

**Closed-form expressions for electron cooling rates with redistribution**S. Seletskiy<sup>\*</sup> and A. Fedotov<sup>✉</sup>*Brookhaven National Laboratory, Upton, New York 11973, USA*

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Electron coolers with high-density electron bunches obtained via rf acceleration provide a feasible path for extending electron cooling to high energies. In such coolers, the dispersive redistribution of cooling rates between the longitudinal and transverse directions is crucial and must be rigorously calculated. This paper presents exact formulas for cooling redistribution in high-energy electron coolers. The most general formulas are derived in terms of Carlson elliptic integrals, which are further reduced to elementary functions for a key special case. These formulas enable an efficient and accurate calculation of cooling rates and were instrumental in designing the electron coolers for the Electron-Ion Collider (EIC) by enabling rapid parameter identification and optimization. Consequently, these results offer a valuable tool for the design and development of future high-energy electron coolers.

DOI: [10.1103/qsdt-6zx1](https://doi.org/10.1103/qsdt-6zx1)**I. INTRODUCTION**

Two common techniques for increasing six-dimensional phase-space density of hadron bunches in the storage rings are electron cooling [1,2] and stochastic cooling [3].

In electron cooling, electron and hadron beams co-travel at the same average velocity in a straight section of a hadron storage ring known as the cooling section (CS). Each hadron interacts with electrons in the CS via Coulomb force, resulting in dynamical friction [4,5]. Over many revolutions, this friction reduces both the transverse and longitudinal emittances of the hadron bunch.

Conventional low-energy coolers employ electrostatic acceleration of a dc beam and utilize beam magnetization; specifically, both the cathode and the cooling section are immersed in a solenoidal field [6]. Operational electron cooling with electrostatic acceleration was successfully extended to relativistic energies ( $\gamma = 9.5$ ) at the Fermilab Recycler Electron Cooler [7–9].

A feasible option to expand electron cooling to higher energies is to utilize rf acceleration of electron bunches and to use a nonmagnetized electron beam. The world's first rf-based nonmagnetized electron cooling was successfully implemented at the Low-Energy RHIC Electron Cooler (LEReC) [10–15]. It was applied to colliding Au<sup>79</sup> bunches and increased RHIC luminosity during 2020–2021 RHIC Run [16]. The LEReC approach is

especially attractive for high-energy coolers because it substantially reduces the average electron current required from the injector, eliminates the need for a CS solenoid with stringent field-quality requirements, and significantly simplifies the engineering aspects of the cooler design. The highest energy electron cooling utilizing rf-accelerated electron bunches was observed at  $\gamma = 19.6$  using the Coherent Electron Cooler accelerator at Brookhaven National Laboratory [17].

A nonmagnetized rf-based electron cooler [18,19] is currently being designed for the Electron Ion Collider [20]. It will operate at protons' injection energy ( $\gamma = 25.4$ ) to reduce the vertical emittance of proton bunches before they are accelerated to collision energy.

At the collision energy, the EIC would strongly benefit from cooling to counteract emittance growth dominated by intrabeam scattering (IBS) and to achieve high average luminosity. The preferred method for this high-energy cooling is again nonmagnetized, rf-based electron cooling [21,22]. We discuss the EIC coolers in more detail in Sec. V.

Typically, at high energies,  $\gamma\sigma_\theta > \sigma_\delta$ , where  $\sigma_\theta$  and  $\sigma_\delta$  are the angular spread and relative momentum spread of a bunch, respectively. This implies that the transverse velocity spread in the beam frame is larger than the longitudinal one; as a result, the transverse cooling rate is substantially smaller than the longitudinal one. This imbalance can be mitigated by redistributing the cooling rates between the longitudinal and transverse degrees of freedom.

While this redistribution has been explored both experimentally and theoretically [23–28], previous derivations relied on the assumption that the cooling force depends linearly on ion velocities. This approach significantly overestimates cooling rates because the linear approximation is

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valid only for small synchrotron and betatron amplitudes, rendering such models inadequate for practical electron cooler design. In this paper, we present rigorous analytic expressions without the small-amplitude approximation, offering precise evaluations of cooling rates and eliminating the need for intensive numerical simulations.

Building on our previous work involving specific cases [29,30], this paper derives general, explicit formulas for cooling redistribution in nonmagnetized electron coolers. Our main results are given in Eqs. (41)–(44) and Eqs. (48)–(49). We reduce the most general expression to Carlson symmetric forms of elliptic integrals. Because these integrals are available in most modern scientific programming libraries, researchers can immediately obtain numerical results for the cooling rate across any combination of beam parameters. Furthermore, we simplify the cooling rate formulas to elementary functions for a key special case.

The derived formulas were critically important in designing the electron coolers for the EIC, enabling rapid parameter identification and the streamlining of optimization algorithms. Given their utility, these formulas should prove valuable to the accelerator physics community for all stages of design of high-energy electron coolers.

The paper is organized as follows: in Sec. II, we provide a qualitative explanation of the redistribution of cooling rates. In Sec. III, we derive exact analytic expressions for the rate of electron cooling with dispersive redistribution. In Sec. IV, we show that these general formulas coincide with previously derived expressions for special cases when proper approximations are applied. Section V provides examples of applying these formulas to the study and optimization of electron coolers for the EIC. Finally, Sec. VI summarizes the presented material.

## II. SIMPLE EXPLANATION OF COOLING REDISTRIBUTION

The concept of redistributing cooling decrements is illustrated in Fig. 1. It shows an ion with an initial relative momentum offset  $\delta_{i0}$  entering a cooling section with nonzero ion dispersion ( $D_i \neq 0$ ). Inside the cooling section, the ion's momentum is reduced by the longitudinal component of the friction force ( $F_z$ ). At the exit of the

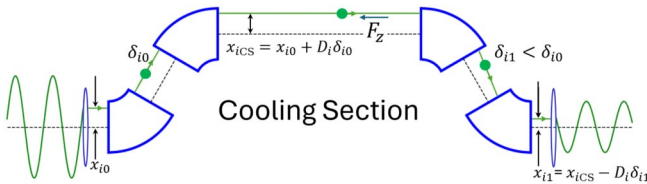


FIG. 1. Schematic illustration of the conversion of longitudinal cooling into an additional reduction in betatron amplitude during a single pass through a cooling section with nonzero ion dispersion.

cooling section, the dispersion is zeroed. Due to the reduced momentum offset ( $\delta_{i1}$ ), an ion with appropriate synchrotron and betatron phases also experiences a reduction in betatron oscillation amplitude.

This “redistributed” reduction in transverse action is in addition to the transverse cooling caused by the transverse component of the friction force. Naturally, this redistribution decreases a transverse momentum for some ions but increases it for others, depending on the ion's synchrotron and betatron phases. If the longitudinal friction force is independent of the transverse coordinate, the average redistribution effect for the whole bunch will zero out. However, by introducing a dependence of  $F_z$  on the transverse coordinate, one can achieve a redistribution of cooling between the longitudinal and transverse phase spaces.

A more rigorous mathematical explanation of the redistribution can be provided using a simple example. Consider a dynamical friction force linearly dependent on an ion's velocity ( $v_i$ ):

$$\begin{aligned} F_x &= -A_x v_{ix} \\ F_z &= -A_z (v_{iz} - \mu_z), \end{aligned} \quad (1)$$

where,  $\mu_z = K \cdot x_i$ ,  $A_x, A_z$  and  $K$  are the coefficients depending on the electron beam parameters only. Such a force corresponds to the cooling experienced by ions with small actions (see Fig. 2 in Sec. III). The term  $\mu_z = Kx_i$  represents the dependence of  $F_z$  on the horizontal position of the ion,  $x_i$ . In all formulas in this paper, both the cooling force and ion velocity are expressed in the beam reference frame.

The change in the ion's velocity during a single pass through the cooling section can be found from Newton's law ( $dp/dt = F$ ):

$$\Delta v_{i(x,y,z)} = \frac{F_{x,y,z} L_{CS}}{m_i \gamma \beta c}, \quad (2)$$

where  $m_i$  is the ion mass,  $L_{CS}$  is the length of the cooling section,  $c$  is the speed of light, and  $\gamma$  and  $\beta$  are the relativistic factors. Substituting Eq. (1) into Eq. (2) and converting the beam-frame velocity  $v_i$  to the ion's relative momentum  $\delta_i$  and angle  $x'_i, y'_i$ :

$$x'_i = \frac{v_{ix}}{\gamma \beta c}, \quad y'_i = \frac{v_{iy}}{\gamma \beta c}, \quad \delta_i = \frac{v_{iz}}{\beta c}, \quad (3)$$

we get for a single pass through the CS:

$$\begin{aligned} \Delta x'_i &= -c_x x'_{i0} \\ \Delta \delta_i &= -c_z (\delta_{i0} - k \cdot x_{iCS}) \end{aligned} \quad (4)$$

where  $c_x = \frac{A_x L_{CS}}{\gamma \beta c m_i}$ ,  $c_z = \frac{A_z L_{CS}}{\gamma \beta c m_i}$ ,  $k = \frac{K}{\beta c}$ , the quantities upstream of the cooling section are denoted with the

index “0,” and  $x_{iCS}$  is an ion’s horizontal coordinate inside the cooling section.

We define the horizontal action of an ion in the CS as

$$J_x = \frac{1}{2} \left( \frac{x_i^2}{\beta_x} + \beta_x x_i'^2 \right), \quad (5)$$

where  $\beta_x$  is the horizontal Twiss  $\beta$  function averaged over the length of the CS; we assume that it does not change significantly throughout the cooling section. We further assume that the Twiss parameter  $\alpha_x = 0$  in the CS. These assumptions are consistent with the practical design of nonmagnetized coolers and simplify the calculations without affecting the final results.

Let us denote the ion’s horizontal coordinate and relative momentum downstream of the CS as  $x_{i1}$  and  $\delta_{i1}$ . Then the following relations hold:

$$\begin{aligned} x_{iCS} &= x_{i0} + D_i \delta_{i0} \\ \delta_{i1} &= \delta_{i0} - c_z (\delta_{i0} - k x_{iCS}) \\ x_{i1} &= x_{iCS} - D_i \delta_{i1}, \end{aligned} \quad (6)$$

and for the change over a pass through the CS, we obtain:

$$\begin{aligned} \Delta x_i &= -c_z D_i k x_{i0} + c_z D_i (1 - D_i k) \delta_{i0} \\ \Delta \delta_i &= -c_z (1 - D_i k) \delta_{i0} + c_z k x_{i0}. \end{aligned} \quad (7)$$

Then, an ion’s horizontal action after a single pass through the CS ( $J_{x1}$ ) is given by

$$\begin{aligned} J_{x1} &= \frac{1}{2} \left( \frac{(x_{i0} + \Delta x_i)^2}{\beta_x} + (x_{i0}' + \Delta x_i')^2 \beta_x \right) \approx \\ &\approx J_{x0} + \frac{x_{i0} \Delta x_i}{\beta_x} + \beta_x x_{i0}' \Delta x_i', \end{aligned} \quad (8)$$

where  $J_{x0} = \frac{1}{2} \left( \frac{x_{i0}^2}{\beta_x} + x_{i0}'^2 \beta_x \right)$ . Finally, substituting Eqs. (4) and (7) into (8), we get the change in the action over a single pass through the CS:

$$\Delta J_x = -c_x \beta_x x_i'^2 - \frac{c_z D_i k}{\beta_x} x_i^2 + \frac{c_z D_i (1 - D_i k)}{\beta_x} \delta_i x_i. \quad (9)$$

Here, we have dropped 0 index for convenience.

Following a similar approach, we introduce the longitudinal action,  $J_z$ :

$$J_z = \frac{1}{2} \left( \frac{z_i^2}{\beta_z} + \beta_z \delta_i^2 \right), \quad (10)$$

where the longitudinal  $\beta$  function  $\beta_z = \sigma_{zi0}/\sigma_{\delta i0}$ , and  $\sigma_{zi0}$  and  $\sigma_{\delta i0}$  are the initial root mean square (rms) longitudinal size and relative momentum spread of an ion bunch, respectively. The change in longitudinal action after a

pass through the CS is

$$\Delta J_z = -c_z (1 - D_i k) \beta_z \delta_i^2 + c_z k \beta_z x_i \delta_i. \quad (11)$$

The change in the ions’ emittances over a pass through the cooling section is

$$\Delta \varepsilon_{x,y,z} = \int \cdots \int \Delta J_{x,y,z} f_i dx_i dy_i dz_i dx_i' dy_i' d\delta_i, \quad (12)$$

where  $f_i = n_i \rho_i$  is the ions’ distribution function. We assume that an ion bunch has a 6D Gaussian distribution:

$$\begin{aligned} n_i &= \frac{1}{(2\pi)^{3/2} \sigma_{xi} \sigma_{yi} \sigma_{zi}} \exp \left( -\frac{x_i^2}{2\sigma_{xi}^2} - \frac{y_i^2}{2\sigma_{yi}^2} - \frac{z_i^2}{2\sigma_{zi}^2} \right), \quad (13) \\ \rho_i &= \frac{1}{(2\pi)^{3/2} \sigma_{\theta ix} \sigma_{\theta iy} \sigma_{\delta i}} \exp \left( -\frac{x_i'^2}{2\sigma_{\theta ix}^2} - \frac{y_i'^2}{2\sigma_{\theta iy}^2} - \frac{\delta_i^2}{2\sigma_{\delta i}^2} \right), \end{aligned} \quad (14)$$

where  $\sigma_{\theta ix}$  and  $\sigma_{\theta iy}$  are the rms horizontal and vertical angular spreads, and  $\sigma_{\delta i}$  is the rms relative momentum spread of an ion bunch.

Equation (12) gives

$$\begin{aligned} \frac{\Delta \varepsilon_x}{\varepsilon_x} &= -c_x - c_z D_i k \\ \frac{\Delta \varepsilon_z}{\varepsilon_z} &= -c_z + c_z D_i k. \end{aligned} \quad (15)$$

For the “small amplitudes cooling force” given by Eq. (1), the cooling rate ( $\lambda$ ) in the laboratory reference frame can be found similarly to Eq. (2):

$$\lambda_{x,y,z} = \frac{F_{x,y,z} \eta}{\gamma m_i v_{i(x,y,z)}}, \quad (16)$$

where the duty factor  $\eta = L_{CS}/C_R$  and  $C_R$  is the storage ring circumference. On the other hand, by definition, the cooling rate is

$$\lambda_{x,y,z} = \frac{1}{\varepsilon_{x,y,z}} \frac{d\varepsilon_{x,y,z}}{dt} = \frac{1}{T_{rev}} \frac{\Delta \varepsilon_{x,y,z}}{\varepsilon_{x,y,z}}, \quad (17)$$

where the revolution period  $T_{rev} = \frac{C_R}{\beta c}$ . Substituting Eq. (1) into (16) and substituting the expressions for  $c_x$ ,  $c_z$ , and  $k$  into Eq. (15), then dividing both sides of it by  $T_{rev}$ , we get:

$$\begin{aligned} \lambda_x &= \lambda_{x(D_i=0)} + D_i k \lambda_{z(D_i=0)} \\ \lambda_z &= \lambda_{z(D_i=0)} - D_i k \lambda_{x(D_i=0)}, \end{aligned} \quad (18)$$

where  $\lambda_{x(D_i=0)}$  and  $\lambda_{z(D_i=0)}$  are the horizontal and longitudinal cooling rates at  $D_i = 0$ .

As one can see, net nonzero redistribution occurs only when  $k = K/(\beta c) \neq 0$ ; that is, when the longitudinal component of the cooling force depends on the transverse (in our case, the horizontal) coordinate.

There are various ways to provide a dependence of  $F_z$  on the transverse coordinate [28]. The most robust method, and the focus of this paper, is the introduction of electron beam dispersion into the cooling section (see, for example, [26,29,30]). Furthermore, it was recently shown [27] that even without electron dispersion, a dependence leading to cooling redistribution can arise from a nonuniformity of the electron beam's transverse distribution. This finding is significant for our analysis, as rf-based nonmagnetized electron coolers can have beams with substantially non-uniform transverse profiles.

In the following chapter, we will derive exact, practical formulas for longitudinal-to-transverse cooling redistribution in nonmagnetized electron coolers in the presence of ion and electron dispersion in the cooling section.

### III. EXACT FORMULAS FOR ELECTRON COOLING WITH $z-x$ REDISTRIBUTION

In this section, we assume the approximation of binary collisions. Furthermore, we assume that both electron and ion bunches have a Maxwell-Boltzmann velocity distribution and a Gaussian spatial distribution. Finally, we assume a constant Coulomb logarithm. A more detailed discussion of these assumptions and the justification for their choice is provided throughout the section.

#### A. Friction force

The friction force acting on an ion co-traveling with an electron bunch is given by [23,31]:

$$\vec{F}(\vec{v}_i) = -\frac{4\pi N_e e^4 Z^2}{m_e} \int \frac{\vec{v}_i - \vec{v}_e}{|\vec{v}_i - \vec{v}_e|^3} \Lambda_C f_{e6} d^3 v_e. \quad (19)$$

Here,  $N_e$  is the number of electrons per bunch,  $e$  is the elementary charge,  $Z \cdot e$  is the ion charge, and  $m_e$  is the electron mass. The terms  $\vec{v}_i$  and  $\vec{v}_e$  denote the ion and electron velocities in the beam frame, respectively, and  $f_{e6}$  is the six-dimensional electron distribution function. The Coulomb logarithm is defined as  $\Lambda_C = \ln(\frac{\rho_{\max}}{\rho_{\min}})$ , where the minimum impact parameter is  $\rho_{\min} = \frac{Ze^2}{m_e u^2}$ , the maximum impact parameter is  $\rho_{\max} = \min[R_{e\perp}, \frac{u L_{CS}}{\gamma \beta c}, \frac{u}{\Omega_e}]$ ,  $u \equiv |\vec{v}_i - \vec{v}_e|$ ,  $R_{e\perp}$  is the characteristic transverse size of the electron bunch, and  $\Omega_e$  is the electron Langmuir frequency. The Coulomb logarithm has a weak dependence on velocity and can thus be moved outside the integral by substituting  $u$  with its rms value.

Assuming a Maxwell-Boltzmann velocity distribution and nonzero horizontal electron dispersion ( $D_e$ ) in the

cooling section,  $f_{e6}$  can be expressed as

$$f_{e6} = \frac{1}{(2\pi)^3 \sigma_{xe} \sigma_{ye} (\gamma \sigma_{ze}) \sigma_{vex} \sigma_{vey} \sigma_{vez}} \times \exp\left(-\frac{(x - D_e \frac{v_{ex}}{\beta c})^2}{2\sigma_{xe}^2} - \frac{y^2}{2\sigma_{ye}^2} - \frac{z^2}{2\sigma_{ze}^2}\right) \times \exp\left(-\frac{v_{ex}^2}{2\sigma_{vex}^2} - \frac{v_{ey}^2}{2\sigma_{vey}^2} - \frac{v_{ez}^2}{2\sigma_{vez}^2}\right), \quad (20)$$

where  $\sigma_{xe}$ ,  $\sigma_{ye}$ , and  $\gamma \sigma_{ze}$  are the horizontal, vertical, and longitudinal rms sizes of the electron bunch in the beam frame;  $\sigma_{vex}$ ,  $\sigma_{vey}$ , and  $\sigma_{vez}$  are rms electrons' velocities in the beam frame; and  $\frac{v_{ex}}{\beta c} = \delta_e$  represents the relative momentum of an electron. To simplify the subsequent calculations, we assume in Eq. (20) that the e-bunch's average Twiss parameter  $\alpha_x = 0$  in the CS, consistent with the assumptions made for the ions. These assumptions are correct for practical cooler designs.

The distribution in (20) can be rewritten as the product of spatial and velocity distributions:

$$f_{e6} = n_e f_{ve} \\ n_e = \frac{\exp\left(-\frac{x^2}{2\sigma_{xe1}^2} - \frac{y^2}{2\sigma_{ye}^2} - \frac{z^2}{2\sigma_{ze}^2}\right)}{\gamma (2\pi)^{3/2} \sigma_{xe1} \sigma_{ye} \sigma_{ze}} \\ f_{ve} = \frac{\exp\left(-\frac{v_{ex}^2}{2\sigma_{vex}^2} - \frac{v_{ey}^2}{2\sigma_{vey}^2} - \frac{(v_{ez} - \mu_z)^2}{2\sigma_{vez1}^2}\right)}{(2\pi)^{3/2} \sigma_{vex} \sigma_{vey} \sigma_{vez1}} \\ \sigma_{xe1} = \sqrt{\sigma_{ex}^2 + D_e^2 \sigma_{\delta e}^2} \\ \sigma_{vez1} = \frac{\sigma_{ex}}{\sqrt{\sigma_{ex}^2 + D_e^2 \sigma_{\delta e}^2}} \sigma_{vez} \\ \mu_z = x \frac{D_e \sigma_{\delta e}}{\sigma_{ex}^2 + D_e^2 \sigma_{\delta e}^2} \sigma_{vez}. \quad (21)$$

The spatial distribution  $n_e$  can then be moved outside the integral in Eq. (19):

$$\vec{F}(\vec{v}_i) = -A_0 n_e \int \frac{\vec{v}_i - \vec{v}_e}{|\vec{v}_i - \vec{v}_e|^3} f_{ve} d^3 v_e, \quad (22)$$

where  $A_0 = 4\pi N_e e^4 Z^2 \Lambda_C / m_e$ .

Next, we introduce an effective potential in velocity space:

$$U = A_0 n_e \int \frac{f_{ve}}{|\vec{v}_i - \vec{v}_e|} d^3 v_e, \quad (23)$$

such that

$$F_{x,y,z} = \partial U / \partial v_{i(x,y,z)}. \quad (24)$$

Using the representation for  $1/|\vec{v}_i - \vec{v}_e|$ :

$$\begin{aligned} & \frac{1}{\sqrt{(v_{ix} - v_{ex})^2 + (v_{iy} - v_{ey})^2 + (v_{iz} - v_{ez})^2}} \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-p^2((v_{ix}-v_{ex})^2 + (v_{iy}-v_{ey})^2 + (v_{iz}-v_{ez})^2)} dp, \end{aligned} \quad (25)$$

we obtain from Eqs. (22)–(25):

$$\begin{aligned} U &= \frac{A_0 n_e}{\sqrt{2\pi^2} \sigma_{vex} \sigma_{vey} \sigma_{vez1}} \int_0^\infty \left[ \int_{-\infty}^\infty e^{-p^2(v_{ix}-v_{ex})^2 - \frac{v_{ex}^2}{2\sigma_{vex}^2}} dv_{ex} \right. \\ &\quad \times \int_{-\infty}^\infty e^{-p^2(v_{iy}-v_{ey})^2 - \frac{v_{ey}^2}{2\sigma_{vey}^2}} dv_{ey} \\ &\quad \left. \times \int_{-\infty}^\infty e^{-p^2(v_{iz}-v_{ez})^2 - \frac{(v_{ez}-\mu_z)^2}{2\sigma_{vez1}^2}} dv_{ez} \right] dp. \end{aligned} \quad (26)$$

Since:

$$\begin{aligned} \int_{-\infty}^\infty e^{-p^2(v_{ix}-v_{ex})^2 - \frac{v_{ex}^2}{2\sigma_{vex}^2}} dv_{ex} &= \frac{\sqrt{2\pi} \sigma_{vex} \cdot e^{-\frac{p^2 v_{ix}^2}{1+2p^2 \sigma_{vex}^2}}}{\sqrt{1+2p^2 \sigma_{vex}^2}} \\ \int_{-\infty}^\infty e^{-p^2(v_{iy}-v_{ey})^2 - \frac{v_{ey}^2}{2\sigma_{vey}^2}} dv_{ey} &= \frac{\sqrt{2\pi} \sigma_{vey} \cdot e^{-\frac{p^2 v_{iy}^2}{1+2p^2 \sigma_{vey}^2}}}{\sqrt{1+2p^2 \sigma_{vey}^2}} \\ \int_{-\infty}^\infty e^{-p^2(v_{iz}-v_{ez})^2 - \frac{(v_{ez}-\mu_z)^2}{2\sigma_{vez1}^2}} dv_{ez} &= \frac{\sqrt{2\pi} \sigma_{vez1} \cdot e^{-\frac{p^2 (v_{iz}-\mu_z)^2}{1+2p^2 \sigma_{vez1}^2}}}{\sqrt{1+2p^2 \sigma_{vez1}^2}} \end{aligned} \quad (27)$$

the effective potential is given by

$$\begin{aligned} U &= \frac{2A_0 n_e}{\sqrt{\pi}} \\ &\quad \times \int_0^\infty \frac{\exp\left(-\frac{p^2 v_{ix}^2}{1+2p^2 \sigma_{vex}^2} - \frac{p^2 v_{iy}^2}{1+2p^2 \sigma_{vey}^2} - \frac{p^2 (v_{iz}-\mu_z)^2}{1+2p^2 \sigma_{vez1}^2}\right)}{\sqrt{(1+2p^2 \sigma_{vex}^2)(1+2p^2 \sigma_{vey}^2)(1+2p^2 \sigma_{vez1}^2)}} dp \end{aligned} \quad (28)$$

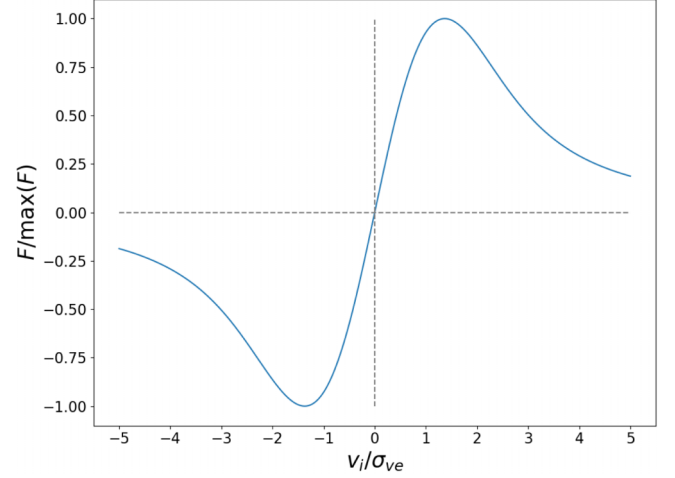


FIG. 2. Friction force as a function of ion velocity. The force is normalized to its maximum value, and the ion velocity is normalized to the electron velocity spread.

Finally, from Eqs. (24) and (28) we obtain:

$$\begin{aligned} F_{x,y} &= -\frac{4A_0 n_e}{\sqrt{\pi}} \int_0^\infty \frac{p^2 v_{i(x,y)}}{1+2p^2 \sigma_{ve(x,y)}^2} \mathcal{G}(p) dp \\ F_z &= -\frac{4A_0 n_e}{\sqrt{\pi}} \int_0^\infty \frac{p^2 (v_{iz} - \mu_z)}{1+2p^2 \sigma_{vez1}^2} \mathcal{G}(p) dp \\ \mathcal{G}(p) &= \frac{\exp\left(-\frac{p^2 v_{ix}^2}{1+2p^2 \sigma_{vex}^2} - \frac{p^2 v_{iy}^2}{1+2p^2 \sigma_{vey}^2} - \frac{p^2 (v_{iz}-\mu_z)^2}{1+2p^2 \sigma_{vez1}^2}\right)}{\sqrt{(1+2p^2 \sigma_{vex}^2)(1+2p^2 \sigma_{vey}^2)(1+2p^2 \sigma_{vez1}^2)}} \end{aligned} \quad (29)$$

Equations (29) coincide with Binney's formulas [32] if one sets  $\mu_z = 0$  and performs the variable substitution  $p^2 = 1/(2q\sigma_{ve(x,y,z)}^2)$  in the respective integrals.

Figure 2 shows how the friction force depends on ion velocity for the simple case of  $\mu_z = 0$  and  $\sigma_{vex} = \sigma_{vey} = \sigma_{vez} \equiv \sigma_{ve}$ . As discussed above, the linear approximation (1) is applicable only in a narrow range where  $v_i \ll \sigma_{ve}$ .

The friction force formulas in the form of Eq. (29) with  $\mu_z \neq 0$  were first derived in [33]. It is notable that while  $\mu_z$  dependent on  $x$  produces a  $z-x$  redistribution of the cooling rates, the case where  $\mu_z$  is independent of the coordinate creates a circular attractor in the longitudinal phase space [15,34–36].

## B. Cooling rates

For a single pass through the CS, from Eqs. (2) and (3), we obtain:

$$\begin{aligned}\Delta x'_i &= \frac{L_{CS}}{m_i(\gamma\beta c)^2} F_x & \Delta J_y &= y'_i \beta_y \frac{L_{CS}}{m_i(\gamma\beta c)^2} F_y \\ \Delta y'_i &= \frac{L_{CS}}{m_i(\gamma\beta c)^2} F_y & \text{And for the longitudinal direction:} \\ \Delta \delta_i &= \frac{L_{CS}}{m_i \gamma (\beta c)^2} F_z. & \Delta J_z &= \delta_i \beta_z \frac{L_{CS}}{m_i \gamma (\beta c)^2} F_z\end{aligned}\quad (30)$$

Then, from Eq. (6), we obtain:

$$\Delta x_i = -D_i \Delta \delta_i = -\frac{D_i L_{CS}}{m_i \gamma (\beta c)^2} F_z. \quad (31)$$

The change in an ion's horizontal action over a pass through the CS then becomes:

$$\Delta J_x = x'_i \beta_x \frac{L_{CS}}{m_i(\gamma\beta c)^2} F_x - x_i \frac{D_i}{\beta_x} \frac{L_{CS}}{m_i \gamma (\beta c)^2} F_z. \quad (32)$$

We have dropped the index 0 here for convenience. Similarly, for the change in vertical action, we get:

Next, we note that an ion “probes” the electron bunch density at the location  $x_{iCS} = x_i + D_i \delta_i$ . Therefore, the proper expression for the electron density in the following formulas is:

$$n_e = \frac{\exp\left(-\frac{(x_i + D_i \delta_i)^2}{2\sigma_{xe1}^2} - \frac{y_i^2}{2\sigma_{ye}^2} - \frac{z_i^2}{2\sigma_{ze}^2}\right)}{\gamma(2\pi)^{3/2} \sigma_{xe1} \sigma_{ye} \sigma_{ze}} \quad (35)$$

Finally, we switch  $x'_i$ ,  $y'_i$  and  $\delta_i$  in Eq. (12) back to the beam-frame ion velocities and introduce new variables  $d_e = \frac{D_e}{\beta c}$ ,  $d_i = \frac{D_i}{\beta c}$ , and  $\nu = \frac{\sigma_{ve z}^2}{\sigma_{xe1}^2}$  to simplify the formulas, and obtain:

$$\begin{aligned}\Delta \varepsilon_{x,y,z} &= \int \cdots \int \Delta J_{x,y,z} f_i dx_i dy_i dz_i dx'_i dy'_i d\delta_i = \\ &= -B_0 \int_0^\infty \frac{\mathcal{S}_{(x,y,z)} p^2}{\sqrt{(1+2p^2\sigma_{vex}^2)(1+2p^2\sigma_{vey}^2)(1+2p^2\sigma_{vez1}^2)}} dp\end{aligned}\quad (36)$$

where

$$B_0 = \frac{A_0 L_{CS}}{4\sqrt{2}\pi^5 m_i \gamma^2 \beta c \sigma_{xe1} \sigma_{ye} \sigma_{ze} \sigma_{xi} \sigma_{yi} \sigma_{zi} \sigma_{vix} \sigma_{viy} \sigma_{viz}} \quad (37)$$

and

$$\begin{aligned}\mathcal{S}_x &= \left( \frac{\beta_x \mathcal{I}_{2v_x} \mathcal{P}}{(\gamma\beta c)^2 (1+2p^2\sigma_{vex}^2)} - \frac{\mathcal{I}_{0v_x} \mathcal{Q}}{\beta_x (1+2p^2\sigma_{vez1}^2)} \right) \mathcal{I}_{0v_y} \mathcal{I}_{0y} \mathcal{I}_{0z} \\ \mathcal{S}_y &= \frac{\beta_y}{(\gamma\beta c)^2 (1+2p^2\sigma_{vey}^2)} \mathcal{I}_{0v_x} \mathcal{I}_{2v_y} \mathcal{I}_{0y} \mathcal{I}_{0z} \mathcal{P} \\ \mathcal{S}_z &= \frac{\beta_z}{(\beta c)^2 (1+2p^2\sigma_{vez}^2)} \mathcal{I}_{2v_x} \mathcal{I}_{2v_y} \mathcal{I}_{0y} \mathcal{I}_{0z} \mathcal{R}\end{aligned}\quad (38)$$

The Gaussian-like integrals  $\mathcal{P}$ ,  $\mathcal{Q}$ , and  $\mathcal{R}$ , as well as the zeroth- and second-order Gaussian integrals  $\mathcal{I}_0$  and  $\mathcal{I}_2$ , are defined by Eqs. (39) and (40). The subscripts of the Gaussian integrals identify their respective integration variables. For example,  $\mathcal{I}_{2v_x}$  denotes the second-order Gaussian integral over the variable  $v_{ix}$

$$\begin{aligned}
\mathcal{P} &= \iint_{-\infty}^{\infty} e^{-\left(\frac{(x_i+d_i v_{iz})^2}{2\sigma_{xe1}^2} + \frac{x_i^2}{2\sigma_{xi}^2} + p^2 \frac{(v_{iz}(1-\nu d_e d_i) - x_i \nu d_e)^2}{1+2p^2\sigma_{vez1}^2} + \frac{v_{iz}^2}{2\sigma_{viz}^2}\right)} dx_i dv_{iz} \\
&= \frac{2\pi\sigma_{xe1}\sigma_{x_i}\sigma_{viz}\sqrt{1+2p^2\sigma_{vez1}^2}}{\sqrt{\sigma_{xe1}^2 + \sigma_{xi}^2 + d_i^2\sigma_{viz}^2}\sqrt{1+2p^2\left(\sigma_{vez}^2 + \sigma_{viz}^2 - \frac{(d_e\sigma_{vez} + d_i\sigma_{viz})^2}{\sigma_{xe1}^2 + \sigma_{xi}^2 + d_i^2\sigma_{viz}^2}\right)}} \\
\mathcal{Q} &= \iint_{-\infty}^{\infty} x_i d_i (v_{iz} - x_i d_e \nu - v_{iz} \nu d_e d_i) e^{-\left(\frac{(x_i+d_i v_{iz})^2}{2\sigma_{xe1}^2} + \frac{x_i^2}{2\sigma_{xi}^2} + p^2 \frac{(v_{iz}(1-\nu d_e d_i) - x_i \nu d_e)^2}{1+2p^2\sigma_{vez1}^2} + \frac{v_{iz}^2}{2\sigma_{viz}^2}\right)} dx_i dv_{iz} \\
&= -\frac{2\pi\sigma_{xe1}\sigma_{x_i}^3\sigma_{viz}(1+2p^2\sigma_{vez1}^2)^{3/2}d_i(d_e\sigma_{vez} + d_i\sigma_{viz})}{(\sigma_{xe1}^2 + \sigma_{xi}^2 + d_i^2\sigma_{viz}^2)^{3/2}\left[1+2p^2\left(\sigma_{vez}^2 + \sigma_{viz}^2 - \frac{(d_e\sigma_{vez} + d_i\sigma_{viz})^2}{\sigma_{xe1}^2 + \sigma_{xi}^2 + d_i^2\sigma_{viz}^2}\right)\right]^{3/2}} \\
\mathcal{R} &= \iint_{-\infty}^{\infty} v_{iz}(v_{iz}(1-d_e d_i \nu) - x_i d_e \nu) e^{-\left(\frac{(x_i+d_i v_{iz})^2}{2\sigma_{xe1}^2} + \frac{x_i^2}{2\sigma_{xi}^2} + p^2 \frac{(v_{iz}(1-\nu d_e d_i) - x_i \nu d_e)^2}{1+2p^2\sigma_{vez1}^2} + \frac{v_{iz}^2}{2\sigma_{viz}^2}\right)} dx_i dv_{iz} \\
&= -\frac{2\pi\sigma_{xe1}\sigma_{x_i}\sigma_{viz}^3(1+2p^2\sigma_{vez1}^2)^{3/2}\left(1 - \frac{d_i^2\sigma_{viz}^2 + d_e d_i \sigma_{vez}}{\sigma_{xe1}^2 + \sigma_{xi}^2 + d_i^2\sigma_{viz}^2}\right)}{\sqrt{\sigma_{xe1}^2 + \sigma_{xi}^2 + d_i^2\sigma_{viz}^2}\left[1+2p^2\left(\sigma_{vez}^2 + \sigma_{viz}^2 - \frac{(d_e\sigma_{vez} + d_i\sigma_{viz})^2}{\sigma_{xe1}^2 + \sigma_{xi}^2 + d_i^2\sigma_{viz}^2}\right)\right]^{3/2}} \quad (39)
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{0v_x} &= \int_{-\infty}^{\infty} e^{-\left(\frac{v_{ix}^2}{2\sigma_{vix}^2} + \frac{v_{ix}^2 p^2}{1+2p^2\sigma_{vex}^2}\right)} dv_{ix} = \frac{\sqrt{2\pi}\sigma_{vix}\sqrt{1+2p^2\sigma_{vex}^2}}{\sqrt{1+2p^2(\sigma_{vex}^2 + \sigma_{vix}^2)}} \\
\mathcal{I}_{2v_x} &= \int_{-\infty}^{\infty} v_{ix}^2 e^{-\left(\frac{v_{ix}^2}{2\sigma_{vix}^2} + \frac{v_{ix}^2 p^2}{1+2p^2\sigma_{vex}^2}\right)} dv_{ix} = \frac{\sqrt{2\pi}\sigma_{vix}^3(1+2p^2\sigma_{vex}^2)^{3/2}}{[1+2p^2(\sigma_{vex}^2 + \sigma_{vix}^2)]^{3/2}} \\
\mathcal{I}_{0v_y} &= \int_{-\infty}^{\infty} e^{-\left(\frac{v_{iy}^2}{2\sigma_{viy}^2} + \frac{v_{iy}^2 p^2}{1+2p^2\sigma_{vey}^2}\right)} dv_{iy} = \frac{\sqrt{2\pi}\sigma_{viy}\sqrt{1+2p^2\sigma_{vey}^2}}{\sqrt{1+2p^2(\sigma_{vey}^2 + \sigma_{viy}^2)}} \\
\mathcal{I}_{2v_y} &= \int_{-\infty}^{\infty} v_{iy}^2 e^{-\left(\frac{v_{iy}^2}{2\sigma_{viy}^2} + \frac{v_{iy}^2 p^2}{1+2p^2\sigma_{vey}^2}\right)} dv_{iy} = \frac{\sqrt{2\pi}\sigma_{viy}^3(1+2p^2\sigma_{vey}^2)^{3/2}}{[1+2p^2(\sigma_{vey}^2 + \sigma_{viy}^2)]^{3/2}} \\
\mathcal{I}_{0y} &= \int_{-\infty}^{\infty} e^{-\frac{y_i^2}{2}\left(\frac{1}{2\sigma_{yi}^2} + \frac{1}{2\sigma_{ye}^2}\right)} dy_i = \frac{\sqrt{2\pi}\sigma_{ye}\sigma_{yi}}{\sqrt{\sigma_{ye}^2 + \sigma_{yi}^2}} \\
\mathcal{I}_{0z} &= \int_{-\infty}^{\infty} e^{-\frac{z_i^2}{2}\left(\frac{1}{2\sigma_{zi}^2} + \frac{1}{2\sigma_{ze}^2}\right)} dz_i = \frac{\sqrt{2\pi}\sigma_{ze}\sigma_{zi}}{\sqrt{\sigma_{ze}^2 + \sigma_{zi}^2}} \quad (40)
\end{aligned}$$

Next, we substitute Eqs. (37)–(40) into Eq. (36), change the integration variable to  $q = \frac{1}{p^2(\beta c)^2}$ , and substitute the following:  $\sigma_{vex} = \gamma\beta c\sigma_{\theta ex}$ ,  $\sigma_{vey} = \gamma\beta c\sigma_{\theta ey}$ ,  $\sigma_{vix} = \gamma\beta c\sigma_{\theta ix}$ ,  $\sigma_{viy} = \gamma\beta c\sigma_{\theta iy}$ ,  $\sigma_{vez} = \beta c\sigma_{\delta e}$ , and  $\sigma_{viz} = \beta c\sigma_{\delta i}$ . Then, in accordance with Eq. (17), we obtain:

$$\begin{aligned}\lambda_x &= -\frac{C_0}{\sigma_x\sigma_y\sigma_z}(\Psi_x + \kappa\Psi_z) \\ \lambda_y &= -\frac{C_0}{\sigma_x\sigma_y\sigma_z}\Psi_y \\ \lambda_z &= -\frac{C_0}{\sigma_x\sigma_y\sigma_z}(\Psi_z - \kappa\Psi_x) \\ \Psi_x &= R_D((\gamma\sigma_{\theta y})^2, \sigma_{\delta}^2, (\gamma\sigma_{\theta x})^2) \\ \Psi_y &= R_D((\gamma\sigma_{\theta x})^2, \sigma_{\delta}^2, (\gamma\sigma_{\theta y})^2) \\ \Psi_z &= R_D((\gamma\sigma_{\theta x})^2, (\gamma\sigma_{\theta y})^2, \sigma_{\delta}^2).\end{aligned}\quad (41)$$

Here,  $C_0 = \frac{2cr_e r_i}{3\pi} \frac{\eta N_e \Lambda_c}{\gamma^2 \beta^3}$ ,  $r_e = \frac{e^2}{m_e c^2}$  and  $r_i = \frac{e^2 Z_i^2}{m_i c^2}$  are the classical electron and ion radii, respectively, coefficient  $\kappa$  is given by

$$\kappa = \frac{D_e D_i \sigma_{\delta e}^2 + D_i^2 \sigma_{\delta i}^2}{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2 + \sigma_{xi}^2 + D_i^2 \sigma_{\delta i}^2} \quad (42)$$

and effective “composite spreads” are given by

$$\begin{aligned}\sigma_x &= \sqrt{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2 + \sigma_{xi}^2 + D_i^2 \sigma_{\delta i}^2} \\ \sigma_y &= \sqrt{\sigma_{ye}^2 + \sigma_{yi}^2} \\ \sigma_z &= \sqrt{\sigma_{ze}^2 + \sigma_{zi}^2} \\ \sigma_{\theta x} &= \sqrt{\sigma_{\theta xe}^2 + \sigma_{\theta xi}^2} \\ \sigma_{\theta y} &= \sqrt{\sigma_{\theta ye}^2 + \sigma_{\theta yi}^2} \\ \sigma_{\delta} &= \sqrt{\sigma_{\delta e}^2 + \sigma_{\delta i}^2 - \frac{(D_e \sigma_{\delta e}^2 + D_i \sigma_{\delta i}^2)^2}{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2 + \sigma_{xi}^2 + D_i^2 \sigma_{\delta i}^2}}.\end{aligned}\quad (43)$$

The function  $R_D$  is a degenerate case of the Carlson symmetric elliptic integral of the third kind:

$$R_D(u_1, u_2, u_3) = \frac{3}{2} \int_0^\infty \frac{dq}{\sqrt{(q+u_1)(q+u_2)(q+u_3)^3}}. \quad (44)$$

It is readily available in numerous scientific programming libraries. For example, in PYTHON [37], it is implemented as “`scipy.special.elliprd`,” and in *Mathematica* [38], it is named “`CarlsonRD`.”

Note that in the formula for  $\sigma_{\delta}$ , the radicand is always positive, as shown by

$$\begin{aligned}\sigma_{\delta e}^2 + \sigma_{\delta i}^2 - \frac{(D_e \sigma_{\delta e}^2 + D_i \sigma_{\delta i}^2)^2}{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2 + \sigma_{xi}^2 + D_i^2 \sigma_{\delta i}^2} &= \\ &= \frac{(\sigma_{\delta e}^2 + \sigma_{\delta i}^2)(\sigma_{xe}^2 + \sigma_{xi}^2) + \sigma_{\delta e}^2 \sigma_{\delta i}^2 (D_e - D_i)^2}{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2 + \sigma_{xi}^2 + D_i^2 \sigma_{\delta i}^2}.\end{aligned}\quad (45)$$

Another important point is that the coefficient  $\kappa$  must not be treated as a “redistribution coefficient.” That is, one cannot simply calculate cooling rates in the absence of dispersion,  $\lambda_{x0}$  and  $\lambda_{z0}$ , and then assume that the redistributed rates will be  $\lambda_x = \lambda_{x0} + \kappa\lambda_{z0}$  and  $\lambda_z = (1 - \kappa)\lambda_{z0}$ . Equations (43) and (45) clearly show that this is not the case, since  $\sigma_{\delta}$ , and consequently functions  $\Psi$  in Eq. (41), are functions of  $D_e$  and  $D_i$ .

Equations (41)–(44) provide general closed-form expressions for cooling rates with dispersive redistribution. These exact formulas yield the redistributed cooling rates for any possible combination of electron and ion beam parameters. These formulas are expressed in terms of tabulated special functions available via modern scientific programming libraries and do not require any additional numerical integration.

In the following section, we will consider various special cases that further simplify the expressions for the redistributed cooling rates.

## IV. SPECIAL CASES AND COMPARISON WITH PREVIOUS RESULTS

### A. Symmetric angular distribution

The integrals  $R_D$  given by Eq. (44) can be evaluated analytically in an important case of:

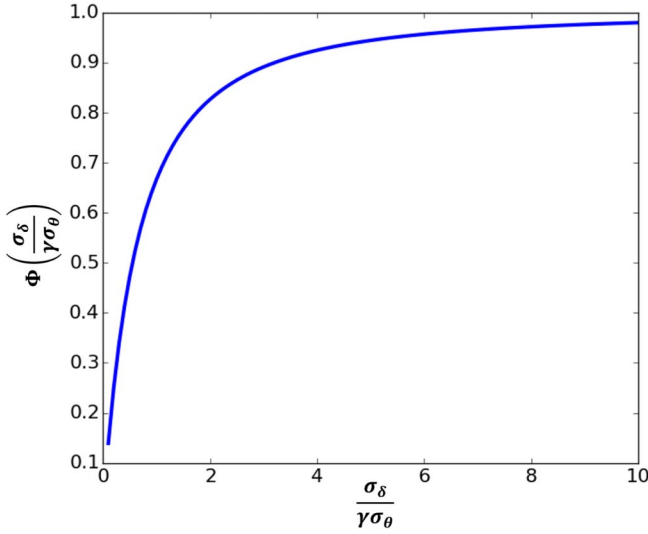
$$\sigma_{\theta} \equiv \sqrt{\sigma_{\theta xe}^2 + \sigma_{\theta xi}^2} = \sqrt{\sigma_{\theta ye}^2 + \sigma_{\theta yi}^2}. \quad (46)$$

A clear example of such a case occurs when both the electron and ion beams have symmetric angular distributions:

$$\sigma_{\theta xe} = \sigma_{\theta ye} \equiv \sigma_{\theta e}; \quad \sigma_{\theta xi} = \sigma_{\theta yi} \equiv \sigma_{\theta i}. \quad (47)$$

The cooling rates then become:

$$\begin{aligned}\lambda_x &= \lambda_{\perp} + \kappa\lambda_{\parallel} \\ \lambda_y &= \lambda_{\perp} \\ \lambda_z &= \lambda_{\parallel} - \kappa\lambda_{\parallel} \\ \lambda_{\perp} &= -\frac{C_1}{\sigma_x\sigma_y\sigma_z(\gamma\sigma_{\theta})^2\sigma_{\delta}}\Phi\left(\frac{\sigma_{\delta}}{\gamma\sigma_{\theta}}\right) \\ \lambda_{\parallel} &= -\frac{2C_1}{\sigma_x\sigma_y\sigma_z(\gamma\sigma_{\theta})^2\sigma_{\delta}}\left[1 - \Phi\left(\frac{\sigma_{\delta}}{\gamma\sigma_{\theta}}\right)\right].\end{aligned}\quad (48)$$

FIG. 3. Function  $\Phi(\frac{\sigma_\delta}{\gamma\sigma_\theta})$ .

Here,  $C_1 = \frac{cr_e r_i \eta N_e \Lambda_C}{\pi \gamma^2 \beta^3}$ ,  $\kappa$  is given by Eq. (42),  $\sigma_\theta$  is given by Eq. (46), and the remaining composite parameters are defined in Eq. (43).

The function  $\Phi$  is defined as

$$\Phi(r) = \begin{cases} \frac{r}{1-r^2} \left( \frac{\arccos(r)}{\sqrt{1-r^2}} - r \right), & r < 1 \\ \frac{2}{3}, & r = 1 \\ \frac{r}{r^2-1} \left( \frac{\log(r-\sqrt{r^2-1})}{\sqrt{r^2-1}} + r \right), & r > 1 \end{cases} \quad (49)$$

and is plotted in Fig. 3.

Equations (48) and (49) provide an exact solution for the redistributed cooling rates in the case of beams with symmetric angular spreads. These results can be compared to the approximate formulas derived in Ref. [27] under the assumption of a linear dependence of the friction force on the ion's velocity. As can be seen, the coefficient  $\kappa$  is identical in both the exact and approximate formulas. However, the approximate formulas fail to capture the strong nonlinear dependence of  $\lambda_\perp$  and  $\lambda_\parallel$  on the ion bunch parameters introduced via the function  $\Phi(\frac{\sigma_\delta}{\gamma\sigma_\theta})$ .

## B. Redistribution formulas for small-amplitude ions

In previous works, the redistribution formulas were derived under the assumption of a linear dependence of the friction force on the ion's velocity, which is correct only for small-amplitude ions:  $v_{i(x,y,z)} \ll \sigma_{ve(x,y,z)}$ . Another standard assumption was that the electron velocity distribution is transversely symmetrical, with:

$$\sigma_{vex} = \sigma_{vey} \equiv \sigma_{ve\perp} = \gamma\beta c \sigma_{\theta e}, \quad (50)$$

where  $\sigma_{\theta e}$  is defined in Eq. (47).

For small amplitudes, the function  $\mathcal{G}(p)$  in Eq. (29) can be approximated by

$$\mathcal{G}(p) \approx \frac{1}{\sqrt{(1+2p^2\sigma_{vex}^2)(1+2p^2\sigma_{vey}^2)(1+2p^2\sigma_{vez1}^2)}}. \quad (51)$$

Note that we also assume  $\mu_z \ll \sigma_{vez1}^2$ , which is equivalent to  $x_i \ll \sigma_{xe}$  for the case of non-negligible redistribution requiring  $D_e \sigma_{\delta e}$  to be comparable to  $\sigma_{xe}$ . Without this assumption, the friction force cannot be represented as a linear function of ion velocity with a coefficient depending solely on electron beam parameters.

Then, integrals (29) can be evaluated analytically:

$$\begin{aligned} F_{s.a.(x,y)} &= -\frac{A_0 n_e}{\sqrt{2\pi}} \frac{v_{i(x,y)}}{\sigma_{ve\perp}^2 \sigma_{vez1}} \Phi\left(\frac{\sigma_{vez1}}{\sigma_{ve\perp}}\right) \\ F_{s.a.(z)} &= -\frac{\sqrt{2} A_0 n_e}{\sqrt{\pi}} \frac{(v_{iz} - \mu_z)}{\sigma_{ve\perp}^2 \sigma_{vez1}} \left[ 1 - \Phi\left(\frac{\sigma_{vez1}}{\sigma_{ve\perp}}\right) \right]. \end{aligned} \quad (52)$$

Here, the index “s.a.” emphasizes that the formulas are for the case of ions with small amplitudes.

Finally, we follow the procedure in Sec. III B arriving at the expression for redistributed cooling rates for small-amplitude ions ( $\lambda_{s.a.(x,y,z)}$ ):

$$\begin{aligned} \lambda_{s.a.(x)} &= \lambda_{s.a.\perp} + \kappa_{s.a.} \lambda_{s.a.\parallel} \\ \lambda_{s.a.(y)} &= \lambda_{s.a.\perp} \\ \lambda_{s.a.(z)} &= \lambda_{s.a.\parallel} - \kappa_{s.a.} \lambda_{s.a.\parallel} \\ \lambda_{s.a.\perp} &= -\frac{C_1}{\sigma_{xe} \sigma_{ye} \sigma_{ze} (\gamma \sigma_{\theta e})^2 \sigma_{\delta e}} \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{\sigma_{xe}}{\sqrt{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2}}\right) \\ \lambda_{s.a.\parallel} &= -\frac{2C_1}{\sigma_{xe} \sigma_{ye} \sigma_{ze} (\gamma \sigma_{\theta e})^2 \sigma_{\delta e}} \\ &\quad \times \left[ 1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{\sigma_{xe}}{\sqrt{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2}}\right) \right] \end{aligned} \quad (53)$$

where the coefficient  $\kappa_{s.a.}$  is given by

$$\kappa_{s.a.} = \frac{D_i D_e \sigma_{\delta e}^2}{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2} \quad (54)$$

and the function  $\Phi$  is defined in Eq. (49).

Naturally, Eq. (53) can be obtained directly from Eq. (48) by assuming  $\sigma_{xi,yi,zi} \ll \sigma_{xe,ye,ze}$ ,  $\sigma_{\theta i} \ll \sigma_{\theta e}$ , and  $\sigma_{\delta i} \ll \sigma_{\delta e}$ . Yet, it is important to provide a clear discussion of what the “smallness” of various parameters entails. Hence, starting with the friction force formulas is a valuable exercise.

Comparing the redistribution Eq. (53) to previous works (see [26], for example), we notice that the general forms of the expressions, including the redistribution coefficient  $\kappa_{s,a}$ , are identical. However, our formula provides explicit expressions for  $\lambda_{s,a,\perp}$  and  $\lambda_{s,a,\parallel}$ , whereas in previous works they were simply given as constants. Moreover, it is important to note that we show the dependence of  $\lambda_{s,a,\perp}$  and  $\lambda_{s,a,\parallel}$  on  $D_e$ . This means that  $\kappa_{s,a}$  redistributes the rates “disturbed” by the presence of electron dispersion, rather than the cooling rates calculated in the absence of both electron and ion dispersion.

### C. Cooling formulas without redistribution

The cooling rates without redistribution ( $\lambda_{0(x,y,z)}$ ) are obtained from Eqs. (41)–(43) by setting  $D_e = 0$  and  $D_i = 0$ :

$$\begin{aligned} \lambda_{x0,y0,z0} &= -\frac{C_0}{\sqrt{(\sigma_{xe}^2 + \sigma_{xi}^2)(\sigma_{ye}^2 + \sigma_{yi}^2)(\sigma_{ze}^2 + \sigma_{zi}^2)}} \Psi_{x0,y0,z0} \\ \Psi_{x0} &= R_D(\gamma^2(\sigma_{\theta ye}^2 + \sigma_{\theta yi}^2), \sigma_{\delta e}^2 + \sigma_{\delta i}^2, \gamma^2(\sigma_{\theta xe}^2 + \sigma_{\theta xi}^2)) \\ \Psi_{y0} &= R_D(\gamma^2(\sigma_{\theta xe}^2 + \sigma_{\theta xi}^2), \sigma_{\delta e}^2 + \sigma_{\delta i}^2, \gamma^2(\sigma_{\theta ye}^2 + \sigma_{\theta yi}^2)) \\ \Psi_{z0} &= R_D(\gamma^2(\sigma_{\theta xe}^2 + \sigma_{\theta xi}^2), \gamma^2(\sigma_{\theta ye}^2 + \sigma_{\theta yi}^2), \sigma_{\delta e}^2 + \sigma_{\delta i}^2). \end{aligned} \quad (55)$$

The exact same expressions for the cooling rates without redistribution were derived in [39,40]. They coincide with Eq. (55) when the integrals in those references are reduced to Carlson’s symmetric form by substituting the integration variable  $t$  with  $q = 1/t^2$  and converting all beam parameters to the laboratory frame.

For the case of symmetrical angular distribution,  $\sigma_{\theta xe} = \sigma_{\theta ye} \equiv \sigma_{\theta e}$  and  $\sigma_{\theta xi} = \sigma_{\theta yi} \equiv \sigma_{\theta i}$ , the cooling rates  $\lambda_{\perp 0} = \lambda_{x0} = \lambda_{y0}$  and  $\lambda_{\parallel 0} = \lambda_{z0}$  can once again be expressed through elementary functions:

$$\begin{aligned} \lambda_{\perp 0} &= -\frac{C_1}{\mathbb{S}\gamma^2(\sigma_{\theta e}^2 + \sigma_{\theta i}^2)\sqrt{\sigma_{\delta e}^2 + \sigma_{\delta i}^2}} \Phi\left(\frac{\sqrt{\sigma_{\delta e}^2 + \sigma_{\delta i}^2}}{\gamma\sqrt{\sigma_{\theta e}^2 + \sigma_{\theta i}^2}}\right) \\ \lambda_{\parallel 0} &= -\frac{2C_1}{\mathbb{S}\gamma^2(\sigma_{\theta e}^2 + \sigma_{\theta i}^2)\sqrt{\sigma_{\delta e}^2 + \sigma_{\delta i}^2}} \left[1 - \Phi\left(\frac{\sqrt{\sigma_{\delta e}^2 + \sigma_{\delta i}^2}}{\gamma\sqrt{\sigma_{\theta e}^2 + \sigma_{\theta i}^2}}\right)\right] \\ \mathbb{S} &= \sqrt{(\sigma_{xe}^2 + \sigma_{xi}^2)(\sigma_{ye}^2 + \sigma_{yi}^2)(\sigma_{ze}^2 + \sigma_{zi}^2)}. \end{aligned} \quad (56)$$

Similar equations in [39,40] coincide with Eq. (56) after all parameters are converted to the laboratory frame.

There are two special cases that simplify the expressions for the cooling rates even further. We consider them below.

### 1. Spherically symmetrical velocity distribution

When the velocity distribution of the bunches is spherically symmetrical,  $\sigma_{\delta e} = \gamma\sigma_{\theta e}$  and  $\sigma_{\delta i} = \gamma\sigma_{\theta i}$ , one gets  $\Phi\left(\frac{\sqrt{\sigma_{\delta e}^2 + \sigma_{\delta i}^2}}{\gamma\sqrt{\sigma_{\theta e}^2 + \sigma_{\theta i}^2}}\right) = \Phi(1) = \frac{2}{3}$ . In that case, all three cooling rates become equal,  $\lambda_{x0} = \lambda_{y0} = \lambda_{z0} \equiv \lambda_0$ , and from Eq. (56), we immediately obtain:

$$\lambda_0 = -\frac{2cr_e r_i}{3\pi} \frac{\eta N_e \Lambda_C}{\gamma^2 \beta^3 \mathbb{S}(\zeta_e^2 + \zeta_i^2)^{3/2}}, \quad (57)$$

where, for equal rms spreads, we introduced variables  $\zeta_e \equiv \sigma_{\delta e} = \gamma\sigma_{\theta e}$  and  $\zeta_i \equiv \sigma_{\delta i} = \gamma\sigma_{\theta i}$ .

From Eq. (57), the small-amplitude ions cooling rate ( $\lambda_{s,a,0}$ ) becomes:

$$\lambda_{s,a,0} = -\frac{2cr_e r_i}{3\pi} \frac{\eta N_e \Lambda_C}{\gamma^2 \beta^3 \sigma_{xe} \sigma_{ye} \sigma_{ze} \zeta_e^3}. \quad (58)$$

The corresponding friction force components,  $F_{s,a,0(x,y,z)}$ , can be obtained from Eq. (52) by setting  $\mu_z = 0$ :

$$F_{s,a,0(x,y,z)} = -\frac{4\sqrt{2\pi} \rho_e r_e^2 m_e c^4 Z^2 \Lambda_C}{3 \sigma_{ve}^3} v_{i(x,y,z)} \quad (59)$$

where  $\rho_e$  is the electron bunch density in the beam frame, and  $\sigma_{ve} \equiv \sigma_{vex} = \sigma_{vey} = \sigma_{vez}$ . Equation (59) is a well-known formula (see, for example, Eq. (1.12) in [23]).

### 2. Pancake velocity distribution

The final case we consider is that of small-amplitude ions cooled by an electron bunch with a pancake velocity distribution. This means that the velocity distribution is transversely symmetric and the longitudinal rms velocity is much smaller than the transverse one, i.e.,  $\sigma_{vez} \ll \sigma_{ve\perp}$  or  $\sigma_{\delta e} \ll \gamma\sigma_{\theta e}$ .

For small values of the argument  $r$ ,  $\Phi(r)$  asymptotically approaches the following function:

$$\lim_{r \rightarrow 0} \Phi(r) = \frac{\pi}{2} r \quad (60)$$

Then, from Eq. (56), we get:

$$\begin{aligned} \lambda_{p.c.\perp} &= -\frac{cr_e r_i \eta N_e \Lambda_C}{2\gamma^2 \beta^3 \sigma_{xe} \sigma_{ye} \sigma_{ze} (\gamma\sigma_{\theta e})^3} \\ \lambda_{p.c.\parallel} &= -\frac{cr_e r_i \eta N_e \Lambda_C}{\gamma^2 \beta^3 \sigma_{xe} \sigma_{ye} \sigma_{ze} (\gamma\sigma_{\theta e})^2 \sigma_{\delta e}}, \end{aligned} \quad (61)$$

where the subscript “p.c.” indicates that the formulas are derived for a pancake velocity distribution.

Similar to the considerations above, the corresponding friction force is

$$F_{p.c.(x,y)} = -\pi\sqrt{2\pi}\frac{\rho_e r_e^2 m_e c^4 Z^2 \Lambda_C}{\sigma_{ve\perp}^3} v_{i(x,y)}$$

$$F_{p.c.(z)} = -4\sqrt{2\pi}\frac{\rho_e r_e^2 m_e c^4 Z^2 \Lambda_C}{\sigma_{ve\perp}^2 \sigma_{vez}} v_{i(z)}. \quad (62)$$

Once again, we arrive at well-known asymptotic formulas (see, for example, Eqs. (1.49) and (1.50) in [23]).

## V. PRACTICAL APPLICATIONS OF THE FORMULAS

### A. EIC ring electron cooler

The EIC Ring Electron Cooler (REC) [21,22] aims to provide cooling of proton bunches at top energy ( $\gamma = 293.1$ ) to counteract IBS and preserve proton emittance and energy spread. The REC employs an electron storage ring where electrons are reutilized for millions of turns. The stored electrons' phase-space volume tends to increase, mostly due to IBS and proton-electron beam-beam scattering (BBS) in the cooling section. To counteract this heating effect, the storage ring includes dedicated damping wigglers. As a result, the equilibrium emittances of the e-bunches are defined by the balance between radiation damping, IBS, and BBS, as well as other, much weaker heating effects.

The schematic layout and Twiss functions of the REC are shown in Figs. 4 and 5. The electron storage ring is a trapezoid-shaped “racetrack,” with the cooling section and damping wigglers occupying most of it.

The REC parameters relevant to the following discussion are given in Table I.

The REC is a high-current, low-energy electron storage ring. Its design and optimization constitute an elaborate, multistep process [21,41–43]. Here, we present a simplified example of one of the final steps in the REC cooling optimization.

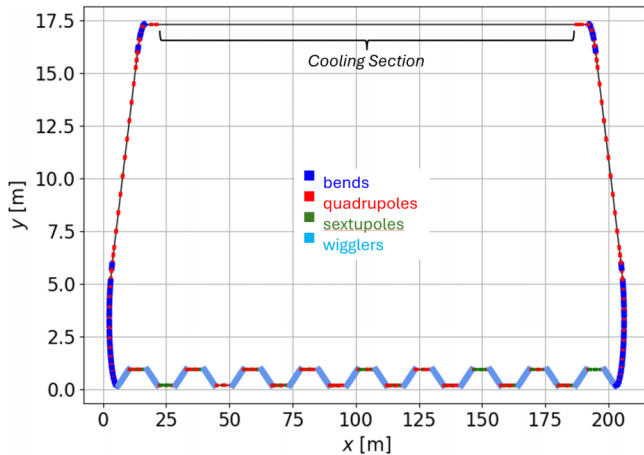


FIG. 4. Layout of the Ring Electron Cooler.

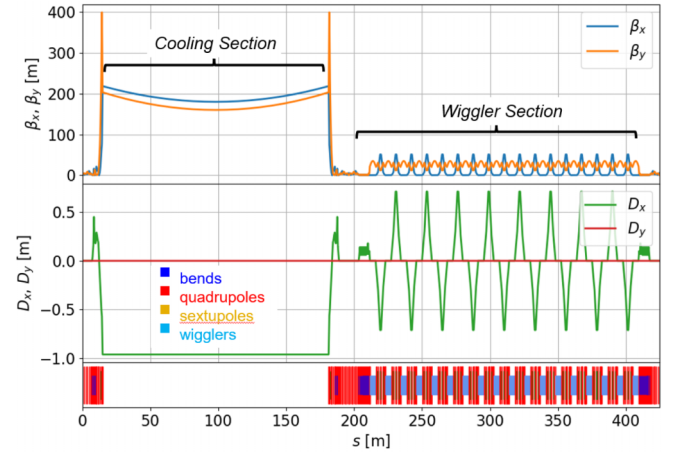


FIG. 5. Lattice of the Ring Electron Cooler.

The EIC requires the electron cooler at top energy to provide a horizontal cooling time of  $\tau_x = 1/\lambda_x = 2$  hours and a longitudinal cooling time of  $\tau_z = 1/\lambda_z = 3$  hours. Given the realistically achievable equilibrium parameters of electron bunches, such cooling times can only be obtained by redistributing the cooling rates from the  $z$  to the  $x$  direction.

As is evident from Eqs. (41) to (44), the redistributed cooling rates depend on the emittances and Twiss parameters of both the electrons and ions. These equations allowed us to develop a multiparametric optimizer that finds the best combination of electron and proton  $\beta$  functions and dispersions while minimizing the electron bunch charge and beam-beam effects in the CS. Because our cooling rate formulas are reduced to tabulated functions, the optimizer operates highly efficiently, producing a solution in less than a minute on a standard laptop. The optimizer was alternated with another dedicated code recalculating the equilibrium electron emittances for the updated parameters of both beams.

The results of running the optimizer are illustrated in Fig. 6. For simplicity, we show a study with a reduced number of parameters, assuming fixed emittances and energy spreads for both beams, as well as fixed electron  $\beta$

TABLE I. The ring electron cooler parameters.

| Parameter                                       | Electrons            | Protons              |
|-------------------------------------------------|----------------------|----------------------|
| Relativistic $\gamma$                           | 293                  |                      |
| Cooling section length (m)                      | 170                  |                      |
| Number of particles per bunch                   | $1.3 \times 10^{11}$ | $6.9 \times 10^{10}$ |
| Geometric emittance ( $x, y$ ) (nm)             | 7.8, 7.8             | 11.3, 1              |
| rms relative momentum spread                    | $9.8 \times 10^{-4}$ | $6 \times 10^{-4}$   |
| rms bunch length (cm)                           | 11                   | 6                    |
| CS $\beta$ function ( $x, y$ ) (m)              | 180, 160             | 300, 700             |
| Horizontal dispersion in the CS (m)             | 0.9                  | 2.1                  |
| <b>Cooling times (<math>x, y, z</math>) (h)</b> | <b>2, 4, 3</b>       |                      |

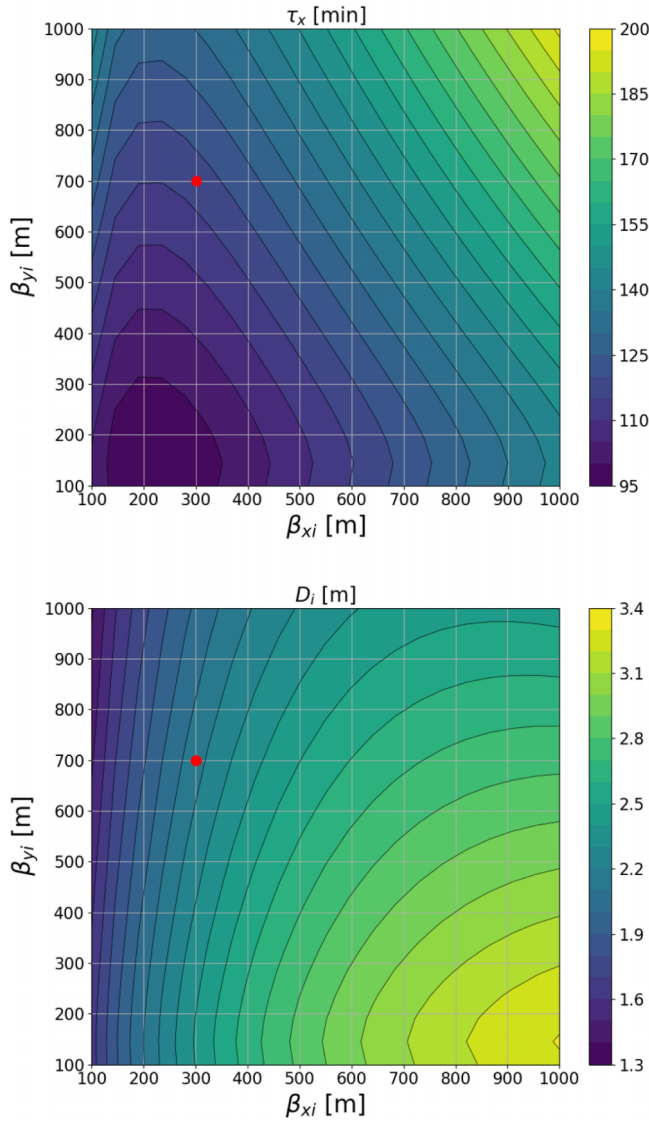


FIG. 6. Optimization of proton parameters in the REC cooling section. The upper plot shows the best horizontal cooling time obtained for each pair of  $(\beta_{xi}, \beta_{yi})$ . The lower plot shows the  $D_i$  corresponding to the fastest horizontal cooling for each  $(\beta_{xi}, \beta_{yi})$  combination. The red dot shows the choice of  $(\beta_{xi}, \beta_{yi})$ , which was driven, among other things, by considerations of beam-beam effects in the cooling section.

functions and dispersion (as listed in Table I). In this study, just three parameters within the CS are scanned: the ion average  $\beta$  functions  $(\beta_{xi}, \beta_{yi})$  and the horizontal dispersion  $D_i$ . During the scan, the longitudinal cooling time is held constant at  $\tau_z = 3$  hours; for each combination of  $(\beta_{xi}, \beta_{yi})$ , the horizontal cooling time is minimized by varying  $D_i$ . The red dot in Fig. 6 indicates the choice of Twiss parameters that maximizes the cooling rate while maintaining a tolerable BBS rate and minimizing the beam-beam focusing effect in the CS.

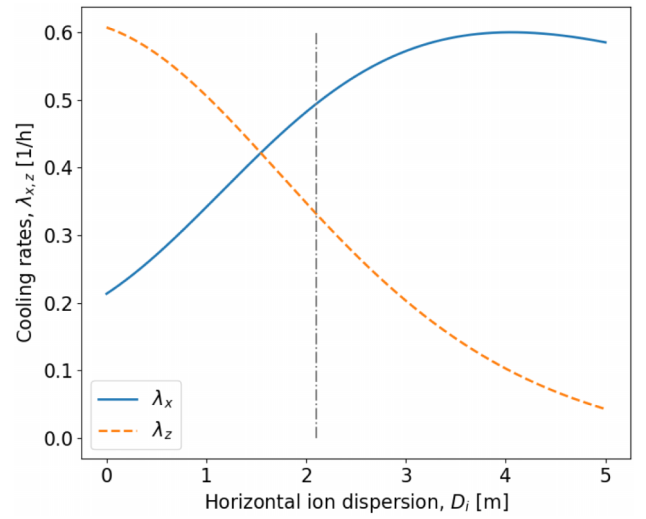


FIG. 7. Horizontal (solid blue line) and longitudinal (dashed orange line) cooling rates in the REC as a function of ion dispersion. All other REC parameters are fixed at the values specified in Table I. The vertical dash-dot gray line corresponds to  $D_i = 2.1$  m.

Finally, the discussion of the Ring Electron Cooler provides an opportunity to elaborate further on the redistribution effect.

In the absence of dispersive redistribution, a plot of the friction force versus ion velocity provides a clear, intuitive picture of the cooling process, as illustrated in Fig. 2. However, when nonzero electron and ion dispersions in the cooling section introduce  $x$ - $z$  coupling, the peak value, shape, and offset of the friction force components become dependent on both the betatron and synchrotron phases of the ion, as well as on its longitudinal and transverse actions. This behavior can be seen from Eq. (29) with inputs from Eqs. (6) and (21).

A clear picture of cooling redistribution is instead provided by the cooling rates, which are obtained by fully integrating the changes in the ions' actions, which depend on the friction force components in accordance with Eqs. (32)–(34), over the 6D ion distribution.

Figure 7 shows how the longitudinal and transverse cooling rates, given by Eq. (41), change with increasing ion dispersion in the REC cooling section. The chosen parameters allow for an increase in the horizontal cooling rate by a factor of 2.5.

### B. EIC low-energy cooler

At the injection energy ( $\gamma = 25.4$ ), the EIC requires active cooling of proton bunches to reduce the normalized vertical emittance from 2 to 0.3  $\mu\text{m}$ . This job is done by the EIC Low-Energy Cooler (LEC) [18,19]: a single-pass, rf-based, nonmagnetized electron cooler.

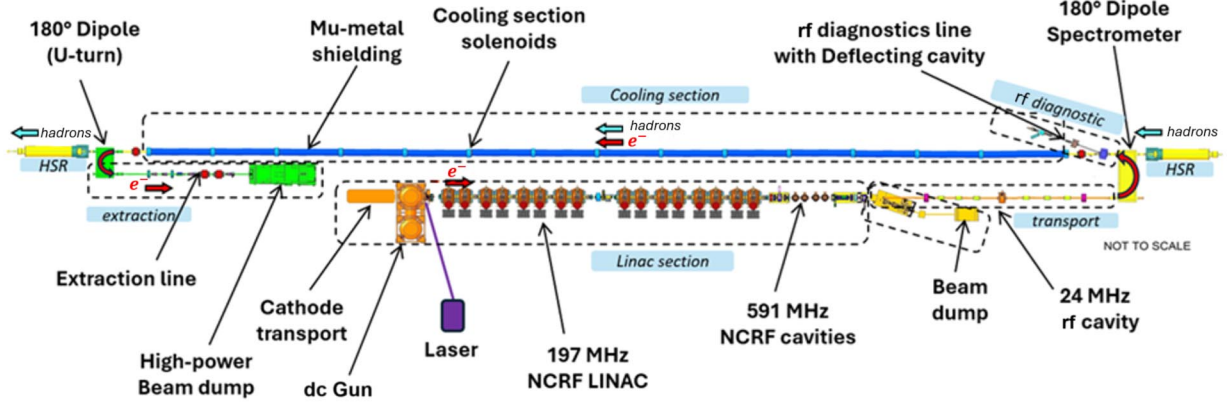


FIG. 8. Layout of the EIC low-energy cooler.

The LEC electrons are produced in a dc gun by illuminating a photocathode with a high-frequency laser. The electrons are accelerated to 12.5 MeV, transported to the 170 m long cooling section, and dumped after a single pass through the CS.

The LEC layout is schematically shown in Fig. 8. The LEC parameters relevant to the following considerations are listed in Table II.

The LEC does not utilize redistributed cooling, and its cooling rates can be calculated using Eq. (55). At the start of the cooling cycle, the calculated cooling times are  $\tau_x = 33.1$  min,  $\tau_y = 33.1$  min, and  $\tau_z = 34.9$  min. For the parameters at the end of the cycle,  $\tau_x = 9.3$  min,  $\tau_y = 8.7$  min, and  $\tau_z = 13.7$  min. The provided cooling is adequate for reducing the transverse emittance throughout the cycle and for balancing the IBS at the end of it.

One of the questions that had to be answered during the LEC design was its tolerance to proton horizontal dispersion in the cooling section. While the LEC does not require cooling redistribution between the longitudinal and horizontal planes, the optics of the EIC Hadron Storage Ring (HSR) at the injection energy is simplified if there is a nonzero dispersion in the CS region. As Eqs. (41)–(44) show, the presence of ion dispersion affects all three cooling rates. Figure 9 illustrates an example of a study on the effect of  $D_{ix}$  on LEC cooling times. This study was performed using the initial proton bunch parameters at

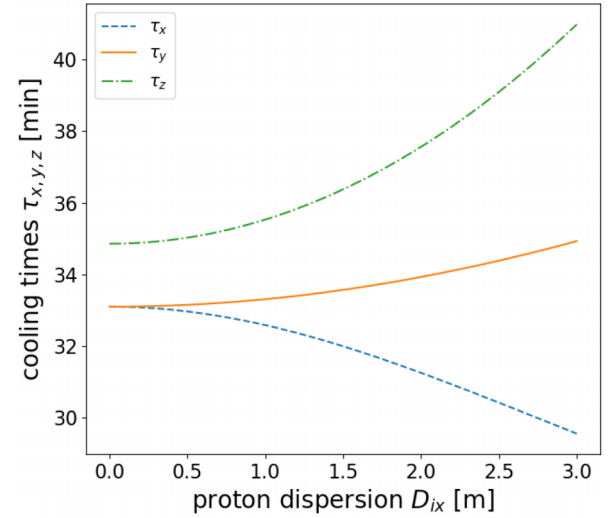


FIG. 9. Dependence of the LEC cooling times on proton horizontal dispersion.

injection energy, which are given in Table II. Based on obtained results, the tolerance for HSR dispersion in the cooling section was set to  $D_{ix} \leq 2$  m.

Note that according to Fig. 9, the longitudinal cooling rate in the LEC is slightly smaller than the transverse rate, even in the absence of redistribution. This is because, for the given LEC parameters, the ratio of the effective

TABLE II. The low-energy cooler parameters.

| Parameter                                         | Electrons                         | Protons                        |
|---------------------------------------------------|-----------------------------------|--------------------------------|
| Relativistic $\gamma$                             |                                   | 25.4                           |
| Cooling section length (m)                        |                                   | 170                            |
| Bunch charge (nC)                                 | 1 ( $\times 3$ e-bunches/p-bunch) | 44.8                           |
| Normalized emittance ( $x, y$ ) ( $\mu\text{m}$ ) | 1.5, 1.5                          | Initial: 2, 2; Final: 0.5, 0.3 |
| rms relative momentum spread                      | $4 \times 10^{-4}$                | $7 \times 10^{-4}$             |
| rms bunch length (cm)                             | 4                                 | 70                             |
| CS $\beta$ function ( $x, y$ ) (m)                | 150, 150                          | 150, 150                       |

“combined” longitudinal-to-transverse velocity spreads in the beam frame is slightly larger than unity. Indeed, using the parameters given in Table II, at the start of the cooling process:  $r \equiv \frac{\sqrt{\sigma_{se}^2 + \sigma_{di}^2}}{\gamma \sqrt{\sigma_{se}^2 + \sigma_{di}^2}} \approx 1.05$ .

Since both the electron and ion bunches have a symmetrical angular distribution at the start of cooling, the cooling rates can be calculated according to Eq. (56). Therefore, the ratio between the longitudinal and transverse cooling rates can be immediately found as

$$\frac{\lambda_{||0}}{\lambda_{\perp 0}} = 2 \left( \frac{1}{\Phi(r)} - 1 \right) \approx 0.95. \quad (63)$$

This choice of LEC parameters is driven by the need to cool only the vertical emittance of the injected proton bunches. Longitudinal cooling in the LEC is undesirable because it increases the proton peak current, which can potentially cause instabilities. If needed, this longitudinal cooling can be counteracted by a dedicated longitudinal stochastic heating of the proton bunch.

## VI. CONCLUSION

In this report, we present the derivation of exact formulas for the cooling rates with dispersive redistribution. The most general analytic expression is given in terms of Carlson symmetric elliptic integrals. Because these special functions are readily available in modern scientific programming libraries, our formulas can be immediately applied to the design of electron coolers, both for the initial identification of requirements for cooler parameters and for building fast, highly effective multi-parametric optimizers.

Following the derivation of the general expressions, we consider special cases. In the case of bunches with a symmetric angular spread, the general exact formulas can be reduced to elementary functions. We further demonstrate that in the absence of redistribution, our formulas coincide with previously derived expressions for cooling rates. Finally, we show that under the approximation of small-amplitude ions, our formulas yield the well-known asymptotic expressions.

In the final section of the paper, we demonstrate the application of the derived formulas to the electron coolers at the Electron Ion Collider.

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