

Beam instability induced by the rf cavity accelerating modes and its suppression in the Super Tau-Charm Facility

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The Super Tau-Charm Facility (STCF), a proposed high-luminosity electron-positron collider in China, requires a beam current of 2 A to achieve its design luminosity. Such intense current induces severe coupled-bunch instability driven by the rf cavity accelerating modes. This paper presents a systematic analysis of these instabilities across the STCF operational energy range of 1–3.5 GeV. Without active control, the low-order coupled-bunch modes, particularly the mode -1 , exhibit growth rates significantly exceeding the radiation-damping rate. We demonstrate that the instability can be effectively suppressed using only the proportional-integral feedback loop of the baseline low-level rf system. The key lies in setting a unit proportional gain and a loop delay of approximately half the revolution period. This configuration creates a notch-filter response that attenuates the most dangerous mode -1 . By combining analytical calculations with particle-tracking simulations, we show that this simple and robust approach maintains beam stability for the baseline design parameters across all design energies, thereby enabling reliable high-current operation without requiring additional dedicated feedback measures, such as direct or one-turn-delay feedback. A stability assessment based on the Nyquist criterion confirms the closed-loop stability of the system and indicates sufficient gain margins. These results provide a reliable foundation for the STCF rf and feedback system design, enabling stable high-current operation under realistic constraints. The proposed approach and the associated analysis framework also offer a practical reference for other high-current storage rings facing similar instability challenges.

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I. INTRODUCTION

The Super Tau-Charm Facility (STCF) is a next-generation circular electron-positron collider proposed in China, designed to operate at center-of-mass energies from 2 to 7 GeV [1]. At 4 GeV center-of-mass energy, STCF targets a peak luminosity exceeding $5 \times 10^{34} \text{ cm}^{-2} \text{ s}^{-1}$ to enable precision studies of charm and tau physics (see Fig. 1 for the overall layout). Achieving this design luminosity requires beam currents up to 2 A, making impedance-driven collective instabilities a major concern in the machine and rf-system design. Among these, longitudinal coupled-

bunch instability (LCBI) driven by the accelerating-mode impedance of the rf cavities poses a particularly significant threat to beam stability [2,3].

To mitigate this risk at the source, STCF adopts normal-conducting 500 MHz TM_{020} cavity design [1], leveraging the successful development and operation of compact TM_{020} cavities at the NanoTerasu 3 GeV light source [4]. Compared to conventional TM_{010} cavities at the same frequency, the TM_{020} design exhibits an unloaded quality factor Q_0 approximately twice as high and a normalized shunt impedance R/Q approximately halved. Consequently, the effective beam-cavity coupling is reduced by approximately half while maintaining a comparable total shunt impedance [5]. Although this reduction directly alleviates the driving term of LCBI, detailed calculations show that the growth rate of the fastest-growing $\mu = -1$ mode still significantly exceeds the longitudinal radiation-damping rate. Thus, active feedback remains essential to suppress the instability at the STCF design current.

In high-current storage rings, suppressing accelerating-mode-driven LCBI requires a sufficient reduction of the

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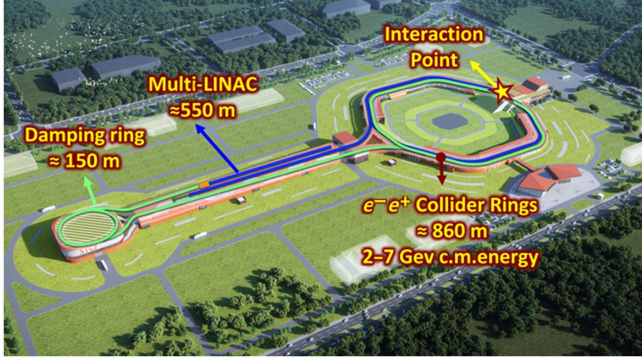


FIG. 1. Conceptual layout of the STCF complex, including the damping ring (~ 150 m), the multisection linac (~ 550 m), and the e^+e^- collider rings (~ 860 m) operating at 2–7 GeV center-of-mass energy.

effective cavity impedance so that intrinsic damping mechanisms—radiation damping or Landau damping—can overcome the instability growth [3]. Besides these intrinsic mechanisms, beam-based feedback systems can also provide extra damping for this instability. Such a feedback typically uses the rf cavity itself as a kicker, demanding low group delay, as exemplified by the Low Group Delay Woofer feedback at PEP-II [6]. By contrast, conventional bunch-by-bunch feedback systems using dedicated kickers are generally designed to combat coupled-bunch instabilities driven by higher-order modes [7]. In modern colliders such as the LHC and FCC-ee, the necessary impedance control to enable intrinsic damping is achieved through low-level rf (LLRF) feedback loops [8,9]. A typical approach combines a wideband direct rf feedback loop to suppress the impedance peak near the accelerating-mode resonance [10] with a one-turn-delay feedback that provides high gain at the relevant revolution harmonics [11]. For example, the CEPC main-ring design employs this combination to mitigate accelerating-mode-driven LCBI to a level manageable by other damping mechanisms [12]. In PEP-II, a one-turn-delay, double-peaked comb filter was integrated into the direct rf feedback to lower the effective impedance and damp low-order coupled-bunch modes, at the cost of increased hardware and commissioning complexity [11,13]. At SuperKEKB, a dedicated narrow-band LCBI damper operates alongside the baseline LLRF system to suppress the dominant unstable modes [14]. These existing solutions, while effective, all rely on additional hardware or specialized feedback paths that increase system complexity and commissioning effort.

In contrast to the approaches described above, we propose that the baseline LLRF system alone can provide sufficient stabilization for the STCF, using only a proportional-integral (PI) feedback loop with appropriately chosen parameters. By setting the proportional gain to unity and the loop delay to approximately half the revolution period, the PI loop shapes the closed-loop

response into an effective notch filter. This configuration successfully suppresses the most dangerous $\mu = -1$ mode and maintains the growth rates of all low-order modes below the radiation-damping threshold. Importantly, this solution addresses accelerating-mode-driven instability without requiring any additional feedback hardware or complex control schemes.

In this paper, we present a systematic analysis of this scheme for STCF at beam energies of 1, 2, and 3.5 GeV. By combining analytical impedance modeling with self-consistent multiparticle tracking simulations that include the full coupled beam-cavity-LLRF dynamics, we map the stable operating region in the proportional-gain-delay plane. The resulting stability region is broad, indicating strong robustness to moderate uncertainties in these parameters. Furthermore, Nyquist analysis confirms that the feedback loop itself is stable at the operating point, with an ample gain margin (GM). We also assess practical implementation aspects, including the effects of sampling frequency, achievable loop delay, residual loop-phase errors, and realistic bunch-filling patterns.

This work thereby establishes a simplified, cost-effective, and reliable solution for longitudinal instability control in the STCF, forming a critical input for the final technical design of its rf and feedback systems. The proposed scheme and the associated analysis framework also offer a valuable reference for the design of feedback systems in other high-current colliders facing similar challenges. The paper is organized as follows. Section II establishes a theoretical framework for the beam-cavity-feedback system, including the notch-filter mechanism achieved with unit gain and half-revolution delay, and introduces the analytical and particle-tracking methods. Section III provides the stability analysis, maps the robust parameter region, and verifies loop stability via Nyquist analysis. Practical implementation considerations are addressed in Secs. IV, and V presents the conclusions.

II. THEORETICAL FRAMEWORK AND METHODS

The longitudinal beam dynamics, rf, and cavity parameters for the STCF at its operating energy points are summarized in Table I, based on the conceptual design report [1]. To compensate for the increasing synchrotron radiation loss at higher energies, the total rf voltage and number of cavities are scaled correspondingly, while the intrinsic parameters of the individual 500 MHz normal-conducting TM_{020} cavity remain fixed, consistent with its proposed design. The key parameters listed in Table I, such as the optimal coupling factor β_{opt} and the loaded shunt impedance R_L , are derived following the beam-loading compensation formulas detailed in Appendix A.

This section introduces the cavity impedance model and details the notch-filter mechanism enabled by a PI feedback loop configured with unit gain and half-turn delay.

TABLE I. Key STCF parameters related to the longitudinal dynamics analysis at different beam energies.

Parameter	Symbol	Unit	1 GeV	2 GeV	3.5 GeV
Circumference	C	m		860.321	
rf frequency	f_{rf}	MHz		499.7	
Revolution period	T_0	ns		2870	
Harmonic number	h	...		1434	
Bunch filling factor	...	%		48	
Beam current	I	A	1.1	2	2
Momentum compaction factor	α_c	10^{-4}	12.63	13.49	13.73
Energy spread (with damping wigglers)	σ_δ	10^{-4}	6.18	7.82	10.02
Energy loss per turn (with damping wigglers)	U_0	keV	106	543	1494
Power loss per turn	P	kW	117	1086	2988
Longitudinal damping time	τ_z	ms	27	11	6.7
Total rf voltage	V_{rf}	MV	0.75	2.5	6
Voltage per cavity	V_c	kV	375	417	400
Number of rf cavities	N_{cav}	...	2	6	15
Power loss per cavity	P_c	kW	27	34	31
Beam power per cavity	P_b	kW	58	181	199
Optimal coupling factor	β_{opt}	...	3.1	6.3	7.4
R/Q of TM ₀₂₀ cavity	R/Q	Ω		42	
Unloaded quality factor	Q_0	...		61,000	
Shunt impedance	R_s	M Ω		2.562	
Adjustable coupling factor	β	...		0.99–10	
Detuning range	...	kHz		± 200	
Maximum input power	...	kW		300	
Maximum cavity voltage	...	kV		600	
Loaded quality factor	Q_L	...	14,790	8308	7280
Loaded shunt impedance	R_L	M Ω	0.62	0.35	0.31
Optimal detuning frequency	$\Delta f_{r,\text{opt}}$	kHz	−60.9	−98.3	−101.6
Detuning frequency ($\theta_L = -10^\circ$)	Δf_r	kHz	−65.5	−107.5	−112.3
Generator power per cavity ($\theta_L = 0^\circ$)	P_g	kW	85.7	214.9	230.4
Generator power per cavity ($\theta_L = -10^\circ$)	P_g	kW	88.4	221.6	237.6

We then describe both the analytical formulation and particle-tracking methods employed to evaluate the LCBI growth rates.

A. Closed-loop cavity impedance

The beam-cavity interaction is generally modeled using an equivalent parallel R-L-C circuit. Throughout this paper, we adopt the rf-engineering sign convention, with a time dependence of $e^{i\omega t}$. Under this convention, a pure time delay τ corresponds to a phase factor of $e^{-i\omega\tau}$. Accordingly, the open-loop cavity impedance (i.e., without feedback) as seen by the beam can be written as [15]

$$Z_{\text{cav}}(\omega) = \frac{R_L}{1 - iQ_L(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r})} \simeq \frac{R_L}{1 - iQ_L \frac{2\Delta\omega}{\omega_r}}, \quad (1)$$

where R_L and Q_L are the loaded shunt impedance and loaded quality factor, respectively, and ω_r is the resonant angular frequency. The detuning is defined as $\Delta\omega = \omega_r - \omega$, where ω is the frequency of interest, typically an rf harmonic or its neighboring revolution sidebands. The approximation in Eq. (1) is valid under the narrow-band condition $|\Delta\omega| \ll \omega_r$.

When a PI controller is used in the LLRF system, the feedback modifies the effective impedance seen by the beam [8]. The closed-loop cavity impedance is

$$Z_{\text{cl}}(\omega) = \frac{Z_{\text{cav}}(\omega)}{1 + e^{-i\omega\tau_d} G(\omega) Z_{\text{cav}}(\omega) e^{i\phi}}, \quad (2)$$

where τ_d is the overall loop delay, ϕ represents the loop-phase correction (ideally adjusted to give zero phase at the fundamental rf frequency), and $G(\omega)$ is the loop gain, approximated as $G(\omega) \approx K_p/R_L$, with K_p the proportional gain of the PI controller and R_L the cavity loaded shunt impedance. This simplified form highlights the dominant role of the proportional gain while neglecting the integral term, a simplification deliberately adopted to facilitate the analytical discussion. It should be emphasized that in the subsequent tracking simulations and analytical calculations of the instability growth rate, the integral gain is fully retained (though its effect on the LCBI is, in practice, very small).

For convenience, we take the loop delay τ_d to be an integer multiple of the rf period. With this choice, ideal

phase correction yields $\phi = 0$, and the phase factor $e^{i\phi}$ appearing in Eq. (2) can be regarded as a residual phase offset arising from nonideal correction. In the following, we primarily analyze the ideal case $\phi = 0$; the influence of a nonzero residual phase is examined separately in Sec. IV C.

To facilitate the analysis of LCBI, we introduce a detuning angle θ defined by

$$\theta = \tan^{-1} \left(\frac{2Q_L \Delta\omega}{\omega_r} \right). \quad (3)$$

Using the identity $1 - i \tan(\theta) = e^{-i\theta}/\cos(\theta)$, Eq. (1) can be expressed in the compact form

$$Z_{\text{cav}}(\theta) = R_L \cos(\theta) e^{i\theta}. \quad (4)$$

Substituting Eq. (4) into Eq. (2) yields the closed-loop cavity impedance in terms of the detuning angle

$$Z_{\text{cl}}(\theta) = \frac{R_L \cos(\theta) e^{i\theta}}{1 + K_p \cos(\theta) e^{i(\theta - \omega\tau_d)}}. \quad (5)$$

Since the coupled-bunch growth rate is governed primarily by the real part of the cavity impedance at the revolution harmonics [16], it is necessary to identify parameter settings that minimize $\text{Re}[\mathbf{Z}_{\text{cl}}(\theta)]$. Equation (5) provides an effective method for analyzing the closed-loop impedance characteristics based on a phasor diagram, which can be directly applied to this minimization. A graphical interpretation is illustrated in Fig. 2. A circle of radius $\cos(\theta)$ is centered at point **O** (unity on the real axis). Phasors are defined as **AB** [numerator of Eq. (5)], **OC** ($\cos(\theta)e^{i\theta}$), and **AT** (tangent to the circle). Given that **AT** $\perp$ **OT** and from triangle **AOT** ($|\mathbf{AO}| = 1$, $|\mathbf{OT}| = \cos(\theta)$), angle $\angle\text{AOT} = \theta$. Therefore, **OT** $\parallel$ **OC** (phase difference $(2k + 1)\pi$), and since **AB** $\perp$ **AT**, it follows that $\text{Re}[\mathbf{AB}/\mathbf{AT}] = 0$, i.e., $\text{Re}[\mathbf{Z}_{\text{cl}}(\theta)] = 0$.

In summary, $\text{Re}[\mathcal{Z}_{\text{cl}}(\theta)]$ vanishes for $K_p = 1$ and $\omega\tau_d = (2k+1)\pi$ ($k \in \mathbb{Z}$). For the unstable low-order modes of concern, such as $\mu = -1$ and -3 , this condition is automatically satisfied by setting $\tau_d = T_0/2$.

B. Notch filter with the setting of unit gain and half-turn delay

To validate the phasor-based interpretation presented above, we examine the dependence of the closed-loop impedance on the proportional gain and loop delay, using the STCF 2 GeV parameters as a representative case. Figure 3 compares the frequency dependence of the real and imaginary parts of the cavity impedance for representative feedback settings. The real part is directly related to the coupled-bunch growth or damping rate, whereas the imaginary part contributes to the coherent modal frequency shift. In particular, with a unit proportional gain

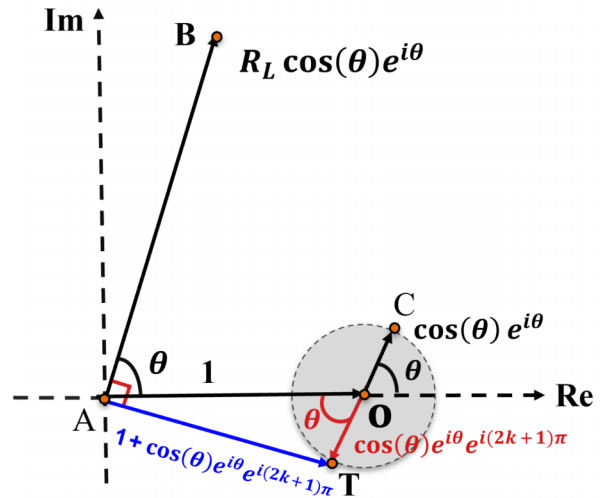


FIG. 2. Phasor diagram for the condition of $\text{Re}[Z_{\text{cl}}] = 0$ under the case of $\phi = 0$.

and a loop delay close to half the revolution period, the effective impedance at the revolution harmonics associated with the $\mu = -1$ and $\mu = -3$ modes is strongly reduced.

In contrast, at the frequency corresponding to $\mu = -2$ mode, the value of $\text{Re}[Z_{\text{cl}}]$ is found to be somewhat higher. This behavior is consistent with the phase condition discussed in Sec. II A: a near half-turn delay produces an odd- π phase shift at the relevant spectral lines for odd- μ modes, while the condition is not necessarily satisfied for even- μ modes. This operating point remains acceptable provided that the resulting growth rate of the $\mu = -2$ mode stays below the radiation-damping rate [17].

Moreover, the impedance spectra indicate that variations of K_p and τ_d within the parameter ranges explored in Fig. 3 primarily affect the depth and frequency alignment of the suppression features in $\text{Re}[Z_{\text{cl}}]$. Over a broad vicinity of the nominal setting $(K_p, \tau_d) \approx (1, T_0/2)$, the reduction of the impedance driving the $\mu = -1$ and $\mu = -3$ modes remains pronounced, while the change near the $\mu = -2$ mode is comparatively modest.

This impedance-spectrum analysis provides a qualitative explanation for the mode-selective suppression achieved by the PI feedback. In the following sections, we quantify the growth rates and coherent oscillation frequencies of the coupled-bunch modes using both analytical calculations and particle-tracking simulations, thereby verifying the effectiveness of the proposed scheme for suppressing accelerating-mode-driven LCBI in the STCF.

C. Instability analysis methods

The growth rates of accelerating-mode-driven LCBI are evaluated using two complementary approaches: (i) an analytical method based on the coupled-bunch dispersion relation and (ii) multibunch particle-tracking simulations.

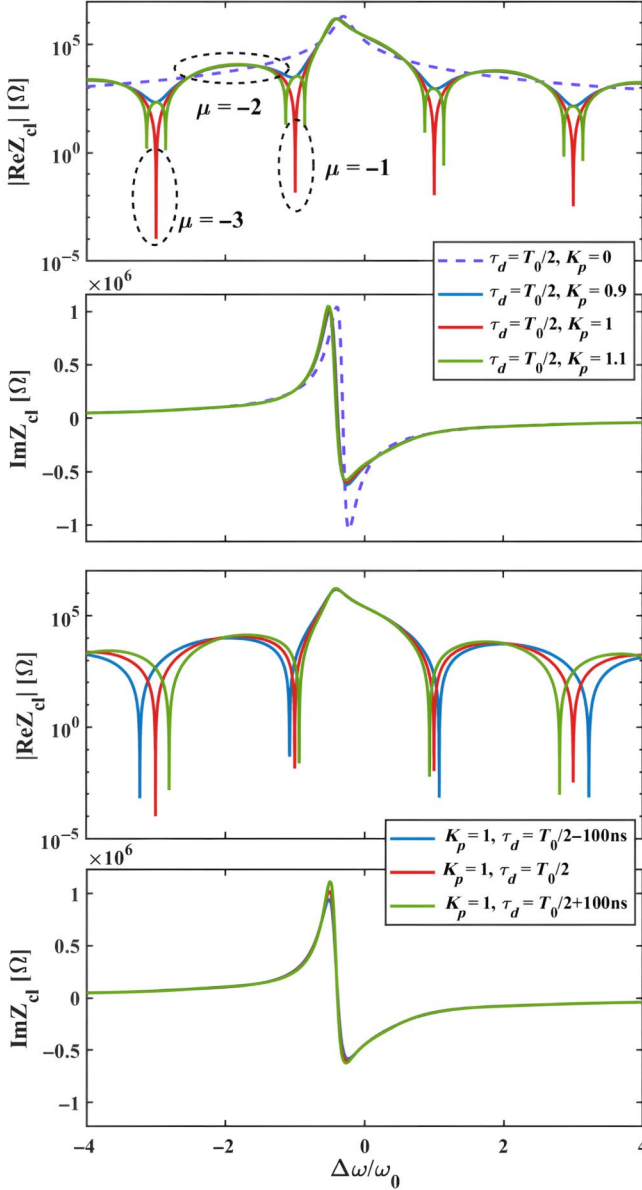


FIG. 3. Closed-loop cavity impedance under different feedback parameters given in the legend. The first and third panels show the real part, and the second and fourth panels show the imaginary part. ω_0 denotes the angular revolution frequency.

The two approaches are used to cross-validate the calculated growth rates.

1. Root-finding numerical method

For a given cavity impedance (open-loop or closed-loop), the complex frequency $\Omega = \Omega_r + i\Omega_i$ of a coupled-bunch mode μ is determined by solving the dispersion relation [3,16,18]. Given that the impedance is dominated by the cavity accelerating mode, the dispersion relation can be simplified by keeping the pair of spectral lines closest to the cavity resonance (see Appendix B for derivation details), i.e., those near $\pm h\omega_0$:

$$\Omega^2 - \omega_s^2 + i\xi[\omega_p^+ Z_{cl}(\omega_p^+) + \omega_p^- Z_{cl}(\omega_p^-)] = 0, \quad (6)$$

where ω_s is the unperturbed synchrotron frequency, $Z_{cl}(\omega)$ is the impedance (open-loop or closed-loop), ξ is a real constant, and $\omega_p^\pm = \pm h\omega_0 + \mu\omega_0 + \Omega$ are the frequencies of the upper and lower synchrotron sidebands for mode μ . Here, Ω_r is the coherent mode angular frequency, and $|\Omega_i|$ gives the growth rate ($\Omega_i < 0$) or damping rate ($\Omega_i > 0$).

Equation (6) can be solved numerically using the root-finding procedure described in Appendix B, which yields reliable values of Ω_r and Ω_i over the parameter range of interest. It should be noted, however, that Eq. (6) is a simplified version of Eq. (B1). When the influence of a residual loop-phase offset is considered, the simplifying assumptions underlying Eq. (6) are no longer satisfied for the $\mu = 0$ mode instability. Consequently, Eq. (6) gives an incorrect result for the mode-0 instability in that case, and one must directly use the unsimplified Eq. (B1) for the analysis.

2. Macroparticle tracking simulations

We validate the analytical framework using the GPU-accelerated tracking code STABLE [19,20], which employs a more realistic, I/Q-based PI feedback model commonly used in modern digital LLRF systems, as opposed to a traditional amplitude/phase control model [21,22]. In this model, as shown in Fig. 4, rf voltages and currents are represented as complex phasors, with the real and imaginary parts corresponding to the in-phase (I) and quadrature (Q) components, respectively. By emulating the actual digital signal-processing chain, the simulation delivers a high-fidelity representation of the coupled beam-cavity-feedback dynamics.

Five parameters define the digital PI feedback implementation: (i) Proportional gain K_p . (ii) Integral gain K_i . (iii) Loop delay d (in buckets), with the corresponding time delay given by $\tau_d = dT_b$, where T_b denotes the rf bucket period. (iv) Averaging-window length m (in buckets), defined as the number of buckets over which the sampled cavity-voltage phasor is averaged. Its value is set by system parameters, including the sampling rate and the finite-impulse-response (FIR) filter taps, and is assigned directly in the simulation [21]. (v) Residual loop-phase offset ϕ , which represents a static phase offset in the complex I/Q feedback signal. In the tracking simulation, this phase shift is implemented by multiplying the feedback signal by $e^{i\phi}$, equivalently rotating the I/Q feedback vector. The nominal setting used in the baseline analysis corresponds to $\phi = 0$.

In the parameter scans, the averaging window is fixed at 10 buckets and the integral gain at 2×10^{-5} ; proportional gain and loop delay are varied. Each bunch is represented by 10,000 macroparticles, and the simulation time step is one rf period, so all temporal parameters are in units of rf buckets. The number of tracking turns is adjusted

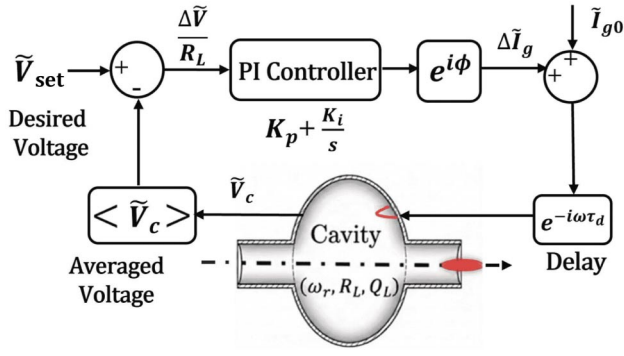


FIG. 4. Block diagram of the standard I/Q-based PI cavity-voltage regulation loop implemented in the STABLE tracking code. The PI controller is represented by the transfer function of $K_p + K_i/s$.

dynamically to ensure a clean exponential-growth signal. Beam stability is evaluated by identifying the fastest-growing coupled-bunch mode μ via a discrete Fourier transform (DFT), constructing its analytic signal with a Hilbert transform [23,24], and extracting the growth rate by an exponential fit to the modal amplitude. The mode-identification and fitting procedures are described in Appendix C.

It should be noted that the PI loop considered here is the standard cavity-voltage regulation loop of the digital I/Q LLRF system, not an additional feedback loop dedicated solely to instability damping. The PI loop serves a dual purpose: it regulates the cavity field against beam-loading perturbations and reference errors, while simultaneously shaping the closed-loop cavity impedance near the relevant revolution harmonics to suppress accelerating-mode-driven LCBI. Auxiliary functions such as slow tuning control, set-point adaptation, and feed-forward may coexist with this loop, but they lie outside the scope of the present study [25].

III. INSTABILITY ANALYSIS AND FEEDBACK PERFORMANCE

A. Instability growth without any feedback

We first evaluate the LCBI growth rates in the absence of rf feedback. The open-loop condition is simulated with $K_p = K_i = 0$, and growth rates are calculated for a uniform fill at the three main STCF energies: 1, 2, and 3.5 GeV. All calculations throughout this paper use the design beam currents and the corresponding machine and rf parameters listed in Table I.

The results are summarized in Table II. For each energy, the table provides the fastest-growing mode index, its growth rate from tracking and analytical methods, and the radiation-damping rate. The analytical results include both an exact numerical solution of the dispersion relation and an approximate simple formula (see Appendix B for details).

In all cases, the fastest-growing mode is $\mu = -1$, and its growth rate substantially exceeds the radiation-damping rate. For example, at 2 GeV, the tracking simulation gives 407 s^{-1} , more than four times the damping rate of 91 s^{-1} . This confirms that, despite the reduced accelerating-mode impedance of the TM_{020} design, the beam is unstable at the STCF design current without any active feedback.

Furthermore, the close agreement among tracking, exact analytical, and approximate results validates the modeling approach, providing a solid basis for the feedback studies that follow.

B. PI configuration with unit gain and half-turn delay

For the PI configuration with unit gain and half-turn delay ($K_p = 1$, $d = 750$ buckets), the resulting growth rates are also listed in Table II. At all energies, the fastest-growing mode shifts from $\mu = -1$ to $\mu = -2$; the growth rate of the new-dominant $\mu = -2$ mode, however, remains below the radiation-damping rate, confirming that the accelerating-mode-driven LCBI is effectively suppressed.

TABLE II. Fastest-growing coupled-bunch mode and its complex eigenvalue with and without PI feedback. For each beam energy, the table lists the fastest-growing mode index μ , the coherent oscillation frequency $f_r = \Omega_r/2\pi$ (in Hz), and the growth rate $-\Omega_i$ for the no-feedback case ($K_p = 0$, $K_i = 0$) and for the proposed PI setting ($K_p = 1$, $K_i = 2 \times 10^{-5}$, $d = 750$ buckets, $\phi = 0$). Results are obtained from particle tracking and from the impedance-based analytical model (exact numerical solution). For the approximate analytical solution, only the growth rate is listed, since that approximation assumes $\Omega_r \approx \omega_s$.

Energy (GeV)	Without any feedback						With PI feedback					
	μ	Particle tracking		Analytical (exact)		Analytical (approx.)	μ	Particle tracking		Analytical (exact)		Analytical (approx.)
		f_r (Hz)	Growth (s^{-1})	f_r (Hz)	Growth (s^{-1})	Growth (s^{-1})		f_r (Hz)	Growth (s^{-1})	f_r (Hz)	Growth (s^{-1})	Growth (s^{-1})
3.5	-1	8364	610	8386	616	644	-2	8040	130	8050	132	135
2.0	-1	7074	407	7092	408	426	-2	6821	88	6834	88	91
1.0	-1	5183	58	5188	58	58	-2	5114	11	5120	13	15
												Radiation damping rate (s^{-1})
												149
												91
												37

Note that in tracking simulations, the radiation-damping time was artificially increased to allow the instability to develop visibly, thereby enabling accurate extraction of the growth rate.

At 2 GeV, the approximate formula yields a growth rate of 91 s^{-1} , equal to the radiation-damping rate and thus indicating marginal stability. In contrast, the exact numerical solution gives 88 s^{-1} , which agrees with the particle-tracking result. This agreement validates the exact solution and shows that the actual growth rate lies below the damping threshold.

The results presented so far correspond to the fastest-growing mode for the fixed nominal parameters. In the following, we examine how deviations of the proportional gain and loop delay from these target values influence the coupled-bunch instability.

C. Robustness analysis

The robustness of the proposed feedback scheme is assessed via two-dimensional parameter scans in the

(K_p, d) plane at beam energies of 2 and 3.5 GeV. The scans span K_p from 0.60 to 1.85 and loop delay from 500 to 1000 buckets. At each grid point, we identify the fastest-growing mode and compute its growth rate, thereby mapping the stable operating region.

Figure 5 presents the results, showing close agreement between the analytical calculation and tracking simulation. A consistent feature across all maps is a crescent-shaped contour that delineates the stable region, where the instability growth rate equals the longitudinal radiation-damping rate. The overlaid mode symbols reveal the evolution of the fastest-growing mode across the parameter space. Outside the crescent—e.g., at low K_p or delays far from $T_0/2$ —the $\mu = -1$ mode dominates. Inside the stable region, which is centered on $K_p = 1$ and $\tau_d \approx T_0/2$, the notch-filter mechanism suppresses the odd- μ modes, causing the even $\mu = -2$ mode to become the fastest-growing. At larger gains or delays, the dominant $\mu = -2$ mode shifts to $\mu = +1$ or $\mu = -3$. Both this mode

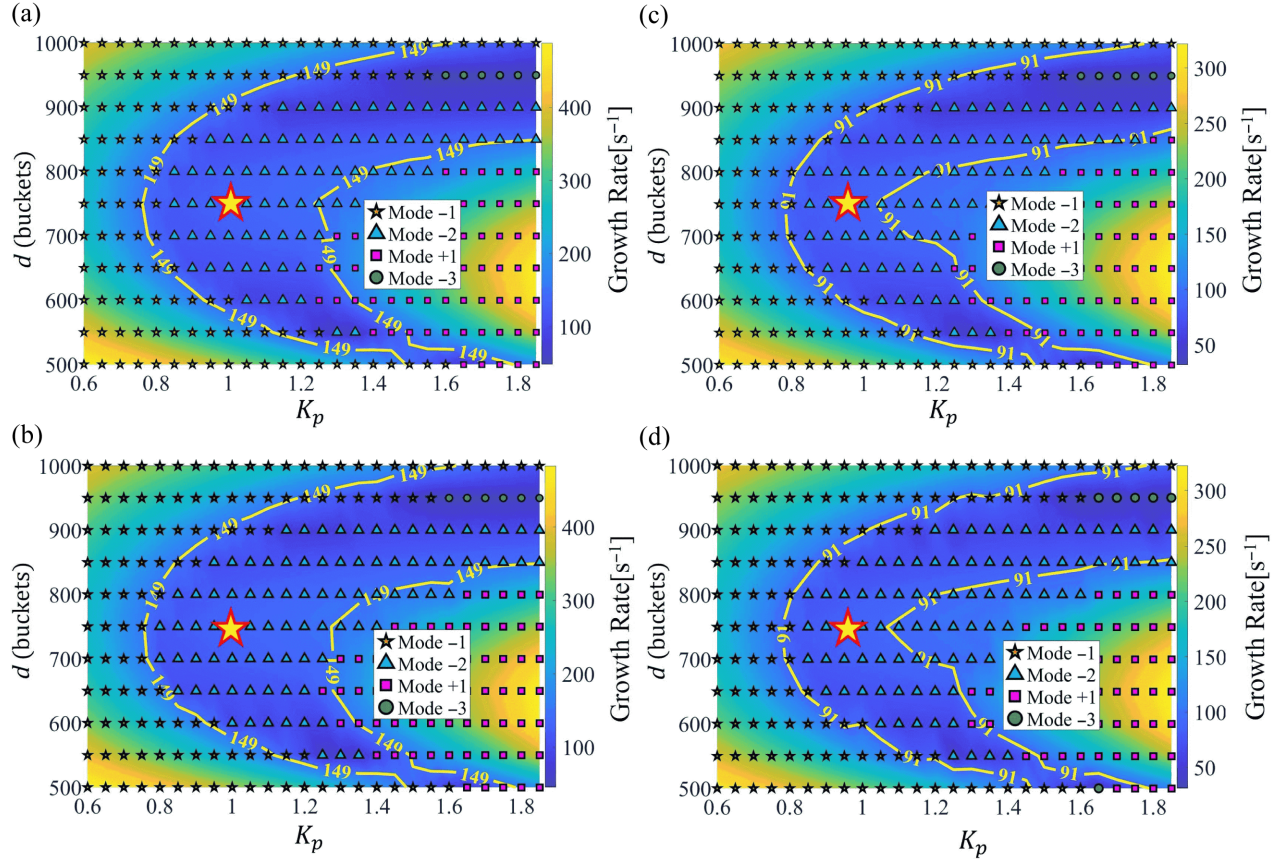


FIG. 5. Two-dimensional parameter scans of the PI feedback at 3.5 and 2 GeV. Panels (a),(b) show analytical and particle-tracking results at 3.5 GeV, respectively; panels (c),(d) show the corresponding results at 2 GeV. The vertical axis is the loop delay in buckets d ($\tau_d = dT_b$). For each (K_p, d) point, the color represents the growth rate of the fastest-growing coupled-bunch mode. The symbols indicate the identity of this mode: mode -1 (black stars), mode -2 (cyan triangles), mode $+1$ (magenta squares), and mode -3 (green circles). The yellow contour marks the longitudinal radiation-damping rate (149 s^{-1} at 3.5 GeV and 91 s^{-1} at 2 GeV); the region inside this contour corresponds to stability (growth rate below damping). The pentagram indicates the chosen operating point: $K_p = 1$ and $d = 750$ buckets ($\tau_d \approx T_0/2$).

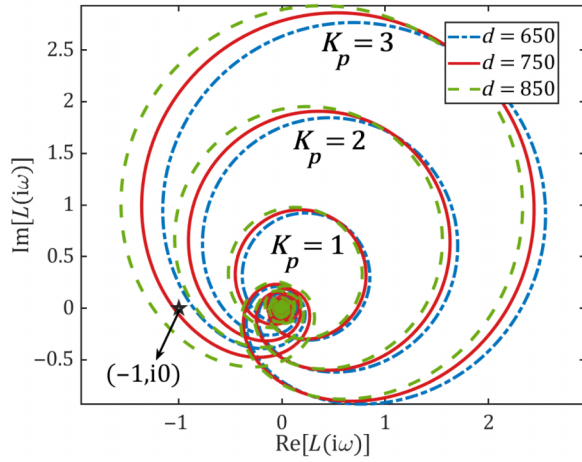


FIG. 6. Nyquist loci of the open-loop transfer function $L(i\omega)$ for three proportional gains ($K_p = 1, 2, 3$) and three loop delays ($d = 650, 750, 850$ buckets). The model includes the cavity impedance and a pure delay $e^{-i\omega\tau_d}$. The critical point $(-1, i0)$ is marked to indicate proximity to instability.

evolution and the crescent-shaped stability boundary are reproducible across the two beam energies.

The chosen operating point ($K_p = 1$, $d = 750$ buckets), marked by a pentagram, lies well within the stable region and far from its boundary. This central placement provides significant tolerance to practical implementation variations, confirming the operational robustness of the feedback scheme.

D. Stability analysis of the PI loop via the Nyquist criterion

The upper bound of the parameter scan in Sec. III C is constrained by the intrinsic stability limit of the LLRF feedback loop itself. In regions where the loop is unstable, simulation results are dominated by controller-induced oscillations rather than by impedance-driven coupled-bunch instability. To distinguish between these two effects, the loop stability at each (K_p, d) point is evaluated using the Nyquist criterion (see Appendix D for details).

Figure 6 illustrates the influence of feedback parameters on stability. Empirically, increasing either K_p or d shifts the Nyquist locus toward the critical point $(-1, i0)$, raising the risk of an encirclement that indicates loop instability. To systematically identify unstable regions, we compute the gain margin from the open-loop transfer function $L(i\omega)$ for each (K_p, d) at 2 and 3.5 GeV.

The combined results of the Nyquist analysis and tracking simulations are shown in Fig. 7. The white-shaded regions mark loop-unstable parameters. The close agreement between the 0 dB gain boundary from the Nyquist criterion and the region of numerical divergence observed in tracking simulations provides mutual validation of the two methods.

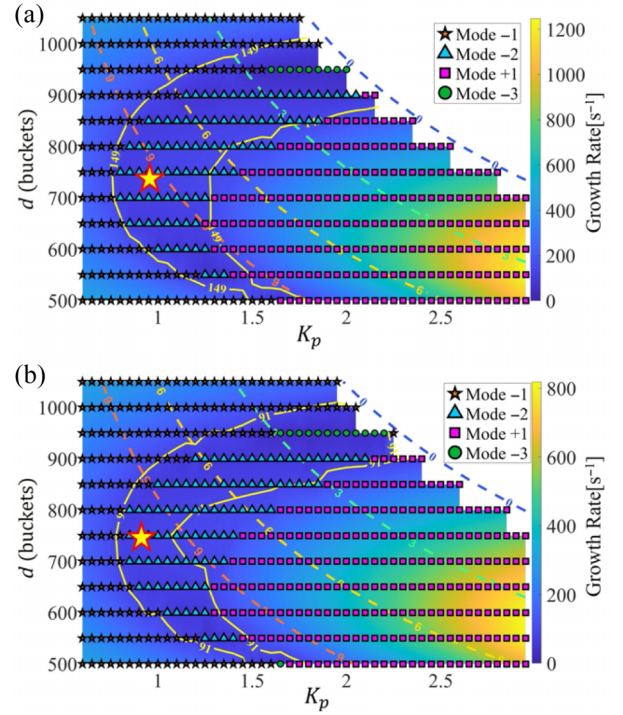


FIG. 7. Loop-stability screening in the (K_p, d) plane for (a) 3.5 and (b) 2 GeV. The white region denotes loop-unstable parameters excluded from analysis. Dashed contours indicate gain margin levels of 0, 3, 6, and 9 dB. For reference, the solid contour (labeled with the damping rate in s^{-1}) shows the instability threshold from coupled-bunch analysis.

The selected operating point ($K_p = 1$, $\tau_d \approx T_0/2$, marked by a pentagram) lies in a region of high stability, with a gain margin exceeding 9 dB at both energies. Although adjusting K_p could, in principle, compensate for a deviation of τ_d from $T_0/2$, the Nyquist analysis shows that stability margins degrade with increasing gain or delay, reducing tolerance to modeling uncertainties. Consequently, the unit gain, half-turn-delay configuration satisfies both the requirement for suppressing accelerating-mode-driven LCBI and the constraints for loop stability, confirming its robustness as an operating condition.

IV. PRACTICAL FEASIBILITY AND IMPLEMENTATION CONSIDERATIONS

For the STCF, the accelerating-mode-driven LCBI is shown to be suppressible below the radiation-damping threshold across the entire energy range solely via the standard I/Q-PI feedback loop configured with $K_p = 1$ and $\tau_d \approx T_0/2$, ensuring stability without complex additional measures. To further evaluate the engineering viability of this simplified scheme, we examine its performance under realistic operational constraints. The assessment includes the feasibility of the required loop delay, the impact of finite sampling rates and FIR-filter taps, the residual loop-phase offset, and the system

response to a realistic fill pattern and transient beam loading. Furthermore, we discuss the residual risk associated with the new-dominant $\mu = -2$ mode, including its narrow stability margin and potential mitigation strategies.

A. Feasibility of the half-turn loop delay

The practical implementation of the feedback scheme is constrained by the total loop delay, comprising contributions from signal transmission, digital processing, and actuator response. In operational storage rings, the total LLRF loop delay typically falls within the microsecond range, most commonly between 1 and 2 μs . Reported values from existing facilities include approximately 1.5 μs at SSRF [26] and 1.9 μs at Sirius [27] and at ALBA [28]. For the STCF, the required delay fits well within this established range. With a circumference of $C \approx 860$ m, the revolution period is $T_0 \approx 2.87$ μs . This gives $T_0/2 \approx 1.44$ μs , a value that lies comfortably within the typical microsecond-scale delays reported above. Therefore, the proposed operating point is not only theoretically grounded but also practically achievable with a standard modern FPGA-based LLRF system.

To achieve $\tau_d \approx T_0/2$, the intrinsic hardware delay must be shorter than the target; excess delay cannot be compensated afterward. Once this condition is met, fine adjustment can be performed by a programmable digital delay line in the FPGA (e.g., a shift-register-based delay [29]) with a step equal to the FPGA clock period (e.g., 8 ns at 125 MHz and 4 ns at 250 MHz). For the STCF, one rf bucket is about 2 ns, and the stable region around the nominal delay extends over ± 150 buckets (± 300 ns). A step of a few nanoseconds is therefore much smaller than this stable range, providing ample resolution to place τ_d at the desired operating point. In practice, the total loop delay can be calibrated via open-loop transfer function or step-response measurements and the digital delay trimmed accordingly, following established procedures in operational digital LLRF systems [30,31]. After calibration, the loop delay becomes a well-controlled, precisely adjustable parameter.

B. Influence of the averaging-window length

In tracking simulations, the averaging-window length corresponds to sampling cavity-voltage phasors over m consecutive buckets and averaging them. This operation mimics the actual cavity-voltage sampling and filtering process; therefore, the value of m is determined primarily by the LLRF system's sampling rate and filter taps and contributes to the total loop delay. In previous analyses, m was set to 10, a value small enough that its delay contribution could be neglected. In practice, however, a larger m is often required to suppress carrier-frequency harmonics and noise [32].

To investigate the impact of m on the stability maps, we decompose the total loop delay into two parts: the

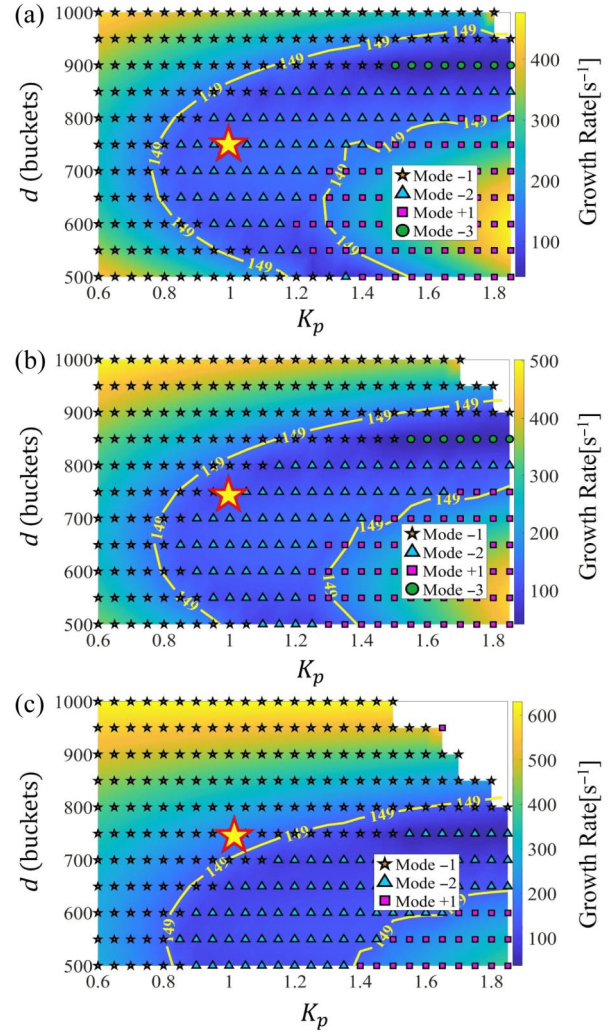


FIG. 8. Stability maps in the (K_p, d) plane for averaging-window lengths of (a) $m = 50$, (b) $m = 100$, and (c) $m = 200$ buckets. The white regions denote intrinsic loop instability. The solid yellow contour marks the radiation-damping rate (149 s^{-1}), defining the beam-stable boundary. The pentagram indicates the nominal operating point ($K_p = 1, \tau_d \approx T_0/2$).

contribution from cavity-voltage sampling and filtering (denoted by m buckets) and the remaining contributions from signal transmission in waveguides and cables, amplifier group delays, etc. (denoted by d buckets). Figure 8 presents the parameter scans for $m = 50, 100$, and 200 buckets. As m increases, the stable-region boundary shifts downward by approximately m buckets. For $m = 200$, this shift places the nominal operating point ($K_p = 1, \tau_d \approx T_0/2$) outside the stable region, demonstrating that the added delay from a large averaging window can destabilize an otherwise stable configuration.

These results reveal a key design trade-off in the LLRF system: increasing m improves signal quality but directly adds to the loop delay. To maintain the same total delay, the remaining delay budget (d) must be correspondingly

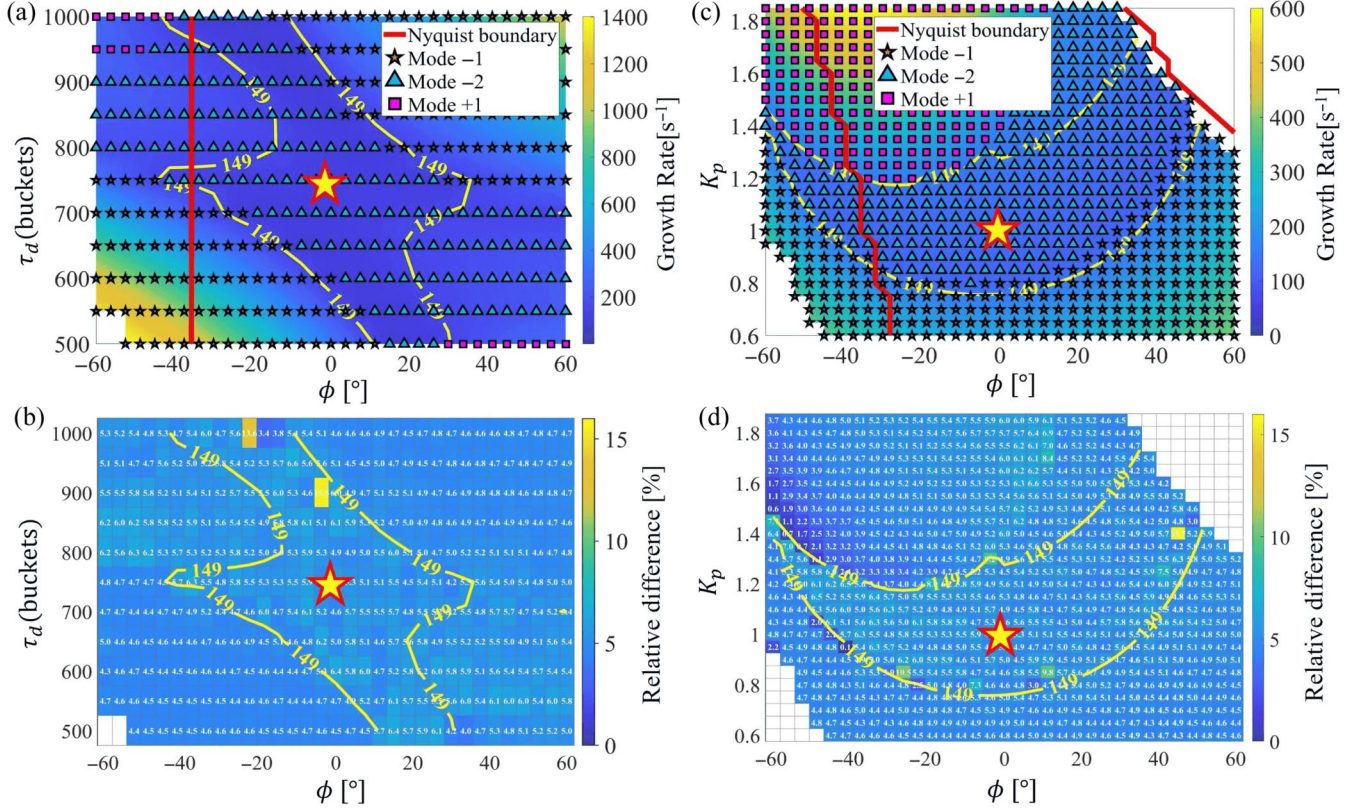


FIG. 9. Sensitivity to the residual loop-phase offset at 3.5 GeV. (a) Tracking growth rates in the (ϕ, τ_d) plane at fixed $K_p = 1$. (c) Tracking growth rates in the (ϕ, K_p) plane at fixed $\tau_d = 750$ buckets. Panels (b),(d) show the relative difference between analytical and tracking results. Yellow contours: radiation-damping rate. Symbols: fastest-growing mode. Red curves: Nyquist-based rf-loop-stability boundary. White region: tracking-unstable points excluded. Star: nominal operating point.

reduced. Hence, the choice of the averaging-window length m must be carefully balanced within the overall system-delay budget to achieve an optimal compromise between noise suppression and stability robustness.

C. Influence of residual loop-phase offset

Due to imperfect phase correction, the residual loop-phase offset ϕ modifies the closed-loop cavity impedance, although its magnitude is normally small. Here, we examine its effect on the baseline PI feedback configuration ($K_p = 1$ and $\tau_d = 750$ buckets), restricting ϕ to the range $[-60^\circ, 60^\circ]$. Figures 9(a) and 9(c) show, respectively, the fastest-mode growth rates obtained from tracking scans in the (ϕ, τ_d) plane at fixed $K_p = 1$ and in the (ϕ, K_p) plane at fixed $\tau_d = 750$ buckets. The stable region is seen to be quite broad, the dominant mode is nearly always mode -2 , and the nominal operating point is located well inside it.

The Nyquist-based stability boundary coincides with the tracking-unstable region on the high- K_p side. On the low-gain side, the integral branch and discrete implementation give rise to instabilities that the simplified Nyquist model does not capture accurately. Consequently, the

tracking-unstable region serves as the conservative exclusion zone for beam stability, while the Nyquist curve is only a reference.

To validate the tracking results, we also computed the fastest-mode growth rate analytically using the numerical root-finding procedure. Figures 9(b) and 9(d) present the relative difference between the analytical and tracking results. The analytical growth rates agree with the tracking values to within a few percent; several larger deviations appear only near mode transitions or loop-stability boundaries, thereby confirming the tracking model. The nominal operating point is well separated from both the high- K_p delay-dominated instabilities and the low-gain integral-dominated instabilities. The proposed configuration is thus robust against realistic residual phase errors.

D. Performance under a realistic fill pattern

For simplicity, previous analyses assumed a uniform fill pattern. For STCF, a more realistic filling must meet the requirements for detector timing and ion instability mitigation. One suitable fill pattern could be

$$(1F + 1E) \times 682 + 1E \times 70, \quad (7)$$

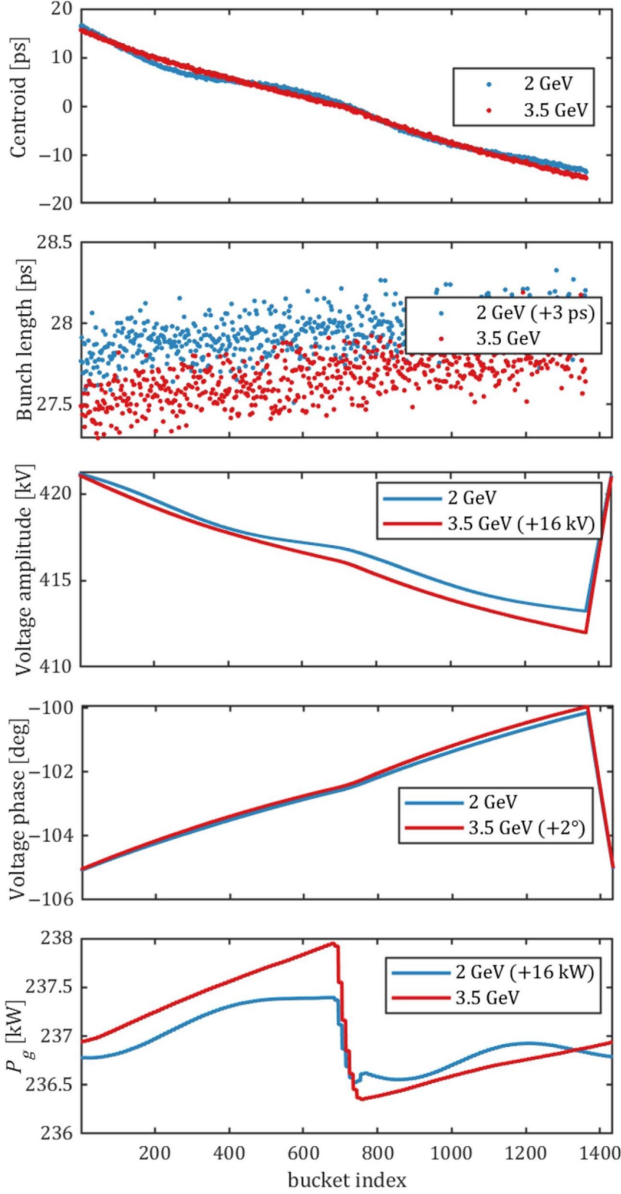


FIG. 10. Simulated transient beam-loading effects under the realistic fill pattern at 2 and 3.5 GeV. Shown are the bunch centroid, rms bunch length, per-cavity-voltage amplitude and phase, and the generator power. For clarity, selected traces are offset: at 2 GeV, bunch length by 3 ps and power by 16 kW; at 3.5 GeV, voltage amplitude by 16 kV and phase by 2° .

where F denotes a filled bucket and E an empty bucket. This filling provides 682 bunches with alternating empty buckets (for detector timing requirements), followed by a long gap of 70 empty buckets (for ion clearing). The long gap would introduce significant transient beam-loading effects [33]. Tracking simulations confirm the longitudinal stability under this fill pattern. Figure 10 shows the steady-state, bucket-by-bucket results at 2 and 3.5 GeV for the PI configuration with unit gain and half-turn delay. The bunch lengths at the two energies are comparable, and the

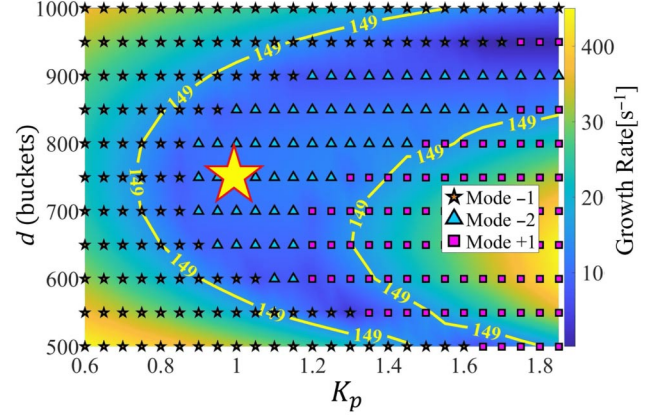


FIG. 11. Stability map for the realistic fill pattern at 3.5 GeV. Symbols indicate the mode index. The solid contour indicates the radiation-damping rate. The pentagram indicates the nominal operating point ($K_p = 1$, $\tau_d \approx T_0/2$).

transient profiles of the bunch centroid and cavity voltage are closely matched. The required generator power stays below its design limit, with fluctuations confined to within 2 kW.

To assess feedback robustness, we performed a two-dimensional scan over the proportional gain and loop delay. The resulting stability map at 3.5 GeV is shown in Fig. 11. Notably, the stability map for the realistic fill closely resembles that of the ideal uniform fill case: the distribution of the fastest-growing modes and the crescent-shaped stable boundary are virtually unchanged. This indicates that transient beam loading, while modulating bunch phases, does not alter the fundamental coupled-bunch instability mechanism in this parameter regime. Consequently, the nominal operating point ($K_p = 1$, $\tau_d \approx T_0/2$) remains centrally located within the stable domain, confirming that the feedback parameters optimized for a uniform fill are directly applicable to realistic operation.

E. Extra considerations for $\mu = -2$ mode

As detailed in Sec. II B, the notch-filter mechanism implemented with the nominal setting of $K_p = 1$ and $\tau_d \approx T_0/2$ suppresses the odd modes of $\mu = -1$ and $\mu = -3$. Consequently, the even mode of $\mu = -2$ becomes the fastest-growing mode. While the growth rate of $\mu = -2$ mode is thereby increased, it remains below the radiation-damping threshold, thus maintaining overall beam stability.

It should be noted, however, that the growth rate of $\mu = -2$ mode is only marginally below the radiation-damping rate—particularly at 2 GeV, where it is lower by just 3 s^{-1} . Should practical nonidealities (e.g., nonlinearities or noise) erode this narrow margin and render $\mu = -2$ mode unstable, a dedicated damper targeting this mode

could be implemented, following the proven approach used at SuperKEKB [14].

V. CONCLUSION

This study investigates the LCBI driven by rf cavity accelerating modes in the STCF. Without active feedback, the growth rate of the dominant $\mu = -1$ mode substantially exceeds the radiation-damping rate, leading to beam instability. We demonstrate that a simple PI feedback loop within the standard I/Q-based LLRF system can effectively suppress this instability. The key is to set the proportional gain to unity ($K_p = 1$) and the loop delay to approximately half the revolution period ($\tau_d \approx T_0/2$). This configuration creates a spectral notch that selectively damps the odd coupled-bunch modes, in particular the most dangerous $\mu = -1$ mode. Two-dimensional parameter scans confirm that this scheme remains robust across the full STCF operating range from 1 to 3.5 GeV. Nyquist stability analysis further verifies that the chosen operating point provides ample gain margin, ensuring the intrinsic stability of the feedback loop.

These results deliver valuable guidance for the STCF rf-system design, providing clear constraints on the allowable loop delay and proportional gain. It is important to note that this parameter setting is particularly suited to the STCF, where the required cavity detuning is substantially lower than the revolution frequency. For machines that require a detuning approaching or even exceeding the revolution frequency, the same simple $K_p = 1$, $\tau_d \approx T_0/2$ configuration may no longer be adequate. Nevertheless, the combined analytical and tracking framework developed in this work establishes a general methodology for analyzing beam-cavity-LLRF coupled dynamics in other high-current storage rings.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: ANALYTICAL FORMULAS FOR BEAM LOADING COMPENSATION

To minimize reflected power, the coupler of an rf cavity must be optimized for the given beam power and cavity losses. The optimal coupling coefficient is given by [34,35]

$$\beta_{\text{opt}} = 1 + \frac{P_b}{P_c}, \quad (\text{A1})$$

where $P_b = I_0 U_0 / N_{\text{cav}}$ is the beam power per cavity, $P_c = V_c^2 / (2R_s)$ is the resistive wall loss per cavity, V_c is the cavity-voltage amplitude, and N_{cav} is the number of cavities. With β_{opt} , the loaded quality factor and shunt impedance become

$$Q_L = \frac{Q_0}{1 + \beta_{\text{opt}}}, \quad R_L = \frac{R_s}{1 + \beta_{\text{opt}}}, \quad (\text{A2})$$

where Q_0 and R_s are the unloaded quality factor and shunt impedance, respectively. The geometry-dependent R/Q remains unchanged.

In electron storage rings, cavities are typically operated with negative detuning ($\omega_r < \omega_{\text{rf}}$) to compensate reactive beam loading, reduce generator power, and suppress Robinson instability [2,35]. The detuning angle φ_d is related to the detuning $\Delta\omega_r = \omega_r - \omega_{\text{rf}}$ by

$$\varphi_d = \tan^{-1} \left(\frac{2Q_L \Delta\omega_r}{\omega_{\text{rf}}} \right), \quad (\text{A3})$$

so $\Delta\omega_r < 0$ implies $\varphi_d < 0$.

Voltages and currents are treated as phasors in the following. The total cavity voltage is the vector sum of the beam-induced voltage and the generator-driven voltage (Fig. 12). Aligning the beam image current with the positive real axis, the beam-loading voltage is

$$\tilde{V}_b = 2I_0 R_L \cos(\varphi_d) e^{i\varphi_d}. \quad (\text{A4})$$

The cavity-voltage phasor is written as

$$\tilde{V}_c = V_c \exp \left[i \left(\varphi_s + \frac{\pi}{2} \right) \right], \quad (\text{A5})$$

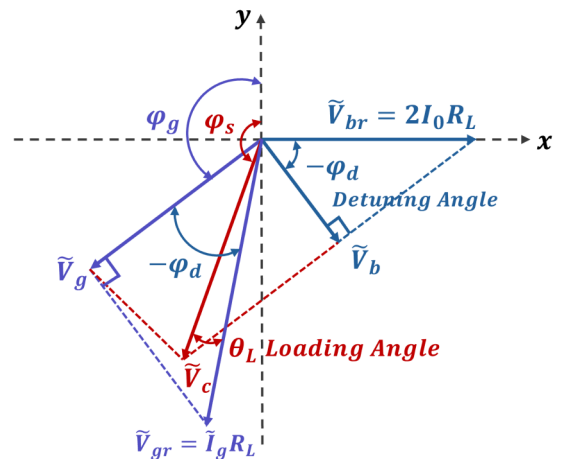


FIG. 12. Phasor representation of rf cavity voltages.

where the synchronous phase $\varphi_s \in (\pi/2, \pi)$ satisfies

$$V_c \sin(\varphi_s) = U_0. \quad (\text{A6})$$

The required generator voltage is therefore

$$\tilde{V}_g = \tilde{V}_c - \tilde{V}_b, \quad (\text{A7})$$

and the corresponding generator-current amplitude is

$$I_g = \frac{|\tilde{V}_g|}{R_L \cos(\varphi_d)}. \quad (\text{A8})$$

The generator power follows as

$$P_g = \frac{I_g^2 R_s}{8\beta_{\text{opt}}}. \quad (\text{A9})$$

The loaded angle θ_L is defined as the angle from the generator-current phasor to the cavity-voltage phasor (positive counterclockwise). Let the generator-voltage synchronous angle be

$$\varphi_g = \angle(\tilde{V}_g e^{-i\pi/2}), \quad (\text{A10})$$

where $\angle$ denotes the phasor argument in $(-\pi, \pi]$. Then

$$\theta_L = \varphi_s - \varphi_g + \varphi_d. \quad (\text{A11})$$

For normal-conducting cavities, θ_L is usually kept between -15° and 0° to increase the Robinson stability margin while avoiding large reflected power. The special case $\theta_L = 0$ minimizes reflected power and defines the optimum detuning. In this condition, the cavity voltage and generator current are aligned, leading to the phasor diagram of Fig. 13. Applying the law of sines gives

$$\frac{V_c}{\sin(\varphi_s - \pi/2)} = \frac{2I_0 R_L \cos(\varphi_d)}{\sin(-\varphi_d)}, \quad (\text{A12})$$

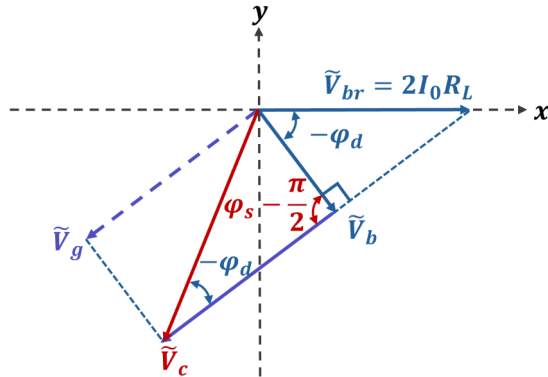


FIG. 13. Phasor diagram for the optimum-detuning condition ($\theta_L = 0$).

which, together with Eq. (A4), yields the optimum detuning

$$\Delta\omega_{r,\text{opt}} = \frac{I_0(R/Q) \cos(\varphi_s) \omega_{\text{rf}}}{V_c}. \quad (\text{A13})$$

In the simulations presented in this paper, a realistic loaded angle of about -10° is used.

APPENDIX B: ANALYTICAL CALCULATION OF COUPLED-BUNCH INSTABILITY DRIVEN BY THE ACCELERATING MODE

Let the impedance seen by the beam be denoted as $Z(\omega)$. For M equally spaced bunches with equal charge, the growth rate of coupled-bunch instability can be evaluated using the following expression [15]

$$\Omega^2 - \omega_g^2 = i \frac{I_0 \alpha_c}{T_0 E / e} \sum_{p=-\infty}^{\infty} \{ p M \omega_0 Z(p M \omega_0) - (p M \omega_0 + \mu \omega_0 + \Omega) Z(p M \omega_0 + \mu \omega_0 + \Omega) \}, \quad (\text{B1})$$

where $\Omega = \Omega_r + i\Omega_i$ is the complex frequency to be solved, where the real part Ω_r represents the oscillation angular frequency and the imaginary part Ω_i represents the growth rate ($\Omega_i < 0$ indicates an unstable, growing mode), i is the imaginary unit, I_0 is the beam average current, α_c is the momentum compaction factor, T_0 is the revolution time, E is the beam energy, e is the electron charge, p is an integer taken from $-\infty$ to $+\infty$, ω_0 is the angular revolution frequency, μ is the coupled-bunch mode number taken from 0 to $M - 1$, $\omega_g = \sqrt{h\omega_0 \alpha_c V_g |\cos(\varphi_g)| / (T_0 E / e)}$ is the angular synchronous frequency under beam loading, V_g is the generator-induced cavity-voltage amplitude, φ_g is the corresponding phase, and h is the harmonic number.

For the accelerating mode, only the two terms with $pM = \pm h$ contribute significantly. Summing the first term on the right-hand side of the equation yields

$$i \frac{I_0 \alpha_c h \omega_0}{T_0 E / e} [Z(h\omega_0) - Z(-h\omega_0)] = \frac{h \alpha_c \omega_0 V_b |\sin(\varphi_d)|}{T_0 E / e}, \quad (\text{B2})$$

where V_b is the beam-induced voltage and φ_d is the cavity detuning angle. By taking $\omega_p^\pm = \pm h\omega_0 + \mu\omega_0 + \Omega$ and substitute Eqs. (B2) into Eq. (B1), Eq. (B1) can be simplified to

$$\Omega^2 - \omega_s^2 + i\xi[\omega_p^+ Z(\omega_p^+) + \omega_p^- Z(\omega_p^-)] = 0, \quad (\text{B3})$$

where the newly introduced two parameters are given by

$$\omega_s^2 = \omega_g^2 + \frac{h\alpha_c\omega_0 V_b |\sin(\varphi_d)|}{T_0 E/e} = \frac{\xi h\omega_0 V_{rf} |\cos(\varphi_s)|}{I_0} \quad (\text{B4})$$

and

$$\xi = \frac{I_0 \alpha_c}{T_0 E/e}. \quad (\text{B5})$$

It should be noted that to derive Eq. (B4), the relation $V_g |\cos(\varphi_g)| + V_b |\sin(\varphi_d)| = V_{rf} |\cos(\varphi_s)|$ is utilized.

To solve Eq. (B3), a common approximate solution assumes $\Omega_r \approx \omega_s$ and neglects the influence of the imaginary part of the complex frequency Ω on the impedance summation, leading to

$$\Omega_i \approx -\frac{\xi}{2\omega_s} \text{Re}[\omega_p^+ Z(\omega_p^+) + \omega_p^- Z(\omega_p^-)], \quad (\text{B6})$$

where $\omega_p^\pm = \pm h\omega_0 + \mu\omega_0 + \omega_s$. Here, an alternative solution method is introduced, which does not require assuming $\Omega_r \approx \omega_s$ and can account for the influence of the imaginary part of the complex frequency on the impedance summation. Rewrite Eq. (B3) as a function of the complex frequency Ω

$$f(\Omega) = \Omega^2 - \omega_s^2 + i\xi[\omega_p^+ Z(\omega_p^+) + \omega_p^- Z(\omega_p^-)]. \quad (\text{B7})$$

The solution to Eq. (B7) must satisfy $f(\Omega) = 0$.

It is anticipated that Ω_r lies near ω_s , and the absolute value of Ω_i is not very large. Therefore, the solution must reside within a rectangular domain as shown in the following Fig. 14. For STCF parameters, typically $X_r = 1000$ and $Y_r = 2000$ are chosen. This rectangular domain is discretized, with a step size of $\Delta\Omega_r = 4\pi$ along the real axis and $\Delta\Omega_i = 2 \text{ s}^{-1}$ along the imaginary axis. The function value $|f(\Omega)|$ at each discrete point is computed, and the complex frequency point corresponding to the smallest $|f(\Omega)|$ is taken as the solution. Since the step size along the imaginary axis is 2 s^{-1} , the growth rate of each

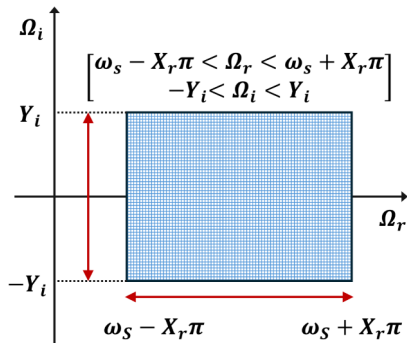


FIG. 14. Schematic of the rectangular domain where the solution may exist.

unstable mode is computed with a precision of 2 s^{-1} . If higher computational accuracy is required, an approximate solution can first be determined with a coarse step size. Then, by narrowing the rectangular domain around this approximate solution and using a finer step size within the reduced domain, theoretically, any desired accuracy can be achieved.

APPENDIX C: MODE IDENTIFICATION AND GROWTH-RATE EXTRACTION FROM TRACKING DATA

This appendix details the postprocessing method used to identify the fastest-growing coupled-bunch mode and to extract its growth rate from macroparticle tracking simulations.

Let $q_{m,n}$ denote the longitudinal centroid of bunch m ($m = 0, \dots, M-1$) recorded at sample index n ($n = 1, \dots, N$). The data are down-sampled by a fixed interval of 10 turns. To ensure the instability is well developed, only the final third of the recorded data is used for the subsequent analysis.

At the last recorded sample, we remove the zero-mode component to highlight the higher-order coupled-bunch mode

$$q_m = q_{m,N} - \frac{1}{M} \sum_{k=0}^{M-1} q_{k,N}. \quad (\text{C1})$$

A candidate coupled-bunch mode is determined by searching for the maximum magnitude of the DFT for q_m

$$\mu_{\text{cand}} = \arg \max_{\mu} \left| \frac{1}{M} \sum_{m=0}^{M-1} q_m \exp \left(-i \frac{2\pi\mu}{M} m \right) \right|. \quad (\text{C2})$$

Because q_m is real-valued, its DFT is conjugate symmetric; therefore, both $+\mu$ and $-\mu$ cannot be distinguished. To determine the sign of μ and to obtain the evolution of the mode envelope, we construct, for each bunch, an analytic signal by applying a Hilbert transform $\mathcal{H}_n[\cdot]$ along the sample index n [24]

$$\tilde{q}_{m,n} = q_{m,n} - i \mathcal{H}_n[q_{m,n}]. \quad (\text{C3})$$

The complex amplitude of mode μ at sample n is then obtained by projecting $\tilde{q}_{m,n}$ onto the corresponding spatial harmonic

$$A_n(\mu) = \frac{1}{M} \sum_{m=0}^{M-1} \tilde{q}_{m,n} \exp \left(-i \frac{2\pi\mu}{M} m \right). \quad (\text{C4})$$

The envelope $|A_n(\mu)|$ is used to compare the two candidates $\pm\mu$. The mode with the larger mean envelope is identified as the most unstable mode μ^* . The growth rate,

$1/\tau_{\mu^*}$, is obtained by fitting an exponential to the exponential-growth phase. Finally, the intrinsic growth rate is given by $1/\tau_{\mu^*} + 1/\tau_{\text{rad}}$, with τ_{rad} being the radiation-damping time.

APPENDIX D: STABILITY ASSESSMENT OF THE LLRF FEEDBACK LOOP VIA THE NYQUIST CRITERION

The LLRF system precisely regulates the amplitude and phase of the cavity field. This regulation can modify the effective cavity impedance experienced by the beam, thereby influencing coupled-bunch effects [36]. However, an improper choice of the PI loop parameters can destabilize the feedback loop itself. In tracking simulations, such loop instability rapidly drives the beam beyond the longitudinal acceptance, leading to particle loss. To prevent this, each candidate parameter pair (K_p, d) is prescreened for loop stability using the Nyquist criterion before running tracking simulations.

In the tracking code, the feedback delay is specified in integer rf buckets. With bucket spacing $T_b = T_0/h$, the loop delay is

$$\tau_d = d T_b, \quad (\text{D1})$$

where d is the delay in buckets. The open-loop transfer function used for Nyquist criterion analysis is [37]

$$L(i\omega) = \exp(-i\omega\tau_d) G(i\omega) Z_{\text{cav}}(i\omega), \quad (\text{D2})$$

where $Z_{\text{cav}}(i\omega)$ is the cavity impedance and $G(i\omega)$ is the controller response. We evaluate $L(i\omega)$ over the frequency range relevant to the cavity accelerating mode, consistent with the tracking model.

Given that the integral gain employed in tracking is very small ($K_i \sim 10^{-5}$) and has a negligible impact on the LCBI, the controller can thus be approximated by a pure proportional term

$$G(i\omega) = \frac{K_p}{R_L}, \quad (\text{D3})$$

where R_L is the loaded shunt impedance. We apply the Nyquist criterion to $L(i\omega)$. Let

$$z(\omega) = 1 + L(i\omega). \quad (\text{D4})$$

In our model, $L(i\omega)$ has no right-half-plane poles. Hence, the closed-loop system is stable if the Nyquist curve of $L(i\omega)$ does not encircle the critical point $(-1, i0)$ [37,38]. Numerically, we compute the winding number of $z(\omega)$ around the origin over $\omega \in [\omega_{\min}, \omega_{\max}]$

$$N = \frac{1}{2\pi} \int_{\omega_{\min}}^{\omega_{\max}} \frac{d}{d\omega} \arg z(\omega) d\omega. \quad (\text{D5})$$

An LLRF setting is classified as internally stable if

$$N = 0. \quad (\text{D6})$$

This yields a binary stability mask over the (K_p, d) parameter scan; only stable points are passed to the subsequent beam-instability tracking analysis.

For each stable point, we also compute the GM to quantify the distance to instability. Let ω_π be the phase-crossover frequency where the open-loop-phase offset reaches $-\pi$, i.e., $\angle L(i\omega_\pi) = -\pi$. At this frequency, the feedback is on the verge of positive feedback. The gain margin in decibels is [39]

$$\text{GM [dB]} = -20 \log_{10} |L(i\omega_\pi)|. \quad (\text{D7})$$

If $\text{GM} > 0$, then $|L(i\omega_\pi)| < 1$: even at the most critical phase condition, the loop gain remains below unity, guaranteeing that the loop does not self-oscillate.

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