

Arclike bunch compressor design with coherent synchrotron radiation effects

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Arclike bunch compressors are important components in recirculating accelerator linacs, energy recovery linacs, and high-brightness light sources. However, their design is complicated by longitudinal nonlinear optics and coherent synchrotron radiation (CSR), including both steady-state CSR (ss CSR) in bending magnets and CSR in the subsequent drifts (dr CSR). These effects can degrade the longitudinal compression quality, generate current spikes, and induce transverse emittance growth. In this work, we develop an analytical model for evaluating CSR effects in complex arclike compressors without relying on the small-angle approximation. The proposed method includes both ss and dr CSR contributions and is applied to the mitigation of CSR-induced transverse displacements as well as to longitudinal matching with nonlinear compression. Based on these methods, a two-stage arclike compressor using an asymmetric quadruple-bend-achromat structure is designed. Numerical simulations show that the proposed compressor can compress a 100 pC bunch to nearly 50 fs while maintaining negligible emittance growth. The analytical model is further used to study CSR-induced modifications to the compression process, including rms bunch length and peak-current behavior. Benchmarking against Bmad simulations demonstrates good agreement, confirming the model as an efficient tool for compressor design and optimization.

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I. INTRODUCTION

Arclike bunch compressors are widely used in circular machines, such as Recirculating Accelerator Linacs (RALs), Energy Recovery Linacs (ERLs), and high-brightness synchrotron radiation sources [1–4]. During the process of beam compression, two major issues need to be addressed: the longitudinal nonlinear optics and the coherent synchrotron radiation (CSR) effects within the deflection beamline. After passing through the accelerating section, the bunch develops a nonlinear energy spread. This nonlinear energy spread will degrade the longitudinal quality of the bunch, adversely affecting the final compression factor and leading to the formation of bunch spikes [5]. For conventional linacs, the beam can be linearized by introducing high-order harmonic cavities, thereby mitigating the effects caused by the nonlinear energy spread [6]. However, for machines with recirculating lines, especially ERLs, the decelerating high-order

harmonic cavities can affect the design of linac optics and energy recovery efficiency. A more common approach is to adjust the sextupole magnets in the arc section to regulate the longitudinal quality of the compressed bunches [4,7]. For the CSR effects, the degradation of the bunch quality in the transverse plane primarily stems from the growth in emittance induced by the displacements caused by CSR [8]. There are two primary approaches to suppressing the variance in emittance caused by CSR: (i) Aligning the phase space at the exit of the deflection beamline with the displacement caused by CSR effects [9–11]. (ii) Minimizing the displacements induced by CSR effects through specialized beamline design [12,13]. In most cases, CSR-induced displacements cannot be completely canceled, so both methods are often used together.

CSR-induced transverse displacements are commonly evaluated using the kick point method [11] or the integration method [14], which have enabled several CSR-suppressed compressor designs [12,13,15]. However, their application to arclike compressors still faces important limitations. Simple DBA-based layouts are generally inadequate for both nonlinear optics correction and strong longitudinal compression, making more complex MBA or multi-DBA structures necessary. In such lattices, especially with large bending angles, the small-angle approximation can lead to non-negligible errors and should be avoided. Moreover, previous arclike compressor studies

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mainly considered steady-state CSR (ss CSR), while CSR in the subsequent drifts (dr CSR) may become important for ultrashort bunches. Although dr CSR has been studied in chicane compressors [16,17], its treatment in arclike compressors remains more involved due to the complex lattice geometry.

Another important issue is the CSR-induced longitudinal effect in bunch compressors. Generalized longitudinal matching models are widely used for rapid linac optimization [18] and have been extended to complex systems such as multipass ERLs [19]. However, these models usually neglect CSR, and a well-defined analytical model for CSR-modified longitudinal compression is still lacking. Conventional studies therefore rely on multi-particle tracking codes, such as ELEGANT [20], OCELOT [21], IMPACT [22], and Bmad [23]. Although these tools provide high accuracy and can handle complex lattices, they are computationally expensive, especially above 100 MeV, where transverse and longitudinal dynamics are largely decoupled but full 3D tracking still evolves the complete transverse phase space. To improve efficiency, the 1D code LITRACK was developed for longitudinal matching with optical compression and ss CSR effects [24]. Fast 1D routines, including all CSR effects, as well as polynomial-expansion-based models with potential back-tracking capability, have also been proposed and benchmarked against 3D simulations [25,26]. Nevertheless, these 1D models are mainly designed for line-type linacs with conventional C-type chicanes, and they generally fail to reproduce 3D simulation results for more complex arclike lattices.

Based on the above discussion, the arclike compressor design in this study contains two main parts. The first focuses on suppressing CSR-induced transverse dilution. The compressor is divided into a prestage and a final-stage section. Since CSR degradation becomes stronger for shorter bunches, only the final-stage compressor is included in the transverse CSR optimization, with the bunch length constrained to decrease monotonically along the reference path.¹ Following Ref. [27], we propose a method to calculate the transverse displacements induced by ss CSR and dr CSR in general compressor lattices. The prestage compressor mainly provides the required R_{56} with a smaller compression factor, so its transverse CSR effects are negligible compared with those in the final stage.

The second part concerns longitudinal matching with nonlinear optics and CSR effects. We propose an analytical model for evaluating CSR-induced modifications to longitudinal compression, including both ss CSR and dr CSR, with reduced dependence on numerical tracking and good agreement with simulations. Combined with the generalized longitudinal matching model in Ref. [18], this

analytical model is used to study nonlinear compression in the proposed compressors.

The paper is organized as follows. Section II introduces the transverse CSR-mitigation method. Section III gives the design of a two-stage asymmetric-QBA arclike compressor and verifies, through simulations, negligible emittance growth for a 100 pC bunch compressed to nearly 50 fs. Section IV presents the CSR-based longitudinal matching model. Section V applies the analytical model to nonlinear compression, using both Gaussian bunches and realistic beam distributions and benchmarks the results against simulations. Section VI summarizes the work and presents the recommended design workflow.

II. CSR EFFECTS IN THE COMPRESSOR

In this section, the ss CSR and the dr CSR are considered. For any longitudinal position z in the bunches, the displacements caused by CSR can be expressed as

$$x_{\text{csr}}(z, s_f) = \int_{s_i}^{s_f} \delta'_{\text{csr}}(z, s) R_{16}^{s \rightarrow s_f} ds, \quad (1)$$

$$x'_{\text{csr}}(z, s_f) = \int_{s_i}^{s_f} \delta'_{\text{csr}}(z, s) R_{26}^{s \rightarrow s_f} ds. \quad (2)$$

Here, s denotes the position along the reference path, the subscripts i and f denote the beamline entrance and exit, respectively, R_{ij} denotes a transfer-matrix element, and the superscript $s \rightarrow s_f$ denotes transport from s to s_f . The CSR-induced relative energy deviation is denoted by $\delta_{\text{csr}}(z, s)$. Its local change per unit path length is $\delta'_{\text{csr}}(z, s) = d\delta_{\text{csr}}(z, s)/ds$. The magnitude of δ'_{csr} generally increases as the bunch is compressed because the local line density increases. In subsequent discussions, the left-hand sides of Eqs. (1) and (2) are referred to as "CSR kicks." Then, to simplify the optimization process, based on the symplectic property of the transfer matrix, Eqs. (1) and (2) can be replaced by [27,28]

$$x_{\text{csr}}(z, s_f) = \int_{s_f}^{s_i} \delta'_{\text{csr}}(z, s') R_{52}(s') ds', \quad (3)$$

$$x'_{\text{csr}}(z, s_f) = - \int_{s_f}^{s_i} \delta'_{\text{csr}}(z, s') R_{51}(s') ds'. \quad (4)$$

Notice that s' is the position index for the reverse beamline in this study; it represents the relative distance from any point to the beamline exit s_f . It satisfies $s' = L - s$, where L is the total length of the beamline. Using $R^{s_f \rightarrow s}$ avoids repeatedly computing the residual forward matrices $R^{s \rightarrow s_f}$, reducing the computational complexity of the CSR-weight evaluation from $O(N^2)$ to $O(N)$, which can significantly improve the efficiency of the optimization process [29].

¹We restrict the bunch length to decrease monotonically along the beamline reference path.

Furthermore, the index s for any longitudinal slice in the bunch varies during the compression process. Therefore, we use the slice index $u = \frac{z}{\sigma_z}$, where σ_z is the root-mean-square (rms) bunch length. The same substitution is applied for s' . It is straightforward to prove that the slice index u (and u') is invariant when only linear compression is considered.

A. ss CSR-kick suppression

Under the substitution from z to u , the δ'_{csr} induced by ss CSR is

$$\delta'_{\text{csr}}(u) = \frac{2N_e r_c}{3^{4/3} \rho^{2/3} \gamma \sigma_z^{4/3}} \Gamma(u), \quad (5)$$

where

$$\Gamma(u) = \int_{u_{\text{tail}}}^u \frac{\lambda(\hat{u}) - \lambda(u)}{(u - \hat{u})^{4/3}} d\hat{u}. \quad (6)$$

Equation (5) can be expressed in a more commonly used form following a convolution integral. Here, r_c is the classical electron radius, N_e is the number of electrons in the bunch, ρ is the radius of the dipole, γ is the relativistic factor, u_{tail} is the slice index of the tail of the bunch, and $\lambda(u)$ is the normalized longitudinal distribution of the bunch, with $\int \lambda(u) du = 1$. Thus, $N_e \lambda(u) / \sigma_z$ gives the electron-number line-density distribution of the bunch. The function $\Gamma(u)$ defined in Eq. (6) remains unchanged under linear compression and is determined solely by the longitudinal distribution of the bunch. By substituting Eq. (5) into Eqs. (3) and (4), we can calculate the ss CSR kick at any u at the exit of the beamline. During the design process, it is common practice to set the magnetic field of the dipoles in the beamline to be uniform, while Eqs. (3) and (4) now involve only three variables related to s' : the $R_{51,2}(s')$ and $\sigma_z(s')$. When only linear compression is considered, the $\sigma_z(s')$ is

$$\sigma_z(s') = [1 + h_i R_{56}(L - s')] \sigma_{z_i}. \quad (7)$$

Here, $h = \frac{\sigma_z}{\sigma_z^2}$ is the energy chirp, where σ_z is the rms energy spread of the bunch. Moreover, for achromatic structures, $R_{56}(s)$ satisfies the following relation:

$$R_{56}(s) = R_{56,\text{tot}} - R_{56}(s'). \quad (8)$$

$R_{56,\text{tot}}$ is the total R_{56} of the beamline. The proof for the relationship described in Eq. (8) can be found in Appendix A. Depending on the chosen reference bunch length, the compression factor $C(s')$ is defined as follows:

$$C(s') = \begin{cases} \frac{1}{1 - \frac{\sigma_z}{\sigma_f} R_{56}(s')} & \text{Case A,} \\ \frac{1}{1 - \frac{\sigma_z}{\sigma_{z_i}} [R_{56,\text{tot}} - R_{56}(s')]} & \text{Case B.} \end{cases} \quad (9)$$

These cases correspond to different optimization scenarios. Case A is suitable for situations where the target bunch length is fixed. Case B is the preferred form when the initial bunch length is fixed or when $R_{56,\text{tot}}$ is strictly constrained, for example, in decompressor design.

For Case A, the ss CSR kicks can be expressed as

$$x_{\text{csr}}(u, s_f) = -k_{\text{csr},B} \Gamma(u) \int_B^{s_f \rightarrow s_0} \frac{C(s')^{4/3} R_{52}(s')}{\rho^{2/3}} ds', \quad (10)$$

$$x'_{\text{csr}}(u, s_f) = -k_{\text{csr},B} \Gamma(u) \int_B^{s_f \rightarrow s_0} \frac{C(s')^{4/3} R_{51}(s')}{\rho^{2/3}} ds', \quad (11)$$

where

$$k_{\text{csr},B} = \frac{2N_e r_c}{3^{4/3} \gamma \sigma_f^{4/3}}.$$

The $\int_B ds'$ indicates that the integral is confined to the region within the bending magnets, as ss CSR exists only inside the dipoles. The coefficients of the integral terms in Eqs. (10) and (11) are constant along the beamline and depend solely on the lattice parameters and the target bunch length. Although relevant studies have demonstrated that CSR-induced effects can be suppressed by specific longitudinal distributions [30], in most practical scenarios, these coefficients cannot be reduced to zero.

To make the ss CSR kick close to zero for any u at s_f , the following conditions need to be met:

$$I_1 = \int_B^{s_f \rightarrow s_0} \frac{R_{52}(s') C(s')^{4/3}}{\rho^{2/3}} ds' \rightarrow 0. \quad (12)$$

$$I_2 = \int_B^{s_f \rightarrow s_0} \frac{R_{51}(s') C(s')^{4/3}}{\rho^{2/3}} ds' \rightarrow 0. \quad (13)$$

B. dr CSR-kick suppression

For the dr CSR, only cases with short bunch lengths are considered. This means that the bunch length satisfies the condition: $\sigma_z \ll \frac{\rho \theta_z^2}{24}$. Under this approximation, the additional energy spread caused by CSR in the subsequent drift can be expressed as [27]

$$\frac{d\delta_{\text{csr},D}(u)}{ds} = \frac{4N_e r_c \lambda(u)}{\gamma \sigma_z \left(\frac{4l^2}{\rho \phi_z} + 2l \right)}. \quad (14)$$

Here, l denotes the distance from the bend exit to the observation point in the downstream drift, and ϕ_z is

defined as

$$\phi_z = \sqrt[3]{\frac{6\sigma_z}{\rho}}. \quad (15)$$

Meanwhile, the velocity compression in the drift is neglected, so the R_{5i} in the nonbend region remains constant. And the σ_z in Eqs. (14) and (15) remains unchanged. Then from the Eqs. (3) and (4), the dr CSR kick can be expressed as

$$\hat{x}_{\text{csr},D}(u) = k_{\text{csr},D}(u) \sum_{i=0}^n R_{52,i} C_i \cdot \ln\left(\frac{2L_{D,i}}{\rho\phi_{i+1}} + 1\right), \quad (16)$$

$$\hat{x}'_{\text{csr},D}(u) = k_{\text{csr},D}(u) \sum_{i=0}^n R_{51,i} C_i \cdot \ln\left(\frac{2L_{D,i}}{\rho\phi_{i+1}} + 1\right). \quad (17)$$

$k_{\text{csr},D} = \frac{2N_e r_c}{\gamma\sigma_f}$. And the subscript i represents the physical quantity after the i th bend of the reverse beamline. Notice that since we are optimizing the reverse beamline, the reference orbit of the reverse beamline is opposite to the actual beam trajectory. Therefore, the parameter ϕ_z should be determined by the $(i+1)$ th bend of the reverse beamline. The details regarding the reverse beamline will be introduced in later sections.

Similarly to the two cases in Eq. (9), the C_i in the above equations can also be divided into two cases:

$$C(s') = \begin{cases} \frac{1}{1 - \frac{\sigma_e}{\sigma_f} R_{56,i}} & \text{Case A,} \\ \frac{1}{1 - \frac{\sigma_e}{\sigma_z} [R_{56,\text{tot}} - R_{56,i}]} & \text{Case B.} \end{cases} \quad (18)$$

According to Eqs. (16) and (17), in order to completely cancel the dr CSR kick, the reverse beamline needs to meet the following two conditions:

$$\Lambda_1 = \sum_{i=0}^n R_{51,i} C_i \ln\left(\frac{2L_{D,i}}{\rho\phi_{i+1}} + 1\right) \rightarrow 0. \quad (19)$$

$$\Lambda_2 = \sum_{i=0}^n R_{52,i} C_i \ln\left(\frac{2L_{D,i}}{\rho\phi_{i+1}} + 1\right) \rightarrow 0. \quad (20)$$

For the beamline that satisfies the above conditions, the impact of the dr CSR kick will be significantly reduced.

In summary, this section has extended the CSR-kick suppression method of Ref. [27] to bunch compressors. The generalized mitigation conditions for both ss CSR and dr CSR kicks were derived within a reverse beamline optimization framework. Unlike the isochronous structures considered in Ref. [27], bunch compressors must satisfy the additional constraint of a prescribed exit bunch length. Therefore, anchoring the reverse optimization to

the exit bunch parameters offers a more appropriate and efficient strategy for compressor design.

III. LINEAR OPTICS DESIGN FOR THE ARCLIKE COMPRESSOR

In this section, the focus is placed on suppressing CSR kicks in the compressors. Under the convention adopted in this study, a chicane compressor has $R_{56} > 0$, and compression requires an initial chirp for which the bunch head has lower energy than the tail. Head-tail interactions, such as CSR and space charge (SC), increase the energy of the head and decrease that of the tail, thereby inducing a local chirp around the bunch center that opposes the initial chirp. The resulting local compression factor and peak current are therefore reduced. In contrast, an arclike compressor can operate with $R_{56} < 0$. In this case, the local chirp induced by the same head-tail interactions enhances the compression, thereby increasing the local compression factor and peak current, as illustrated in Fig. 1.

Although the R_{56} of an arclike compressor can be adjusted to a positive value, this configuration is not recommended because it produces significant decompression in the final bend, especially for structures with large deflection angles, and can consequently induce strong and avoidable collective effects. Governed by the conditions in Eqs. (12), (13), (19), and (20), the optimization process strictly targets linear optics. For any compressor, the compression factor can be expressed as

$$C = \frac{\sigma_{z,i}}{\sigma_{z,f}}, \quad (21)$$

where the subscript i or f represents the parameters at the entrance or exit of beamlines. $C > 1$ corresponds to compression, while $C < 1$ corresponds to decompression. For multistage compressors,² the total compression factor C_{all} can be expressed as

$$C_{\text{all}} = \prod_{i=1}^n C_i, \quad (22)$$

C_i is the compression factor of the i -stage compressor. In most cases, the final-stage compressor has a larger compression factor, which implies a more significant impact from CSR in this stage. Previous studies have demonstrated that asymmetric DBA lattices possess a remarkable capability to suppress ss CSR kicks [12,13]. However, owing to structural limitations, the DBA architecture cannot effectively mitigate dr CSR kicks within a single DBA cell. Furthermore, the second-order term T_{566}

²While conventional multistage compression typically involves distributing compressors at various locations along the main linac, in this study, the term refers to distinct achromatic structures in a single arc section with nonzero R_{56} .

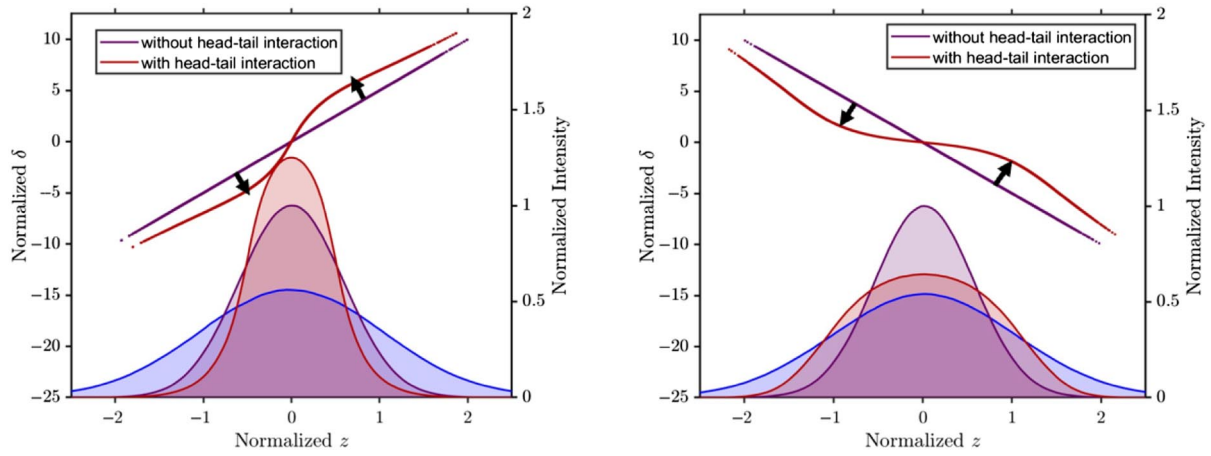


FIG. 1. Effect of head-tail interactions on compression for opposite signs of R_{56} . Left: an arclike compressor with $R_{56} < 0$; right: a chicane compressor with $R_{56} > 0$ under the convention adopted in this study. The blue curve is the initial longitudinal profile, while the purple and red curves are the compressed profiles without and with head-tail interactions, respectively. The interactions increase the local compression for $R_{56} < 0$ and reduce it for $R_{56} > 0$.

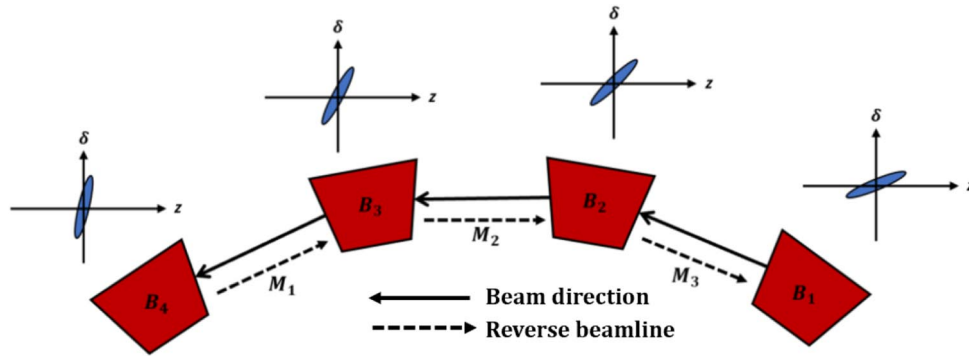


FIG. 2. Schematic diagram of the beamline. The solid line indicates the direction of optimization (reverse beamline), while the dashed line represents the actual physical beam direction.

is inherently fixed for second-order achromatic DBA cells (see Appendix B). Consequently, mitigating the adverse effects of nonlinear compression requires modifying the linear optics design of the DBA. Based on previous studies [27], QBA structures offer a high degree of flexibility, enabling them to simultaneously fulfill specific R_{56} requirements³ and achieve the suppression of both ss CSR and dr CSR kicks, while satisfying various other lattice constraints. Furthermore, unlike isochronous structures, the CSR kicks in the sequence sections exhibit a significantly larger magnitude. Therefore, similar to prior DBA designs, an asymmetric-QBA configuration is adopted in this work. The arclike compressor is divided into two sections. The second section, referred to as the

³Although R_{56} is not strictly fixed to a single value during the design work and is allowed to vary within a small range, practical design considerations dictate that the structure must consistently satisfy the CSR-kick suppression conditions at any arbitrary R_{56} to accommodate different compression factors.

final-stage compressor, is responsible for compressing the bunch to its target length, a regime where the effects induced by CSR become highly pronounced. Conversely, the prestige compressor provides the initial precompression while compensating for the required total deflection angle. Since the bunch-length variation in this stage is relatively small, CSR-kick cancellation is treated under the isochronous approximation. In the present design, a double triple-bend achromat (TBA) with $-I$ matrix is adopted, which can, in principle, fully suppress CSR kicks in an isochronous lattice [31].

As discussed in Sec. II, the proposed design focuses on the reverse beamline, as illustrated in Fig. 2. In contrast to the actual direction of beam propagation (dashed line), the bunch is tracked in reverse during the optimization process (solid line). Consequently, the bunch length increases along the reverse beamline, and its value at any longitudinal position s can be expressed by Eqs. (9) and (18). This reversal approach not only improves computational efficiency but also aligns with practical

design considerations; the bunch length at the compressor exit is of primary concern, as it is strictly constrained by the downstream insertion sections. Therefore, even if the upstream bunch parameters vary, practical operations typically adjust the upstream compression (corresponding to a lower compression factor and a longer bunch length) to maintain fixed exit parameters. As a result, reverse optimization and its associated calculations more authentically reflect the actual compressor design process.

The target final bunch length σ_f was set to 16 μm (800 A at 100 pC). The entrance bunch parameters are taken from Ref. [28], with an initial bunch length of approximately 0.7 mm. According to Eq. (33), an accelerating phase exceeding 10° is required, corresponding to a lower limit of 0.3% for the projected energy spread. The upper limit is capped at 0.5% purely to avoid excessive energy spread, which could otherwise induce higher-order effects in the beamline or degrade the FEL spectrum. The optimization process for the final-stage compressor is divided into two distinct steps. In the first step, the primary focus is placed on the ss CSR kicks within dipoles B_4 to B_2 and the dr CSR kicks in the sequential sections M_1 to M_2 , with the objective functions defined by Eqs. (12), (13), (19), and (20). The imposed constraints include maintaining horizontal and vertical focusing, as well as keeping $R_{16}^{s_f \rightarrow s} > 0$ to prevent oscillations of R_{56} , particularly within B_3 . The optimization variables for this phase are the matching sections M_1 and M_2 , alongside the total length from M_1 to M_3 . In the second step, the optimization objective is to close the dispersion, utilizing the matching section M_3 as the tuning variable.

Figure 3 illustrates the optimization results. To accommodate upstream beam variations, the chirp parameter $h_f = -\sigma_e/\sigma_f$ is considered within the range of -320 to -220 , corresponding to σ_e values from 0.5% to 0.3%. Under the constraints of Eqs. (12) and (13), the values decrease as $|h_f|$ is reduced, with I_1 and I_2 reaching local minima at $h_f = -230$. Regarding Eqs. (19) and (20), $|\Lambda_1|$ reaches a minimum near $h_f = -290$, whereas $|\Lambda_2|$ decreases as h_f approaches -220 . Since I_2 simultaneously approaches zero, the induced CSR kick in x' is effectively suppressed. According to the ss CSR emittance growth model in Eq. (20) of Ref. [14], any remaining emittance dilution can be suppressed by matching to a smaller β function at the beamline exit.

Following the completion of the second step, the parameters for the final-stage compressor are shown in Table I. Bmad simulations including all CSR effects are conducted with varying h_f to verify the capability of the QBA lattice to suppress CSR kicks [23]. In these simulations, the CSR kick is calculated from the Liénard-Wiechert fields after subtracting the SC contribution [32–34]. Because this section focuses exclusively on the impact of CSR kicks

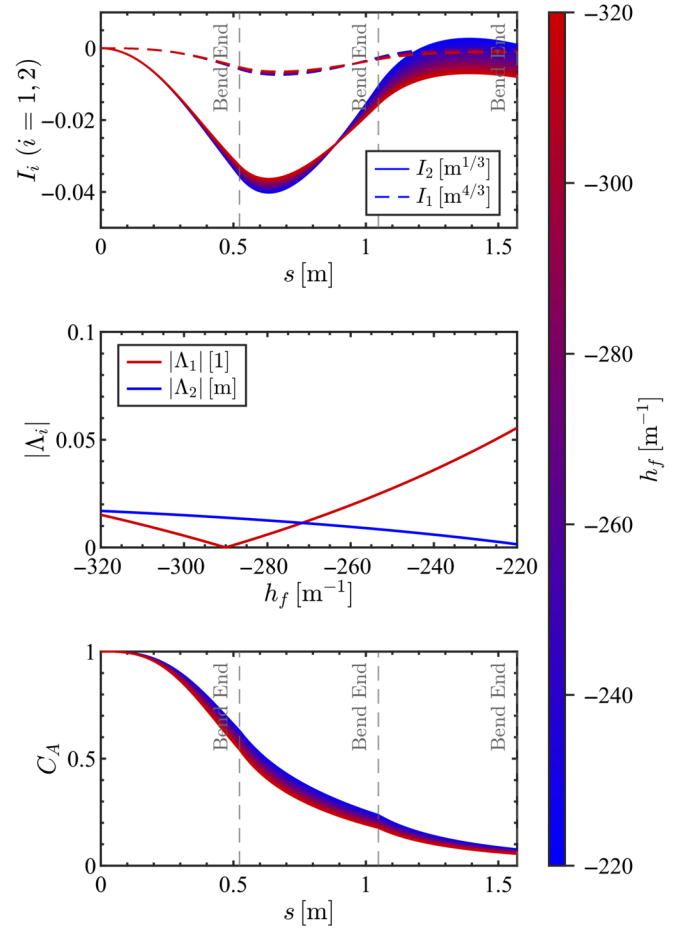


FIG. 3. Optimization results of the first step with varying h_f . Top panel: the integral terms I_1 and I_2 defined in Eqs. (12) and (13). Middle panel: the absolute values of Λ_1 and Λ_2 defined in Eqs. (19) and (20). Bottom panel: the compression factor given by Case A of Eq. (9).

on transverse emittance, the simulations assume an ideal initial Gaussian bunch. Furthermore, sextupole magnets are incorporated to ensure that the second-order terms T_{166} , T_{266} , and T_{566} equal zero. Since σ_f is held constant during optimization, varying h_f requires simultaneous

TABLE I. The parameters of the final-stage compressor.

Parameter	Value [Unit]
Bending angle	$10 \times 4^\circ$
Bending radius	3(m)
Reference energy	1(GeV)
M_1 length	5(m)
M_2 length	4 (m)
M_3 length	5 (m)
M_1 matrix ($R_{11,12,22}$)	-4.51, 1.506 (m), 0.571
M_2 matrix ($R_{11,12,22}$)	-1.6, 2.03 (m), 2.22
$R_{56,\text{tot}}$	-0.057 (m)

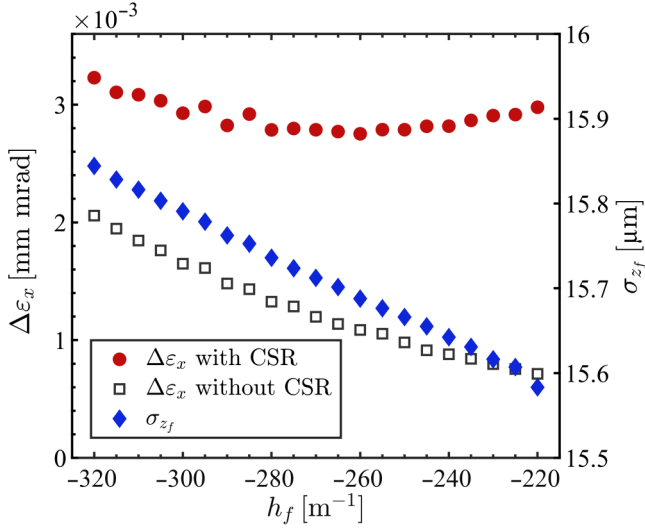


FIG. 4. Exit normalized horizontal emittance changes and final rms bunch length for the final-stage compressor as functions of the final chirp. Here, $\Delta\epsilon_x = \epsilon_{x,f} - \epsilon_{x,i}$. The red circles and black squares show the bmad results with and without CSR effects, respectively, while the blue diamonds show the final rms bunch length.

adjustments to the initial bunch length and energy spread, which must satisfy the following relations:

$$\begin{aligned}\sigma_i &= (1 + h_f R_{56}) \sigma_f, \\ \sigma_e &= |h_f \sigma_f|.\end{aligned}\quad (23)$$

Additionally, the Twiss parameters at the exit of the beamline are fixed at $\alpha = -1.2$ and $\beta = 0.3$ m, and the initial normalized horizontal emittance is 0.5 mm mrad. Figure 4 separately presents the exit emittance changes with and without CSR effects. Over the scanned chirp range, the final rms bunch length remains between 15.58 and 15.84 μm . The difference between the two emittance datasets, representing the additional contribution from CSR effects, remains below 0.0023 mm mrad or less than 0.5% of the initial emittance. Thus, the influence of CSR effects on the horizontal emittance is negligible.

The prestage compressor is responsible for supplementary compression, ensuring that the bunch can be compressed from its upstream bunch length to the target bunch length. Consequently, the R_{56} is

$$R_{56,1} = \frac{\sigma_0 - \sigma_f}{\sigma_e} - R_{56,2}, \quad (24)$$

$R_{56,1}$ and $R_{56,2}$ denote the R_{56} values of the prestage and final-stage compressors, respectively. And σ_e also can be expressed as the formula in Eq. (23). In the prestage compressor, the bunch length is relatively long and the compression factor is much lower than that in the final-stage compressor, resulting in mild CSR kicks.

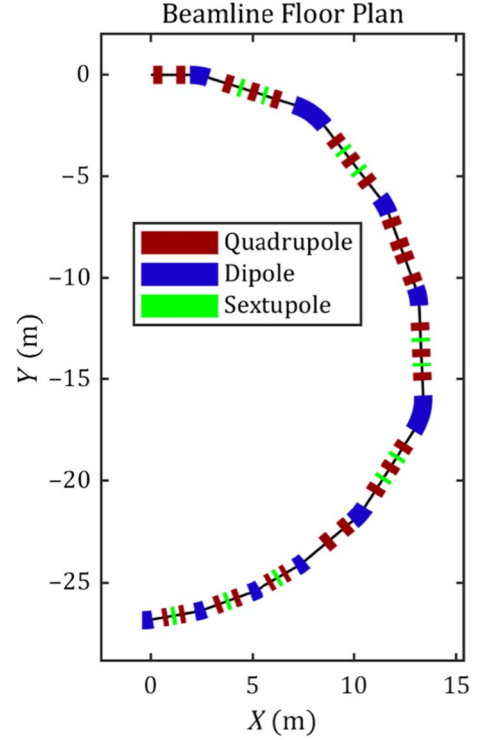


FIG. 5. The floor plan of the 180° arc section.

Consequently, CSR suppression schemes established for isochronous structures can be directly referenced. As discussed above, a double TBA with a $-I$ transfer matrix is employed. Under the constant bunch length approximation, this configuration not only completely suppresses the CSR kicks but also effectively restricts the growth of geometric aberrations, thereby significantly alleviating the matching constraints on the upstream sections.

The floor plan for the total arc section is shown in Fig. 5. The deflection angle of the prestage compressor is set to 140°. Similarly, simulations were performed using an ideal Gaussian bunch. The simulation setup incorporates both the double TBA and the QBA lattice configurations as detailed in Table I. In these simulations, the initial bunch length σ_0 is fixed at 0.7 mm, and the initial emittance is correspondingly set to 0.5 mm mrad. The R_{56} of the prestage compressor is dynamically tuned as a function of the varying chirp parameter h_f :

$$R_{56,1} = \frac{\sigma_0 - \sigma_f}{-h_f \sigma_f} - R_{56,2} \quad (25)$$

Figure 6 separately presents the exit emittance changes with and without CSR effects for the total arc section. Over the scanned chirp range, the emittance changes remain below 0.0471 mm mrad with CSR effects and 0.0399 mm mrad without CSR effects, while the final rms bunch length ranges from 15.09 to 17.58 μm . The separation between the two emittance datasets represents the additional contribution from CSR effects and remains

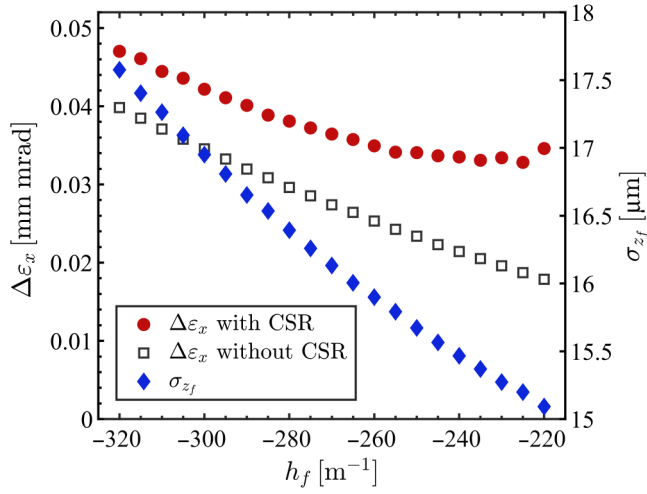


FIG. 6. Exit normalized horizontal emittance changes and final rms bunch length for the total arc section as functions of the final chirp. Here, $\Delta\epsilon_x = \epsilon_{x,f} - \epsilon_{x,i}$. The red circles and black squares show the Bmad results with and without CSR effects, respectively, while the blue diamonds show the final rms bunch length.

below 0.0168 mm mrad or 3.4% of the initial emittance. Although this contribution is larger than that obtained for the final-stage compressor alone, it remains small over the scanned range.

In this section, we presented the linear optics design of the arc section using the reverse beamline approach and the CSR-mitigation conditions derived in Sec. II. The design satisfies the required linear compression factor while strongly suppressing CSR-induced emittance growth. Specifically, for a 100 pC bunch, the additional CSR-induced normalized horizontal emittance change is kept below 0.02 mm mrad over the scanned chirp range. Nonlinear compression effects will be further discussed in the following sections.

IV. LONGITUDINAL NONLINEAR OPTICS IN COMPRESSORS

In the previous section, we introduced methods to mitigate the transverse emittance dilution caused by CSR. However, during the compression process, longitudinal matching also plays a critically important role. Various factors affect nonlinear bunch compression, primarily originating from nonlinear effects in single-particle dynamics—such as rf cavity curvature and nonlinear optics in the arc sections—as well as nonlinear modulations in the longitudinal phase space caused by collective effects, particularly those induced by CSR effects. In what follows in this section, we will investigate these two primary influences and utilize the established model to effectively assess the final bunch compression quality.

A. Nonlinear effects in single-particle dynamics

For any bunch, the energy spread can be expressed as

$$\delta_0 = \sum_{i=1}^n a_{0i} z_0^i. \quad (26)$$

The coefficients a_{0i} can be obtained by fitting the longitudinal distribution of the bunches. And the slice energy spread corresponding to the peak current is set to zero in this study. When $a_{0i} = 0$ for $i \neq 1$, the initial chirp is linear. However, the linear chirp is an overly idealized assumption. In particular, incorporating a harmonic cavity into an ERL can be challenging, leaving a strongly nonlinear energy chirp at the main-linac exit. A longitudinal matching model is introduced to quantify the quality of the compressed bunches [18,19]. The longitudinal position at the exit of the i th element is

$$z_i = z_{i-1} + r_i \cdot \delta_{i-1}(z_0) + t_i \cdot \delta_i(z_0)^2 + u_i \cdot \delta_i(z_0)^3, \quad (27)$$

where

$$\delta_i = \frac{(1 + \delta_{i-1})E_{i-1} + \Delta E_i}{E_i} - 1. \quad (28)$$

E_i is the energy at the exit of the element i . And the compression factor in this model is

$$C_i(z_0) = \frac{1}{z'_i(z_0)}, \quad z'_i(z_0) = \frac{\partial z_i}{\partial z_0}(z_0). \quad (29)$$

z_0 is the longitudinal position of the initial bunch. Further differentiation of Eq. (29) yields the current gradient of the compressed bunch, as detailed in Ref. [18]. Then, the compressed current for any slice is

$$I_i[z_i(z_0)] = I_0(z_0) \cdot C_i(z_0). \quad (30)$$

The overlap of longitudinal slices described here is known as caustic formation in longitudinal phase space. Caustics arise when neighboring particle trajectories merge and the longitudinal mapping becomes multivalued, producing folds in longitudinal phase space and characteristic current spikes. This phenomenon and its connection to catastrophe theory have been established in previous theoretical and experimental studies [35–38]. Such current spikes are typically accompanied by enhanced energy spread and an extremely narrow FWHM and can consequently degrade beam quality.

According to the compression model described in Eqs. (27) and (28), caustic formation occurs when distinct initial positions z_{01} and z_{02} are mapped to the same final position, i.e., $z_i(z_{01}) = z_i(z_{02})$. To maintain a one-to-one longitudinal mapping and avoid caustic formation, we impose the criterion

$$z'_i(z_0) > 0, \quad (31)$$

for any z_0 . Various factors affect nonlinear bunch compression, primarily originating from nonlinear effects in single-particle dynamics—such as rf cavity curvature and nonlinear optics in the arc sections—as well as nonlinear modulations in the longitudinal phase space caused by collective effects, particularly those induced by CSR effects. In what follows in this section, we will investigate these two primary influences and utilize the established model to effectively assess the final bunch compression quality.

In this study, we focus on recirculating machines with a single rf accelerating structure and a single arc section, such as single-pass ERLs and RALs. After performing a Taylor expansion and retaining terms up to the third order, the energy spread at the exit of linac is

$$\begin{aligned} \delta_1 &= [-k_1 k_{\text{rf}} \tan \phi_{\text{rf}} + (1 - k_1) a_{01}] z_0 \\ &\quad - \left[\frac{1}{2} k_1 k_{\text{rf}}^2 - (1 - k_1) a_{02} \right] z_0^2 \\ &\quad + \left[\frac{1}{6} k_1 k_{\text{rf}}^3 \tan \phi_{\text{rf}} + (1 - k_1) a_{03} \right] z_0^3 + O(z_0^4) \\ &= a_{11} z_0 + a_{12} z_0^2 + a_{13} z_0^3 + O(z_0^4). \end{aligned} \quad (32)$$

Here, k_{rf} is the rf wave number, E_0 and E_1 are the reference energies before and after rf acceleration, respectively, and $k_1 = (E_1 - E_0)/E_1$. For a compressor with the third-order momentum compaction close to zero, the minimum bunch length of the compressed bunch is

$$\sigma_{z_2, \min 2} = \sigma_{z_0}^3 \sqrt{\frac{60a_{12}^4}{a_{11}^4} - \frac{60a_{12}^2 a_{13}}{a_{11}^3} + \frac{15a_{13}^2}{a_{11}^2}}. \quad (33)$$

Under this condition, the arc section satisfies $R_{56} = -\frac{1}{a_{11}}$ and $T_{566} = \frac{a_{12}}{a_{11}^2}$. Equation (33) indicates that for a fixed compressed bunch length, the acceleration phase must deviate from the crest by a certain amount when the initial bunch length is predetermined and vice versa. This minimum phase deviation can be determined by solving this equation.

B. Nonlinear modulation induced by ss CSR and dr CSR effects

In this subsection, the ss CSR and dr CSR effects are included in the analysis of bunch compression. Analogous to the transverse displacements induced by CSR effects, the longitudinal offset at the compressor exit is given by

$$\Delta z_{\text{csr}}(u, s_f) = \int_{s_i}^{s_f} \delta'_{\text{csr}}(u, s) R_{56}^{s \rightarrow s_f} ds. \quad (34)$$

Analogous to the calculation of CSR kicks, by replacing the standard mapping with the reverse beamline

calculation, we can determine the longitudinal position at the end of the arc section, $z_i(z_{i-1})$, from Eqs. (5) and (6) and (8) and (9):

$$z_i(z_{i-1}) = z_{i, \text{Eq. (27)}} + k_{\text{csr}, B} \Gamma(u) \int_B^{s_f \rightarrow s_0} \frac{R_{56}(s')}{\rho^{2/3}} C^{4/3} ds', \quad (35)$$

where $z_{i, \text{Eq. (n)}}$ denotes the expression given in Eq. (n). And the energy spread for any z at the exit of this section is

$$\delta_i(z_{i-1}) = \delta_{i-1}(z) + k_{\text{csr}, B} \Gamma(u) \int_B^{s_f \rightarrow s_0} \frac{1}{\rho^{2/3}} C^{4/3} ds'. \quad (36)$$

Similar to ss CSR, the longitudinal modification caused by dr CSR can be formulated as

$$\begin{aligned} z_i(z_{i-1}) &= z_{\text{Eq. (35)}} \\ &\quad + k_{\text{csr}, D} \lambda(u) \sum_{i=0}^n R_{56, i} C_i \ln \left(\frac{2L_{D, i}}{\rho \phi_{i+1}} + 1 \right), \end{aligned} \quad (37)$$

$$\delta_i(u) = \delta_{\text{Eq. (36)}} + k_{\text{csr}, D} \lambda(u) \sum_{i=0}^n C_i \ln \left(\frac{2L_{D, i}}{\rho \phi_{i+1}} + 1 \right). \quad (38)$$

In the fast computational evaluation, $k_{\text{csr}, B \& D}$ is treated as a constant throughout the process. It can be derived from numerical calculations or directly fitted from simulation data, satisfying $k_n = \frac{k_{n,0}}{R_\sigma^{\frac{4}{3}+n}}$ for the fitting coefficients of each order, where k_n is the n th order fitting coefficient, and R_σ is the ratio of the actual bunch length to the reference bunch length used for fitting. It is worth noting that this provides merely an estimation of the longitudinal impact of ss CSR under linear transport. Under more extreme conditions, significant distortion may occur, such as when the bunch shape changes substantially in regions with a large local $R_{56}^{s \rightarrow s_f}$ or a large T_{566} . To mitigate this, the lattice design should avoid oscillations of R_{56} to ensure that the most severe CSR effects occur at the beamline exit, where the local R_{56} and T_{566} are relatively small. Alternatively, k_{csr} must be recalculated when the bunch shape undergoes drastic changes.

In this section, we developed the analytical model for longitudinal matching used in this study. By extending previous models [18], new matching criteria were established to incorporate CSR-induced effects. In addition, a fast computational model was proposed to evaluate the impacts of ss CSR and dr CSR on longitudinal compression in lattices with complex R_{56} distributions.

Benchmarking results against Bmad simulations will be presented in the following section.

V. LONGITUDINAL MATCHING INCLUDING CSR EFFECTS

Building upon the discussion in Sec. IV, this section explores longitudinal matching in bunch compressors, accounting for ss CSR and dr CSR effects. The investigation is structured into two parts. First, an analytical model based on an initial Gaussian bunch is established to derive general theoretical expressions for preliminary evaluations. Second, a start-to-end (S2E) simulation is presented to demonstrate a practical compressor design tailored to realistic beam distributions.

A. Gaussian bunches

1. Compression factor at the initial peak current

Although Sec. IV addresses nonlinear compression without CSR, the compression factor and rms bunch length are modified by CSR. Neglecting the third-order transport of the arc section, the longitudinal mapping can be written as

$$z_i = P(z) + S_B \Gamma(u) + S_D \lambda(u), \quad (39)$$

where $u = z/\sigma_0$, $\lambda(u)$ is the normalized Gaussian profile, and $\Gamma(u)$ is the ss CSR wake function. The non-CSR mapping is

$$P(z) = z + r_i \delta + t_i \delta^2 = \sum_{n=1}^6 c_n z^n, \\ P(u) \equiv P(\sigma_0 u) = \sum_{n=1}^6 d_n u^n, \quad d_n = c_n \sigma_0^n. \quad (40)$$

where the coefficients c_n are listed in Appendix C 1. The ss CSR strength is

$$S_B = k_{\text{csr},B} \mathcal{I}_B, \quad \mathcal{I}_B = \int_B^{s_f \rightarrow s_0} \frac{R_{56}(s)}{\rho^{2/3}} C^{4/3} ds. \quad (41)$$

The dr CSR strength is

$$S_D = k_{\text{csr},D} \mathcal{I}_D, \quad \mathcal{I}_D = \sum_{j=0}^n R_{56,j} C_j \ln \left(\frac{2L_{D,j}}{\rho \phi_{j+1}} + 1 \right). \quad (42)$$

For a Gaussian bunch, the initial peak-current slice is at $z = 0$.⁴ Because $\lambda'(0) = 0$, the dr CSR term does not change the compression factor of this slice, although it contributes to the rms bunch length. The detailed

derivation is given in Appendix C 1. The symbol Γ_E below denotes the Euler gamma function and is distinguished from the CSR wake $\Gamma(u)$. The reciprocal of the center-slice compression factor is

$$\left. \frac{dz_i}{dz} \right|_{z=0} = 1 + r_i a_{11} + \frac{S_B}{\sigma_0} \frac{2^{-2/3} \Gamma_E(1/3)}{\sqrt{2\pi}}. \quad (43)$$

From Eq. (43), the local compression factor for the slice corresponding to the initial peak current can be calculated. To verify the accuracy of this expression, a comparison with the 1D analytical model was performed. In actual multiparticle tracking, accurately determining the local compression factor for a specific region is challenging. Therefore, we compare the expression with the 1D model rather than with direct macroparticle tracking. We also developed an OCTAVE-based code to rapidly simulate bunch compressors, including ss CSR and dr CSR wakes. A brief introduction is given in Appendix D. This code enables the rapid evaluation of longitudinal profiles with CSR effects for any initial distribution. The benchmarks in the following sections also demonstrate the reliability of the 1D model code.

In this comparison, an ideal Gaussian bunch with $a_{0i} = 0$ and $\sigma_0 = 0.7$ mm is utilized. The bunch passes through an rf structure that accelerates it from 10 to 1000 MeV. The accelerating phase is set to 13.3° , corresponding to $a_{11} = 6.37$ derived from Eq. (32). The beam-line configuration follows the structure described in Sec. III, where R_{56} is -0.1539 m. To avoid caustic formation, T_{566} is dynamically adjusted for different bunch charges. The comparison results presented in Fig. 7 demonstrate excellent agreement. The minor discrepancies arise because the final bunch length σ_f is not a strict constant due to the presence of CSR effects, which deviates from the assumption of a fixed σ_f in the analytical formula. At higher bunch charges, caustic formation can no longer be avoided because of the severe CSR effects;

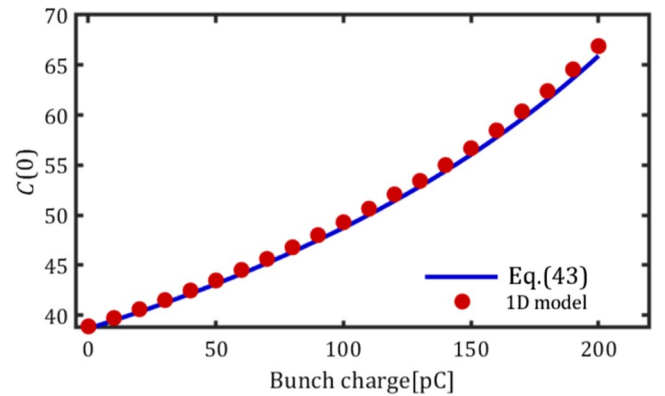


FIG. 7. Comparison of the local compression factor at $z_0 = 0$ obtained from the code-based 1D analytical model and Eq. (43).

⁴For a realistic bunch, the centroid position and energy deviation can also be shifted such that $z = 0$ and $\delta = 0$.

consequently, the local compression factor is unpredictably amplified.

2. Peak current of the compressed bunches

In the previous section, the local compression factor for the slice with the initial peak current is discussed. Since it depends solely on ss CSR effects, it can be rapidly estimated. However, after compression, the slice corresponding to the peak current of the compressed bunch may no longer be the same slice with peak current. In this section, we attempt to utilize a semianalytical solution to determine the postcompression peak current. From Eqs. (31) and (39), the current of the compressed bunch is

$$I_{\text{out}}(u) = Qc \frac{\lambda(u)}{|z'(u)|}. \quad (44)$$

Therefore, the maximum current is not necessarily located at $u = 0$, but satisfies

$$\frac{d}{du} I_{\text{out}}(u) = 0. \quad (45)$$

For a normalized Gaussian bunch profile, this condition becomes

$$-u - \frac{z''(u)}{z'(u)} = 0. \quad (46)$$

The complete peak-current equation can then be written as

$$u[P'(u) + S_B \Gamma'(u)] + P''(u) + S_B \Gamma''(u) - S_D \lambda(u) = 0. \quad (47)$$

Here, the Gaussian identities $\lambda'(u) = -u\lambda(u)$ and $\lambda''(u) = (u^2 - 1)\lambda(u)$ have been used. The derivation is given in Appendix C 2. The dr CSR contribution introduces an exponential term, so the peak-current position should generally be obtained by solving Eq. (47) numerically when dr CSR is included.

For a rapid estimate with only ss CSR, $S_D = 0$, a local expansion around $u = 0$ and one Newton step are used. Define

$$\begin{aligned} G_0 &= Z'(0) = d_1 + S_B \Gamma'_0, \\ G_1 &= Z''(0) = 2d_2 + S_B \Gamma''_0, \\ G_2 &= Z'''(0) = 6d_3 + S_B \Gamma'''_0, \end{aligned} \quad (48)$$

where $d_n = c_n \sigma_0^n$, as defined in Eq. (40); the wake derivatives are given in Appendix C 2. The peak-current location is approximated by

$$u_{\text{pk}} \simeq -\frac{G_1 G_0}{G_0^2 + G_2 G_0 - G_1^2}. \quad (49)$$

The corresponding peak current is

$$I_{\text{pk}} \simeq Qc \frac{\lambda(u_{\text{pk}})}{|P'(u_{\text{pk}}) + S_B \Gamma'(u_{\text{pk}})|}. \quad (50)$$

Furthermore, we evaluate the peak current of the compressed bunch using different methods. The bunch charge is fixed at 100 pC, and the other parameters are identical to those used in the preceding center-slice benchmark. To compute the slice current while avoiding caustic formation, T_{566} is scanned over a specific range. The benchmark results are shown in Fig. 8. In the top panel, the approximate solution for ss CSR agrees well with the 1D analytical model and the simulations of bend CSR in bmad. However, its accuracy degrades when dr CSR is included. Numerically solving Eq. (47) instead yields excellent agreement with both the analytical model and

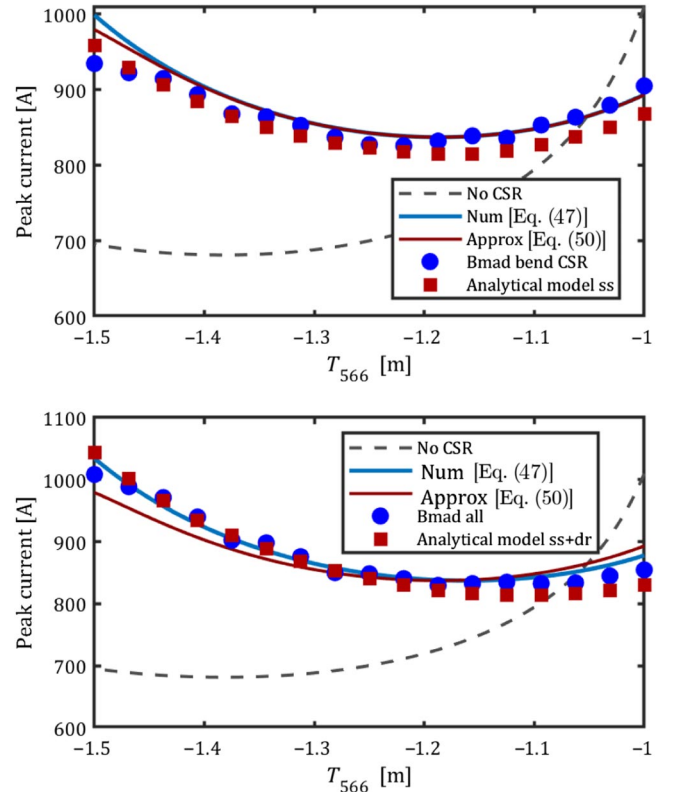


FIG. 8. Peak current of the compressed bunches calculated using different methods. The top panel compares the ss CSR calculations with simulations of bend CSR in bmad, whereas the bottom panel additionally includes dr CSR. Here, “bmad bend CSR” includes CSR in the bending magnets throughout the evolution from the transient to the steady state. The gray dashed curves denote the case without CSR, the blue solid curves denote the numerical solution of Eq. (47), and the red solid curves denote the approximate solution of Eq. (50). The blue circles show the bmad results, and the red squares show the results from the 1D analytical model.

simulations. Nevertheless, because the full equation is more complex, considering only ss CSR is recommended for preliminary estimates.

3. Rms bunch length of the compressed bunches

In addition to the peak current, the rms bunch length represents another vital figure of merit that characterizes the longitudinal quality of the compressed bunches. The rms bunch length at the exit of the beamline is

$$\sigma_{z_i}^2 = \langle \tilde{P}^2 \rangle + 2S_B \langle \tilde{P}\tilde{\Gamma} \rangle + 2S_D \langle \tilde{P}\tilde{\lambda} \rangle + S_B^2 \langle \tilde{\Gamma}^2 \rangle + S_D^2 \langle \tilde{\lambda}^2 \rangle + 2S_B S_D \langle \tilde{\Gamma}\tilde{\lambda} \rangle. \quad (51)$$

Here,

$$\tilde{P} = P - \langle P \rangle, \quad \tilde{\Gamma} = \Gamma - \langle \Gamma \rangle, \quad \tilde{\lambda} = \lambda - \langle \lambda \rangle. \quad (52)$$

The last three terms in Eq. (51) are second order in the CSR strengths. They become important at high charge and explain the breakdown of the weak-CSR approximation. When CSR is treated as a first-order perturbation, these terms are neglected. For compact notation, define

$$H_1 = \frac{\Gamma_E(1/3)}{2^{7/3}\sqrt{\pi}}, \quad H_2 = \frac{\Gamma_E(5/6)}{2^{4/3}\sqrt{\pi}}. \quad (53)$$

The Gaussian moments and the intermediate integrations are derived in Appendix C 3. The resulting first-order expression, including both ss CSR and dr CSR, is

$$\begin{aligned} \sigma_{z_i}^2 \simeq & \langle \tilde{P}^2 \rangle + 2S_B \left[d_1 H_1 + 2d_2 H_2 + \frac{11}{6} d_3 H_1 \right. \\ & \left. + \frac{26}{3} d_4 H_2 + \frac{211}{36} d_5 H_1 + \frac{440}{9} d_6 H_2 \right] \\ & + 2S_D \left[-\frac{d_2}{4\sqrt{\pi}} - \frac{9d_4}{8\sqrt{\pi}} - \frac{105d_6}{16\sqrt{\pi}} \right]. \end{aligned} \quad (54)$$

Figure 9 compares the predictions of Eq. (54), the analytical model, and Bmad simulations. In this benchmark, T_{566} is fixed at -1.1 m, while R_{56} and the rf parameters are the same as in the previous cases. The bunch charge is scanned from 0 to 800 pC.⁵ The weak-CSR approximation in Eq. (54) reproduces the CSR-induced bunch-length variation well below 300 pC, but loses accuracy at higher charges because the higher-order coupling terms in Eq. (51) become non-negligible. The slice-based analytical model, by contrast, tracks the bunch-length evolution through a weighted slice-wise calculation and remains more accurate than the weak-CSR approximation

⁵Caustic formation does not need to be avoided when calculating the bunch length.

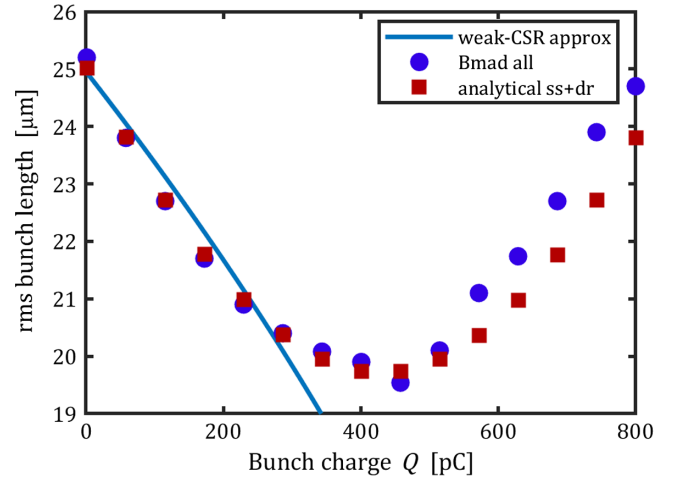


FIG. 9. Bunch length calculated using different methods with varying bunch charge, with $T_{566} = -1.1$ m. The R_{56} and rf cavity parameters are kept fixed at the values used in the other benchmarks in this section.

in stronger CSR regimes. At extremely high charge, however, the local CSR wake modifies the subsequent longitudinal evolution of the bunch. Consequently, the modeled CSR wake can no longer be represented as a direct linear superposition of the individual wake contributions, because more complex coupling develops between the CSR wake and the evolving bunch distribution. This effect leads to a larger discrepancy between the analytical model and the simulation above 600 pC than in the low-charge regime. Nevertheless, the analytical model still provides a useful reference for the overall trend in this regime. It can therefore be used to identify suitable parameter ranges and initial solutions for subsequent optimization based on simulations.

Although the weak-CSR approximation is limited to charges below 300 pC for this lattice, it is still useful in practical designs. In ERLs with high compression factors, the nominal bunch charge is usually below 100 pC, where the approximation remains valid. In high-charge linacs, chicane-type compressors typically exhibit a smaller R_{56} evolution, especially in the final compressor stage, which reduces $\mathcal{I}_B$ and can also preserve the validity of the weak-CSR approximation.

B. Actual bunches

During actual bunch compression, the beam profile frequently deviates from an ideal Gaussian distribution. This complex longitudinal structure can induce unpredictable dynamics, including the emergence of multi-spike structures. Consequently, relying exclusively on the central slice compression factor or the rms bunch length is inadequate. Based on the 1D analytical model, the current-weighted rms bunch length can be formulated as follows:

TABLE II. The initial and compressed bunch parameters.

Parameter	Value	Unit
Bunch charge	100	pC
Bunch energy	1000	MeV
Initial rms bunch length	690	μm
Target rms bunch length	16	μm
Initial peak current	15	A
Normalized horizontal emittance	0.47	mm mrad
Normalized vertical emittance	0.45	mm mrad

$$\sigma_z = \sqrt{\frac{\sum_{i=1}^N (z_i - \bar{z})^2 I_i}{\sum_{i=1}^N I_i}}. \quad (55)$$

Other statistical quantities take similar forms.

The realistic bunch distribution at the merger exit is obtained from start-to-end simulations using ASTRA [39] and Bmad. Based on the previous work in Ref. [28], the bunch parameters can be found in Table II. During the optimization process, we employed a multiobjective

genetic algorithm (MOGA), specifically nondominated sorting genetic algorithm II [40]. In the optimization procedure, the peak current (or the target bunch length) is treated as a constraint, necessitating a direct assessment of the longitudinal shape. Given the absence of a standard formulation to quantify longitudinal distribution quality, we introduce a metric defined as the rms difference between the compressed bunch profile and its equivalent-bunch-length Gaussian distribution. Consequently, the objective functions in this process are

$$f_1 = |\sigma_f - 16 \mu\text{m}|, \quad f_2 = \sqrt{\frac{\sum_{i=1}^N [\lambda_i - \lambda(u_i * \sigma_f)]^2 I_i}{\sum_{i=1}^N I_i}}. \quad (56)$$

During the optimization process, the origin of the longitudinal coordinate system is automatically fixed at the peak-current slice, setting $u_i = 0$ for this slice. The optimization variables include the acceleration phase in the range of $[13^\circ, 20^\circ]$, T_{566} within $[-3, 3]$ m, and $dR56$ within $[-0.02, 0.02]$ m. The baseline of the $R_{56, \text{tot}}$ is -0.15 .

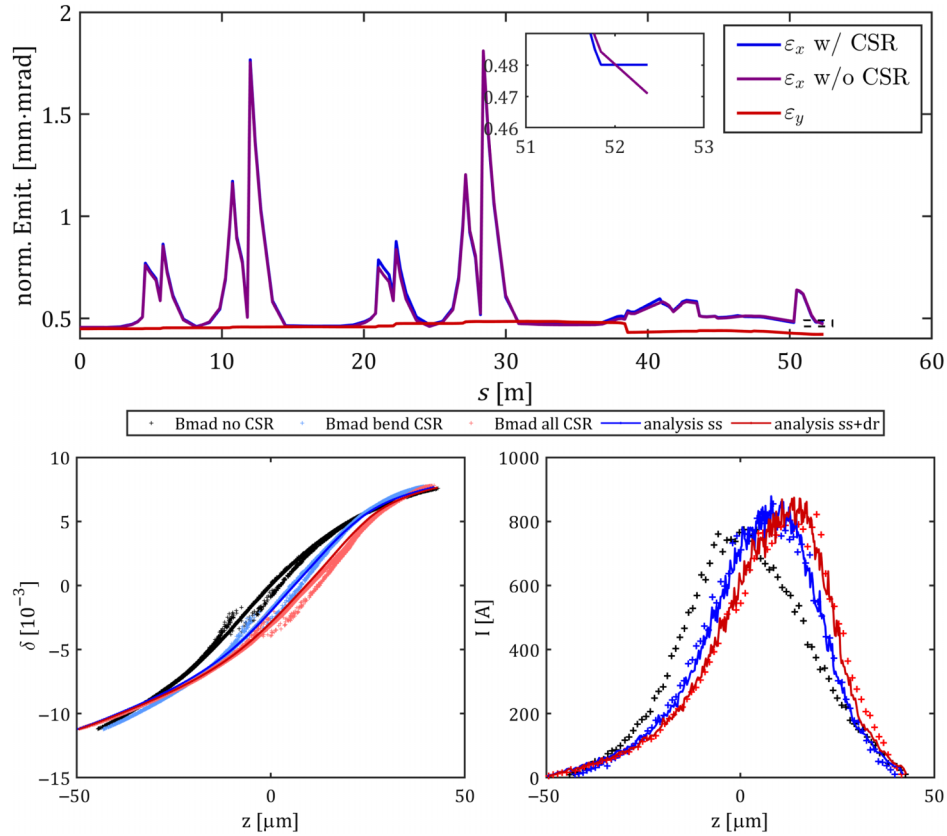


FIG. 10. Results from Bmad simulations and the 1D analytical model. The top panel illustrates the evolution of the horizontal emittance from Bmad, presenting two cases: with and without CSR effects. An inset is provided in the upper section to compare the emittance difference near the beamline exit, thereby isolating the emittance growth induced purely by CSR. The two plots in the bottom panel display the longitudinal phase space (left) and the longitudinal profile (right) obtained from Bmad simulations and the 1D analytical model. The simulation results are presented for three scenarios: without CSR, with bend CSR, and with all CSR effects included.

Crucially, when the condition in Eq. (31) is not met, the optimization routine terminates and returns a large penalty value.

Benefiting from the analytical model introduced in this study, the compression dynamics, including ss CSR and dr CSR effects, can be computed within milliseconds. The corresponding simulation results are illustrated in Fig. 10. First, the initial Twiss parameters are scanned with the objective of minimizing emittance growth. Although CSR effects do not directly alter the vertical emittance, for a realistic bunch, the mismatch among individual slices can still degrade the final vertical emittance [41]. Therefore, matching the entrance Twiss parameters in the vertical plane is also essential. The emittance evolution is illustrated in the top panel of Fig. 10. The horizontal emittance at the exit of the compressor is 0.47 and 0.48 mm mrad for the cases without and with CSR effects, respectively. Consequently, the total horizontal emittance growth along the arc section is less than 0.2 mm mrad, with the growth induced purely by CSR effects being restricted to less than 0.1 mm mrad. This behavior is highly consistent with the simulation results obtained for an ideal Gaussian bunch.

The longitudinal phase space (left) and longitudinal profile (right) are displayed in the bottom panel. The left plot compares three cases for the compressed bunches: no CSR, bend CSR in Bmad, and all CSR in Bmad. Additionally, predictions from the 1D analytical model are provided, specifically incorporating purely ss CSR [Eqs. (35) and (36)] and combined ss CSR and dr CSR effects [Eqs. (37) and (38)]. The 1D analytical model exhibits excellent agreement with the Bmad simulations, effectively validating and explaining the phase space distortions induced by ss CSR and dr CSR. Similarly, as predicted, the bunch is compressed to a peak current of nearly 1 kA, and the CSR-induced changes in the longitudinal profile are also well predicted. The localized discrepancies in the current profile primarily arise from the effects of high-order transverse-longitudinal coupling terms and differences in the smoothing methods. Overall, this 1D analytical model provides an accurate and rapid prediction of CSR effects in compressors, even for complex arc structures.

VI. CONCLUSION

In this study, we proposed an arclike compressor design with integrated CSR mitigation. For the transverse plane, objective functions were introduced to suppress both ss CSR and dr CSR effects. Based on these criteria, an asymmetric QBA lattice was designed to provide sufficient degrees of freedom for satisfying multiple constraints. Numerical simulations demonstrated that a 100 pC bunch can be compressed to 16 μm , corresponding to a peak current above 1 kA, while maintaining negligible emittance growth.

For the longitudinal plane, we developed a fast analytical model by extending previous longitudinal compression models to include ss CSR and dr CSR effects. The model shows good agreement with simulations including all CSR effects and provides an efficient tool for rapid longitudinal matching in complex arclike compressors. It was first used to derive generalized CSR-induced compression behavior for ideal Gaussian bunches and then combined with MOGA to optimize realistic beam distributions. During this optimization, a noncaustic criterion was imposed to maintain a one-to-one longitudinal mapping and avoid artificially high peak currents with poor longitudinal beam quality. Applied to a realistic beam distribution at the merger exit, the proposed framework produces a compressed bunch with high transverse and longitudinal beam quality.

Crucially, we propose a design strategy for arclike compressors capable of drastically suppressing CSR effects based on the reverse beamline approach. Furthermore, we provide a set of auxiliary codes to assist with the design process, as described in Appendix D. In summary, the design procedure for the arc section can be outlined in the following steps: (i) Define the required parameters postcompression, including but not limited to the final bunch length, energy spread, accelerating phase [e.g., the constraints given in Eq. (33)], and the compression factor corresponding to the peak current described in Eq. (43), as well as the initial bunch parameters at the compressor entrance. (ii) For a multibend achromat (MBA, specifically a QBA in this study), the optimization should ensure that the first through the $(M - 1)$ th dipoles in the reverse beamline satisfy the conditions in Eqs. (12), (13), (19), and (20), and that the compression factor aligns with case A in Eqs. (9) and (18). During this process, oscillations of the R_{56} in the final stage must be strictly avoided to mitigate the adverse effects associated with ultrashort bunches. (iii) Utilize the final dipole of the reverse beamline and the matching section between the final and penultimate dipoles to render the beamline completely achromatic. (iv) The primary objective is to determine the auxiliary parameters, including the bending angles and R_{56} . The configuration of the precompressor (double TBA structure with minus I in this study) is established via Eq. (24). (v) Sextupoles are integrated to ensure that the second-order dispersion terms T_{166} and T_{266} vanish at the exit of each compressor stage while simultaneously tuning the second-order longitudinal dispersion (T_{566}). Taking into account the nonlinear optics described in Eqs. (27) and the CSR effects in Eqs. (37) and (38), T_{566} is optimized to satisfy the condition defined in Eq. (31). Additional constraints, such as the skewness and kurtosis of the longitudinal distribution, may also be applied to evaluate the final bunch compression quality. (vi) A systematic parameter scan is conducted to minimize the transverse emittance growth caused by CSR. The optimization

variables include the initial Twiss functions and the parameters of the matching sections between successive compressor stages.

A notable limitation is that the longitudinal model in this study focuses exclusively on ss CSR and dr CSR effects. For the Gaussian bunches considered in Sec. III, the full-arc CSR-induced energy spread predicted by the analytical model differs by less than 8% from the results including transient CSR (tr-CSR, not shown). Nevertheless, tr-CSR may become non-negligible under other operating conditions, for example, when a relatively long bunch enters a bend and the transition angle θ_T is comparable to the bend angle θ_B [42]. To address such cases, tr-CSR needs to be incorporated into the analytical model; related research based on semianalytical solutions for the CSR wake is currently underway. Furthermore, for the low-energy sections of ERLs, longitudinal space charge must also be taken into account, along with other relevant collective effects. Finally, the stability of this method for large-scale facilities, such as multipass ERLs with multiple arc sections, requires further investigation.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: PROOF OF THE RELATIONSHIP DESCRIBED IN EQ. (8)

For any point s in the beamline with nonzero dispersion, the $R_{56}(s)$ can be expressed as

$$\begin{aligned} R_{56}(s) &= \int_{s_i}^s \frac{R_{16}(s')}{\rho} ds' \\ &= \int_{s_i}^{s_f} \frac{R_{16}(s')}{\rho} ds' - \int_s^{s_f} \frac{R_{16}(s')}{\rho} ds', \end{aligned} \quad (\text{A1})$$

where

$$R_{56,\text{tot}} = \int_{s_i}^{s_f} \frac{R_{16}(s')}{\rho} ds'. \quad (\text{A2})$$

Notice that the s' in this section is unrelated to the s' in the main text and is merely an intermediate variable used in the proof. Then for the total beamline, the transfer matrix is

$$R^{s_i \rightarrow s_f} = R^{s \rightarrow s_f} R^{s_i \rightarrow s}. \quad (\text{A3})$$

For the achromat structures, we have $R_{16}^{s_i \rightarrow s_f} = 0$ and $R_{26}^{s_i \rightarrow s_f} = 0$. In the absence of coupling between the x and y directions in the beamline, Eq. (A3) gives

$$-R_{16}^{s \rightarrow s_f} = R_{11}^{s \rightarrow s_f} R_{16}^{s_i \rightarrow s} + R_{12}^{s \rightarrow s_f} R_{26}^{s_i \rightarrow s}, \quad (\text{A4})$$

$$-R_{26}^{s \rightarrow s_f} = R_{21}^{s \rightarrow s_f} R_{16}^{s_i \rightarrow s} + R_{22}^{s \rightarrow s_f} R_{26}^{s_i \rightarrow s}. \quad (\text{A5})$$

Meanwhile, for the reverse beamline of the achromat structures, we also have $R_{51}^{s_f \rightarrow s_i} = 0$ and $R_{52}^{s_f \rightarrow s_i} = 0$. Then

$$-R_{51}^{s_f \rightarrow s} = R_{51}^{s \rightarrow s_i} R_{11}^{s_f \rightarrow s} + R_{52}^{s \rightarrow s_i} R_{21}^{s_f \rightarrow s}, \quad (\text{A6})$$

$$-R_{52}^{s_f \rightarrow s} = R_{51}^{s \rightarrow s_i} R_{12}^{s_f \rightarrow s} + R_{52}^{s \rightarrow s_i} R_{22}^{s_f \rightarrow s}. \quad (\text{A7})$$

The relationship $R^{-1} = -SR^T S$ is applied in the derivation above, where S is the standard symplectic matrix and the superscript T denotes matrix transposition. A concise demonstration of this transformation is also provided in Eqs. (6)–(8) of Ref. [28]. From Eqs. (A4)–(A7), it is easy to derive:

$$R_{16}^{s_i \rightarrow s} = R_{16}^{s_f \rightarrow s}, \quad (\text{A8})$$

$$R_{26}^{s_i \rightarrow s} = -R_{26}^{s_f \rightarrow s}. \quad (\text{A9})$$

Substituting Eq. (A8) into Eq. (A1), the $R_{56}(s)$ meets:

$$\begin{aligned} R_{56}(s) &= \int_{s_i}^{s_f} \frac{R_{16}^{s_i \rightarrow s'}}{\rho} ds' - \int_s^{s_f} \frac{R_{16}^{s_i \rightarrow s'}}{\rho} ds' \\ &= \int_{s_i}^{s_f} \frac{R_{16}^{s_i \rightarrow s'}}{\rho} ds' - \int_s^{s_f} \frac{R_{16}^{s_f \rightarrow s'}}{\rho} ds' \\ &= R_{56,\text{tot}} - R_{56}^{s_f \rightarrow s}. \end{aligned} \quad (\text{A10})$$

Equation (8) is proven.

APPENDIX B: T_{566} FOR SECOND-ORDER ACHROMATIC DBA CELLS

For an off-energy particle, the horizontal dispersive orbit can be expanded as

$$x(s) = R_{16}(s)\delta + T_{166}(s)\delta^2 + \dots, \quad (\text{B1})$$

$$x'(s) = R_{26}(s)\delta + T_{266}(s)\delta^2 + \dots \quad (\text{B2})$$

In a bending magnet, the path-length increment can be approximated by

$$dz \simeq \frac{x(s)}{\rho(s)} ds + \frac{1}{2} x'^2(s) ds. \quad (\text{B3})$$

Substituting Eqs. (B1) and (B2) into Eq. (B3), and keeping terms up to second order in δ , gives

$$dz \simeq \frac{R_{16}(s)}{\rho(s)} \delta ds + \left[\frac{T_{166}(s)}{\rho(s)} + \frac{1}{2} R_{26}^2(s) \right] \delta^2 ds. \quad (\text{B4})$$

Therefore, R_{56} is

$$R_{56} = \int \frac{R_{16}(s)}{\rho(s)} ds, \quad (\text{B5})$$

and the second-order longitudinal dispersion is

$$T_{566} = \int \left[\frac{T_{166}(s)}{\rho(s)} + \frac{1}{2} R_{26}^2(s) \right] ds. \quad (\text{B6})$$

Eq. (B6) shows that, once the linear optics is fixed, the geometric term

$$\frac{1}{2} \int R_{26}^2(s) ds, \quad (\text{B7})$$

is fixed. Therefore, sextupoles can affect T_{566} only indirectly through the second-order horizontal dispersion $T_{166}(s)$.

For an isolated DBA cell, denote the second-order dispersion vector at the entrance of the second bend as

$$U_a = \begin{bmatrix} T_{166,a} \\ T_{266,a} \end{bmatrix}, \quad (\text{B8})$$

and that at the exit as

$$U_f = \begin{bmatrix} T_{166,f} \\ T_{266,f} \end{bmatrix}. \quad (\text{B9})$$

The second bend transports the second-order dispersion vector as

$$U_f = R_2 U_a + q_2, \quad (\text{B10})$$

where R_2 is the horizontal first-order transfer matrix of the second bend, and q_2 is the fixed second-order term generated by the second bend. The second-order achromatic condition requires

$$U_f = 0. \quad (\text{B11})$$

Combining Eqs. (B10) and (B11) gives

$$R_2 U_a + q_2 = 0, \quad (\text{B12})$$

and therefore

$$U_a = -R_2^{-1} q_2. \quad (\text{B13})$$

Thus, under the second-order achromatic condition, the required values of $T_{166,a}$ and $T_{266,a}$ at the entrance of the second bend are uniquely fixed. Since the second bend itself is fixed, the evolution of $T_{166}(s)$ inside the second bend is also fixed. Substituting the fixed $T_{166}(s)$ into Eq. (B6) shows that T_{566} is also fixed. Therefore, for an isolated DBA cell with fixed dipoles and fixed linear optics, once T_{166} and T_{266} are closed, T_{566} is not an independent adjustable quantity.

APPENDIX C: DERIVATIONS OF THE GAUSSIAN-BUNCH FORMULAS

1. Center-slice compression factor

Substituting Eq. (32) into the non-CSR part of Eq. (39) gives Eq. (40), with

$$\begin{aligned} c_1 &= 1 + r_i a_{11}, \\ c_2 &= r_i a_{12} + t_i a_{11}^2, \\ c_3 &= r_i a_{13} + 2t_i a_{11} a_{12}, \\ c_4 &= t_i (2a_{11} a_{13} + a_{12}^2), \\ c_5 &= 2t_i a_{12} a_{13}, \\ c_6 &= t_i a_{13}^2. \end{aligned} \quad (\text{C1})$$

For the initial peak-current slice, the derivative of Eq. (39) is

$$\left. \frac{dz_i}{dz} \right|_{z=0} = 1 + r_i a_{11} + \frac{S_B}{\sigma_0} \Gamma'(0) + \frac{S_D}{\sigma_0} \lambda'(0). \quad (\text{C2})$$

The term $t_i \delta^2$ does not contribute because $\delta(0) = 0$. For a finite u_{tail} , the derivative of the regularized CSR wake is

$$\Gamma'(0) = \int_{u_{\text{tail}}}^0 \frac{\lambda'(\hat{u}) - \lambda'(0)}{(-\hat{u})^{4/3}} d\hat{u} + \frac{\lambda(u_{\text{tail}}) - \lambda(0)}{(-u_{\text{tail}})^{4/3}}. \quad (\text{C3})$$

For a normalized Gaussian distribution and $u_{\text{tail}} \rightarrow -\infty$, $\lambda'(0) = 0$, and

$$\begin{aligned} \Gamma'(0) &= \int_0^\infty \frac{\lambda'(-t)}{t^{4/3}} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty t^{-1/3} \exp\left(-\frac{t^2}{2}\right) dt. \end{aligned} \quad (\text{C4})$$

Using

$$\int_0^\infty t^{a-1} e^{-\beta t^2} dt = \frac{1}{2} \beta^{-a/2} \Gamma_E\left(\frac{a}{2}\right), \quad (\text{C5})$$

with $a = 2/3$ and $\beta = 1/2$ gives

$$\Gamma'(0) = \frac{2^{-2/3} \Gamma_E(1/3)}{\sqrt{2\pi}}, \quad (\text{C6})$$

which yields Eq. (43).

2. Compressed peak current

Differentiating Eq. (39) with respect to u gives

$$\begin{aligned} z'(u) &= P'(u) + S_B \Gamma'(u) + S_D \lambda'(u), \\ z''(u) &= P''(u) + S_B \Gamma''(u) + S_D \lambda''(u). \end{aligned} \quad (\text{C7})$$

Substitution into Eq. (46) gives

$$\begin{aligned} u[P'(u) + S_B \Gamma'(u) + S_D \lambda'(u)] + P''(u) \\ + S_B \Gamma''(u) + S_D \lambda''(u) = 0. \end{aligned} \quad (\text{C8})$$

For a Gaussian distribution, $\lambda'(u) = -u\lambda(u)$ and $\lambda''(u) = (u^2 - 1)\lambda(u)$, so that

$$uS_D \lambda'(u) + S_D \lambda''(u) = -S_D \lambda(u). \quad (\text{C9})$$

This gives Eq. (47). If only ss CSR is retained, $S_D = 0$, and the wake derivatives near $u = 0$ are

$$\begin{aligned} \Gamma'(u) &\simeq \Gamma'_0 + \Gamma''_0 u + \frac{1}{2} \Gamma'''_0 u^2, \\ \Gamma''(u) &\simeq \Gamma''_0 + \Gamma'''_0 u, \end{aligned} \quad (\text{C10})$$

where

$$\begin{aligned} \Gamma'_0 &= \frac{2^{-2/3} \Gamma_E(1/3)}{\sqrt{2\pi}}, \\ \Gamma''_0 &= \frac{2^{4/3} \Gamma_E(5/6)}{\sqrt{\pi}}, \\ \Gamma'''_0 &= -\frac{7}{3} \frac{2^{-7/6} \Gamma_E(1/3)}{\sqrt{\pi}}. \end{aligned} \quad (\text{C11})$$

Applying one Newton step to Eq. (46) from $u = 0$, then gives the peak-current location and current stated in the main text.

3. Rms bunch length

Under the weak-CSR approximation, the terms that are second order in S_B and S_D are omitted from Eq. (51), giving

$$\sigma_{z_i}^2 \simeq \langle \tilde{P}^2 \rangle + 2S_B \langle \tilde{P} \tilde{\Gamma} \rangle + 2S_D \langle \tilde{P} \tilde{\lambda} \rangle. \quad (\text{C12})$$

For a normalized Gaussian distribution,

$$\langle u \rangle = 0, \quad \langle u^2 \rangle = 1, \quad \langle u^4 \rangle = 3, \quad \langle u^6 \rangle = 15. \quad (\text{C13})$$

The mean of P and the bunch-length variance without CSR are

$$\langle P \rangle = d_2 + 3d_4 + 15d_6, \quad (\text{C14})$$

$$\begin{aligned} \langle \tilde{P}^2 \rangle &= d_1^2 + (6d_1 d_3 + 2d_2^2) \\ &+ (30d_1 d_5 + 24d_2 d_4 + 15d_3^2) \\ &+ (180d_2 d_6 + 210d_3 d_5 + 96d_4^2) \\ &+ (1800d_4 d_6 + 945d_5^2) + 10170d_6^2. \end{aligned} \quad (\text{C15})$$

The first-order ss CSR correction can be expressed as

$$\langle \tilde{P} \tilde{\Gamma} \rangle = \sum_{n=1}^6 d_n M_n, \quad (\text{C16})$$

where

$$M_n = \langle (u^n - \langle u^n \rangle) (\Gamma(u) - \langle \Gamma(u) \rangle) \rangle. \quad (\text{C17})$$

With $u_{\text{tail}} \rightarrow -\infty$,

$$\Gamma(u) = \int_0^\infty \frac{\lambda(u-t) - \lambda(u)}{t^{4/3}} dt. \quad (\text{C18})$$

Defining

$$\begin{aligned} I_n &= \langle u^n \Gamma(u) \rangle = \int_0^\infty t^{-4/3} [J_n(t) - J_n(0)] dt, \\ J_n(t) &= \int_{-\infty}^\infty u^n \lambda(u) \lambda(u-t) du, \end{aligned} \quad (\text{C19})$$

and using the shifted Gaussian identity

$$\lambda(u) \lambda(u-t) = \frac{1}{2\pi} \exp \left[-\left(u - \frac{t}{2}\right)^2 - \frac{t^2}{4} \right], \quad (\text{C20})$$

one has

$$J_n(t) = \frac{e^{-t^2/4}}{2\sqrt{\pi}} P_n(t), \quad (\text{C21})$$

where

$$\begin{aligned}
P_0(t) &= 1, \\
P_1(t) &= \frac{t}{2}, \\
P_2(t) &= \frac{t^2}{4} + \frac{1}{2}, \\
P_3(t) &= \frac{t^3}{8} + \frac{3t}{4}, \\
P_4(t) &= \frac{t^4}{16} + \frac{3t^2}{4} + \frac{3}{4}, \\
P_5(t) &= \frac{t^5}{32} + \frac{5t^3}{8} + \frac{15t}{8}, \\
P_6(t) &= \frac{t^6}{64} + \frac{15t^4}{32} + \frac{45t^2}{16} + \frac{15}{8}.
\end{aligned} \tag{C22}$$

The required integrations follow from

$$\int_0^\infty t^{a-1} e^{-t^2/4} dt = 2^{a-1} \Gamma_E \left(\frac{a}{2} \right), \tag{C23}$$

and

$$\int_0^\infty t^{-4/3} (e^{-t^2/4} - 1) dt = 2^{-4/3} \Gamma_E \left(\frac{5}{6} \right). \tag{C24}$$

In particular,

$$\langle \Gamma(u) \rangle = -\frac{3\Gamma_E(5/6)}{2^{4/3}\sqrt{\pi}}. \tag{C25}$$

Using H_1 and H_2 defined in the main text, one obtains

$$\begin{aligned}
M_1 &= H_1, & M_2 &= 2H_2, & M_3 &= \frac{11}{6}H_1, \\
M_4 &= \frac{26}{3}H_2, & M_5 &= \frac{211}{36}H_1, & M_6 &= \frac{440}{9}H_2.
\end{aligned} \tag{C26}$$

Similarly, the dr CSR correction is

$$\langle \tilde{P} \tilde{\lambda} \rangle = \sum_{n=1}^6 d_n Q_n, \tag{C27}$$

where

$$Q_n = \langle (u^n - \langle u^n \rangle) (\lambda(u) - \langle \lambda(u) \rangle) \rangle \tag{C28}$$

and

$$\langle \lambda(u) \rangle = \int_{-\infty}^\infty \lambda^2(u) du = \frac{1}{2\sqrt{\pi}}. \tag{C29}$$

Thus,

$$\begin{aligned}
Q_1 &= 0, & Q_2 &= -\frac{1}{4\sqrt{\pi}}, & Q_3 &= 0, \\
Q_4 &= -\frac{9}{8\sqrt{\pi}}, & Q_5 &= 0, & Q_6 &= -\frac{105}{16\sqrt{\pi}}.
\end{aligned} \tag{C30}$$

Therefore,

$$\langle \tilde{P} \tilde{\lambda} \rangle = -\frac{d_2}{4\sqrt{\pi}} - \frac{9d_4}{8\sqrt{\pi}} - \frac{105d_6}{16\sqrt{\pi}}. \tag{C31}$$

Substitution of these moments into the first-order part of Eq. (51) gives Eq. (54).

APPENDIX D: AN OCTAVE-BASED CODE FOR ACCELERATOR COMPUTATION [43]

This program can independently calculate accelerator-physics quantities, including Twiss-parameter evolution, linear and nonlinear optics, the beamline floor plan shown in Fig. 5, and the CSR kicks discussed in Sec. II.

Based on Sec. IV, a longitudinal matching code was developed. The command is `out = csr_calc_slice(z_slice, delta_slice, I_slice, beamline, gamma_rel, T566, NAME, VALUE, ...)`. Here, `z_slice`, `delta_slice`, and `I_slice` are the slice longitudinal position, correlated energy deviation, and initial current, respectively. The beamline input follows the same format as in `CSRkickscal()`. The parameter `gamma_rel` is the reference relativistic factor, and `T566` is the second-order longitudinal dispersion of the arc section. Additional optional inputs are specified by NAME-VALUE pairs and are listed in Table III.

The optional parameter `sig` overrides the default value of σ_e , which is calculated as the current-weighted standard deviation of `delta_slice`. Similarly, `sigzf` overrides σ_{zf} , which is otherwise estimated from Eq. (55) without CSR effects. The parameter `smooth_win` controls the Gaussian smoothing window applied to $I(z)$. The flags `plot_flag` and `drift_csr` enable diagnostic plots and dr CSR calculation, respectively. The parameter `dR56` denotes an additional R_{56} applied during compression. Since it is applied only at the exit of the reverse

TABLE III. Optional name-value parameters.

Parameter	Default	Unit
<code>sig</code>	[]	
<code>sigzf</code>	[]	m
<code>smooth_win</code>	5	
<code>drift_csr</code>	true	
<code>plot_flag</code>	1	
<code>dR56</code>	0	m

TABLE IV. Output of `csr_calc_slice`.

Column	Unit	Expression
z_0	m	input slice z sorted in order
$z_{\text{out_noCSR}}$	m	Eq. (27)
$z_{\text{out_wCSR}}$	m	Eq. (37)
δ_{noCSR}		Eq. (28)
δ_{wCSR}		Eq. (38)
dI_{smooth}/dz	A/m	$\partial I_{\text{smooth}}/\partial z$

beamline, where the bunch is longer, its influence on the CSR wake is negligible. Therefore, `dR56` only modifies the linear compression in Eq. (27), while the CSR integrals in Eqs. (37) and (38) remain unchanged.

The output variables are summarized in Table IV. Among them, `dIdz_smooth` is the derivative of the smoothed current profile and directly affects the smoothness of the calculated CSR wake. Since the output longitudinal distribution depends on z' through Eqs. (29)–(31), numerical noise in the input profile can be amplified after differentiation and may generate artificial current spikes. Therefore, diagnostic plots are provided, as shown in Fig. 11, to check the quality of the smoothed distribution.

Smoothing is not only a numerical treatment but also a physical requirement for CSR wake evaluation. In the two-particle CSR model [44], the CSR wake cannot be obtained directly from the Lienard-Wiechert potentials alone. A singular term must be canceled, which requires

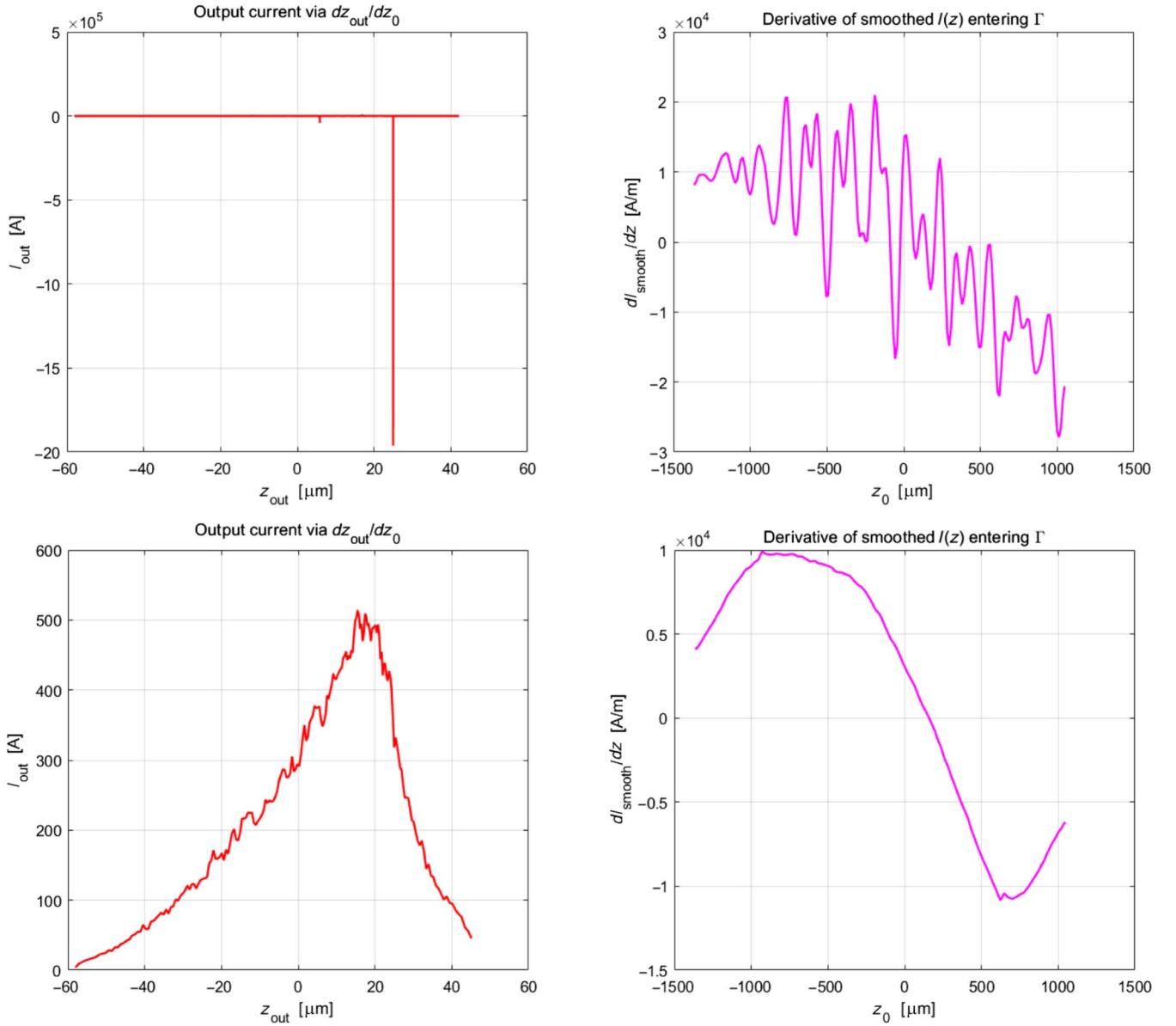


FIG. 11. Compressed current profile and `dIdz_smooth` for 200 slices with smoothing window sizes of 5 (top) and 40 (bottom).

the local charge density to vary slowly over an infinitesimal region:

$$\lim_{\delta \rightarrow 0} \int_{z-\delta}^z w(z-z') \lambda(z') dz' = \lambda(z) \int_{z-\delta}^z w(z-z') dz'.$$

This implies that the variation in $\lambda(z)$ is infinitesimally small over a small region δ . Therefore, discontinuities or statistical spikes in $\lambda(z)$ would invalidate this cancellation. A smooth reconstruction of the charge density is thus required before evaluating the CSR wake.

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