

Analysis of the transverse phase space diagnostic in laser wakefield accelerators using the principal planes

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The focus-scan scheme has been adopted to diagnose the emittance of the electron beams produced from laser wakefield accelerators. The spot size of the electron beam has an energy dependence after propagating through a beamline consisting of two or more quadrupoles, which can be used to fit the beam emittance and the Courant-Snyder (C-S) parameters. We use the concept of the principal planes to unify the analysis of the focus-scan scheme. The fitting procedures and the effects of the length variation between the source, magnets, and screen on the diagnostic are presented. Particle-in-cell simulations using plasmas with different downramp lengths are conducted to study their effects on the measurements of the emittance and C-S parameters. Our analysis provides a unified understanding of the focus-scan scheme and can be applied to beamlines consisting of more quadrupoles.

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I. INTRODUCTION

Manipulation and measurement of the transverse phase space of charged particle beams are essential for modern accelerators and their applications [1,2]. The transverse phase space distribution is usually quantified using the emittance and Courant-Snyder (C-S) parameters [3]. Different schemes have been developed to measure these quantities of charged particle beams on conventional radio-frequency (rf) accelerators, e.g., the mask method [4,5] and the focus-scan method [6,7]. As a high-gradient acceleration scheme, laser wakefield acceleration (LWFA) [8] delivers hundreds of MeV electron beams in millimeter- to centimeter-long plasmas, which holds the potential to shrink the size and reduce the cost of electron accelerators [9]. Meanwhile, the electron beams from LWFA are characterized by micron spot sizes, milliradian divergences, and pointing jitters, and usually $\geq$ MeV absolute energy spread [10]. These unique properties pose challenges for the phase space diagnostic of LWFA beams. Furthermore, a single-shot measurement scheme is preferred for LWFA due to the large jitters of the beam properties.

A mask can intercept a beam into narrow beamlets whose divergences can be derived by measuring their sizes after a proper drift [4,5]. Due to the small spot size and large divergence of the LWFA beams, a thin line distribution is formed in the $y - y'$ phase space quickly after the plasma, where y and y' represent the transverse coordinate and angle of the electrons, respectively. The divergence of an individual beamlet is thus dominated by the $y - y'$ correlation, not by the y -slice angular width [11]. The mask method has been implemented in LWFA experiments to give an upper limit of the beam emittance as several mm mrad [12–14]. The betatron radiation scheme can derive the electron oscillation amplitude inside the ion channel by measuring the betatron spectrum and the beam energy distribution [15,16]. Assumptions are needed to know the beam spot size inside the ion channel from the oscillation amplitude. It is also challenging to know the beam divergence and the $y - y'$ correlation inside the ion channel, since the plasma-vacuum transition region can significantly alter them [17,18].

The focus-scan scheme can give the beam's emittance and C-S parameters by measuring the spot sizes under different strengths of the quadrupole [6,7]. This multishot scheme is adapted as a single-shot approach in LWFA by leveraging the inherent large energy spread of their beams [19]. Electrons with different energies are focused by permanent magnetic quadrupoles (PMQs) with different spot sizes, which can be measured by placing a screen after a dipole. This single-shot scheme has been demonstrated in

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LWFAs to measure the sub-mm mrad normalized emittance of ~ 100 MeV electron beams from self-injection, ionization injection, and density downramp injection [19–23]. An MeV energy range is typically needed for a reliable measurement, which is naturally satisfied in LWFAs. By replacing PMQs with a high-gradient active plasma lens [24], this scheme measures the emittance of GeV-class electron beams of several microns [25,26]. Nonideal issues that affect the measurement of the emittance, including space-charge effects, high-order fields, angular alignment, and position alignment accuracy, were analyzed for their beamlines [19,21,27,28].

The focus-scan beamline usually consists of two or more thick quadrupoles. In this work, we introduce the concepts of the principal planes, which are widely used in optics [29], to unify the analysis of the transverse phase space diagnostic. In Sec. II, the fitting procedures are discussed using the principal planes. In real experiments, the distance between the source, magnets, and the imaging screen may deviate from the designed values. We apply the principal planes to analyze their impact on the imaging energy in Sec. III and find the distance between the source and the first quadrupole has the most critical effect. The conclusions are then applied to discuss the effects of different downramps of gas targets on the transverse phase space diagnostics. Examples of a beamline with two thick quadrupoles are given in each section. The discussion presented using the principal planes can be easily extended to beamlines consisting of ≥ 3 quadrupoles, which is useful for the transverse phase space diagnostic in LWFAs.

II. BEAMLINE DESIGN AND THE TRANSVERSE PHASE SPACE DIAGNOSTIC

The setup of the focus-scan method is illustrated in Fig. 1. A 100-MeV electron beam generated by shooting an ultrashort and ultraintense laser pulse on a gas target propagates successively through two PMQs (labeled as “Q1” and “Q2”) and one magnetic dipole, which disperses electrons along the x direction. Its energy-dependent y direction spot size is recorded by a screen behind the dipole. The electron beam from LWFA is usually characterized by a large energy spread, and we first consider electrons with the same energy (a part of the beam). The evolution of their C-S parameters can be expressed using the transfer matrix $\mathbf{R}$ as [3]

$$\begin{pmatrix} \beta(z) \\ \alpha(z) \\ \gamma(z) \end{pmatrix} = \begin{pmatrix} R_{11}^2 & -2R_{11}R_{12} & R_{12}^2 \\ -R_{11}R_{21} & 1+2R_{12}R_{21} & -R_{12}R_{22} \\ R_{21}^2 & -2R_{21}R_{22} & R_{22}^2 \end{pmatrix} \begin{pmatrix} \beta_i \\ \alpha_i \\ \gamma_i \end{pmatrix}, \quad (1)$$

where the subscript “i” indicates the initial values immediately after the plasma. Note that $|\mathbf{R}| = 1$ and the C-S parameters always satisfy $1 + \alpha^2 = \beta\gamma$. Their spot size can

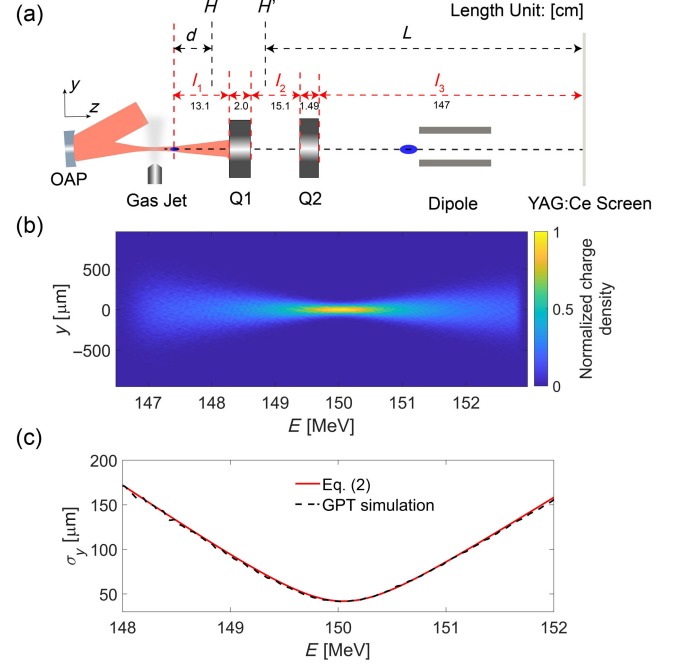


FIG. 1. (a) Schematic setup of the transverse phase diagnostic using the focus-scan method (not to scale). A laser pulse is focused by an off-axis parabolic mirror (OAP) on the gas jet, and the blue ellipse represents the electron bunch. The principal planes of this beamline (H and H') are also shown. (b) The normalized charge density distribution of an electron beam on the screen from GPT simulations. This plot is generated using 20,000 particles. (c) The energy-dependent spot size of the electron beam on the screen. The red line is calculated using Eq. (2) and the black dashed line represents the rms value of the GPT simulation result.

thus be known as

$$\sigma_y(z) = \sqrt{\epsilon_g \beta(z)} = \sqrt{\epsilon_g (R_{11}^2 \beta_i - 2R_{11}R_{12} \alpha_i + R_{12}^2 \gamma_i)}, \quad (2)$$

where ϵ_g is the geometric emittance in the $y - y'$ phase space of these monoenergetic electrons, and its definition is $\epsilon_g = \sqrt{\langle (y - \langle y \rangle)^2 \rangle \langle (y' - \langle y' \rangle)^2 \rangle - \langle (y - \langle y \rangle)(y' - \langle y' \rangle) \rangle^2}$, where $\langle \rangle$ represents the ensemble average over the electrons. It is a constant during propagation if the space-charge force or any other nonlinear force is neglected, and a discussion on the space-charge effect can be found in Appendix D.

Since the transfer matrix of quadrupoles depends not only on the quadrupole parameters but also on the electron energy E , an energy-dependent spot size can be obtained on the screen after the dipole disperses the electrons. The emittance and the C-S parameters of the electron beam can be fitted using the measured $\sigma_y - E$ curve after knowing the configuration of the entire beamline and the relation between $\epsilon_g, \beta_i, \alpha_i$, and E . The beamline is usually designed

to include a beam waist on the screen to improve the fitting reliability [6], i.e., a “tie figure” is formed on the screen as shown in Fig. 1(b). Note that the fitted phase space quantities only represent the properties of the electrons that contribute to the adopted $\sigma_y - E$ data. In other words, this focus-scan scheme measures the properties of electrons around a particular energy that has a waist on the screen, not the whole beam.

After choosing the energy of interest E_0 , there are several design rules of the beamline [20,22]. First, the transfer matrix from the source to the imaging plane satisfies $R_{12y} = R_{12x} = 0$, i.e., a point-to-point imaging system is used. As explained later, $R_{12y} = 0$ ensures a “tie figure” on the screen. A small x direction spot size on the screen ensured by $R_{12x} = 0$ can improve the energy resolution and the accuracy of the measured $\sigma_y - E$ curve [21]. Furthermore, these two conditions can eliminate the effect of $\sim$ mrad electron beam pointing jitters on the measured beam distribution [30]. Second, the spot size needs to be resolved well on the screen. For a point-to-point imaging system, the beam spot size on the screen is $\sigma_y = |R_{11y}| \sigma_{yi}$. Considering that the beam size at the exit of the plasma is $\sim \mu\text{m}$ and the resolution of the optics imaging system (including the screen, the imaging system, and the CCD) is $\sim \mu\text{m}$, we usually choose $|R_{11y}| = 10\text{--}20$ to resolve the beam distribution well [19,27]. Finally, it is beneficial to design a compact beamline.

A numerical example with $E_0 = 150$ MeV is presented. As shown in Fig. 1(a), two PMQs with $l_{Q1} = 2.0$ cm, $l_{Q2} = 1.49$ cm, $K_1 = 516.1 \text{ m}^{-2}$, $K_2 = -350.1 \text{ m}^{-2}$ are chosen (details of the PMQs can be found in Appendix A), where l_Q is the effective length of the quadrupoles, $K = \frac{ecG}{E_0}$ is the focusing ($K > 0$) or defocusing ($K < 0$) strength of the quadrupoles, G is the magnetic field gradient, e is the electron charge, and c is the speed of light in vacuum. Using $R_{12x} = R_{12y} = 0$ and $R_{11y} = -20$, we obtain $l_1 = 13.1$ cm, $l_2 = 15.1$ cm, $l_3 = 1.47$ m, where l_1 is the distance from the source to the left edge of Q1, l_2 is the distance from the right edge of Q1 to the left edge of Q2, and l_3 is the distance from the right edge of Q2 to the imaging screen (see Fig. 1). Three or more quadrupoles can give extra freedoms [20,23], but it is also challenging to align them in experiments.

We then use the three-dimensional particle tracking code, Generated Particle Tracer (GPT) [31], to simulate the propagation of electrons. An electron beam, uniformly distributed between 147 and 153 MeV, is initialized at the source. The beam has a transverse Gaussian distribution with $\sigma_{xi} = \sigma_{yi} = 2 \mu\text{m}$ and a flat-top longitudinal distribution with a length of $2 \mu\text{m}$. The total charge is assumed to be 4 pC. The initial normalized emittance and C-S parameters are $\epsilon_{ni} = 2$ mm mrad, $\beta_i = 0.59$ mm, $\alpha_i = 0$, respectively. A rectangular magnetic dipole with a length of 23 cm and a width of 25 cm is placed 1 m (central position) from the source. The magnetic field within the dipole is uniform, with a magnitude of $B = 0.8$ T. The

deflection angle dependence on the energy of this dipole is $\frac{d\theta_x}{dE} \approx 1.5$ mrad/MeV when $E = 150$ MeV, which is converted to $\frac{dx}{dE} \approx 0.92$ mm/MeV on the screen. The simulated charge distribution of the electron beam on the screen is shown in Fig. 1(b), where a $4 \mu\text{m}$ CCD pixel size and an optical system with a magnification factor of 10 are assumed. A $12 \mu\text{m}$ pixel blurring kernel is used to simulate the point spread function of the optical imaging system. The energy-dependent spot size from simulations is shown in Fig. 1(c) by the black dashed line, which agrees well with Eq. (2) (red solid line). The electrons reach a minimum spot size of $41.8 \mu\text{m}$ at 150.06 MeV.

As shown in Fig. 1(a), we use the concepts of the principal planes in optics [29] to unify the analysis of the beam diagnostic line, which usually consists of two or more thick quadrupoles. In optics, a lens system is fully characterized by four parameters: object focal point, image focal point, object principal plane H , and image principal plane H' . The formulas of these parameters for a magnetic focusing system consisting of N quadrupoles are presented in Appendix B. Then the transfer matrix from the source to the screen can be unified as

$$\mathbf{R} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \\ = \begin{pmatrix} 1 - \frac{L}{f} & L + d - \frac{Ld}{f} \\ -\frac{1}{f} & -\frac{d}{f} + 1 \end{pmatrix},$$

where f is the focal length of the compound system, d is the distance between the source and the object's principal plane H , and L is the distance between the image's principal plane H' and the screen. The energy-dependent spot size [Eq. (2)] can be approximated as

$$\sigma_y^2(E) \approx \epsilon_g \left\{ R_{11}^2 \beta_i + 2R_{11} \left(\frac{dR_{11}}{dE} \beta_i - \frac{dR_{12}}{dE} \alpha_i \right) \delta_E \right. \\ \left. + \left[\left(\frac{dR_{11}}{dE} \beta_i^{\frac{1}{2}} - \frac{dR_{12}}{dE} \frac{\alpha_i}{\beta_i^{\frac{1}{2}}} \right)^2 + \left(\frac{dR_{12}}{dE} \right)^2 \frac{1}{\beta_i} \right] \delta_E^2 \right\}, \quad (3)$$

where $\delta_E = E - E_0 \ll E_0$ is assumed. We want to emphasize that $R_{12} = 0$ is used to get Eq. (3). We neglect the subscript “y” for most quantities to simplify the symbols. Using the energy-dependence analysis in Appendix C, we have the following relations for the parameters of a compound system,

$$\frac{\delta_f/f}{\delta_E/E_0} = \kappa_1, \\ \frac{\delta_d/d}{\delta_E/E_0} = \frac{\delta_{l_H}/l_H}{\delta_E/E_0} \frac{l_H}{d} = \kappa_2, \\ \frac{\delta_L/L}{\delta_E/E_0} = \frac{\delta_{l_{H'}}/l_{H'}}{\delta_E/E_0} \frac{l_{H'}}{L} = \kappa_3, \quad (4)$$

where κ_1, κ_2 , and κ_3 represent the dependence of the relative change of the focal length, the object length, and the image length on the relative change of the particle energy, respectively. More details can be found in Appendix C. Since $d \ll L$ in most magnification imaging systems and $\frac{\delta_H/l_H}{\delta_E/E_0}$ is on the same order of $\frac{\delta_{l_H}/l_H}{\delta_E/E_0}$, we can neglect the term with κ_3 when calculating $\frac{dR_{11}}{dE}$ and $\frac{dR_{12}}{dE}$. We have

$$\frac{dR_{11}}{dE} \approx \frac{\kappa_1 L}{f E_0}, \quad \frac{dR_{12}}{dE} \approx \frac{(\kappa_1 - \kappa_2) L d}{f} \frac{1}{E_0} + \frac{\kappa_2 d}{E_0}. \quad (5)$$

The coefficients of δ_E^0, δ_E^1 , and δ_E^2 in Eq. (3) can thus be known. The aforementioned procedures can be applied to a compound system consisting of arbitrary quadrupoles.

For the two quadrupoles system used in this work, $d = 7.72$ cm, $L = 1.6$ m, $f = 7.37$ cm, $l_H = -6.39$ cm, $l_{H'} = -12.45$ cm, and $\kappa_1 \approx 1.64, \kappa_2 \approx -0.53$. If $\epsilon_g, \beta_i, \alpha_i$ are assumed to be independent of E , they can be known by fitting the measured $\sigma_y - E$ curve. The fitted emittance and C-S parameters using the simulated spot sizes between 148 and 152 MeV are $\epsilon_{ni} = 1.92$ mm mrad, $\beta_i = 0.62$ mm, $\alpha_i = -0.14$, which are close to the setting values. We want to emphasize that the magnification factor of the optical system has a critical effect on the fitting, e.g., the fitted results are $\epsilon_{ni} = 2.6$ mm mrad, $\beta_i = 0.85$ mm, $\alpha_i = -0.16$ when the factor is 1. We also try an electron beam with a small emittance as $\epsilon_{ni} = 0.2$ mm mrad, $\beta_i = 0.56$ mm, $\alpha_i = 0$ and the fitted results are $\epsilon_{ni} = 0.203$ mm mrad, $\beta_i = 0.55$ mm, $\alpha_i = -0.07$. This good agreement shows the beamline has the ability to measure submicron emittance.

Since the electrons are dispersed by the dipole and an energy-dependent spot size curve is used to fit the emittance and the C-S parameters, we only need to consider the chromatic emittance growth of electrons, which are treated as having the same energy on the imaging screen. The measured uncertainty of the energy is dominated by overlap of electrons with similar energies on the screen and the point spread function of the optical imaging system, which is 14 keV. GPT simulations show that the chromatic emittance growth induced by this energy uncertainty in our beamline is negligible. The chromatic emittance growth for 1 MeV energy measurement uncertainty is 82%. This indicates a high energy resolution is critical for this emittance measurement method.

III. THE EFFECTS OF VARIOUS LENGTH DEVIATIONS

Previous works have analyzed nonideal issues in the focus-scan methods, e.g., space-charge effects, high-order fields, angular alignment, and position alignment accuracy [19,21,27]. In practical experiments, the source position and the distance between the source, magnets, and screen

may deviate from the designed values and vary. It is important to understand how these deviations affect the transverse phase space diagnostic, especially the imaging energy. In addition, the relationship between these deviations and the imaging energy can be used to adjust and fine-tune the interested electron energy.

Before studying the impact of the length deviations on the transverse phase space diagnostic, we note that the imaging plane ($R_{12} = 0$) is not the waist plane ($\alpha = 0$) for the chosen energy E_0 , as the latter depends on both the beamline and the initial C-S parameters, while the imaging plane is only determined by the beamline configuration. We can calculate the energy $E_0 + \delta_E$ that reaches the minimum spot size at the imaging plane using Eq. (3). The focused energy shift δ_E is

$$\begin{aligned} \delta_E &\approx -\frac{R_{11}\beta_i\left(\frac{dR_{11}}{dE}\beta_i - \frac{dR_{12}}{dE}\alpha_i\right)}{\left(\frac{dR_{11}}{dE}\beta_i - \frac{dR_{12}}{dE}\alpha_i\right)^2 + \left(\frac{dR_{12}}{dE}\right)^2} \\ &\approx -\frac{\beta_i}{(\alpha_i + \alpha_i^{-1})(\kappa_1 - \kappa_2)d} E_0, \end{aligned} \quad (6)$$

where the linear optics and the wake chromatic effects are used as fundamental assumptions, and $\frac{dR_{12}}{dE} \approx \frac{(\kappa_1 - \kappa_2)Ld}{f} \frac{1}{E_0}$, $|\frac{dR_{11}}{dE}\beta_i| \ll |\frac{dR_{12}}{dE}\alpha_i|$, and $1 - (L/f) \approx -(L/f)$ are used to simplify the expression. For a fixed beamline, δ_E is affected by the initial C-S parameters of the beam.

We find the focused energy shift for different β_i and α_i for our beamline and show the results in Fig. 2(a), while Fig. 2(b) shows the δ_E predicted using Eq. (6). A good agreement is achieved between Figs. 2(a) and 2(b). Clearly, $\delta_E < 0$ when $\alpha_i > 0$ and vice versa, which can be seen from Eq. (6). At each β_i , $|\delta_E|$ reaches its maximum when $|\alpha_i| = 1$, which is shown by the white dashed lines in Fig. 2(b). When α_i is fixed, $|\delta_E|$ increases with an increase in β_i . For $\beta_i \sim 1$ mm of typical ~ 100 MeV electrons

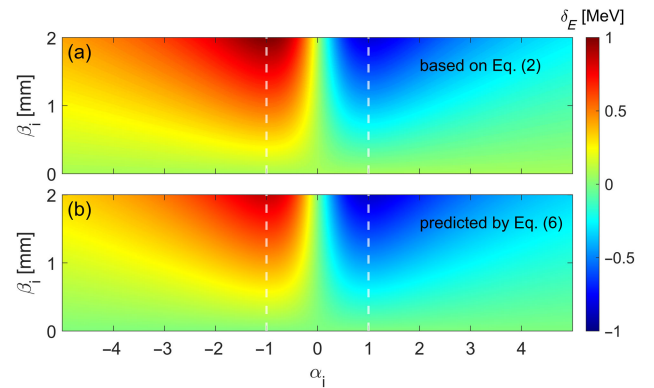


FIG. 2. Focused energy shift δ_E for beams with different initial C-S parameters: (a) calculated based on Eq. (2) and (b) predicted using Eq. (6). The beamline is the same as that in Fig. 1 and $E_0 = 150$ MeV. The white dashed lines represent $\alpha_i = \pm 1$.

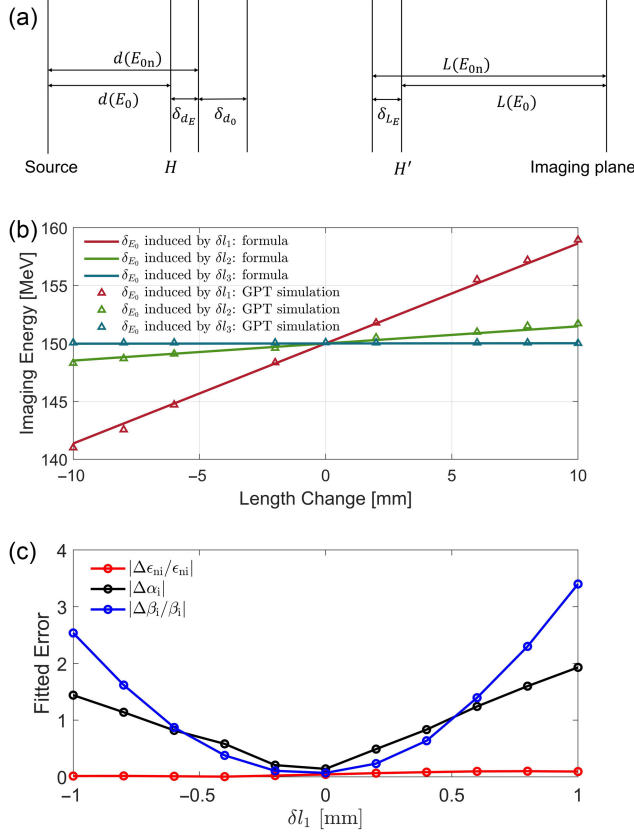


FIG. 3. Imaging energy shift δ_{E_0} induced by various length changes. (a) A diagram showing the definitions of $d(E_0)$, $d(E_{0n})$, δ_{d_E} , δ_{d_0} , and $L(E_0)$, $L(E_{0n})$, δ_{L_E} . (b) A comparison between the GPT simulation results (triangles) and the formulas (solid lines). (c) The fitted error in the emittance and the C-S parameters caused by the axial length inaccuracy of l_1 . The beamline and electron beam parameters are identical to those shown in Fig. 1.

produced from current LWFA experiments, the focused energy shift $|\delta_E|$ is less than 1 MeV. Thus, the curve $\sigma_y - E$ measured in the imaging plane of the chosen energy E_0 can include a minimum, which is beneficial for the fitting, especially when considering various noises in real experiments [6].

Now we use the principal planes of a compound magnetic system to study the change of d , f , and L on the imaging energy. As shown in Fig. 3(a), we consider a δ_{d_0} change for the chosen imaging energy E_0 , the new imaging energy $E_{0n} = E_0 + \delta_{E_0}$ satisfies

$$\frac{1}{d(E_{0n}) + \delta_{d_0}} + \frac{1}{L(E_{0n})} = \frac{1}{f(E_{0n})}. \quad (7)$$

Assuming $\delta_{d_0} \ll d(E_0)$, $\delta_{d_E} \equiv d(E_{0n}) - d(E_0) \ll d(E_0)$, $\delta_{L_E} \equiv L(E_{0n}) - L(E_0) \ll L(E_0)$, and $\delta_{f_E} \equiv f(E_{0n}) - f(E_0) \ll f(E_0)$, we then have $\frac{1}{d^2}(\delta_{d_0} + \delta_{d_E}) + \frac{1}{L^2}\delta_{L_E} = \frac{1}{f^2}\delta_{f_E}$. Substituting Eq. (4), we get

$$\delta_{E_0} \approx \frac{f}{\kappa_1 d - \kappa_2 f} \frac{\delta_{d_0}}{d} E_0. \quad (8)$$

Similarly, the imaging energy change induced by δ_{f_0} and δ_{L_0} is

$$\delta_{E_0} \approx \frac{d}{\kappa_1 d - \kappa_2 f} \frac{\delta_{f_0}}{f} E_0, \quad (9)$$

$$\delta_{E_0} \approx \frac{f d}{L(\kappa_1 d - \kappa_2 f)} \frac{\delta_{L_0}}{L} E_0, \quad (10)$$

where the terms with κ_3 are neglected. For the two quadrupoles beamline, we have $\frac{\delta_{E_0}/E_0}{\delta_{d_0}/d} \approx 0.44$, $\frac{\delta_{E_0}/E_0}{\delta_{f_0}/f} \approx 0.47$, $\frac{\delta_{E_0}/E_0}{\delta_{L_0}/L} \approx 0.02$. The main reason why d has a large impact on the energy shift is that we choose d to be close to f to achieve a large magnification factor.

Any changes in the relative positions between the source, magnets, and the screen can be converted as δ_{d_0} , δ_{f_0} , and δ_{L_0} , and then their impact on the imaging energy can be known through Eqs. (8)–(10). The change of the magnetic field gradients G of the quadrupoles can also be converted as δ_{d_0} , δ_{f_0} , and δ_{L_0} , and then their impact on the imaging energy can be known similarly.

We show some examples of the two quadrupole beamline. A change in the position of the source results in a change in l_1 . Since $\delta_{d_0} \approx \delta_{l_1}$ and $\delta_{f_0} = \delta_{L_0} = 0$, the imaging energy change corresponding to δ_{l_1} is $\delta_{E_{0,l_1}} \approx \frac{f}{\kappa_1 d - \kappa_2 f} \frac{\delta_{l_1}}{d} E_0$.

A change of l_2 can be achieved by moving Q2 and the screen together while keeping a constant distance between them. When l_2 changes, it results in a change on d , L , and f . Since $d = l_1 + l_H + h_1$, we have $\delta_{d_0} \approx \delta_{l_H}$, where $\delta_{h_1} \approx 0$ is used. Using the expression of l_H for two quadrupoles system, $l_H = \frac{f_1 d_1}{f_1 + f_2 - d_1}$, we have $\delta_{l_H} \approx \frac{\partial l_H}{\partial d_1} \delta_{d_1} = \frac{\partial l_H}{\partial d_1} \delta_{l_2}$, where d_1 is the distance from H_1 of Q1 to H'_2 of Q2. Similarly, we have $\delta f_0 = \frac{\partial f}{\partial d_1} \delta_{l_2}$ and $\delta L_0 = \frac{\partial L}{\partial d_1} \delta_{l_2}$. Thus, the imaging energy change corresponding to δ_{l_2} is $\delta_{E_{0,l_2}} = [\frac{f}{\kappa_1 d^2 - \kappa_2 f d} \frac{\partial l_H}{\partial d_1} + \frac{f d}{L^2(\kappa_1 d - \kappa_2 f)} \frac{\partial l_H}{\partial d_1} + \frac{d}{\kappa_1 d f - \kappa_2 f^2} \frac{\partial f}{\partial d_1}] \delta_{l_2} E_0$.

A change of the screen position leads to a δ_{l_3} , which can be converted to $\delta_{d_0} = \delta_{f_0} = 0$, $\delta_{L_0} = \delta_{l_3}$. Thus, the imaging energy change induced by δ_{l_3} is $\delta_{E_{0,l_3}} \approx \frac{f d}{L(\kappa_1 d - \kappa_2 f)} \frac{\delta_{l_3}}{L} E_0$.

We scan δ_{l_1} , δ_{l_2} , and δ_{l_3} in GPT simulations and show the imaging energy shift on the screen in Fig. 3(b). The simulation results (triangles) and the formulas (solid lines) agree well. Figure 3(b) indicates that δ_{l_1} has a significant impact on the imaging energy (0.86 MeV/mm) and δ_{l_2} has a mild effect (0.15 MeV/mm) while δ_{l_3} affects δ_{E_0} little. It is thus efficient to adjust the imaging energy by introducing a controllable δl_1 , which can be achieved by simultaneously moving Q1, Q2, and the screen.

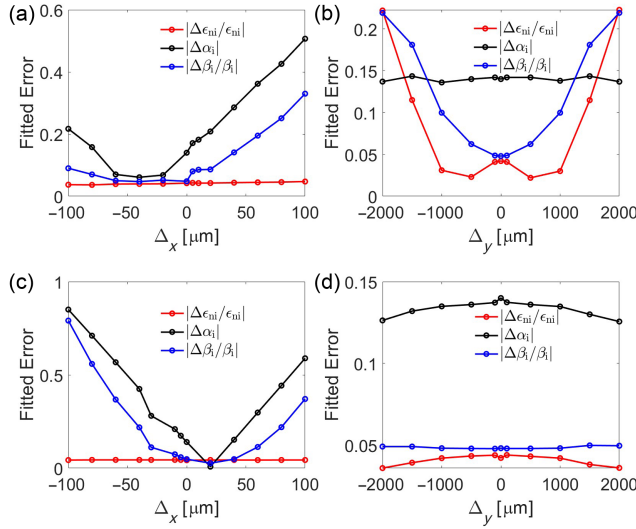


FIG. 4. Fitted error in the emittance and the C-S parameters caused by the transverse offsets of Q1 and Q2 relative to the beam propagation axis. Panels (a) and (b) correspond to Q1, while panels (c) and (d) correspond to Q2.

In real experiments, the error in l_1 , l_2 , and l_3 primarily results from distance measurements. Figure 3(b) indicates that a 1 mm measurement error can cause an imaging energy shift of ≤ 0.86 MeV. This significantly affects the fitted C-S parameters while minimally impacts the fitted emittance. We take l_1 as an example and show the results in Fig. 3(c). To limit the fitted error of the C-S parameters to $|\frac{\Delta\beta_i}{\beta_i}|, |\Delta\alpha_i| \sim 10\%$, the error in l_1 needs to be less than 10 s of μm . Improving the accuracy of distance measurements can reduce the imaging energy shift and be beneficial for the transverse phase space diagnostics.

The transverse misalignment between the PMQs and the beam propagation axis effectively introduces a dipole component into the beamline, which is critical for accurate transverse phase space diagnostics. A misalignment along the y direction causes the tie figure to tile [27], while a misalignment in the x direction introduces an error in the energy measurement. We analyze the impact of misalignments in Q1 and Q2 using GPT simulations, as shown in Fig. 4. A misalignment of $\Delta_y \sim 1$ mm in the y direction causes a $\sim 10\%$ error in the emittance and the C-S parameters ($|\frac{\Delta\epsilon_{ni}}{\epsilon_{ni}}|, |\frac{\Delta\beta_i}{\beta_i}|$, and $|\Delta\alpha_i|$). The tolerance for the x direction misalignment is more stringent, with a $\sim 10\%$ error of the fitted results requiring an x direction misalignment of $\Delta_x \sim 10$ s of μm . The accuracy of the fitted emittance is less affected by the misalignments compared to the fitted C-S parameters. We also note that a proper Δ_x can compensate the imaging energy shift caused by the initial C-S parameters and achieve a low fitted error.

IV. C-S PARAMETERS EVOLUTION DURING THE PLASMA DOWNRAMP

The evolution of the C-S parameters in the plasma-vacuum transition regions is critical for high-quality plasma-based acceleration [17,32]. In this section, we study the feasibility of diagnosing the C-S parameters of the injected electrons after plasma downramps with different lengths using the focus-scan scheme. As shown in Fig. 5(a), an 800 nm laser pulse with an FWHM duration of $\tau = 30$ fs is incident on an underdense plasma with a peak density of $n_p = 2 \times 10^{18} \text{ cm}^{-3}$. The laser is focused at $z_p = 0.5$ mm with a spot size of $w_0 = 15 \mu\text{m}$ and a peak normalized vector potential $a_0 = 2$, where $a_0 \equiv \frac{eA}{m_e c^2}$, m_e is the electron mass, and A is the vector potential of the laser pulse. The laser is polarized along the y direction. The plasma consists of a 0.5-mm-long upramp, a 1.5-mm-long plateau, and a downramp whose length is changed in simulations. The ramps have a half-Gaussian profile, and we use L_{ramp} to label the ramps, where L_{ramp} is three times the rms length of the Gaussian profile. Nitrogen atoms with $n_N = 0.01n_p$ are also initialized to supply the injected electrons.

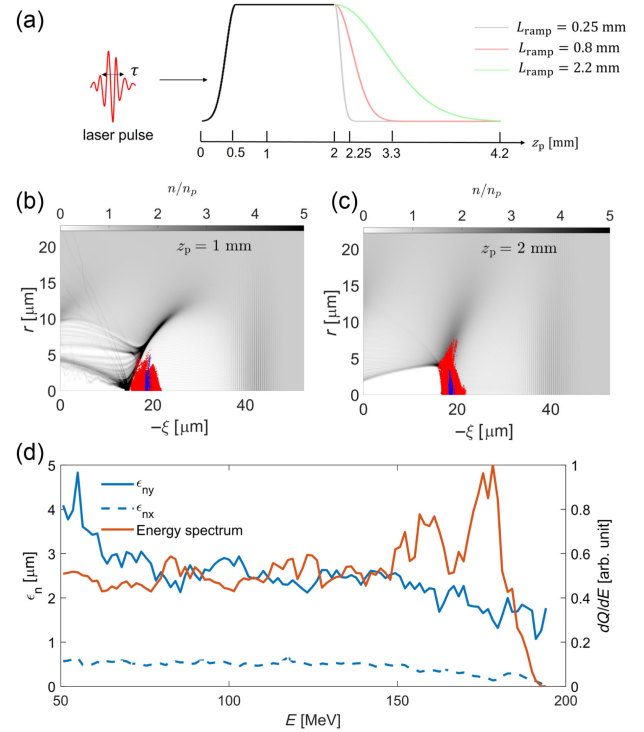


FIG. 5. (a) Schematic of the laser plasma wakefield accelerator. The plasma density distribution in the $\xi - r$ space at $z_p = 1$ mm (b) and $z_p = 2$ mm (c). Here $\xi \equiv z_p - ct$ is the coordinate in a frame moving with the speed of light. The blue dots are the electrons whose energy after the plasma is between 148 and 152 MeV and the red dots represent other trapped electrons. (d) The normalized emittance (blue lines) and the energy spectrum of the injected electron at the end of the plateau.

We use the quasi-three-dimensional (Q3D) version of the fully relativistic PIC code OSIRIS [33] with two modes to model the electron beam injection and acceleration in LWFA. A moving window propagating at the speed of light in vacuum c is used. The window size is $56.4 \times 45.1 \mu\text{m}$ and there are 2700×240 cells along the z and r directions, respectively. The time step is chosen as $dt = 60$ as to satisfy the Courant-Friedrichs-Lewy condition. There are 8 and 128 macroparticles in each cell to represent the preionized plasma electrons and nitrogen atoms, respectively. An advanced Maxwell solver [34] is used to suppress the numerical fields [34–36] and model the injected electrons with high fidelity.

The intense laser driver pushes the plasma electrons forward and outward, and creates a nonlinear plasma wake as shown in Fig. 5(b). The inner shell electrons of nitrogen with high ionization potentials are released inside the wake and accelerated when they slip back relative to the wake. If they obtain a forward velocity higher than the plasma wake before the wake tail, they are trapped and stay inside the wake [red and blue dots in Fig. 5(b)] to be accelerated to high energies [37,38]. Transverse phase space mixing between injected electrons ionized at different times happens and the injected beam is approximately matched with the linear focusing force of the nonlinear plasma wake [39],

i.e., $\beta \approx \sqrt{2g^{-1}\gamma c/\omega_p}$ and $\alpha = 0$, where $\omega_p = \sqrt{\frac{n_p e^2}{m_e \epsilon_0}}$ is the plasma frequency. In a fully blowout plasma wake, the comoving relativistic electrons are confined by a linear focusing force as $F_r = (r/2)m_e \omega_p^2$. Here we use the parameter g to quantify the ratio of the focusing force experienced by the electrons to that in a fully blown-out wake. As shown in Fig. 5(c), the drive laser is weak at the end of the density plateau ($z_p = 2$ mm) and $g = 0.46$. In Fig. 5(d), we show the energy spectrum (red line) and the emittance (blue lines) of the injected beam at $z = 2$ mm. The emittance parallel to the laser polarized direction is higher than that of the other direction due to the large residual momentum imparted by the drive laser [40]. The total injected charge is 65 pC.

At the plasma density downramp, the focusing force of the plasma wake decreases as the density decreases ($F_p \propto n_p$) and modifies the C-S parameters of the injected beam. A properly designed downramp can enlarge the β -function by a factor of 10–100 while keeping $|\alpha| \sim 1$, which is beneficial for the subsequent beam transport [17]. We choose three different downramp lengths, $L_{\text{ramp}} = 0.25, 0.8, \text{ and } 2.2$ mm, to study their impact on the transverse phase space diagnostic using the focus-scan scheme.

As an example, we show the evolution of a_0 of the drive laser in the downramp when $L_{\text{ramp}} = 0.8$ mm in Fig. 6(a). The laser intensity decreases from $a_0 \approx 2$ to $a_0 \approx 1$ during the ramp (blue line) while $g \approx 0.5$ (red line). We choose electrons whose energy after the plasma is between 148 and 152 MeV and show the evolution of the ratio between

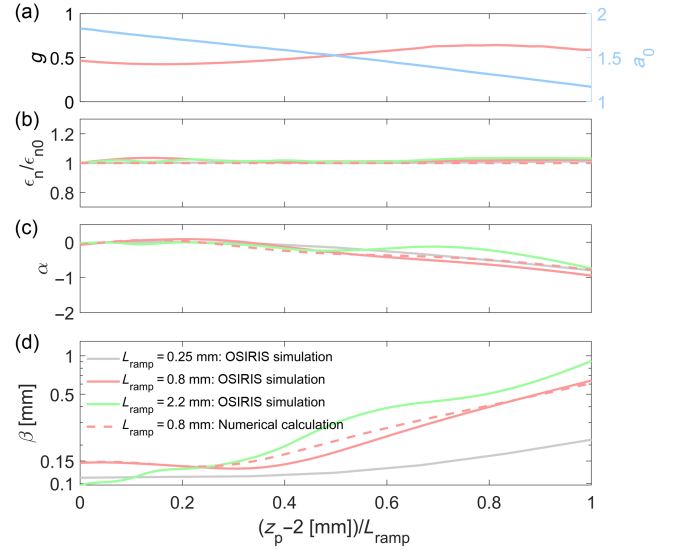


FIG. 6. Evolution of different quantities during the plasma downramp. (a) g and the drive laser a_0 when $L_{\text{ramp}} = 0.8$ mm. ϵ_n/ϵ_{n0} (b) and α (c), β (d) of the chosen electrons in three cases. The red dashed line in (b)–(d) represents the numerical calculation results using the parameter g shown in (a).

normalized emittance ϵ_n and ϵ_{n0} , and C-S parameters in Figs. 6(b)–6(d), where ϵ_{n0} is the normalized emittance of the chosen electrons at the start of the ramp. When $L_{\text{ramp}} = 0.25$ mm, α decreases to -0.8 and β grows by a factor of 2 from 0.11 to 0.22 mm [gray lines in Figs. 6(c) and 6(d)]. When $L_{\text{ramp}} = 0.8$ mm, α decreases to -0.95 and β grows by a factor of 4.6 from 0.14 to 0.64 mm [red lines in Figs. 6(c) and 6(d)]. When $L_{\text{ramp}} = 2.2$ mm, α decreases to -0.75 and β grows by a factor of 9 from 0.1 mm to 0.9 mm [green lines in Figs. 6(c) and 6(d)]. We also use the g extracted from simulations when $L_{\text{ramp}} = 0.8$ mm [red line in Fig. 6(a)] to calculate the evolution of the C-S parameters [red dashed lines in Figs. 6(c) and 6(d)], which agrees well with the simulation results. The emittance of the chosen electrons in three cases is $\epsilon_n \approx 2$ mm mrad when $L_{\text{ramp}} = 0.25$ mm, $\epsilon_n \approx 2.6$ mm mrad when $L_{\text{ramp}} = 0.8$ mm, and $\epsilon_n \approx 2.4$ mm mrad when $L_{\text{ramp}} = 2.2$ mm. They are preserved well during the downramps as shown in Fig. 6(b).

The 6D phase space coordinates of the injected electrons from OSIRIS simulations are imported into GPT to model their propagation in the beamline. The energy-dependent spot sizes on the screen for three different ramp lengths are shown in Fig. 7. The initial C-S parameters and the distance l_1 between the source and Q1 are different in three cases, leading to a shift of the focus energy. The imaging energy shift induced by $\delta_{l_1} = -L_{\text{ramp}}$ can be calculated using Eq. (8) and the results are $\delta_{E_0} = -0.22, -0.69, -1.89$ MeV for $L_{\text{ramp}} = 0.25, 0.8, 2.2$ mm, respectively. The deviation between the imaging energy and the focusing energy is determined by the initial C-S parameters of the beam

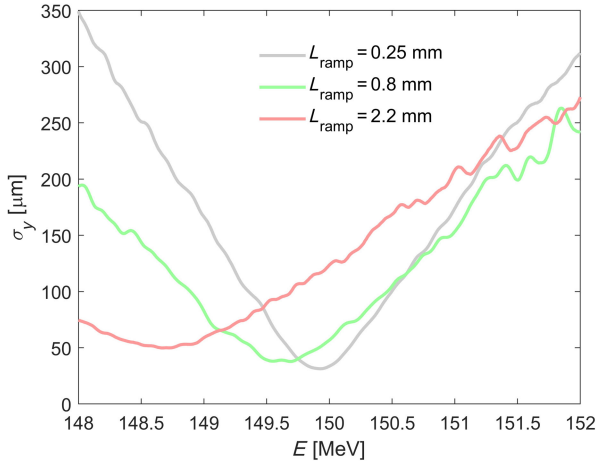


FIG. 7. Energy-dependent spot size on the screen in three different downramp lengths.

[Eq. (6)], which are 0.16, 0.32, 0.41 MeV, respectively. Thus, the predicted beam waist energies for three cases are 149.94, 149.63, 148.52 MeV, which agree well with the GPT simulations as shown in Fig. 7.

We fit the emittance and the C-S parameters using the $\sigma_y - E$ data. When $L_{\text{ramp}} = 0.25$ mm, the fitted error is $|\frac{\Delta \epsilon_n}{\epsilon_n}| \approx 1\%$, $|\Delta \alpha| \approx 33\%$, $|\frac{\Delta \beta}{\beta}| \approx 16\%$ (fitted energy range: 149–151 MeV); when $L_{\text{ramp}} = 0.8$ mm, the fitted error is $|\frac{\Delta \epsilon_n}{\epsilon_n}| \approx 0.5\%$, $|\Delta \alpha| \approx 5\%$, $|\frac{\Delta \beta}{\beta}| \approx 17\%$ (fitted energy range: 149–151 MeV); when $L_{\text{ramp}} = 2.2$ mm, the fitted error is $|\frac{\Delta \epsilon_n}{\epsilon_n}| \approx 2\%$, $|\Delta \alpha| \approx 12\%$, $|\frac{\Delta \beta}{\beta}| \approx 20\%$ (fitted energy range: 148–151 MeV). The used energy range is reduced to avoid the effects of the fluctuations of the curves around 152 MeV, which may be caused by random and microstructures of the injected electrons. It should be noted that when $L_{\text{ramp}} = 2.2$ mm, $R_{12} \approx 0.05$ m due to the decrease of l_1 and we need to replace the coefficient of δ_E^0 in Eq. (3) from $R_{11}^2 \beta_i$ to $(R_{11} - R_{12} \frac{\alpha_i}{\beta_i})^2 \beta_i + \frac{R_{12}^2}{\beta_i}$, and replace the coefficient of δ_E^1 from $2R_{11}(\frac{dR_{11}}{dE} \beta_i - \frac{dR_{12}}{dE} \alpha_i)$ to $2R_{11}(\frac{dR_{11}}{dE} \beta_i - \frac{dR_{12}}{dE} \alpha_i) - 2R_{12}(\frac{dR_{11}}{dE} \alpha_i - \frac{dR_{12}}{dE} \frac{1}{\beta_i} - \frac{dR_{12}}{dE} \frac{\alpha_i^2}{\beta_i})$. After considering this nonzero R_{12} , the fitting results agree well with the initial values.

V. CONCLUSION

In conclusion, we introduce the principal planes of a magnetic compound system consisting of N thick quadrupoles to analyze the transverse phase space diagnostic using the focus-scan scheme. The general fitting procedures of the emittance and C-S parameters are given. We then use the principal planes to analyze the effect of the distance variations between the source, magnets, and screen on the imaging energy and apply the conclusions to gas targets with different downramp lengths. A beamline with two quadrupoles is designed and analyzed as an example. The

analysis can be applied to beamlines consisting of more quadrupoles and is useful for the transverse phase diagnostic using the focus-scan scheme.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available [41].

APPENDIX A: DESIGN AND PARAMETERS OF THE QUADRUPOLES

Miniature permanent magnetic quadrupoles with $G \sim 100$ T/m have been implemented in the focus-scan scheme for electron beams from LWFA [19–23]. We adopt the Halbach array, which can achieve large magnetic fields in a compact configuration [42]. There are M small trapezoidal magnetic sectors made of neodymium iron boride (NdFeB) assembled in a single PMQ, where M is an integer that is a multiple of 4. According to the formula given by Halbach [42], the magnetic field gradient of a permanent quadrupole is $G = 2a_2 B_r (1/r_2 - 1/r_1)$, where B_r is the remanent magnetic strength, $a_2 = \cos^2((\pi/M)) \sin(\frac{2\pi}{M}) \frac{M}{2\pi}$ is the magnetic strength coefficient, and r_1 , r_2 represent the inner radius and outer radius of quadrupole, respectively. The typical values of a_2 are 0.32, 0.77, 0.89, 0.94, 0.96, 0.97 for $M = 4, 8, 12, 16, 20, 24$. For the PMQs used in Sec. II, we choose $M = 16$, $B_r = 1.42$ T, $r_1 = 12.3$ mm, $r_2 = 49$ mm for Q1, and $M = 16$, $B_r = 1.42$ T, $r_1 = 14.2$ mm, $r_2 = 49$ mm for Q2. We have fabricated these two PMQs and the measured field gradients using the Hall probe are $G_1 = -258.5$ T/m for Q1 and $G_2 = 175.4$ T/m for Q2. We plan to conduct the transverse phase space diagnostics experiment using the focus-scan scheme for electron beams produced by the 200 TW Ti:Sapphire laser system available in our laboratory [43].

APPENDIX B: THE FOCAL LENGTH AND THE PRINCIPAL PLANES OF QUADRUPOLES

We illustrate how to calculate the positions of the object principal plane (H), the image principal plane (H'), and the focal length f of a thick magnet quadrupole. The transfer matrix of a thick focusing quadrupole is

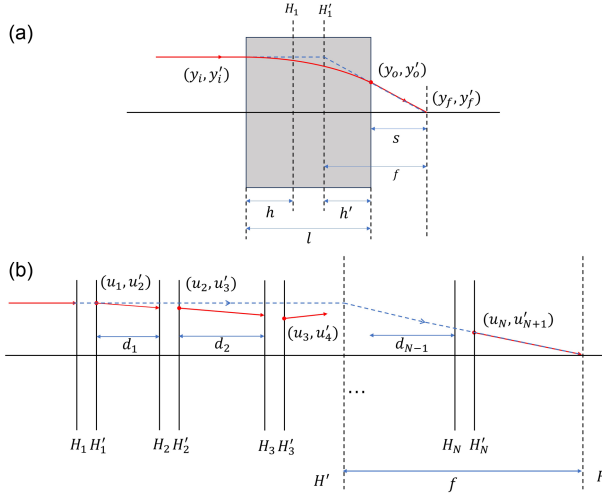


FIG. 8. Object principal plane (H), the image principal plane (H'), and the focal length f of a thick quadrupole (a) and a magnetic compound system (b).

$$\mathbf{R}_Q = \begin{pmatrix} \cos \sqrt{K}l & \frac{1}{\sqrt{K}} \sin \sqrt{K}l \\ -\sqrt{K} \sin \sqrt{K}l & \cos \sqrt{K}l \end{pmatrix},$$

where l is the length of the quadrupole and $K = \frac{eG}{\gamma mc} > 0$ is the focusing strength of the quadrupole. As shown in Fig. 8(a), the distance from the right edge of the quadrupole to the focal point can be known by finding where an incident particle with $y' = 0$ reaches $y = 0$, which is $s = \frac{\cos \sqrt{K}l}{\sqrt{K} \sin \sqrt{K}l}$. The focal length f , defined as the distance from the image principal plane H' to the focal point, can be known as

$$f = \frac{1}{\sqrt{K} \sin \sqrt{K}l}. \quad (\text{B1})$$

We use h' to represent the distance from H' to the right edge of the quadrupole and its value is

$$h' = f - s = \frac{1 - \cos \sqrt{K}l}{\sqrt{K} \sin \sqrt{K}l}. \quad (\text{B2})$$

Similarly, h which represents the distance between the left edge of the quadrupole and H can be known as

$$h = \frac{1 - \cos \sqrt{K}l}{\sqrt{K} \sin \sqrt{K}l}. \quad (\text{B3})$$

For a defocusing quadrupole with $K < 0$, we can similarly get

$$h = h' = \frac{\cosh \sqrt{K}l - 1}{\sqrt{K} \sinh \sqrt{K}l}. \quad (\text{B4})$$

For the two quadrupoles in Appendix A, we have $h_1 = h'_1 = 0.99$ cm, $f_1 = 9.98$ cm and $h_2 = h'_2 = 0.74$ cm, $f_2 = -19.42$ cm. Using the principal planes, a magnetic quadrupole can be treated as all changes in the particles' angles occur at H , and particles travel parallel to the axis between the planes (H and H'), shown by the blue dashed line in Fig. 8(a).

We then generalize the formulas of the focal length and the positions of the principal planes to a magnetic compound system composed of N thick quadrupoles. As shown in Fig. 8(b), we define the distance between the one's image principal plane to the next object plane to be d_j , where $j = 1, 2, 3, \dots, N-1$. Considering an incident particle, we label its transverse position at H_j as u_j and the tangent value of the angle at H'_j as u'_{j+1} , where $j = 1, 2, 3, \dots, N$. The recurrence relation of u_j and u'_j is [29]

$$u_j = u_{j-1} - d_{j-1} \sum_{k=1}^{j-1} \frac{u_k}{f_k},$$

$$u'_{j+1} = \sum_{k=1}^j \frac{u_k}{f_k}.$$

The focal length f of the compound system can be known as $f = \frac{u_1}{u'_{N+1}}$ when setting $u'_1 = 0$. The position of the imaging principal plane of the compound system relative to H'_N is $l_{H'} = \frac{u_N - u_1}{u'_{N+1}}$. Similarly, the position of the object principal plane of the compound system relative to H_1 can be known by considering a particle with $u'_{N+1} = 0$. It should be noted that $l_{H'} < 0$ means H' is at the left side of H'_N , $l_{H'} > 0$ means H' is at the right side of H'_N , which is the same for l_H to H_1 , and $f > 0$ means a focusing compound quadrupole system, and vice versa.

We give the expressions of f , l , and l' for $N = 2$ and 3,

$$N=2: f = \frac{f_1 f_2}{f_1 + f_2 - d_1}, l_{H'} = -\frac{f d_1}{f_1}, l_H = \frac{f d_1}{f_2};$$

$N=3$:

$$f = \frac{f_1 f_2 f_3}{f_1 f_2 + f_1 f_3 + f_2 f_3 - d_1(f_2 + f_3) - d_2(f_1 + f_2) + d_1 d_2},$$

$$l_{H'} = \frac{(d_1 d_2 - d_2 f_1 - d_1 f_2 - d_2 f_2) f_3}{d_1 d_2 - d_2 f_1 - d_1 f_2 - d_2 f_2 + f_1 f_2 - d_1 f_3 + f_1 f_3},$$

$$l_H = -\frac{(d_1 d_2 - d_1 f_3 - d_2 f_2 - d_1 f_2) f_1}{d_1 d_2 - d_1 f_3 - d_2 f_2 - d_1 f_2 + f_1 f_2 - d_2 f_1 + f_1 f_3}.$$

APPENDIX C: ENERGY DEPENDENCE OF THE FOCAL LENGTH AND THE POSITIONS OF THE PRINCIPAL PLANES

We first study the energy dependence of the focal length and the positions of the principal planes for one thick

quadrupole. Consider electrons with a small energy deviation as $\gamma = \gamma_0 + \delta_\gamma$ and $\delta_\gamma \ll \gamma_0$, then $K \approx K_0 + \delta_K$, where $\delta_K = -\frac{\delta_\gamma}{\gamma_0} K_0$. The relative change of the focal length can be known from Eq. (B1) as

$$\frac{\delta_f}{f} \approx \frac{\sin \sqrt{K_0} l + \sqrt{K_0} l \cos \sqrt{K_0} l \frac{\delta_\gamma}{\gamma_0}}{2 \sin \sqrt{K_0} l} \frac{\delta_\gamma}{\gamma_0},$$

where $\frac{\sqrt{K_0} l \delta_K}{2 \gamma_0} \ll 1$ is used to simplify the expression. Similarly, we have

$$\frac{\delta_h}{h} = \frac{\delta_{h'}}{h'} \approx \frac{\sqrt{K_0} l - \sin \sqrt{K_0} l \frac{\delta_\gamma}{\gamma_0}}{2 \sin \sqrt{K_0} l} \frac{\delta_\gamma}{\gamma_0}.$$

The betatron phase advance $\sqrt{K_0} l$ in one quadrupole is usually a small quantity in the focus-scan scheme,

e.g., $\sqrt{K_0} l \approx 0.37$ for Q1 and 0.34 for Q2. Assume $\sqrt{K_0} l \ll 1$, we have

$$\frac{\delta_f/f}{\delta_\gamma/\gamma_0} \approx 1, \frac{\delta_h/h}{\delta_\gamma/\gamma_0} = \frac{\delta_{h'}/h'}{\delta_\gamma/\gamma_0} \approx \frac{(\sqrt{K_0} l)^2}{12} \ll 1. \quad (C1)$$

This indicates that we can neglect the change of h and h' when considering particles with slight energy deviation. This conclusion greatly simplifies the energy-dependence analysis of a compound system.

We then calculate the energy dependence of the focal length and the positions of the principal planes for a magnetic compound system. Using the expression of f in Appendix B and Eq. (C1), the energy dependence of f can be obtained, and we define $\kappa_1 \equiv \frac{\delta_f/f}{\delta_\gamma/\gamma_0}$. We show the expressions for $N = 1, 2$, and 3,

$$\begin{aligned} N=1: \kappa_1 &\approx 1; \\ N=2: \kappa_1 &\approx \frac{f_1 + f_2 - 2d_1}{f_1 + f_2 - d_1}; \\ N=3: \\ \kappa_1 &\approx \frac{f_1 f_2 + f_1 f_3 + f_2 f_3 - 2d_1(f_2 + f_3) - 2d_2(f_1 + f_2) + 3d_1 d_2}{f_1 f_2 + f_1 f_3 + f_2 f_3 - d_1(f_2 + f_3) - d_2(f_1 + f_2) + d_1 d_2}. \end{aligned} \quad (C2)$$

Similarly, the energy dependence of δ_{l_H} and $\delta_{l_{H'}}$ can be known as

$$\begin{aligned} N=2: \frac{\delta_{l_H}/l_H}{\delta_\gamma/\gamma_0} &= -\frac{\delta_{l_{H'}}/l_{H'}}{\delta_\gamma/\gamma_0} \approx \frac{d_1}{-f_1 - f_2 + d_1}; \\ N=3: \\ \frac{\delta_{l_H}/l_H}{\delta_\gamma/\gamma_0} &\approx \frac{d_2^2(f_3 + f_2)^2 - 2d_3 d_2^2(f_3 + f_2) + d_3 d_2(f_3 f_2 + 2f_2^2) + d_3^2 d_2^2 - 2d_3^2 d_2 + d_3^2(f_2^2 + f_2 f_1)}{d_3 d_2 - d_2 f_3 - d_3 f_2 - d_2 f_2}, \\ \frac{\delta_{l_{H'}}/l_{H'}}{\delta_\gamma/\gamma_0} &\approx -\frac{d_2^2(f_1 + f_2)^2 - 2d_1 d_2^2(f_1 + f_2) + d_1 d_2(f_1 f_2 + 2f_2^2) + d_1^2 d_2^2 - 2d_1^2 d_2 + d_1^2(f_2^2 + f_2 f_3)}{d_1 d_2 - d_2 f_1 - d_1 f_2 - d_2 f_2}. \end{aligned} \quad (C3)$$

In the two quadrupoles setup used in this work, we have $\frac{\delta_f/f}{\delta_\gamma/\gamma_0} \approx 1.64$, $\frac{\delta_{l_H}/l_H}{\delta_\gamma/\gamma_0} = -\frac{\delta_{l_{H'}}/l_{H'}}{\delta_\gamma/\gamma_0} \approx 0.64$.

APPENDIX D: DISCUSSION ON THE SPACE-CHARGE EFFECTS

The space-charge effects may modify the beam spot size evolution and introduce an error in the fitted emittance and C-S parameters. When accounting for the space-charge effects, the beam envelope evolution is governed by $\frac{d^2 \sigma}{dz^2} + K\sigma - \frac{e_n^2}{\gamma^2 \beta^2 \sigma} = \frac{2I}{\gamma^3 \beta^3 I_0 \sigma}$, where I is the beam current and $I_0 = 17$ kA is the Alfven current [3]. Here the

space-charge force is approximately treated as a linear force, which is only strictly valid when the beam has a flat-top transverse distribution. However, due to the complex evolution of the spot size along the beamline, it is challenging to analytically quantify the space-charge effects. We conduct GPT simulations with different beam charges and the results are summarized in Fig. 9. The beamline and other parameters of the electron beam are identical to those shown in Fig. 1. For the 4 pC charge (0.6 kA for the 2 μ m long beam) used in the Sec. II, the fitted errors of the emittance and C-S parameters ($|\frac{\Delta \epsilon_{ni}}{\epsilon_{ni}}|$, $|\frac{\Delta \beta_i}{\beta_i}|$, and $|\Delta \alpha_i|$) are similar to the case without the space-charge forces [dashed lines in (b)]. As the charge

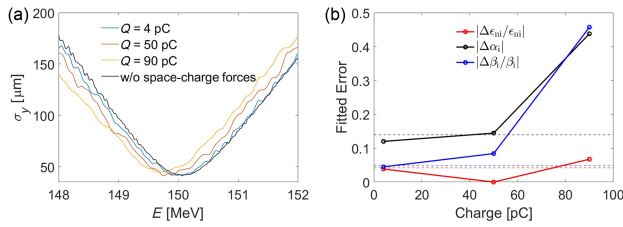


FIG. 9. (a) Energy-dependent spot size curve when accounting for the space-charge forces. (b) The fitted errors of the emittance and C-S parameters with different beam charge. The dashed lines represent the results without the space-charge forces.

increased to 50 pC (7.5 kA), the fitted errors are still similar to the case without the space-charge forces. The errors of the fitted C-S parameters increase significantly as the charge increases to 90 pC (13.5 kA) while the accuracy of the fitted emittance remains similar to the low charge case.

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