoherent tune shift due to the nonlinearity of the rf system. In summary, we use Fourier series expansions for mode decomposition, followed by orthogonal integration to solve the differential equations. This approach naturally gives rise to an eigenvalue problem formulation. We note that similar mathematical structures have been reported in several plasma physics studies employing analogous method [26,27].

TABLE I. Settings for single rf and optimal lengthening condition.

Name	V_1 (V)	ϕ_s (rad)	V_2 (V)	ϕ_{2s} (rad)
Single	1.754×10^6	3.142	0.000	0.000
Optimal	3.640×10^6	3.142	1.213×10^6	0.000

TABLE II. The main parameters of model lattice.

Parameter	Units	Value
Circumference, C	m	1360.4
Energy, E_0	GeV	6
Momentum compaction, α_c		1.56×10^{-5}
Betatron tune, ν_y		106.27
Relative energy spread (normal), σ_δ		1.06×10^{-3}
Relative energy spread (overstretch), σ_δ		5.0×10^{-4}
Harmonic number of main rf (normal), h_1		756
Harmonic number of main rf (overstretch), h_1		2268
Harmonic ratio, h		3

Note that in the case of overstretching, two different elements from matrices [Eqs. (23) and (24)], which correspond to two discretized actions, may share the same action value. Because there are two branches in an overstretching potential well, where particles with the same

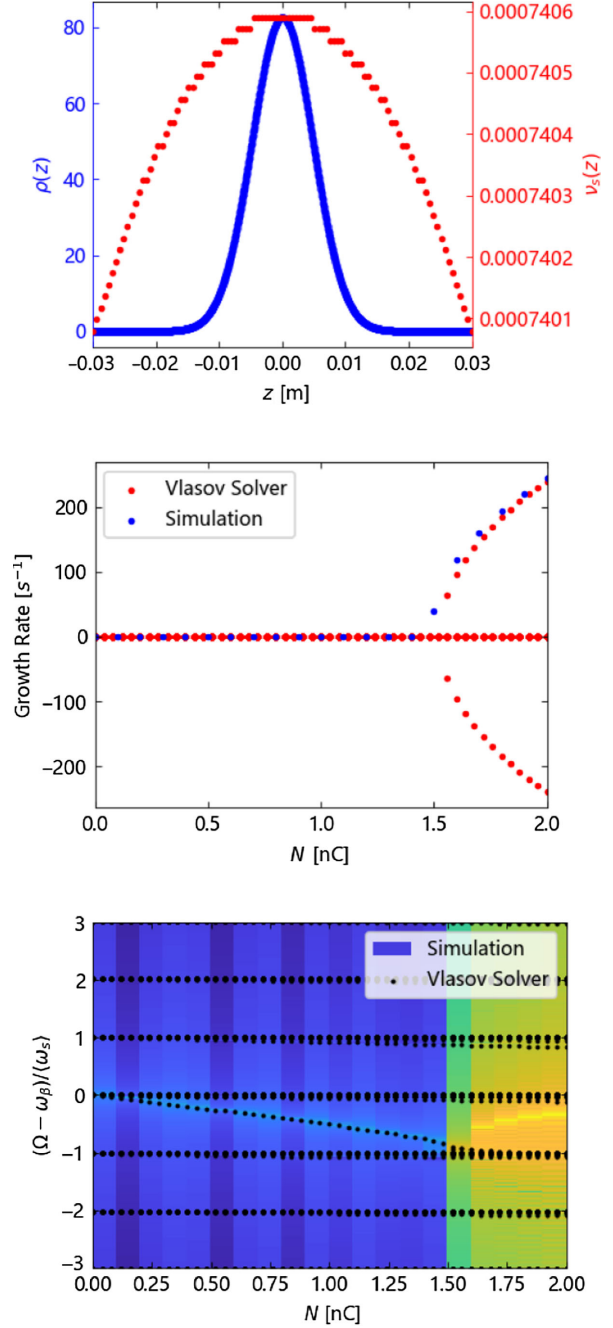


FIG. 2. TMCI analysis of single rf condition. The image above represents the longitudinal density distribution $\rho(z)$ (blue line) and synchrotron tune spread $\nu_s(z)$ (red dots). The image in the middle shows the growth rates. The blue and black dots represent simulation and Vlasov solver results. The lower image displays the tune shifts. The threshold is almost at the intersection of the tune shift of $m = 0$ mode and $m = -1$ mode.

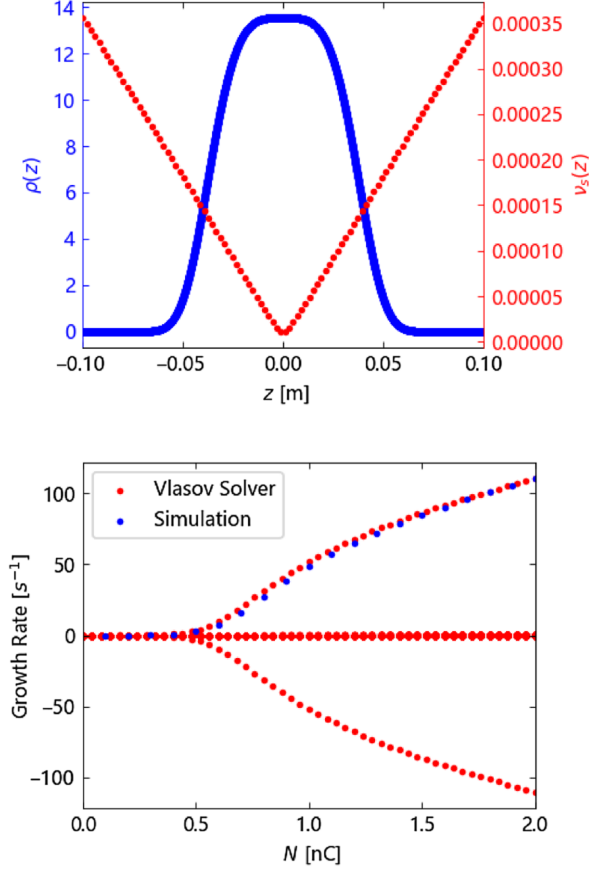


FIG. 3. TMCI analysis of ideal lengthening condition. The image above represents longitudinal density distribution $\rho(z)$ (blue line) and synchrotron tune spread $\nu_s(z)$ (red dots). The lower image shows the growth rates. The blue and black dots represent simulation and Vlasov solver results.

Hamiltonian do not exchange positions, the limits of the integration in Eq. (A3) are different for calculating the action. However, values of such pairs of actions are the same, they resemble different tori, leading to different synchrotron tunes and phase-space densities, as shown in Fig. 6. As a result, one same action value from two different potential wells contributes two elements of the matrix. We arrange elements of the matrix in a particular order, and find agreements between three methods, as shown in Sec. IV.

To calculate the function $S_l(\omega, J)$, Oide also proposed the technique of sampling points within the longitudinal phase space [25], a technique also utilized in Lin's transfer matrix method [28,29], to deal with phase space with

potential well distortion. On the phase-space tori of a certain discretized action J_i , we choose several points to calculate angular variables according to Eq. (A4) from 0 to 2π , obtaining the function $z(J_i, \phi)$ and calculating the integration [Eq. (20)] numerically.

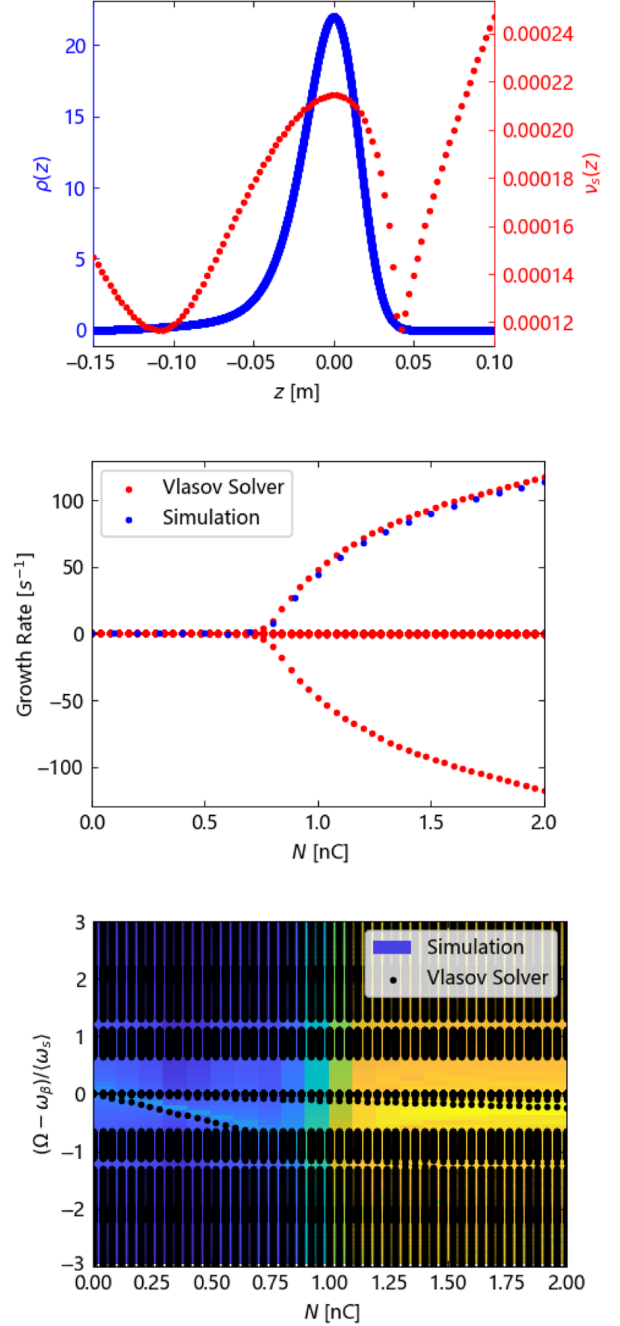


FIG. 4. TMCI analysis of suboptimal lengthening bunch 1. The image above represents the longitudinal density distribution $\rho(z)$ (blue line) and synchrotron tune spread $\nu_s(z)$ (red dots). The image in the middle shows the growth rates. The blue and black dots represent simulation and Vlasov solver results. The lower image displays the tune shifts. The threshold is almost at the intersection of the tune shift of $m = 0$ mode and $m = -1$ mode.

TABLE III. Settings for one SFP condition.

Name	V_1 (V)	ϕ_s (rad)	V_2 (V)	ϕ_{2s} (rad)
Suboptim 1	8.298×10^5	2.918	2.875×10^5	5.588
Suboptim 2	8.298×10^5	3.420	3.000×10^5	0.864

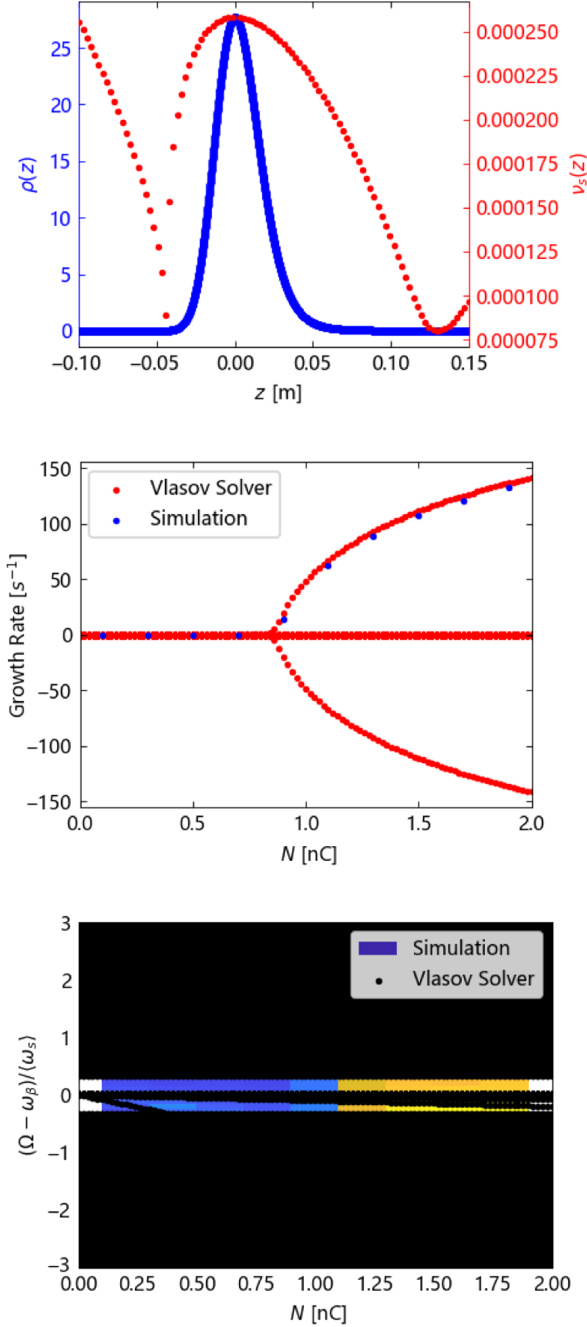


FIG. 5. TMCI analysis of suboptimal lengthening bunch 2. The z of sampling points ranges from -0.064 to 0.204 m. The image above represents the longitudinal density distribution $\rho(z)$ (blue line) and synchrotron tune spread $\nu_s(z)$ (red dots). The rms width of $\rho(z)$, σ_z , is 0.0161 m. The image in the middle shows the growth rates. The blue and black dots represent simulation and Vlasov solver results, respectively. The lower image displays the tune shifts. Because the minimum value of ν_s is very low and the sampling too dense, the tune shift of mode $m = 0$ is difficult to identify.

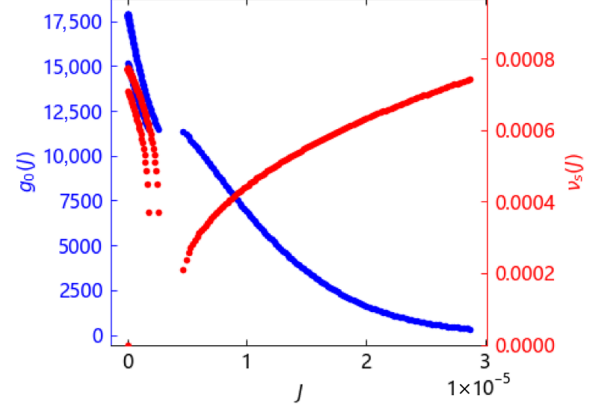


FIG. 6. When we separate two closed orbits of the same Hamiltonian for overstretching bunches, the synchrotron tune $\nu_s(J)$ and radial density distribution $g_0(J)$ are multivalued and discontinuous.

III. VLASOV SOLVER FOR SUBOPTIMAL CONDITION

A. Benchmark

In comparison with the ELEGANT tracking method, we have established a benchmark under two setups: a single rf system and an optimal lengthening condition. The configurations of the rf system for these setups are detailed in Table I. Note that the energy loss per turn is zero for these settings due to the damping effect generated by the combination of ILMATRIX and RFDF in ELEGANT simulations. The parameters of our model lattice, which are based on the HEPS storage ring, are listed in Table II; however, some have been adjusted to achieve a better overstretch bunch shape.

The analysis of TMCI for a single rf system is depicted in Fig. 2. The top figure displays the longitudinal distribution and synchrotron tune. The distribution is not perfectly Gaussian and the synchrotron tune is not constant due to rf nonlinearity. However, the deviation from the Gaussian model is negligible, allowing the small amplitude approximation to remain valid. In the middle figure, the growth rates show good agreement between the Vlasov solver (red dots) and the simulation (blue dots). As observed in the bottom figure, when the bunch charge increases, mode $m = 0$ moves toward and intersects with $m = -1$, which also agrees well with the simulation results. Considering no radiation damping, the threshold current is about 1.5 nC. In the modal analysis, the average tune shift $\langle \omega_s \rangle$ is calculated by

$$\langle \omega_s \rangle = 2\pi \int_0^\infty \omega_s(J) g_0(J) dJ. \quad (25)$$

TABLE IV. Settings for overstretching bunches.

Name	V_1 (V)	ϕ_s (rad)	V_2 (V)	ϕ_{2s} (rad)
Overstretch 1	8.000×10^8	3.142	2.667×10^8	6.283
Overstretch 2	8.000×10^8	3.142	2.676×10^8	6.283
Overstretch 3	8.000×10^8	3.141	2.680×10^8	6.281

The comparison between the theoretical and simulated growth rates for optimal lengthening is shown in Fig. 3, and these results are also consistent.

B. General suboptimal lengthening

Two suboptimal lengthening settings of the double rf system are listed in Table III.

The growth rates and the tune shifts of the Vlasov solver and those of ELEGANT show excellent agreement, as can be seen in Figs. 4 and 5, where two different distributions of forward and backward lean are compared. In both cases, we found that the boundary of mode $m = -1$ falls between single rf and optimal lengthening conditions when $N = 0$, because the double-rf potential well cause a large tune spread, resulting in a difference in the width of the mode spectrum. A clear threshold also emerges, which is lower than that for the single rf condition.

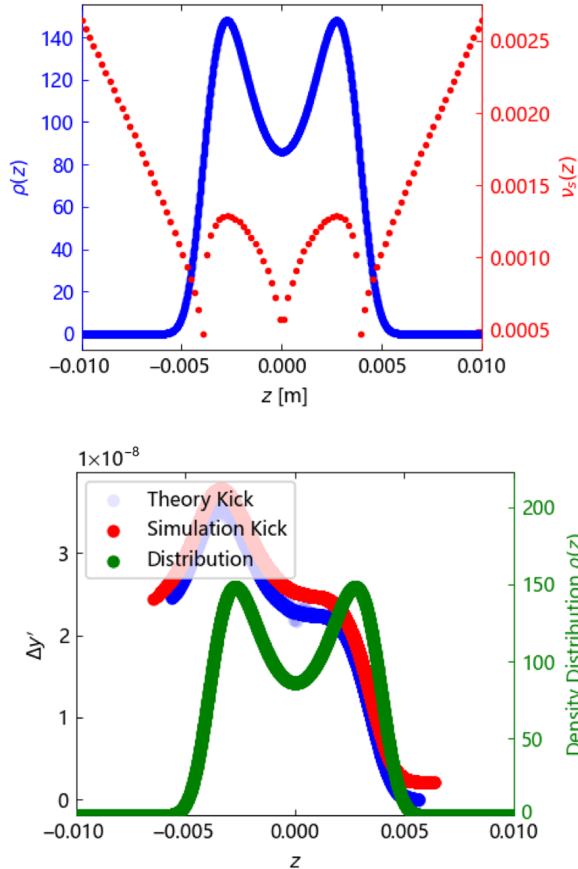


FIG. 7. Density distribution and kick of wakefield under overstretching condition.

IV. VLASOV SOLVER FOR OVERSTRETCHING CONDITION

A. Phase space division method

For the overstretching conditions, we have proposed dividing the Hamiltonian into three parts, as indicated in Fig. 1. Consequently, the $z_{\min}$ and $z_{\max}$ in the definition of J , Eq. (A3), should be the minimum and maximum values of each independent closed torus, respectively.

On the one hand, as seen from Eqs. (23) and (24), the Vlasov solver does not explicitly depend on J but on J_i . Thus, the order of J_H , J_T , and J_O , and whether J is discontinuous, are not important. Therefore, the action variable J in Eqs. (23) and (24) can be reinterpreted as an ordered vector $[J_H, J_T, J_O]^T$, which comprises the J values from the three parts.

$$J \rightarrow \begin{bmatrix} J_H \\ J_T \\ J_O \end{bmatrix} \quad (26)$$

Here J_H , J_T , and J_O denote the action variables of the head, tail, and outside regions, respectively. So, although the J values for $J_H = 0$ and $J_T = 0$ are identical, the two

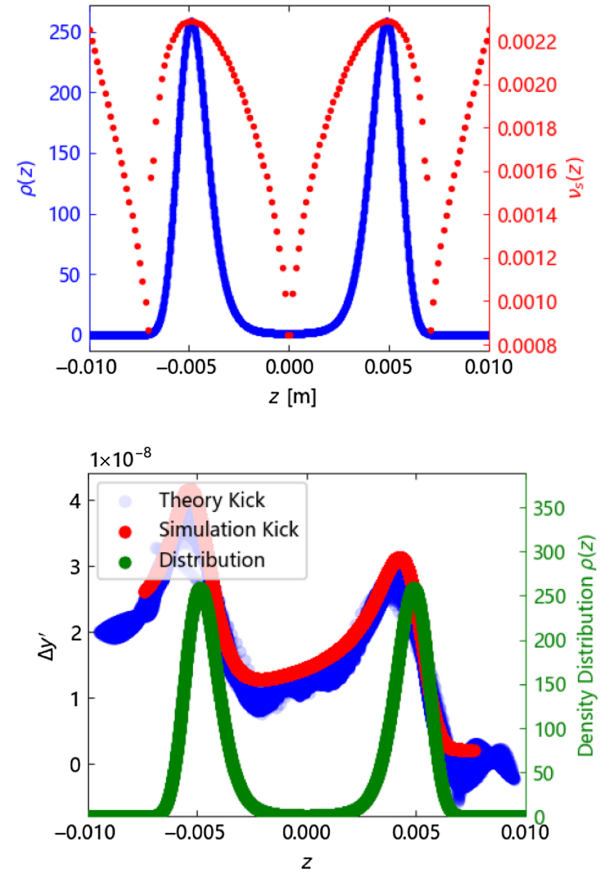


FIG. 8. Density distribution and kick of wakefield under overstretching condition.

action variables differ because they correspond to different longitudinal positions. Similarly, the vectorized $g_0(J)$, $\nu_s(J)$ in Fig. 6 will also correspond one-to-one with $J = [J_H, J_T, J_O]^T$ in Eq. (26), respectively, and there will be no confusion in the calculation process of the Vlasov solver.

On the other hand, by employing the same phase-space division method with transfer matrix method [30], we can verify that this method accurately calculates the kick of wakefield. The transfer matrix method applies the wakefield matrix separately to the transverse phase space $y(J, \phi)$, $y'(J, \phi)$, rather than density perturbations $g_1(J, \phi)$. By assigning different values to $y(J, \phi)$ and $y'(J, \phi)$, we can compare the method with ELEGANT to determine whether the calculation of the kick is correct using this phase-space division method.

Three configurations have been selected for the overstretching conditions, as shown in Table IV. The bunch length was intentionally shortened because, for overstretching bunches, simulation results often fail to converge even when the N_BINS parameter is sufficiently large. Decreasing the bunch length helps to mitigate the high requirements on N_BINS and improves simulation speed.

The longitudinal density distribution $\rho(z)$ and tune spread $\nu_s(z)$ for the overstretching condition are illustrated

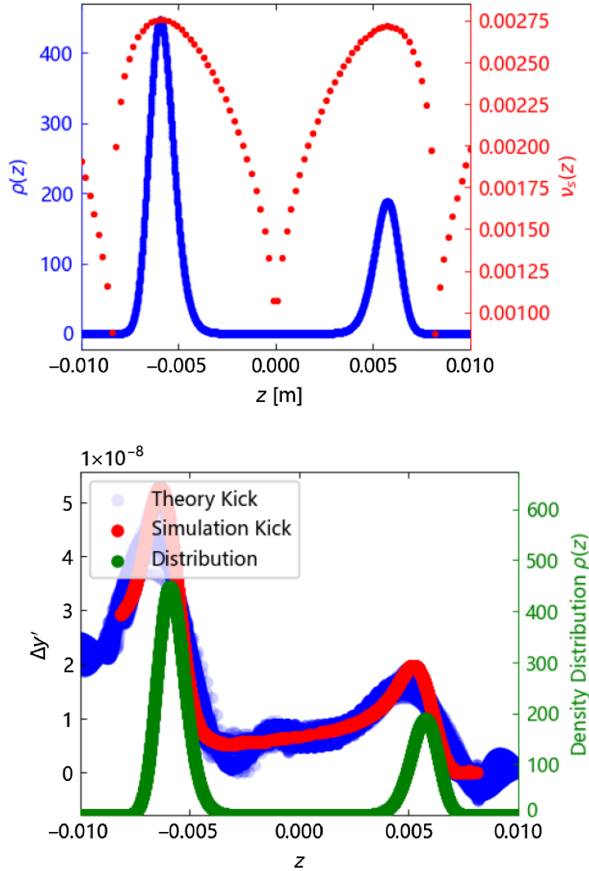


FIG. 9. Density distribution and kick of wakefield under overstretching condition.

in the upper figures in Figs. 7–9. The lower figures show the kick of wake force from the transfer matrix method (blue dots) and ELEGANT (red dots). The blue dots are the projection of $\Delta y'(z, \delta)$ to the $\Delta y' - z$ plane. We can see that for different types of overstretching conditions, the kick of wake force is very similar. So it is reasonable to apply the phase-space division method to the overstretching condition.

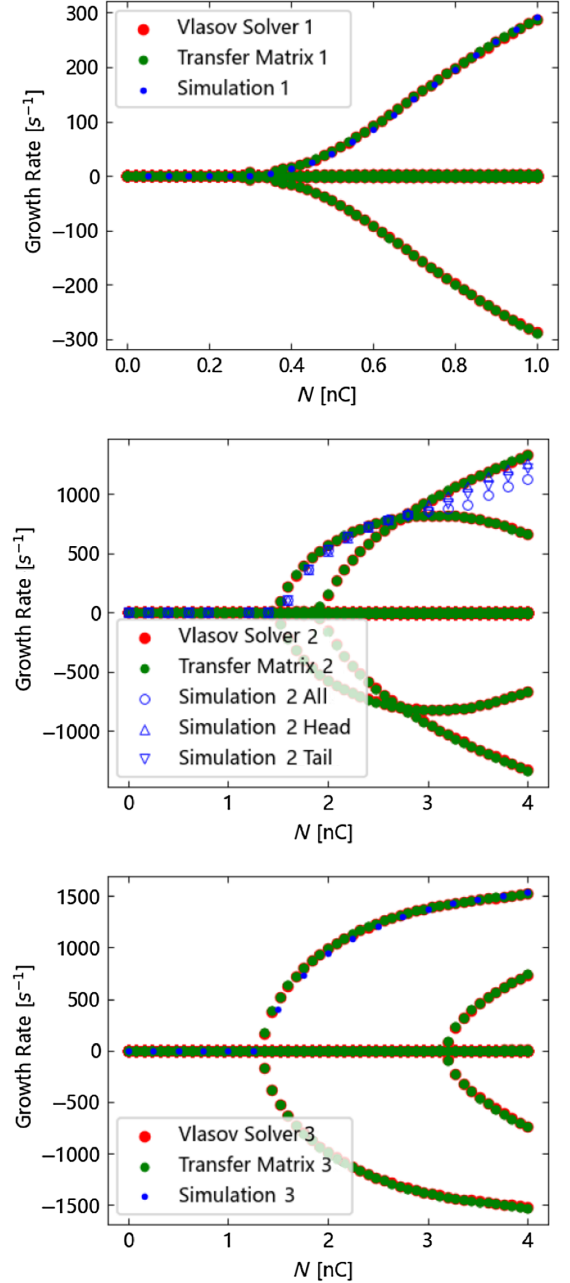


FIG. 10. The growth rates of the overstretching bunch labeled as “Overstretch 1” (top), “overstretch 2” (middle) and “overstretch 3” (bottom) in Table IV. The blue dots/triangle/circle, green dots and red dots represent results from the simulation, transfer matrix, and Vlasov solver, respectively.

B. Comparison of different methods

Figure 10 shows the growth rates of overstretch cases 1–3. The red and green dots in these figures represent the growth rates obtained by the Vlasov solver and transfer matrix method, respectively. Two different theories, using the same phase-space division method, maintained good consistency in the results. The two theories also show good consistency with the simulation results (blue dots/triangle/circle).

V. CONCLUSION

In this paper, we propose a numerical method to solve the Vlasov equation, Eqs. (18)–(20), to analyze TMCI for any double-rf systems.

The process encounters two primary difficulties: one is the asymmetry of the longitudinal potential well, and the other is the presence of two SFPs under the overstretching condition. To address the first issue, we propose using numerical calculations for radial and angular sampling in phase space. For the second one, we suggest dividing the overstretched phase space into three parts. This approach results in the points in phase space not being uniquely identifiable using (J, ϕ) . However, the characteristics of the Vlasov equation allow us to do so. Whether the condition is suboptimal or overstretching, the results of theory and simulation are in good agreement.

In this paper, we have not considered potential well distortion and have assumed that the chromaticity is zero. Our approach can be extended to scenarios involving potential well distortion by employing the Haïssinski distribution [22] and the associated potential well. Furthermore, a self-consistent approach would consider the impact of longitudinal impedance on beam dynamics, leading to a more complex Vlasov equation. The solver we introduced also offers a possible method for handling cases of nonzero chromaticity as demonstrated in Appendix A, although numerical tests are still required.

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DATA AVAILABILITY

The computational codes that support the findings of this article are openly available [31]. The data are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: ACTION-ANGLE VARIABLES, POLAR COORDINATES

In the longitudinal plane, using the Hamiltonian in Eq. (3), we can derive the action-angle variables by canonical transformation [32].

We define the potential well as

$$V_{\text{pw}}(z) = \frac{eV_1}{2\pi h_1 E_0} \left\{ \cos \left(\phi_s - \frac{2\pi h_1}{C} z \right) - \cos \phi_s + \frac{r}{h} \left[\cos \left(\phi_{2s} - \frac{2\pi h_2}{C} z \right) - \cos \phi_{2s} \right] - \left(\frac{2\pi h_1}{C} z \right) (\sin \phi_s + r \sin \phi_{2s}) \right\}. \quad (\text{A1})$$

Then, we write the simplified expression of the canonical transformation

$$F_2[z, J(H)] = \int_z^{z_{\text{max}}} dz \sqrt{-2[H + V_{\text{pw}}(z)]/\eta}, \quad (\text{A2})$$

where the action variable $J(H)$ is given by

$$J(H) = \frac{1}{\pi} \int_{z_{\text{min}}}^{z_{\text{max}}} dz \sqrt{-2[H + V_{\text{pw}}(z)]/\eta}, \quad (\text{A3})$$

and the angle variable ϕ is

$$\phi = \frac{\partial F_2(z, J)}{\partial J}. \quad (\text{A4})$$

We can calculate the incoherent synchrotron tune shift by

$$\nu_s(J) = \frac{C}{2\pi} \frac{dH}{dJ}. \quad (\text{A5})$$

By solving the equation $V_{\text{pw}}(z) = H(J_i)$, which involves the potential well and the specific Hamiltonian corresponding to a discretized action, one can obtain the integration limits z_{min} and z_{max} .

If one takes linear chromaticity into consideration, the first term in Eq. (16) is replaced by

$$i(\Omega - \omega_\beta - \omega_\beta \xi \delta) g_1(J, \phi). \quad (\text{A6})$$

Chromaticity can be canceled out by employing the chain rule of differentiation as performed, using a technique similar to Oide's [25]

$$\delta = -\frac{1}{\eta} \frac{\partial H}{\partial \delta} \quad (\text{A7})$$

$$= -\frac{1}{\eta} \frac{dz}{ds} \quad (\text{A8})$$

$$= -\frac{1}{\eta} \frac{d\phi}{ds} \frac{\partial z}{\partial \phi} \quad (\text{A9})$$

$$= -\frac{\omega_s}{\eta c} \frac{\partial z}{\partial \phi}. \quad (\text{A10})$$

By substituting the relationship into the differentiation term in Eq. (16), we get

$$\omega_s \frac{\partial g_1}{\partial \phi} = \left(il + \frac{i\xi\omega_\beta}{c\eta} \frac{\partial z(J, \phi)}{\partial \phi} \right) g_1 \quad (\text{A11})$$

$$= (il\omega_s - i\xi\omega_\beta\delta)g_1, \quad (\text{A12})$$

where the second term in the bracket ensures that the linear chromaticity cancels, except for the beam spectrum function [Eq. (20)]. As a result, the eigenvalue problem [Eq. (22)] remains applicable even for nonzero chromaticity cases.

In the vertical plane, we use Chao's definition of polar coordinates [33]

$$y = q \cos \theta, \quad (\text{A13})$$

$$p_y = -\frac{\omega_\beta}{c} q \sin \theta. \quad (\text{A14})$$

The physical interpretation is that the beam distribution is considered an undistorted circle in the vertical phase space. In contrast, in the longitudinal phase space, the distribution is affected by a double-rf system, where action-angle variables are more suitable than polar coordinates.

In the longitudinal plane, the Poisson bracket remains unchanged under canonical transformations

$$\begin{aligned} \{H, \Psi\} &= \frac{\partial \Psi}{\partial z} \frac{dz}{ds} + \frac{\partial \Psi}{\partial \delta} \frac{d\delta}{ds} \\ &= \frac{\partial \Psi}{\partial \phi} \frac{d\phi}{ds} + \frac{\partial \Psi}{\partial J} \frac{dJ}{ds} \\ &= \frac{\omega_s(J)}{c} \frac{\partial \Psi}{\partial \phi}. \end{aligned} \quad (\text{A15})$$

In the transverse plane, applying differentiation to Eq. (A13) leads to

$$\frac{\partial \Psi}{\partial \theta} = -q \sin \theta \frac{\partial \Psi}{\partial y} - \frac{\omega_\beta}{c} q \cos \theta \frac{\partial \Psi}{\partial p_y}. \quad (\text{A16})$$

After performing the variable substitution, we obtain

$$\begin{aligned} \frac{\partial \Psi}{\partial y} \frac{dy}{ds} + \frac{\partial \Psi}{\partial p_y} \frac{dp_y}{ds} &= p_y \frac{\partial \Psi}{\partial y} + \left[-\left(\frac{\omega_\beta}{c}\right)^2 y + \frac{F_y}{E_0} \right] \frac{\partial \Psi}{\partial p_y} \\ &= \frac{\omega_\beta}{c} \frac{\partial \Psi}{\partial \theta} + \frac{F_y}{E_0} \frac{\partial \Psi}{\partial p_y}. \end{aligned} \quad (\text{A17})$$

By combining Eqs. (A15) and (A17), we obtain Eq. (7).

APPENDIX B: DERIVATION OF TRANSVERSE DIPOLE WAKE FORCE

The vertical force F_y arises from the $m = 1$ moment of the wakefield. And the $m = 1$ wakefield is excited by dipole oscillations of the particles. This wakefield persists in the vacuum chamber and subsequently drives dipole oscillations in trailing test particles. Thus, it includes the longitudinal density $\rho(z)$ and transverse amplitude $y(z, s)$. The transverse wake force $F_{y,m=1}$ can be expressed as

$$\begin{aligned} F_{y,m=1} &= \frac{\Delta p}{\Delta t} \\ &= -\frac{e}{cT_0} \int_{-\infty}^{+\infty} e \cdot \rho(z') y(z', s) W_{m=1}(z - z') dz'. \end{aligned} \quad (\text{B1})$$

The first e is the test particle, and $e\rho(z)$ is the longitudinal charge distribution. The term W_m represents a wake that generated by a beam with $2m$ -pole moment, as shown in Table 2.2 (p. 63) of Chao's textbook [21]. Therefore, $W_{m=1}$ is the dipole wake, which is linear with the centroid displacement amplitude of the driving beam.

The present analysis excludes closed orbit distortion [21]. Consequently, the vanishing amplitude term $f_0(q)g_0(J)$ does not contribute to $F_{y,m=1}$. Therefore, the driving force comes from the perturbation. When adding a perturbation, the unperturbed distribution $f_0(q)$ is displaced by a small distance D . Thus, the distribution in the transverse phase space is

$$\begin{aligned} f_0(q) &\rightarrow f_0(q - De^{i\theta}) \approx f_0(q) - De^{i\theta} f'_0(q) \\ &= f'_0(q) + f_1(q, \theta). \end{aligned} \quad (\text{B2})$$

Thus, the contribution of g_1 in wake force [Eq. (B1)] is

$$\begin{aligned} y(z', s) \cdot \rho(z') &= y(z') e^{-i\Omega s/c} \rho(z') \\ &= De^{-i\Omega s/c} \rho_1(z') \\ &= e^{-i\Omega s/c} \int_{-\infty}^{\infty} d\delta g_1(r, \phi) \\ &= De^{-i\Omega s/c} \rho_1(z). \end{aligned} \quad (\text{B3})$$

Here, when $y(z') \rightarrow D$, the amplitude dipole oscillation, the term $\rho_1(z)$ serves to scale the oscillation amplitude along the bunch. This leads to the expression for the $m = 1$ transverse driving force $F_{y,m=1}$ given in Eq. (15):

$$F_{y,m=1}(z, s) = -\frac{De^2}{cT_0} e^{-i\Omega s/c} \int_{-\infty}^{\infty} \rho_1(z) W_{m=1}(z - z') dz'. \quad (\text{B4})$$

APPENDIX C: DERIVATION OF INTEGRAL EQUATION

Applying differentiation to Eq. (A14), combining with Eq. (A16), we obtain

$$\frac{\partial \psi}{\partial p_y} = \frac{c \cos \theta}{q \omega_\beta} \frac{\partial \psi}{\partial \theta} + \frac{c \sin \theta}{\omega_\beta} \frac{\partial \psi}{\partial \theta} \quad (\text{C1})$$

Thus, we can rewrite Eq. (7) as

$$\begin{aligned} \frac{\partial \psi}{\partial s} - \frac{c \sin \theta}{E_0 \omega_\beta} F_y(z, s) \frac{\partial \psi}{\partial q} + \left[\frac{\omega_\beta}{c} - \frac{c \cos \theta}{q \omega_\beta E_0} F_y(z, s) \right] \frac{\partial \psi}{\partial \theta} \\ + \frac{\omega_s(J)}{c} \frac{\partial \psi}{\partial \phi} = 0. \end{aligned} \quad (\text{C2})$$

Assuming vanishing chromaticity, we substitute the phase-space density $\psi(q, \theta, J, \phi; s)$ into the Vlasov equation, yielding

$$\begin{aligned} \left[-i \frac{\Omega}{c} f_1 g_1 + \frac{\omega_\beta}{c} \frac{\partial f_1}{\partial \theta} g_1 + \frac{\omega_s}{c} f_1 \frac{\partial g_1}{\partial \phi} \right] e^{-i\Omega s/c} \\ - \frac{c}{E_0 \omega_\beta} \sin \theta F_y f'_0 g_0 = 0. \end{aligned} \quad (\text{C3})$$

Here, the wake force F_y and the derivative f'_0 are first-order quantities, similar to f_1, g_1 . Higher-order terms (second order and above) are neglected. Then substituting the $m = 1$ mode wake force

$$F_y(z, s) = -\frac{De^2}{cT_0} \int_{-\infty}^{\infty} dz' \rho_1(z') W_{m=1}(z - z') e^{-i\Omega s/c} \quad (\text{C4})$$

$$= i \frac{De^2}{2\pi c T_0} e^{-i\Omega s/c} \int_{-\infty}^{\infty} d\omega' \tilde{\rho}_1(\omega') Z_{m=1}^\perp(\omega') e^{i\omega' z/c} \quad (\text{C5})$$

into the Vlasov equation, we get Eq. (16)

$$\begin{aligned} i(\Omega - \omega_\beta) g_1(J, \phi) - \omega_s(J) \frac{\partial g_1(J, \phi)}{\partial \phi} \\ - \frac{Nce^2}{4\pi E_0 T_0 \omega_\beta} g_0(J) \int_{-\infty}^{\infty} e^{i\frac{\omega}{c} z} Z_1^\perp(\omega) \tilde{\rho}_1(\omega) d\omega = 0. \end{aligned} \quad (\text{C6})$$

Here, the $\sin \theta$ is replaced by $e^{i\theta}/2i$, as the frequency shift induced by the wakefield is negligible compared to ω_β [21].

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