

Effects of curved superconducting magnets on beam stability in a compact ion therapy synchrotron

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Superconducting, curved magnets can reduce accelerator footprints by producing strong fields (>3 T) and reducing the total number of magnets through their capability for combined-function multipolar fields, making them an attractive choice for applications such as heavy ion therapy. There exists the problem that the effect of strongly curved harmonics and fringe fields on compact accelerator beam dynamics is not well represented: existing approaches use integrated cylindrical multipoles to describe and model the fields for beam dynamics studies, which are invalid in curved coordinate systems and assume individual errors cancel out over the full machine. In the modeling of these machines, the effects of strongly curved harmonics and fringe fields on compact accelerator beam dynamics need to be properly included. An alternative approach must be introduced for capturing off-axis fields in a strongly curved magnet, which may affect long-term beam stability in a compact accelerator. In this article, we investigate the impacts of deploying a curved canted-cosine-theta (CCT) superconducting magnet in a synchrotron for the first time. We develop a method to analyze and characterize the 3D curved fields of an electromagnetic model of a CCT developed for the main bending magnets of a compact (27 m circumference) carbon ion therapy synchrotron, designed within the Heavy Ion Therapy Research Integration Plus European project (HITRIPlus), and the CERN Next Ion Medical Machine Study (NIMMS). The fields are modelled in the compact synchrotron in MAD-X/PTC to study their effects on beam dynamics and long-term beam stability. Results indicate that the synchrotron is able to operate with the presence of the magnetic field gradients, with considerable improvement to the long-term beam stability in the machine after tuning the higher order field gradients. The insights gained through the methods presented allow the optimisation of both magnet and synchrotron designs, with the potential to impact the operational performance of future ion therapy facilities.

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I. INTRODUCTION

Designs of next generation particle therapy facilities focus on improving patient accessibility and clinical outcomes, while reducing cost and ensuring sustainability

[1–4]. Much attention is given to reducing the physical footprints of the machine, while delivering high intensity beams (1×10^{10} particles per extraction cycle) to enable efficient treatment delivery [5,6]. These critical changes will improve access to particle therapy, which could benefit between 5% and 15% of radiotherapy patients (depending on the specific modality) [7,8], but is currently far less available than conventional radiotherapy [9].

One factor behind the low number of facilities is the high running and building costs associated with the accelerator and supporting beam delivery technology, as their physical and technical demands are too high for retrofitting into existing facilities. The accelerator alone can typically have a large footprint (e.g., 75 m circumference for some medical synchrotrons [3,10]) due to dependency on normal

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conducting magnets. Reducing the footprint requires magnets that exhibit both superconducting (SC) fields (>3 T) and strong curvature (e.g., 1.9 m bending radius for 0.3 m aperture radius).

Introducing curvature introduces complexity to the design and operation of the magnets and accelerator: the additional field components arising from curvature can negatively impact beam quality and stability during acceleration [11], if they are not properly taken into account and compensated. For medical applications, acceptable limits on field quality must be determined by beam dynamics studies to comply with performance and safety measures during operation and treatment. This requires the exact field content of the magnet to be well-known for implementation in the accelerator.

A caveat lies in the method of field characterization: fields inside the magnet body are typically described in terms of 2D transverse cylindrical multipoles [Eq. (3.2) in Sec. III], which are integrated over the magnet length and represented as such in particle tracking codes [12]. A curved magnet necessitates the use of field gradients, as cylindrical multipoles are mathematically invalid in curved coordinate systems (see Ref. [13]) and in the magnet ends, where the field is no longer translation-invariant. As such, integrated multipoles are insufficient to represent 3D curved magnetic fields and to gauge the full effect of curvature on the beam dynamics. As shown in Refs. [13–15], 2D and 3D field derivatives are valid for describing the fields in straight and curved coordinate systems (see Sec. III), and are already specified in beam optics codes [16]. These provide a straightforward translation from the magnetic field content to the accelerator.

The aim of this study is to understand how both the curvature and the field composition of a curved, combined-function (CF), canted-cosine-theta (CCT) coil-based magnet can impact the beam dynamics and stability in a compact medical synchrotron. While the influence of curvature has been analyzed for this magnet type on the field content and quality [17] and on the beam dynamics in a compact ring [18], the two aspects have not been studied together in the context of a medical synchrotron. To explore these effects, we develop general tools (further detailed in Secs. III and IV) that perform field sampling and gradient calculations for implementation in the accelerator.

We investigate the suitability of a strongly curved (90°), CCT coil-dominated magnet with CF quadrupole layers for the bending sections in a compact medical synchrotron: a project within the HITRI-plus initiative [19], which has produced novel accelerator designs for a planned ion therapy facility in South Eastern Europe [20]. The strong fields (3.5 T) and CF layers attributed to the unique geometry of the CCT (detailed in Sec. II) enable reduction of the accelerator circumference to 27 m.

Our method employs finite element software [21] to simulate a model of the magnet for field analysis.

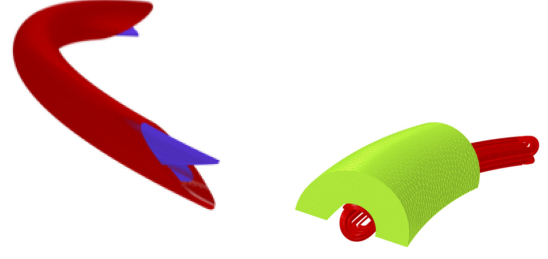


FIG. 1. Left: alternating gradient canted-cosine-theta magnet with 90° bend. Right: combined-function cosine-theta magnet with 30° bend. Models courtesy of L. Brouwer (LBNL) and M. Karpinnen (CERN).

We develop tools ([22,23]) to analyze and characterize the fields using gradients for direct implementation by beam optics codes [16,24], using which we simulate particle tracking in the synchrotron lattice. From the results, we set limits on the higher order gradients that would otherwise drive strong resonances, leading to instabilities in the ring over more turns. Subsequently, the suitability of the accelerator design for future facilities is determined.

II. MAGNET MODELS

A CCT configuration has been established as an attractive solution for a compact medical accelerator [17,20,25], but is yet to be implemented in operation. As such, two SC coil-dominated models are presented for the main bending magnets in the NIMMS SC compact synchrotron: a combined-function (dipole and quadrupole) 30° cosine-theta (CT) magnet and a 90° CCT dipole magnet with alternating-gradient focusing-defocusing (FD) quadrupole layers (Fig. 1). The AG-CCT magnet has been customized for NIMMS by LBNL, based on a similar design for a compact SC proton therapy gantry [26]; the CT is an alternative option in case of engineering complications [27]. The specifications for both magnets are in Table I.

The magnet geometries are chosen due to the simplicity of design, engineering, and “tuning” (adjustment of higher order field gradients) [28]. Additional advantages include good field quality ($\frac{\Delta B}{B} = 10^{-4}$) compared with normal conducting magnets. Both geometries produce pure cosine-theta dipole fields. CT magnets are formed from several turns of one coil; CCT geometries are formed of

TABLE I. Parameters for the 90° AG-CCT and 30° combined-function CT.

Parameter	AG-CCT	CF-CT
Main bending field B_{dip} (T)	3.5	3.0
Max quadrupole gradient (T/m)	± 10	± 3.5
Magnet physical bend ($^\circ$)	90	30
Bending radius (m)	1.9	2.21
Half aperture diameter ($A_{x,y}$) (mm)	30	11.1

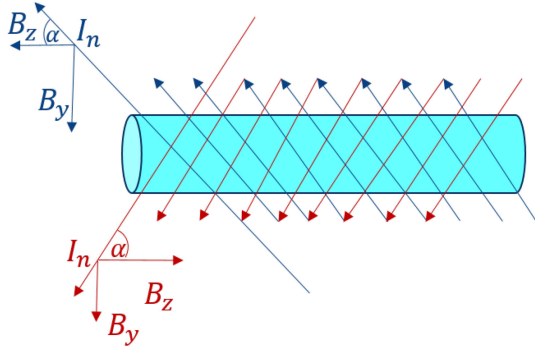


FIG. 2. The current I_n flows on two coil layers tilted at angle α from the horizontal plane z and wound around a mandrel (a central body with grooves). I_n flows in opposite directions, producing an overall main field stored in B_y in the vertical plane and canceling B_z in the z plane.

two (or more) coil layers, superposed with opposite directions of tilt, as shown in Fig. 2.

The reasons for choosing a CCT geometry over the more common CT geometry are due to ease of tunability and stress management. A CCT has an internal structure consisting of a mandrel (the magnet body in Fig. 2), where the Lorentz forces are intercepted by its ribs and transferred to the spar (the center of the magnet, see Fig. 3). As such, no prestress is required for a CCT magnet, unlike the CT [28]. For tunability, the winding pattern of the coil layers in a CCT can be modified to produce different orders of fields (e.g., dipole, quadrupole, or combinations as in the AG-CCT) and to reduce unwanted higher order field gradients. Generally, the field gradients of any magnet must be well-known to determine that the field quality is within design tolerances. This is important for beam dynamics studies: unwanted higher orders can give rise to instabilities in the accelerator over time, leading to beam loss. In our case, due to the compactness of the magnets and synchrotron, the field gradients must be known to high accuracy—

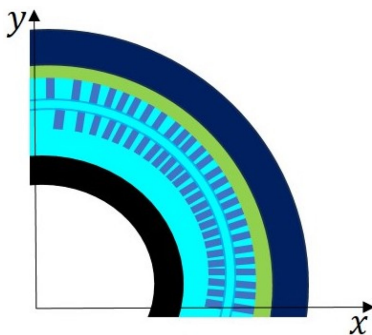


FIG. 3. A quarter cross section of a CCT coil. The conductor is shown in light blue (the widest layer in the diagram), with intercepting ribs shown (darker blue). The ribs intercept Lorentz forces, which are transferred to the spar (the black innermost layer). The outermost layer is the iron yoke, in navy.

especially in the AG-CCT, where the FD layers overlap and where end fringe fields will have a significant effect [29]. As such, OPERA-3D models of the CT and the AG-CCT are studied using a sampling and fitting tool generated in this paper, described in more detail in Sec. IV. While the CF-CT model is not the subject of investigation in this paper, a straight (not shown) and curved version of the CF-CT model is used to validate the calculations of the sampling and fitting tool, before it is applied to the AG-CCT model for implementation in the NIMMS synchrotron for investigations presented in Sec. V. In Sec. III, we present the method for calculating higher order field gradients on and off the magnet midplane.

III. MAGNETIC FIELD THEORY OF STRAIGHT AND CURVED MAGNETS

In practice, the field vector is measured within the aperture of a simulated or physical magnet, where there is no current [30]. A 2D representation of the field can accurately describe the field over the magnet body, away from the ends. To know the field distributed over an entire (coil-based) magnet, we examine the magnetic scalar potential in cylindrical polar coordinates, $V(r, \theta, z)$.¹

A. Case 1: 2D straight system

To find the potential everywhere within the magnet aperture, we need to solve the 2D cylindrical Laplace equation subject to boundary conditions defined for our particular system (see Appendix A). The general form of the Laplace equation in cylindrical coordinates, in the absence of current and longitudinal variation, is

$$\nabla^2 V(r, \theta) = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} = 0. \quad (3.1)$$

r , θ are the radial distance and angular distance, respectively, measured from the nominal beam trajectory where we evaluate the field, depicted in Fig. 4. From Eq. (A1), we require a finite potential as $r \rightarrow 0$. Using separation of variables, the solution to Eq. (3.1) is given by a linear combination of the basis functions, known as cylindrical multipoles, where the radial and azimuthal components of the magnetic field $\mathbf{B}(r, \theta)$ follow as

$$B_{r,\theta}(r, \theta) = \sum_{n=1}^{\infty} \left(\frac{r}{r_0} \right)^{(n-1)} [B_n \sin n\theta \pm A_n \cos n\theta], \quad (3.2)$$

¹It is worth pointing out at this stage that for the purpose of illustrating equations of interest, we use the general form of the 2D cylindrical Laplace equation in terms of a scalar potential, $V(r, \theta)$. Later, we will show solutions to the Laplace equation based on a vector potential, $\mathbf{A}(r, \theta)$, as we will need to consider additional transverse components that arise from the magnet curvature.

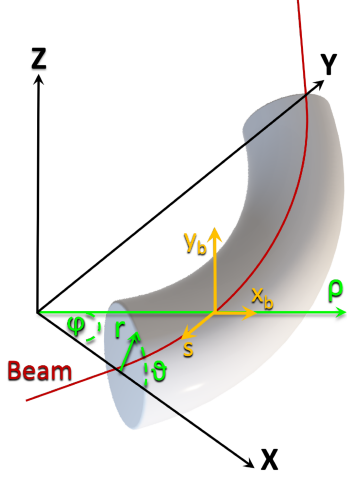


FIG. 4. A diagram depicting a curved magnet with its coordinate systems and variables: global straight Cartesian coordinates (X, Y, Z) and local curvilinear Cartesian coordinates (x_b, y_b, s) (the coordinate system experienced by a beam traveling through the magnetic field). θ is the angular distance measured from the nominal trajectory. ρ is the bending radius of the magnet (same as the local x coordinate of the beam, x_b) while r is the radial distance measured from the beam. ϕ is the angle between the global Cartesian coordinate x and the local (beam) coordinate x_b .

where the integer $n = 1, 2, \dots$ is the multipole order (e.g., $n = 1$ for a dipole in European convention).² The arbitrary reference radius r_0 is typically chosen as $\approx \frac{2}{3}$, the coil radius: the upper limit within the good field region of the magnet [30]. B_n and A_n are the *normal* and *skew* multipole coefficients, respectively, and can be recovered by a discrete Fourier transform (DFT) performed on Eq. (3.2) [29]. Field harmonics are given in terms of the main field expected in the magnet and in “units” scaled by 10^4 . Typically, magnet designers aim for ≈ 0.1 to 2 units to achieve good field quality [31].

In a 2D straight system (Cartesian or polar), multipole expansion is equivalent to a 2D Taylor expansion on the main field along the midplane. It is sufficient to take the derivatives of the main field along the midplane (see Ref. [14]),

$$\left. \frac{\partial^n B_y(x, y)}{\partial x^n} \right|_{y=0} \quad (3.3)$$

The equivalence between the two field expansions in different systems is discussed further in Appendix B; it does not hold in a curved system, as we will demonstrate in Sec. IV.

²NB: this paper uses the European convention for calculating the corresponding equations for the radial (B_r) and the azimuthal (B_θ) components of the field (the summation starts from $n = 1$ instead of $n = 0$).

B. Case 2: 2D curved system

When considering the beam dynamics in an accelerator, we should be in the local (beam) coordinate system, known as curvilinear coordinates, shown in Fig. 4 alongside the previously referred to coordinate systems. Now we present the solutions to the Laplace equation based on a vector potential, $\mathbf{A}(r, \theta)$ as we consider the additional transverse components that arise from magnet curvature. Switching to 2D curvilinear coordinates and satisfying the Coulomb gauge prescription ($\nabla \cdot \mathbf{A} = 0$) results in the general form for the vector Laplacian [32]

$$\begin{aligned} \nabla^2 \mathbf{A}(x, y) &= \frac{\partial^2 A_s}{\partial x^2} + \frac{\partial A_s}{\partial y^2} + \frac{1}{(\rho + x)} \frac{\partial A_s}{\partial x} - \frac{A_s}{(\rho + x)^2} \\ &= 0 \end{aligned} \quad (3.4)$$

We can assume the vector potential to act only along the beam coordinate axis s ; the bending radius ρ is along the length of the magnet, and transverse coordinates (x, y) are centered on the nominal trajectory. Equation (3.4) is not solvable in closed form, due to added constants that depend on x : there is no linear combination of cylindrical multipoles that can form solutions for a curved magnetic field.

We can, however, find a solution [15,33] in the form of a power series and obtain recursion relations between the coefficients for a more generalized case (i.e., off the magnet midplane). The solution for the magnetic field, given by the solution to the vector potential that satisfies Eq. (3.4), can be found by taking a smaller 2D “slice” of the full (global) fieldmap, where the slice is characterized by (n, m) indices (which identify the particular “small-grid” area of the individual slice compared to the global fieldmap) and the local coordinates (x_l, y_l) , which define the position within the individual slice. The solution to the magnetic field can then be found by fitting a polynomial to each field component $B_{i=x,y,z}$ on each 2D slice, as given by Eq. (3.5).

$$\begin{aligned} B_i(j, k, x_l, y_l) &= B_i(j, k, 0, 0) + \sum_{j=1} \sum_{k=1} c_{i,j,k,n,m} (x_l)^n (y_l)^m \\ n, m &= 0, 1, \dots, \quad i = (1, 2, 3) = (x, y, z) \end{aligned} \quad (3.5)$$

where (j, k) are the orders of the 2D polynomial expansion and $c_{i,n,m,j,k}$ represent the (unnormalized) field derivatives, given by recursion relations in Ref. [14]. Conveniently, on each individual slice, we have a 1:1 mapping between the coefficients of Eq. (3.5) $c_{i,j,k}$ of the expansion and the field derivatives, via

$$c_{i,j,k} = \frac{1}{(j, k)!} \frac{d^{j,k} B_y}{dx^{j,k}}. \quad (3.6)$$

The components of $\mathbf{B}$ are expanded along the magnet midplane in Appendix B using Eqs. (3.5) and (3.6). Even in

the ideal case of being on the midplane, the coefficients $c_{i,j,k,n,m}$ now have dependency on the curvature, ρ . Away from the midplane, we cannot assume $A_x = A_y = 0$, therefore, our expansion must be performed on $\mathbf{A}$ and not just a single scalar component of $\mathbf{A}$. The advantage of series expansion is that the solution stays the same; whether there is variation in $\mathbf{A}$ along s , if we are on or off the magnet midplane. We need only to take care when including the dependence on magnet curvature when solving for the coefficients $c_{i,j,k,n,m}$.³ We see shortly how the generalized 2D polynomial expansion is applicable in 3D systems.

C. Case 3: 3D curved system

In close proximity to the magnet's physical ends, there are fringe fields which continue to deflect the beam beyond the ends of the coil [34]. The field here is 3D, meaning the usual 2D multipole expansion is no longer valid. There exist a few different approaches to treat the dependence of the field components on the magnet length (see Ref. [35]). The most straightforward approach we have taken is to assume that at any one sampled measurement of a magnetic field, the beam will be subject to a 2D slice of the magnetic field and will have one z coordinate; the problem is reduced to 2D spatially.

By sampling all three components of the field along (x, y) (one 2D slice) for one particular z and fitting (3.5) to the main field component over the slice, we can step along in z and repeat the fit over each 2D slice defined along the magnet length or region of interest, integrating the gradients at the end of the region.

We have developed a Python tool based on this idea that samples and analyzes 3D fields from *any* magnetic field map, such that the normal and skew field derivatives may be recovered at any point within the magnet. Further detail and application follows in Sec. IV.

IV. FIELD SAMPLING AND ANALYSIS (METHODS)

Reliable methods must be available for calculating both multipole coefficients and field gradients for a given magnetic field, in order to compare—thereby justify—the use of gradients to describe curved fields. In this section, we construct a Python tool that makes use of these methods. Outside of this study, we first tested the tool on a 2D, arbitrary (or “ideal”) straight Cartesian magnet, which is given initial inputs to generate dipole, quadrupole, and higher order fields. Ideal fields are used such that the multipoles and gradients after calculation can be compared

³In a 2D curved system, the terms of the field expansion along the midplane is now attached to both x and y : one can relate the terms of the 2D polynomial expansion to both B_y and B_x , therefore, one should be careful to define the skew components as well as the normal components for use in tracking codes.

against the input fields. The tool was then applied to a more realistic magnet: the straight CT OPERA 3D model, to confirm the matching of field gradients and multipoles after sampling the main field in the magnet body. Using the curved CT, we sample the field and calculate both the gradients and multipoles to demonstrate the mismatch between the two quantities and to reiterate the importance of using field gradients to describe curved fields. Finally, we extend the tool to 3D for off-axis field gradient calculation.

A. Field component calculation in 2D curved systems

We apply the tool to the magnetic fieldmap of a curved magnet (the 30° combined-function CT model shown in Fig. 1) to show that multipoles are not equivalent to gradients in a curved coordinate system. This is such that the tool can be applied to the strongly curved AG-CCT model to sample its field gradients for later performing beam dynamics simulations. We define an ideal path taken by a particle through the center of the magnet and construct vectors perpendicular to the magnet curvature (Fig. 5). The resulting integrated gradients and integrated multipoles, calculated from sampling along the ideal magnet path through the curved CT magnet, are presented in Fig. 6.

From Fig. 6, it is seen that the calculated integrated gradients and multipoles mostly agree within the body region. However, there is a stark difference between the gradients and multipoles at the ends of the curved magnet. While the multipoles are seen to fall to zero at the ends, the polynomial fitting shows that this is not the case. Table V in Appendix C shows the divergence between the two quantities integrated over the whole length of the magnet, emphasizing that these two representations are not equivalent in curved coordinate systems: we must use Taylor expansion instead of Fourier analysis to calculate the magnetic field gradients in a curved system and where there is longitudinal dependence.

B. 3D field component calculation

In practice, higher order field gradients contribute to nonlinear behavior in the beam dynamics of the ring. Especially in the case of a strongly curved magnet (the AG-CCT), due to the geometry, we must take transverse off-axis field components (B_x , B_y) into account, as well as the longitudinal path (z) and field (B_z).

To calculate the strongly curved field gradients from the AG-CCT, we sample the magnet's fieldmap with 10×20 mm spacing on each slice, off the magnet midplane, along 120 points in z (the nominal trajectory), as the integrated gradient values stabilize at this discretization of sampling. To account for the curvature and for the off-axis field contributions, we fit a 2D version of the polynomial [Eq. (3.5)] to the main field over each 2D slice at each constant z , as shown in Fig. 7. Using the 2D polynomial, we can calculate the 2D field gradients that include

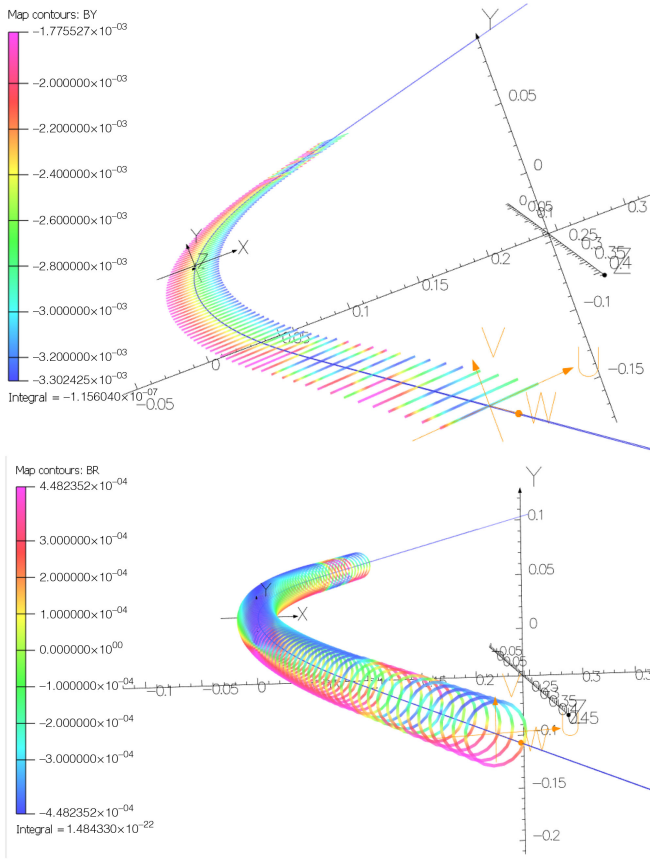


FIG. 5. Main field (represented by color scale) sampled on a nominal trajectory through a curved CT magnetic field. Top: B_y sampled along $y = 0$ perpendicular to each point on the path. Bottom: B_r sampled on circles defined by $r_0 = 0.02$ m ($2/3$ the aperture radius), perpendicular to each point on the path. Local coordinate systems (U, V, W) are defined for both sampling depictions, which are not equal to global (X, Y, Z) .

transverse field dependence (i.e., the skew components), as this has been demonstrated to be a valid approach when sampling the main field off the magnet midplane (see Ref. [15]). By using the gradients, we can reproduce the original field and assess the accuracy of the fit, both graphically (Fig. 7) and analytically, as presented in Table II.

The dipole and quadrupole gradients (b_0, b_1) approximately match their expected design values of 3.5 T, 10 Tm^{-1} (b_1 translated to 2.79 units after integrating throughout the magnet), however, the sextupole and octupole harmonics (b_2, b_3) are outside the usual design limits of 0.1–2 units. As a means of verification of the 3D gradient calculation, we also applied the calculation to the curved and straight CT models and recovered the same results as calculated in Tables IV and V. The large higher order gradients are likely due to the high curvature and additional combined-function layers and will need tuning in the next iteration of the design. It is evident from the results that there are also nonzero skew multipoles (a_n) present due to the magnet curvature, as earlier established in Sec. III.

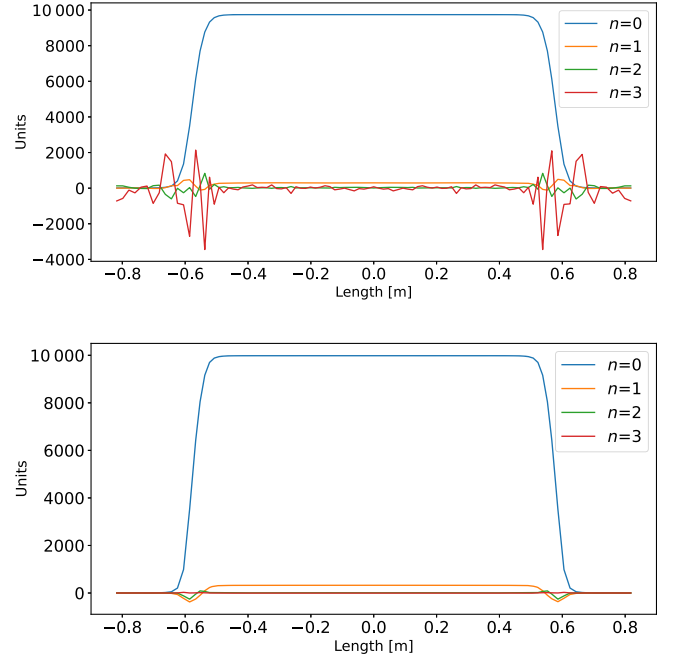


FIG. 6. Magnetic field sampled over the entire length of a 30° CT magnet with CF dipole/quadrupole fields ($z = -0.8$ to 0.8 m). Top: integrated gradient harmonics calculated up to $n = 3$ (octupole). Bottom: integrated multipole harmonics calculated up to $n = 3$. Both quantities are calculated in units $\times 10^4$.

In summary, we have developed a Python tool that is able to calculate field gradients and multipole coefficients in straight and curved systems and have shown that the two representations are not equivalent in curved systems. Having established that a more accurate representation of a curved magnetic field is to use field gradients using polynomial fitting, we have defined both the normal and skew 3D gradients of the AG-CCT magnet in Table II.

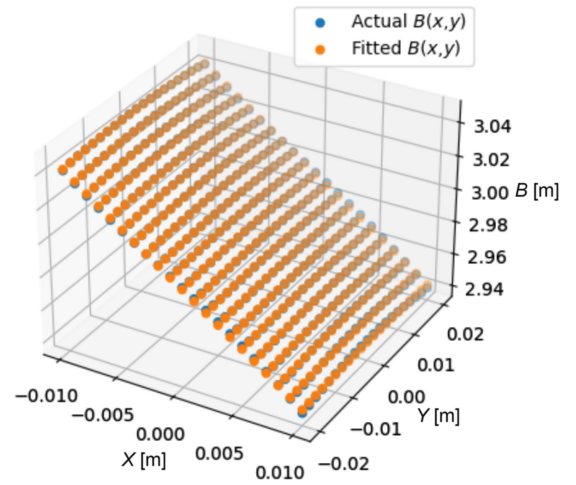


FIG. 7. A comparison between the main magnetic field measured over a 2D slice of the AG-CCT fieldmap and the field reproduced by a 2D polynomial fitted to the data points.

TABLE II. 3D curved system: Integrated gradients converted from the coefficients of a 2D polynomial fitted to main field B_z for orders of $n = (0, \dots, 3)$, between $z = -1.5$ and 1.5 m. The integrated gradients have been scaled by $r_0 = 2/3 \times r = 50$ mm. Each integrated gradient harmonic is calculated via $b_n = B_n/B_{\text{main}} \times 10^4$.

Order (n)	Normal $b_n(x, y)$	Skew $a_n(x, y)$	Unit
0	3.5	-0.002	T
1	2.79	-0.15	10^{-4}
2	14.38	-0.10	10^{-4}
3	28.29	-7.58	10^{-4}

These results are taken forward to the next phase of the study for particle tracking simulations in MAD-X/PTC (Sec. V).

V. BEAM DYNAMICS STUDIES (METHODS)

In this section, we implement the curved magnetic field gradients calculated from the AG-CCT to study the effects in a model of a superconducting, compact carbon ion therapy synchrotron [20], depicted in Fig. 8, with details given earlier in Secs. I and II. The original working point ($q_x = 1.68$, $q_y = 1.13$) is near a third order resonance for extraction [20]. For our study, the working point is moved to ($q_x = 1.71$, $q_y = 1.085$), away from strong coupling resonances in order to examine tune shift due to field gradients. Other parameters of the synchrotron operation and tracking are given in Table III.

A. Optical thin lens element approximation

The synchrotron is represented in MAD-X [16] as a lattice composed of four periodic cells, each containing an

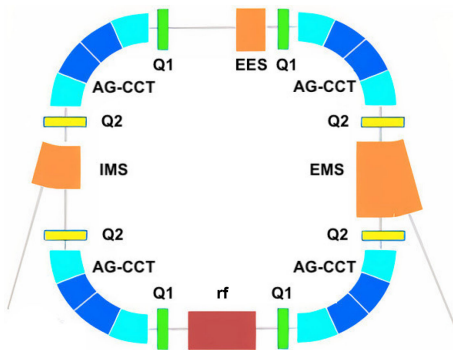


FIG. 8. The NIMMS 27 m superconducting synchrotron, based on four CCT combined-function dipoles (represented as light blue) with alternating-gradient quadrupole layers (represented as dark blue) with 90° curvature. The yellow and green quadrupole families (Q1 and Q2, respectively) are for additional tuning. IMS, EMS, and EES (orange) are the injection magnetic septa, electromagnetic septa, and electrostatic septa, respectively, for injection and extraction of the beam.

TABLE III. Operational and particle tracking parameters for the 27 m NIMMS synchrotron ($\beta_{\text{rel}} = 0.73$, $\gamma_{\text{rel}} = 1.46$).

Parameter	Unit	Value
$E_{k,\text{max}}$	MeV/u	430
$N_{\text{particles}}$	...	2×10^{10}
$\epsilon_{x_N}, \epsilon_{y_N}$	μm	(0.7, 1.0)
$\beta_{x,\text{max}}, \beta_{y,\text{max}}$	m	(5.16, 13.53)
Dispersion $D_{x,\text{max}}$	m	3.03
σ_x, σ_y (injection)	mm	(0.85, 1.78)
Q_x, Q_y (extraction)	...	(1.68, 1.13)
Q_x, Q_y (working conditions)	...	(1.71, 1.085)
Natural chromaticity ξ_x, ξ_y	...	(-0.263, -0.207)

AG-CCT dipole magnet (defined using the optical “sector bend” element). Particle tracking is performed using the PTC [24] module within MAD-X. Approximating the behavior of the beam under influence of “kicks” (magnets) is dependent on how the field gradients are defined and positioned in the lattice. As earlier discussed, a common technique in accelerator physics is to integrate the gradients over an entire magnet and divide by an effective length to obtain the “average” field gradients over the length of the magnet [36]. As the AG-CCT is short (≈ 3 m in length) compared to its fringe field extent [37] and the field gradients near the magnet ends fluctuate and do not fall to zero (as seen in Fig. 6), a more detailed representation is required.

In contrast to the integrated field gradient representation, the field gradients of the AG-CCT are instead represented in MAD-X using discretely sampled fields at each point along the magnet length for performing beam dynamics simulations. The main field is sampled on each 2D slice (as described for the 3D gradient calculation in Sec. IV B) for a total of 64 points⁴ along the magnet length, where the field gradients are calculated for each individual slice and subsequently represented as a thin lens, interleaved with subdivided sector bends within the magnet in MAD-X. The comparison of the tracking results between both representations differ significantly and may lead one to assume a more stable machine than in reality when using the integrated gradient representation (see Sec. VI), justifying our choice of using the discretely sampled gradient representation for performing accurate beam dynamics simulations for the NIMMS synchrotron.

⁴This number of slices/points along the magnet length was determined by calculating the resolution at which the magnetic field gradients stabilize when calculated using the Python tool, with reference to a similar study [38].

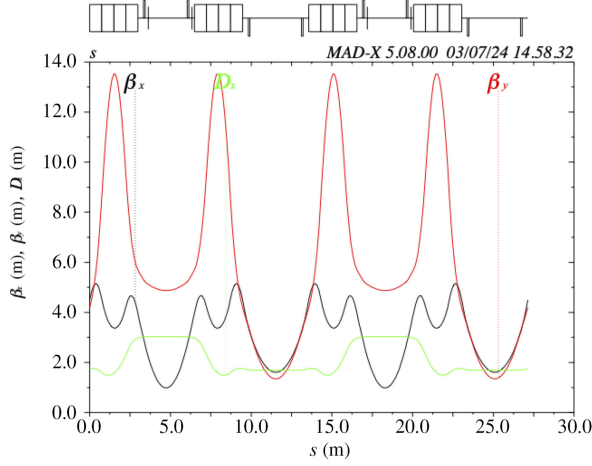


FIG. 9. Optics functions $\beta_{x,y}$ and D_x , using magnetic elements within the NIMMS synchrotron lattice. $s = 0$ is located to the right of the injection location, left of the top left-hand AG-CCT magnet in Fig. 8, going clockwise.

B. Methods: beam dynamics simulations (stability studies)

1. Linear dynamics tools: phase space and tune shift with action

The $\beta_{x,y}$ and D_x optical functions with respect to the synchrotron lattice elements are shown in Fig. 9. Initial (x, y) coordinates are scanned up to the magnet half-aperture ($A_{x,y} = 30$ mm) [20], which accommodates an initial amplitude offset of up to $10\sigma_{x,y}$, where $\sigma_{x,y}$ is the beam size at injection⁵ (see Table III). On-momentum particles are launched in x from 0 to $10\sigma_x$ and scanned separately in y from 0 to $10\sigma_y$. The normalized horizontal (x, x') and vertical (y, y') phase space is then analyzed as a first approach to identify any beam distortion arising from the presence of the gradients.

To gain insight into the effects of the gradients on long-term beam stability, the amplitude-dependent tune shift is examined. Using the same tracking data, fractional tunes $q_{x,y}$ are calculated for each particle launched in a 2D grid from 0 to $10\sigma_x$ and from 0 to $10\sigma_y$ using the NAFF algorithm [39] on the positions of each particle over many turns. We use invariant action-angle variables $(J_{x,y}, \phi_{x,y})$ to examine the change in tune of the particles from the initial starting amplitudes [32].

2. Non-linear dynamics tools: Frequency map analysis and dynamic aperture

We examine the nonlinear effects of the curved CCT field gradients on the beam to determine the overall

performance of the synchrotron with the presence of curved gradients. Frequency map analysis (FMA) is used to study the relationship between initial particle amplitudes and their resulting tunes after 2048 turns to estimate the tune spread and stability of the beam over time [40]. This is viewed on a tune diagram to see how close the particle tunes are to strong coupling resonances. The particles are launched in a 2D grid from 0 to $10\sigma_x$ and from 0 to $10\sigma_y$ for 2048 turns per initial set of amplitudes. Losses are defined as any particles that have amplitude growth outside the magnet half-aperture. The dynamic aperture (DA) can be estimated from the stable region in (x, y) amplitude space, indicated by the diffusion index D , which measures the change in vertical and horizontal tune spread between the first and last 1024 turns.

$$D = \log_{10} \sqrt{(\Delta q_x)^2 + (\Delta q_y)^2}. \quad (5.1)$$

The diffusion index is used to color code the tune space and initial amplitude space diagrams in Fig. 13 onward. More details follow in Sec. VI. A large dynamic aperture (approximately equal to the physical aperture size [41]) in a medical synchrotron is needed for high injection efficiency and a long, stable lifetime of therapeutic beams.

3. Tolerance studies

As well as higher order field gradients, there are sources of errors that may occur in practice which will affect the overall stability of the beam, such as magnet roll errors, manufacturing errors, and magnet misalignments. To emulate a more realistic simulation, we study the tolerance of the synchrotron with respect to tuned sextupole and octupole field gradients (k_2L, k_3L), as well as the inclusion of potential magnet misalignments that may occur in practice, and compare these with an idealized simulation without placement errors. A particle survival study is performed with the same tracking parameters as for the FMA studies described in Sec. V, this time scanning over (k_2L, k_3L) from 0% to 100% of their original values on a 12×12 grid. Following a method in Ref. [42], the number of surviving particles (when tracked from 0– $10\sigma_{x,y}$) is plotted against each (k_2L, k_3L) on the grid, as depicted in Fig. 10.

This is to determine the minimal percentage reduction in (k_2L, k_3L) that is needed (i.e., the minimal amount of tuning of the higher order gradients) that allows $\approx 100\%$ of particles to survive. This tuning study helps to indicate how much tuning is required for the DA to increase. Beam dynamics simulations are then performed with the new tuned values of (k_2L, k_3L) implemented to examine the potentially improved DA. Comparing with studies at other synchrotron facilities (see Ref. [43]), small random magnet misalignments (≈ 50 – 100 μm , equal to a 2.5σ truncated Gaussian), are introduced into the lattice to gauge the effect on the DA of positioning errors that may occur in practice,

⁵The version of the MAD-X lattice studied does not have the different septas implemented, therefore, the injection point is taken as $s = 0$.

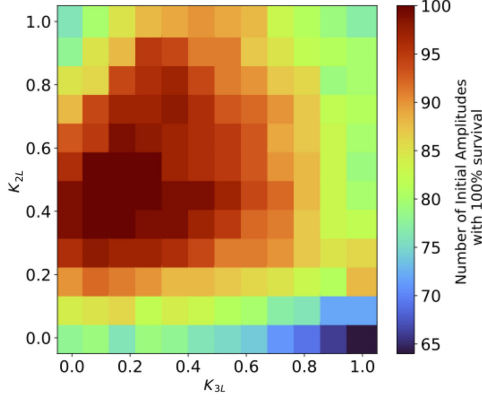


FIG. 10. Particle survival rate corresponding to the tuning of ($k_2L = \text{sextupole}$, $k_3L = \text{octupole}$) for particles tracked from $(-5\sigma_x \text{ to } 5\sigma_x)$ and $(0 \text{ to } 10\sigma_y)$ in a 12×12 grid in the discretely sampled gradient lattice. (k_2L , k_3L) is varied between 0% and 100% of their previous values.

compared to the influence of the higher order magnet gradients present.

VI. RESULTS

Studying the effects of incorporating the AG-CCT 3D magnetic field gradients, Fig. 11 shows a comparison between the normalized phase space for the base lattice without the addition of any magnetic field gradients (top), the normalized phase space for the lattice with the addition of integrated field gradients (middle), and the lattice with additional thin lenses and nonintegrated field gradients (bottom). For both lattices with the gradients implemented, it can be seen that their presence causes some distortion in both the horizontal and vertical phase space, expected due to the effect of higher order fields (e.g., sextupole and octupole); however, no particle losses occur in the integrated gradient lattice for the starting amplitude ranges previously stated. In the nonintegrated gradient lattice, it is clear that there is more distortion in both coordinates' phase spaces, with losses appearing in the vertical phase space from $\approx 5\sigma_y$.

To verify that there are no additional drivers of tune shift or beam instability in the base lattice without the inclusion of field gradients, the horizontal and vertical tune shift with action $J_{x,y}$ was checked for the base lattice, which exhibited a constant tune with increased action (not shown), as expected. This was checked such that any tune shifts occurring in the lattice with the inclusion of field gradients can be solely attributed to the presence of the gradients. For the base lattice with integrated gradients added to the sector bends (Fig. 12, top), there are obvious shifts in tune of $\Delta q_x = 0.005$ and $\Delta q_y = 0.03$. Comparing to the nonintegrated gradient case (Fig. 12, bottom), there is a more significant tune shift of $\Delta q_x = -0.01$ and a shift of $\Delta q_y = 0.01$ before losses occur for $y = 5\sigma_y$. It is clearer

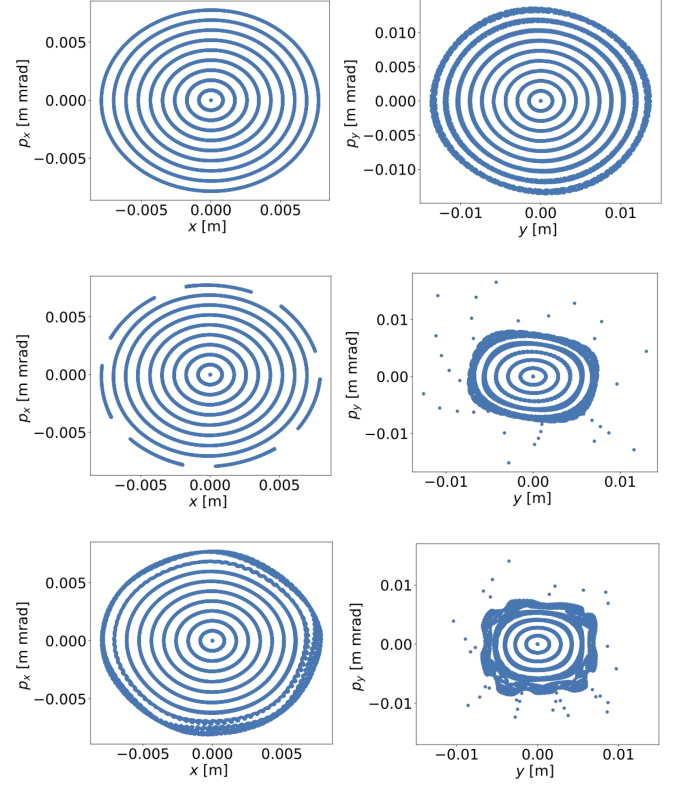


FIG. 11. Phase-space diagrams from tracking particles in x (horizontal) only from 0 to $10\sigma_x$ (left) and in y (vertical) only from 0 to $10\sigma_y$ (right). Top: unmodified lattice without higher order field gradients. Middle: lattice with integrated higher order field gradients as thin lenses between the F-D regions inside the sector bends. Bottom: lattice with discretely sampled (nonintegrated) higher order field gradients as thin lenses distributed throughout the sector bends.

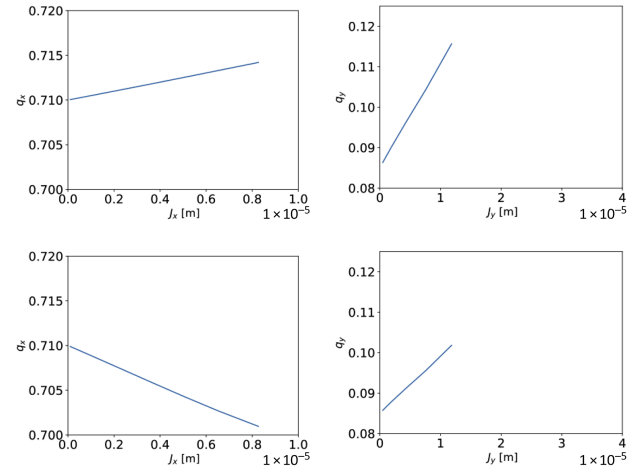


FIG. 12. Tune shift with action ($J_{x,y}$) from tracking particles in $(x, y = 0-10\sigma_{x,y})$. Top: lattice with integrated higher order field gradients defined as thin lenses between the F-D regions inside the sector bends. Bottom: lattice with discretely sampled (nonintegrated) higher order field gradients as thin lenses distributed throughout the sector bends.

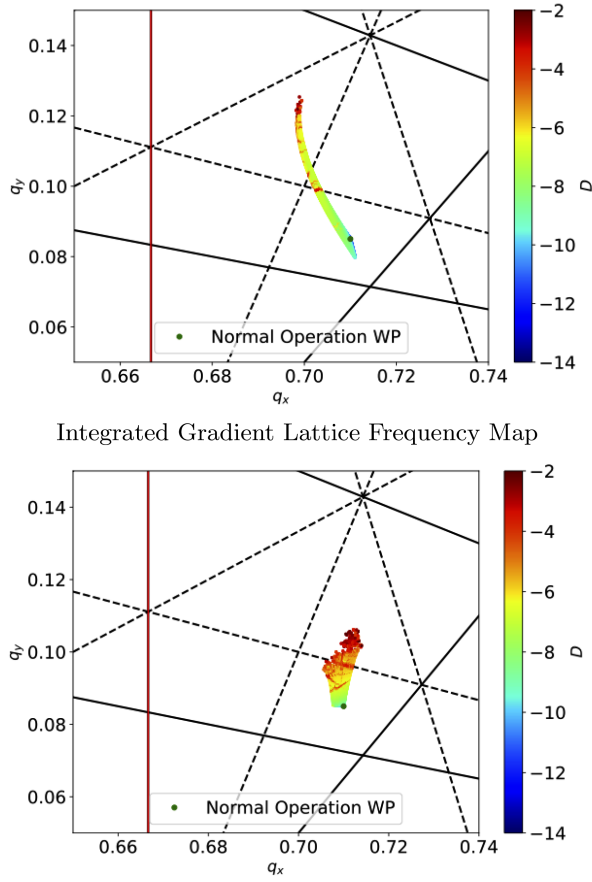


FIG. 13. Tune space diagrams from tracking 2×10^{10} particles at initial amplitudes on a 100×100 grid of $(-5-5\sigma_x)$, $(0-10\sigma_y)$ for the two different lattice representations (integrated gradients (top) and (bottom) discretely sampled gradients).

after cross examination of the phase space diagrams (Fig. 11) and tune shift with action (Fig. 12) that due to the presence of gradients, losses occur due to a large vertical tune shift: particles with a vertical tune of $q_y = 0.095$ become unstable and their vertical amplitudes grow outside the physical magnet half-aperture after excitation.

Examining the calculated tunes in frequency space (Fig. 13), tune spread of the beam over time in each lattice is shown by diffusion index D on the colorbar, where blue indicates more stable tunes, red equates to unstable tunes and white pixels contained within the colored area indicate particles that have been lost during tracking. The representative stable colors correspond to a threshold value of $D < -6$, following similar studies [44,45]. From Fig. 13 (top), it can be seen that there is a large vertical tune spread over several resonances. However, there is little evidence of loss, indicating that the resonance lines crossed ($3q_x - q_y = 2$, $2q_x + 6q_y = 2$) are not strong enough to cause major excitation of particles at the amplitudes corresponding to those tunes. Comparing to Fig. 13 (bottom), the vertical tune spread occurs near the same second order skew resonance ($2q_x + 6q_y = 2$), which appears to

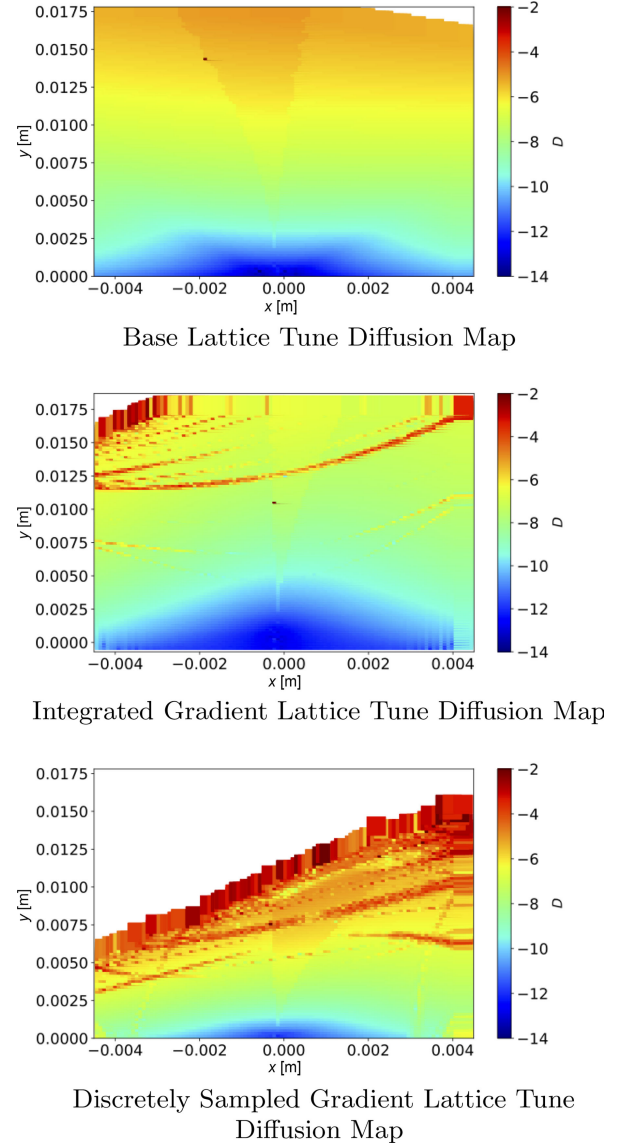


FIG. 14. Initial amplitude space diagrams (tune diffusion maps) constructed from tracking 2×10^{10} particles at initial amplitudes on a 100×100 grid of $(-5-5\sigma_x)$, $(0-10\sigma_y)$ for (top) the base lattice and two different lattice representations (integrated gradients (middle) and (bottom) discretely sampled gradients).

be driving losses in this case. It is interesting to observe from Fig. 13 that, due to the discretely sampled gradients, the beam has become more unstable compared with integrated gradients: the second order skew resonance causes a stronger effect in the bottom figure, with the majority of the tune spread occurring over this resonance line before losses occur. Viewing the same tracking data in (x, y) amplitude space (Fig. 14, middle and bottom), we can estimate the DA from the stable regions in the diagrams and evaluate any shrinkage due to the gradients by comparing with the base lattice without gradients (Fig. 14, top). Similar to Fig. 13, the initial amplitudes of the particles are compared to their tune spread after tracking. Using the

colorbar, the area of the stable region (indicated from blue to yellow) allows estimation of the dynamic aperture. Comparing the top and middle graphs in Fig. 14, it can be seen that the presence of integrated gradients is well tolerated by the synchrotron. The DA is reduced only slightly in the vertical plane from $|y| = 0.012$ to 0.010 compared to that of the base lattice. However, comparing to the discretely sampled (nonintegrated) gradient representation results [Fig. 14 (bottom)], there is a dramatic decrease in the vertical DA to 25% ($|y| = 0.003$, or $\approx 2\sigma_y$). This result justifies our representation of magnetic fields for beam dynamics with discretely sampled field gradients along the entire CCT magnet curvature. Otherwise, higher order field detail due to curvature may be missed using an integrated gradient approach, and a much larger DA might be falsely achieved. Though the presence of the curved field gradients reduces the DA by 75%, the simulated synchrotron is able to store particles with up to $10\sigma_{x,y}$ initial amplitude offset, although with reduced stability, for 2048 turns (as dictated by FMA). Resonances do not appear to cause significant beam loss in this amplitude range for the given number of turns; however, the stability of the beam will likely decrease as the number of turns approaches 10^6 , as required for hadron therapy [5]. There is no set design limit on the DA; however, we aim to maximize it for optimal performance, ideally approaching $10\sigma_{x,y}$ (the magnet half-aperture).

We have demonstrated that the model carbon therapy synchrotron lattice appears able to store $\epsilon_{xN,yN} = 0.7 \mu\text{m}$, $1.0 \mu\text{m}$ carbon ions for 2048 turns. However, for ultimate performance and beam stability, the DA may need further improvement depending upon the operational needs of a treatment facility. We have, therefore, further investigated the intentional tuning of higher order field gradients in order to improve the DA, which may be required for the next iteration of design of the AG-CCT magnet. Figure 10 shows that (k_2L, k_3L) must be tuned to 0.56 and 0.25 times their respective values to yield the highest particle survival and potentially increase the DA. Using these tuned gradients, particles with the same tracking parameters as in Sec. V are tracked to determine the new DA, as shown in Fig. 15. The vertical DA improves from $|y| = 0.3 \text{ m}$ to $|y| = 0.6 \text{ m}$, or $\approx 50\%$ after tuning. This means that in the presence of unwanted higher order gradients arising from the CCT curvature, the DA of the NIMMS synchrotron can be mostly recovered with adjustment of the higher order field gradients.

Further adding to the realism of the simulation, we also investigate the DA when small misalignments (as earlier described in Sec. V) are introduced to both the base lattice and the lattice with discretely sampled, tuned gradients. The effect on the DA of the misalignments is practically negligible in both the base lattice and in the lattice with tuned gradients implemented. As can be seen by comparing Figs. 15 and 16, we conclude that for the range of

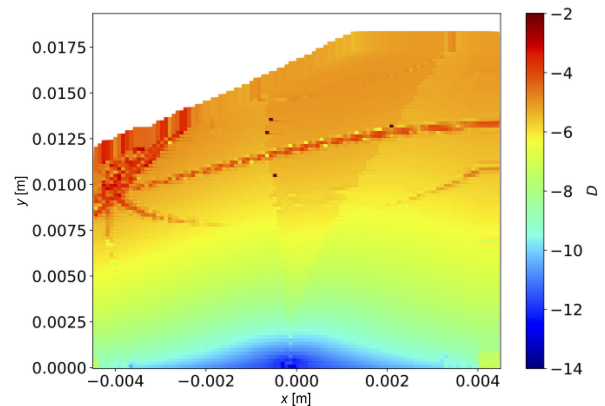


FIG. 15. Initial amplitude space (tune diffusion map) in the discretely sampled gradient lattice after tuning higher order gradients (k_2L, k_3L).

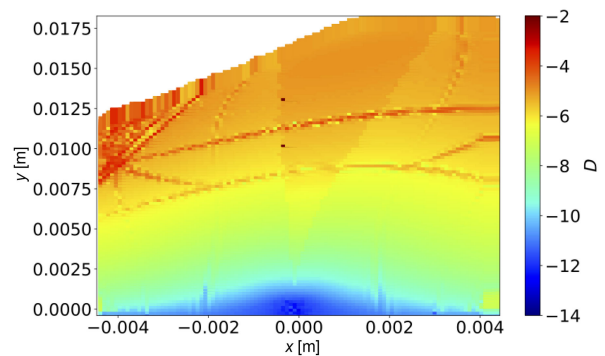


FIG. 16. Initial amplitude space (tune diffusion map) in the lattice with discretely sampled, tuned higher order gradients (k_2L, k_3L) after misalignments of $100 \mu\text{m}$ in the sector bends, $50 \mu\text{m}$ in the quadrupoles and sextupoles.

misalignment values studied, there is little additional detrimental effect on the DA beyond the reduction already caused by the (tuned) magnetic field gradients. For a more relaxed range of misalignments (up to 1 mm), with the addition of rotational misalignments, this may yield a further reduction in the DA. However, this is beyond the scope of this particular study.

The key results from this study show distortion arising in the phase space due to curved magnetic field gradients, leading to a reduction in the vertical DA by 75% using discretely sampled gradients in a thin lens representation, compared to a 17% reduction using an integrated gradient representation. By using integrated gradients, one may be led incorrectly to assume the region of stability in the lattice to be much larger than reality: while this may be a sufficient representation for magnets within larger accelerators [46–48], we have demonstrated this is not the case here. The DA of the synchrotron can be mostly recovered by tuning sextupole and octupole gradients to (0.56, 0.25) times their initial values, which would require additional magnet layers in practice [49]. We found that there was no

significant impact on the DA from small magnet misalignments that may occur in practice.

VII. CONCLUSION

The goals of this paper were to analyze a strongly curved, superconducting, combined-function CCT magnet in context of a compact synchrotron for heavy ion therapy and to assess the impact of the curvature on the field content and the beam dynamics. This required the accurate analysis of nonlinear fields due to curvature and a simulation pipeline capable of modeling the compact accelerator, which is necessary to determine the suitability of the magnet in this application but has not been previously performed.

To address these goals, we devised a tool to sample fields from simulated magnetic fieldmaps in OPERA 3D and calculate magnetic field gradients, as multipole expansion is no longer mathematically valid for field analysis. Our choice of the field gradient representation in the curved coordinate system was found to be justified, overcoming the mismatch between curved gradients and multipoles. An extension of the curved field gradient calculation to 3D was performed for fully representative field modeling in the curved coordinate system of the AG-CCT.

The 3D normal and skew gradients were implemented in the synchrotron lattice in MAD-X for beam dynamics simulations using the in-built PTC module to investigate the gradients' effects on the compact synchrotron operation. It was shown that the commonly used integrated gradient representation did not reflect accurately the phase space distortion and particle losses on resonances due to higher order fields. After improvement of the representation using discrete field sampling along the magnet length, we discovered significantly larger tune shifts due to additional finite field detail, leading to a reduction in the vertical dynamic aperture by 75%. After tuning sextupole and octupole gradients by means of a particle survival study, the vertical dynamic aperture improved to 50%.

Our results show a considerable reduction in the DA compared to the initial particle amplitudes tracked within the physical half-aperture; however, we have assessed in this study that this is tolerable for stable operation of the synchrotron. We further checked the impact of realistic magnet misalignments on beam stability, finding a negligible reduction in DA, concluding that for the range of misalignments introduced, there is little further detrimental effect on the synchrotron performance compared to the impact of the nonlinear fields arising from magnet curvature.

In conclusion, our studies assess that the presence of curved magnetic field gradients do not severely restrict the operation of the medical synchrotron studied. However, the gradients are important to assess for other compact

accelerator scenarios or in the case the achieved DA is insufficient. In this case, the CCT magnet design must be adjusted accordingly by, e.g., adding additional coil layers for tuning. In its current iteration, we assess that the synchrotron based on curved AG-CCT magnets is capable of achieving the design goals laid out in the introduction of this paper, with the potential to enable future heavy ion therapy facilities for improved patient access and clinical outcomes.

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DATA AVAILABILITY

The data supporting this study's findings are available within the article.

APPENDIX A: BOUNDARY CONDITIONS

$$V(r, \theta) = \begin{cases} V(\theta + L) & \text{for } r > 0 \\ < \infty & \text{for } r = 0 \\ V(\theta) & \text{for } r = r_0 \end{cases} \quad (\text{A1})$$

L = an integer multiple of 2π , r is the radial distance measured from the nominal beam trajectory, θ is the angular distance measured from the nominal trajectory, and r_0 is the reference radius.

APPENDIX B: ALTERNATIVE SOLUTION: FIELD DERIVATIVES IN 2D STRAIGHT SYSTEMS

In Cartesian coordinates, the magnet midplane can be chosen along the vertical component $y = 0$ and $\mathbf{B}(x, y) = B_y(x, y)$, where transverse components (x, y) are centered on the nominal trajectory and $B_x = -B_x = B_z = -B_z = 0$. This has the form

$$B_y(x|_{y=0}) = B_y + \frac{dB_y}{dx}x + \frac{1}{2!} \frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!} \frac{d^3B_y}{dx^3}x^3 + \dots, \quad (\text{B1})$$

where all B_y are evaluated at $(x, y) = 0$ and the first term represents the dipole field, the second term represents the

quadrupole field, and so on. By comparison, the multipole expansion of the field in straight Cartesian coordinates is

$$B_y(x|_{y=0}) = B_0 + \frac{1}{r_0} B_1 x + \frac{1}{2!} \frac{1}{r_0^2} B_2 x^2 + \frac{1}{3!} \frac{1}{r_0^3} B_3 x^3 + \dots \quad (\text{B2})$$

The multipole coefficients are evaluated at r_0 as dictated by (A1); the normalization factor requires a reference radius to be defined, due to the increase in the field strength as a polynomial function of the radius, whose order is $(n-1)$. It can be seen that the two representations are equivalent, aside from a normalization factor r_0^{-n} in (B2).

On the magnet midplane, the components of $\mathbf{B}$ follow from Ref. [14] as

$$B_x(x, y, s) = \frac{\partial V(x, y, s)}{\partial x} = c_{11}y + c_{12}xy + \dots \quad (\text{B3})$$

$$B_y(x, y, s) = \frac{\partial V(x, y, s)}{\partial y} = c_{10} + c_{11}x + \frac{1}{2!} c_{12}x^2 + \frac{1}{2!} c_{30}y^2 + \dots$$

$$c_{30} = [h_s c_{11} + c_{12}] \quad (\text{B4})$$

$$B_s(x, y, s) = \frac{1}{h_s} \frac{\partial V(x, y, s)}{\partial s} = \frac{1}{h_s} [c'_{10}y + c'_{11}xy + \dots] \quad (\text{B5})$$

where scale factor $h_s = (1 + \frac{x}{\rho})$ [32]. Here, the prime denotes the total derivative with respect to s . In a rectangular coordinate system, $\rho \rightarrow \infty$ and $h_s = 1$, which makes $y = 0$ and reduces (B4) back to (B1) as before.

APPENDIX C: CALCULATING GRADIENTS AND MULTIPOLES FOR DIFFERENT COORDINATE SYSTEMS

TABLE IV. 2D straight system (realistic magnet): A comparison of integrated gradient harmonics converted from the coefficients of a polynomial fitted to main field B_y and integrated multipole harmonics calculated from sampling the radial field B_r around a circle of radius ($r_0 = 0.02$ m) for a few orders of $n = (0, \dots, 3)$, between $z = -0.4$ and 0.4 m (the magnet body). Each harmonic b_n is given in terms of the main field B_{main} : $b_n = B_n/B_{\text{main}} \times 10^{-4}$

Order (n)	Polynomial fit $b_n(x)$	Fourier analysis $b_n(x)$	Unit
0	2.77	2.77	Tm
1	-331.91	-331.88	10^{-4}
2	15.68	15.56	10^{-4}
3	0.585	0.109	10^{-4}

TABLE V. 2D curved system: A comparison of integrated gradient harmonics converted from the coefficients of a polynomial fitted to main field B_y and integrated multipole harmonics calculated from sampling the radial field B_r around a circle of radius ($r_0 = 0.02$ m) for a few orders of $n = (0, \dots, 3)$, between $z = -0.8$ and 0.8 m (the entire magnet length). As before, each $b_n = B_n/B_{\text{main}} \times 10^{-4}$.

Order (n)	Polynomial fit $b_n(x)$	Fourier analysis $b_n(x)$	Unit
0	3.39	3.39	Tm
1	310.0	264.18	10^{-4}
2	5.16	1.50	10^{-4}
3	16.04	1.26	10^{-4}

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