

Formulation of energy-dependent pathlength variation for a two Double Bend Achromat dogleg

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Recently, the generation of ultrashort x-ray free-electron laser (XFEL) pulses of less than 1 fs has been reported using additional electron bunch compression in front of undulators. In this bunch compression, a self-modulated energy chirp provided by space charge force is used. In the present paper, we discuss an optimal bunch compression scheme using a two-Double Bend Achromat (DBA) dogleg and an electron bunch having the energy chirp which is reversed with respect to that used for a conventional magnetic chicane bunch compressor. After giving analytical expressions of R_{56} and T_{566} of an ordinary DBA section, a method for evaluating R_{56} and T_{566} is presented for varied bending angle cases to achieve additional bunch compression. When changing the bending angles for controlling R_{56} and T_{566} , the electron beam no longer passes the magnetic center of focusing quadrupole magnets of the DBA section, but the electron beam trajectory and transverse envelope remain the same downstream of the two-DBA dogleg, including the undulators. The analytical framework proposed in this paper would be useful to design and finely tune the bunch compression in front of the undulators to generate ultrashort XFEL pulses.

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I. INTRODUCTION

To ensure enough gain of laser amplification in x-ray free-electron lasers (XFELs), an electron bunch is compressed in the traveling direction to increase a six-dimensional phase space density. For the bunch compression, velocity and pathlength dependences on energy are used in relativistic and ultrarelativistic energy regions, respectively [1–3]. In the ultrarelativistic energy region, an energy chirp is given to the electron bunch using off-crest rf acceleration, where head electrons have lower energy. Then the electron bunch is compressed through a magnetic chicane, where the path length decreases with energy; in other words, the first-order energy-dependent pathlength variation R_{56} [4,5] is negative. Various factors, such as intrinsic energy spread and deviation from a design energy chirp caused by inhomogeneous electron beam density and jitter of rf accelerators, limit the achievable minimum bunch length at final bunch compression, which is approximately 10 fs.

To overcome this limit, various methods [6–10] have been proposed and developed.

Recently, the group of Yan *et al.* at DESY proposed and demonstrated a groundbreaking method surpassing this limit [11]. In the process of accelerating a high-density electron bunch after the final bunch compressor, a non-negligible energy modulation is generated along the traveling direction due to longitudinal space charge force proportional to the gradient of a current distribution around its peak. The energy modulation created in this process increases the energy of head electrons and gives a reverse energy chirp [12]. The remarkable feature of this process is the local formation of an energy chirp that directly reflects the temporal structure of the bunch current distribution. At Euro-XFEL (the European x-ray FEL) [13], the generation of subfemtosecond pulses is demonstrated by compressing the electron bunch in front of undulators using the reverse energy chirp and a positive R_{56} [8]. A similar reverse chirp can also be provided by the geometrical wakefields of high-impedance rf accelerating structures, and it further enhances the reverse energy chirp.

At SPring-8 Angstrom Compact free electron LASER (SACLA) [14], the beam energy at the final bunch compressor is 1.4 GeV and XFEL routinely operates with extremely high peak currents of more than 10 kA [15] resulting in high space charge fields. To make the facility compact, the main accelerator system employs C-band

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high-gradient accelerators [16] at SACLA, so its high impedance brings about high wakefields and further increases the reverse energy chirp. The generation of sub-fs XFEL pulses using the reverse energy chirp is under investigation [17].

One of SACLA's two XFEL beamlines, BL2, is configured with a 3° dogleg transport to offset the BL2 beamline by 3 m in parallel to the beam axis of the SACLA linear accelerator [18,19]. To suppress coherent synchrotron radiation (CSR) effects [20–22] caused by high peak currents, the BL2 dogleg is composed of two Double Bend Achromat (DBA) sections with an optimized phase advance [23–27], as shown in Fig. 1. By finely controlling R_{56} and T_{566} (second-order energy-dependent pathlength variation) of the two-DBA dogleg, additional bunch compression would be possible to generate sub-fs short XFEL pulses. The two-DBA dogleg configuration offers an advantage of allowing precise adjustment of R_{56} and T_{566} by changing only the bending magnet fields, while maintaining the same trajectory and transverse envelope of the electron beam downstream of the dogleg. When the bending angle is changed, a dipole kick is provided by quadrupole magnets in the dispersive arcs, so the total deflection angle of DBA is unchanged. This characteristic of the two-DBA dogleg allows adjustments of R_{56} and T_{566} without affecting the beam trajectory and envelope at the undulator section, thereby making its tuning procedure simple and robust.

In Sec. II, we formulate an analytical expression of R_{56} and T_{566} for an ordinary two-DBA dogleg, in which the

electron beam passes the center of the quadrupole magnets in the DBA sections, namely on-axis condition. Section III outlines a method to evaluate R_{56} and T_{566} for a two-DBA dogleg in case of changing the bending angle, so the electron beam passes off center of the quadrupole magnets, namely off-axis condition. Section IV describes a sequential procedure to calculate R_{56} and T_{566} for the off-axis case with an arbitrary bending angle. Finally, Sec. V discusses the higher-order nonlinear effects of the two-DBA dogleg.

II. THE ANALYTICAL EXPRESSION OF R_{56} AND T_{566} FOR THE TWO-DBA DOGLEG UNDER ON-AXIS CONDITIONS

Figure 1 shows the schematic layout and electron beam optics of the two-DBA dogleg of SACLA BL2 [18,26]. Magnets are arranged in a symmetric configuration: the bending magnets B1 (B3) and B2 (B4) are identical rectangular types with the same bending angle θ_0 ($-\theta_0$) and a radius of curvature ρ_0 ($-\rho_0$), and the drift spaces D1, D2, D9, and D10 have the same length L . At SACLA BL2, these values are $\theta_0 = 1.5^\circ$, $\rho_0 = 36.291$ m, and $L = 5.275$ m. The focusing quadrupole magnet Q1 is treated as a thin lens with integrated strength k . Under the on-axis condition, the beam trajectory passes through the center of Q1 and a linear energy dispersion is closed in each DBA section. Since the beam transport line connecting the first and second DBA sections is straight without any dipole fields, R_{56} and T_{566} along the dogleg can be evaluated with sufficiently high precision by simply doubling R_{56} and T_{566} of one DBA section. Note that the second-order energy dispersion leaks from the first DBA into the straight beam transport line due to the energy-dependent dipole component of Q1. This second-order energy dispersion, however, can be closed within the dogleg by setting a horizontal betatron phase advance between the two Q1s in the first and second DBA sections to an even multiple of π . The numerical results shown in this paper assume the optics where this phase difference is finely tuned to 2π to satisfy the above condition.

By geometrically tracing an off-momentum electron trajectory with respect to the on-axis trajectory along the DBA section, one can calculate the path length variation $\Delta l \equiv R_{56}\delta + T_{566}\delta^2$ with $\delta \equiv \Delta p/p$ for each component of the DBA section and obtain the following expressions of R_{56} and T_{566} :

$$R_{56}(\text{B1}) = \rho_0(\theta_0 - \tan \theta_0), \quad (1)$$

$$T_{566}(\text{B1}) = \frac{\rho_0}{2} \tan^3 \theta_0, \quad (2)$$

$$R_{56}(\text{D1}) = \rho_0(\tan \theta_0 - \sin \theta_0), \quad (3)$$

$$T_{566}(\text{D1}) = \frac{1}{2}(L - \rho_0 \tan \theta_0) \tan^2 \theta_0, \quad (4)$$

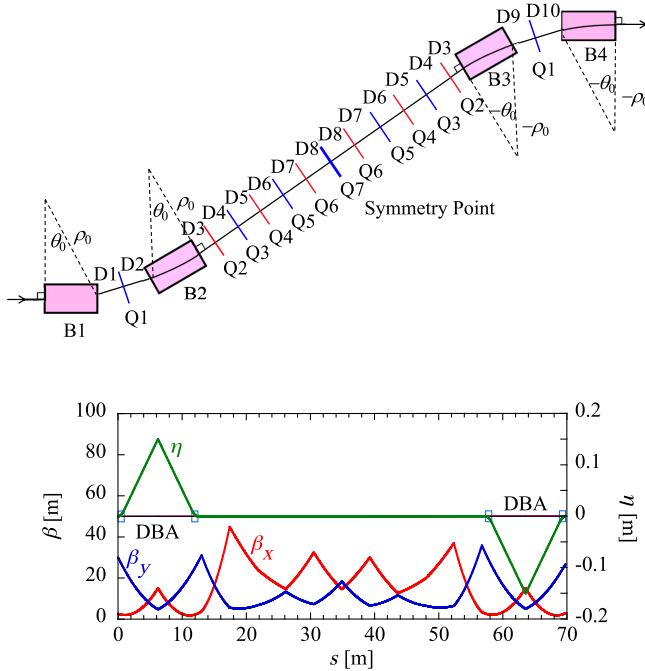


FIG. 1. Schematic layout (upper) and electron beam optics (lower) of a two-DBA dogleg used in SACLA BL2.

$$R_{56}(Q1) = 0, \quad (5)$$

$$T_{566}(Q1) = 0, \quad (6)$$

$$R_{56}(D2) = \rho_0(1-a)(\tan \theta_0 - \sin \theta_0) + L(2-a)\tan^2 \theta_0, \quad (7)$$

$$\begin{aligned} T_{566}(D2) &= \frac{1}{2} \{ (L - \rho_0 \tan \theta_0) \tan^2 \theta_0 \\ &\quad + 2\rho_0 \tan^2 \theta_0 (1 - \cos \theta_0) [2 \tan \theta_0 - b(1 - \cos \theta_0)] \\ &\quad + L(1 + 2\tan^2 \theta_0) [a \tan \theta_0 + b(1 - \cos \theta_0)]^2 \\ &\quad + L \tan^2 \theta_0 [a(2 - 5\tan^2 \theta_0) + b(10 \sin \theta_0 - 9 \tan \theta_0) \\ &\quad - 2(2 - \tan^2 \theta_0)] \}, \end{aligned} \quad (8)$$

$$R_{56}(B2) = \rho_0 [\theta_0 - \tan \theta_0 + (2-a)(\tan \theta_0 - \sin \theta_0) - b(1 - \cos \theta_0)^2], \quad (9)$$

$$\begin{aligned} T_{566}(B2) &= \frac{\rho_0}{2} \{ \tan^3 \theta_0 [1 - 2(1 - \cos \theta_0)] \\ &\quad + [a \tan \theta_0 + b(1 - \cos \theta_0)]^2 \sin \theta_0 \\ &\quad - a(\tan \theta_0 - \sin \theta_0)(1 + \tan^2 \theta_0)(\cos^2 \theta_0 \\ &\quad + 2 \cos \theta_0 - 1) - b(1 - \cos \theta_0) \tan^2 \theta_0 \}, \end{aligned} \quad (10)$$

where $a \equiv kL$, $b \equiv k\rho_0$, L is the length of D1 and D2, k is the integrated strength of Q1 ($k > 0$ for focusing Q), and θ_0 and ρ_0 represent the bending angle and radius of curvature of B1 and B2.

The total values of R_{56} and T_{566} for one DBA section are obtained by adding all the contributions. The energy-dependent pathlength variation Δl along one DBA section is then calculated with the second-order approximation in δ as follows:

$$\Delta l = R_{56}(\text{DBA})\delta + T_{566}(\text{DBA})\delta^2, \quad (11)$$

$$\begin{aligned} R_{56}(\text{DBA}) &= R_{56}(B1) + R_{56}(D1) + R_{56}(Q1) \\ &\quad + R_{56}(D2) + R_{56}(B2), \end{aligned} \quad (12)$$

$$\begin{aligned} T_{566}(\text{DBA}) &= T_{566}(B1) + T_{566}(D1) + T_{566}(Q1) \\ &\quad + T_{566}(D2) + T_{566}(B2). \end{aligned} \quad (13)$$

The analytically obtained pathlength variation, i.e., $\Delta l = 2 \times (R_{56}(\text{DBA})\delta + T_{566}(\text{DBA})\delta^2)$ is plotted in Fig. 2 by the red solid curve. For comparison, we also plot the simulation result for the entire dogleg using the ELEGANT code [28,29] by the blue filled circles. We see very good agreement, indicating the following two things. The first is keeping the terms up to the second order of δ is sufficient for evaluating energy-dependent pathlength variation in a realistic range of energy deviation, and the second is that

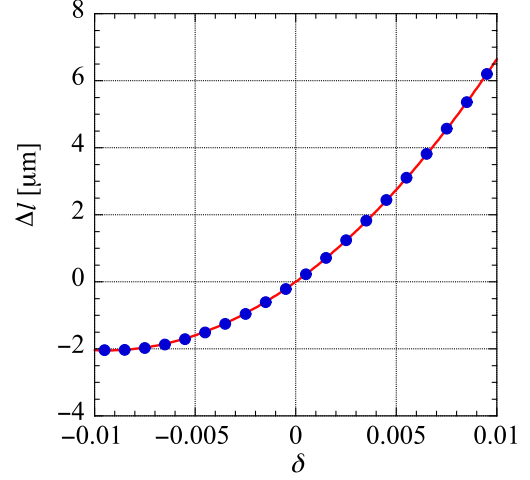


FIG. 2. Comparison of the energy-dependent path length variations obtained from analytical calculations with doubled R_{56} and T_{566} for a single DBA section (the red solid line) and from the simulation code ELEGANT (the filled blue circles) for the entire two-DBA dogleg.

the model using doubled R_{56} and T_{566} of one DBA section accurately represents the energy-dependent pathlength variation for the entire two-DBA dogleg.

III. OVERVIEW OF A METHOD TO DIRECTLY EVALUATE R_{56} AND T_{566} FOR THE TWO-DBA DOGLEG UNDER OFF-AXIS CONDITIONS

When the bending angle of DBA is changed to adjust R_{56} and T_{566} , the beam trajectory does not pass through the center of Q1, and linear energy dispersion leaks downstream from the DBA section. The formula therefore will contain a huge number of terms and becomes so complicated that it cannot be treated by the same approach as in Sec. II. Instead, we adopt the method of calculating the energy dependence of the horizontal beam position x , angle θ , and the path length l of each component of the two-DBA dogleg shown in Fig. 1.

Since we will change the values of R_{56} and T_{566} by changing the bending magnet fields, we need to consider two electron trajectories: one is for the on-axis bending field B and the other is for the off-axis modified bending field $\tilde{B}$. We geometrically trace the electron trajectory [30] for the latter case of the bending field $\tilde{B}$ with $\delta \neq 0$, and then calculate the differences in x , θ , and l from the former case of the on-axis bending field B with $\delta = 0$. This is illustrated in Fig. 3, where Δx , $\Delta \theta$, and Δl on the exit surface are given as a function of those on the entrance surface. The mapping function $F: (\Delta x_{\text{in}}, \Delta \theta_{\text{in}}) \rightarrow (\Delta x_{\text{out}}, \Delta \theta_{\text{out}}, \Delta l)$ can be expressed with a rigorous closed form as shown in Sec. IV and can be expanded up to any order of δ using, e.g., differential algebra [31] if necessary. The trajectory offsets caused by changing the bending field B to $\tilde{B}$ are treated as a constant term independent of δ in this mapping. Once this mapping function is given for each

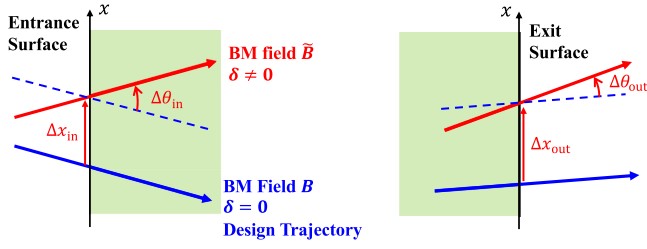


FIG. 3. Beam trajectories at the entrance and the exit of a beamline element.

component as a closed form, one can perform this mapping process successively component-by-component from the entrance of B1 with the initial values $(\Delta x_{\text{in}}, \Delta \theta_{\text{in}}) = (0, 0)$ to the end of the dogleg. After obtaining Δl for the entire dogleg, we obtain R_{56} and T_{566} as the coefficients of δ and δ^2 , respectively. In our calculations, the expansion with respect to δ was carried out by using *Mathematica* [32] and the coefficients of δ and δ^2 were obtained numerically.

IV. A SEQUENTIAL PROCEDURE TO DIRECTLY EVALUATE R_{56} AND T_{566} FOR THE TWO-DBA DOGLEG UNDER OFF-AXIS CONDITIONS WITH AN ARBITRARY BENDING ANGLE

This section describes a sequential procedure to evaluate R_{56} , T_{566} , and higher-order energy-dependent pathlength variations for a two-DBA dogleg under an arbitrary beam trajectory. The maps of horizontal trajectory and pathlength for all components used for the dogleg are shown below in closed forms.

In this section, we will denote the length of the drift space Dn as L_n (see Fig. 1). Note that SACLA BL2 is designed such that $L_1 = L_2 = L_9 = L_{10}$, and in Sec. II these common values were denoted by L for simplicity.

A. Mapping of B1

Let the bending radius corresponding to the modified magnetic field $\tilde{B}$ be $\tilde{\rho}_0$ (see Fig. 4). The distance between the points P_1 and P_2 can be expressed in two ways as follows:

$$\overline{P_1 P_2} = \frac{\rho_0 \sin \theta_0}{\cos(\Delta \theta_{\text{in}} + \frac{\theta}{2})} = 2\tilde{\rho}_0(1 + \delta) \sin \frac{\theta}{2}.$$

From this, we obtain

$$\theta = \sin^{-1} \left(\sin \Delta \theta_{\text{in}} + \frac{\rho_0 \sin \theta_0}{\tilde{\rho}_0(1 + \delta)} \right) - \Delta \theta_{\text{in}},$$

and the difference of the path length, position, and angle at the exit can be calculated as follows:

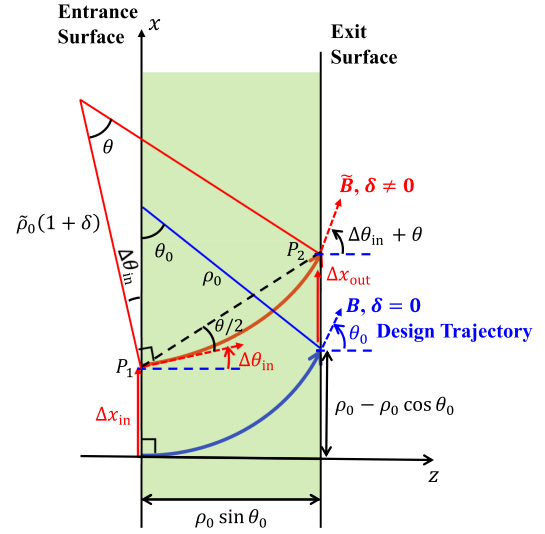


FIG. 4. Beam trajectories through the rectangular bending magnet B1.

$$\Delta l = \tilde{\rho}_0(1 + \delta)\theta - \rho_0\theta_0, \quad (14)$$

$$\Delta x_{\text{out}} = \Delta x_{\text{in}} + \rho_0 \sin \theta_0 \tan \left(\Delta \theta_{\text{in}} + \frac{\theta}{2} \right) - \rho_0(1 - \cos \theta_0), \quad (15)$$

$$\Delta \theta_{\text{out}} = \Delta \theta_{\text{in}} + \theta - \theta_0. \quad (16)$$

B. Mapping of D1

The maps for the drift space between B1 and Q1 are easily obtained by geometric consideration, as shown in Fig. 5. Here, note that the entrance surface of the drift space (i.e., the exit surface of the rectangular bending magnet B1) is not perpendicular to the z axis of the reference coordinate system.

$$\Delta l = \frac{L_1 - \Delta x_{\text{in}} \sin \theta_0}{\cos \Delta \theta_{\text{in}}} - L_1, \quad (17)$$

$$\Delta x_{\text{out}} = \frac{L_1 \sin \Delta \theta_{\text{in}} + \Delta x_{\text{in}} \cos(\theta_0 + \Delta \theta_{\text{in}})}{\cos \Delta \theta_{\text{in}}}, \quad (18)$$

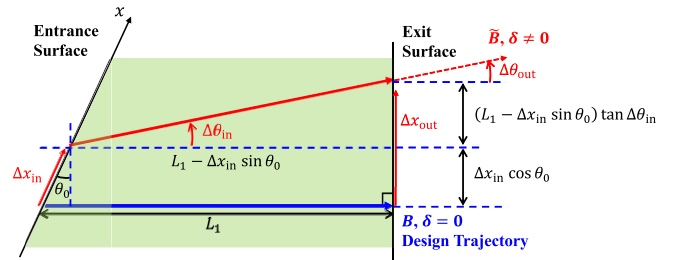


FIG. 5. Beam trajectories through the drift space D1.

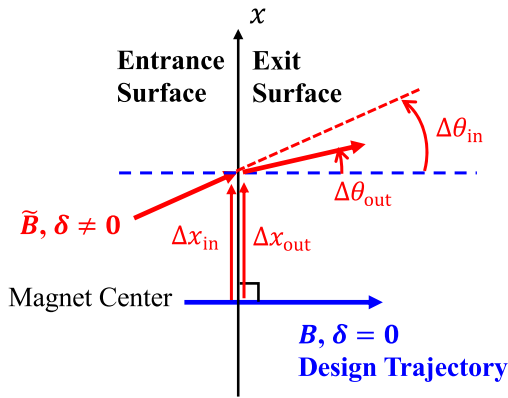


FIG. 6. Beam trajectories through the thin lens quadrupole Q1.

$$\Delta\theta_{\text{out}} = \Delta\theta_{\text{in}}. \quad (19)$$

C. Mapping of Q1

Under the thin lens approximation, the maps for a quadrupole magnet with entrance and exit surfaces perpendicular to the z axis (see Fig. 6) are as follows:

$$\Delta l = 0, \quad (20)$$

$$\Delta x_{\text{out}} = \Delta x_{\text{in}}, \quad (21)$$

$$\Delta\theta_{\text{out}} = \Delta\theta_{\text{in}} - \frac{k}{1+\delta}\Delta x_{\text{in}}, \quad (22)$$

where k is the integrated strength of the quadrupole magnet and $k > 0$ for the focusing quadrupole. These maps are applicable for the other thin lens quadrupoles, Q2 to Q7.

D. Mapping of D2

The maps for the drift space between Q1 and B2 are obtained by geometric consideration, as shown in Fig. 7. Here, note that the exit surface of the drift space (i.e., the entrance surface of the rectangular bending magnet B2) is not perpendicular to the z axis of the reference coordinate system.

$$\Delta l = \frac{L_2 \cos \theta_0 - \Delta x_{\text{in}} \sin \theta_0}{\cos(\Delta \theta_{\text{in}} - \theta_0)} - L_2, \quad (23)$$

$$\Delta x_{\text{out}} = \frac{L_2 \sin \Delta\theta_{\text{in}} + \Delta x_{\text{in}} \cos \Delta\theta_{\text{in}}}{\cos(\Delta\theta_{\text{in}} - \theta_0)}, \quad (24)$$

$$\Delta\theta_{\text{out}} = \Delta\theta_{\text{in}}. \quad (25)$$

E. Mapping of B2

As in the case of B1, taking account of the difference in the off-momentum electron trajectory in the arc (see Fig. 8), we have

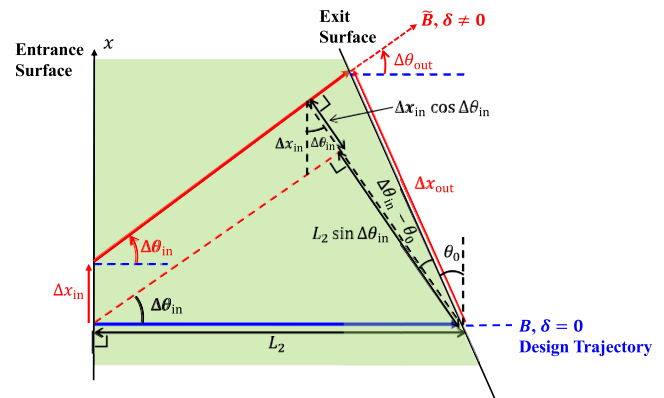


FIG. 7. Beam trajectories through the drift space D2.

$$\overline{P_1 P_2} = \frac{\rho_0 \sin \theta_0}{\cos \left(\theta_0 - \Delta \theta_{\text{in}} - \frac{\theta}{2} \right)} = 2\tilde{\rho}_0(1 + \delta) \sin \frac{\theta}{2},$$

$$\theta = \sin^{-1} \left(-\sin(\theta_0 - \Delta\theta_{\text{in}}) + \frac{\rho_0 \sin\theta_0}{\tilde{\rho}_0(1+\delta)} \right) + \theta_0 - \Delta\theta_{\text{in}},$$

and obtain the following maps:

$$\Delta l = \tilde{\rho}_0(1 + \delta)\theta - \rho_0\theta_0, \quad (26)$$

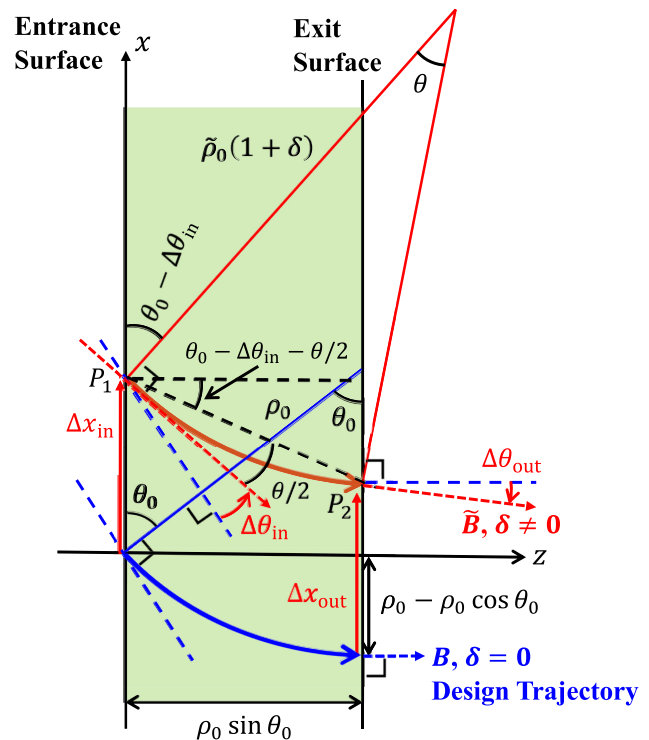


FIG. 8. Beam trajectories through the rectangular bending magnet B2.

$$\Delta x_{\text{out}} = \Delta x_{\text{in}} - \rho_0 \sin \theta_0 \tan \left(\theta_0 - \Delta \theta_{\text{in}} - \frac{\theta}{2} \right) + \rho_0 (1 - \cos \theta_0), \quad (27)$$

$$\Delta \theta_{\text{out}} = \Delta \theta_{\text{in}} + \theta - \theta_0. \quad (28)$$

F. Mapping of Q2 to Q7

As shown in Fig. 1, all quadrupole magnets Q2 to Q7 in the beam transport line between the two DBA sections are aligned along the on-axis orbit. Therefore, the same maps as Q1 [i.e., Eqs. (20)–(22)] can be applied.

G. Mapping of D3 to D8

Under the simple condition of having all the quadrupole magnets aligned along the on-axis orbit, the entrance and exit surfaces of all the drift spaces are perpendicular to the on-axis orbit, as illustrated in Fig. 9. The maps then take the following simple forms ($i = 3$ to 8):

$$\Delta l = \left(\frac{1}{\cos \Delta \theta_{\text{in}}} - 1 \right) L_i, \quad (29)$$

$$\Delta x_{\text{out}} = \Delta x_{\text{in}} + L_i \tan \Delta \theta_{\text{in}}, \quad (30)$$

$$\Delta \theta_{\text{out}} = \Delta \theta_{\text{in}}. \quad (31)$$

H. Mapping of B3

The maps for B3 are obtained by changing the sign of ρ_0 , θ_0 , and $\tilde{\rho}_0$ in the case of B1.

$$\Delta l = \tilde{\rho}_0 (1 + \delta) \theta - \rho_0 \theta_0, \quad (32)$$

$$\Delta x_{\text{out}} = \Delta x_{\text{in}} + \rho_0 \sin \theta_0 \tan \left(\Delta \theta_{\text{in}} - \frac{\theta}{2} \right) + \rho_0 (1 - \cos \theta_0), \quad (33)$$

$$\Delta \theta_{\text{out}} = \Delta \theta_{\text{in}} - \theta + \theta_0, \quad (34)$$

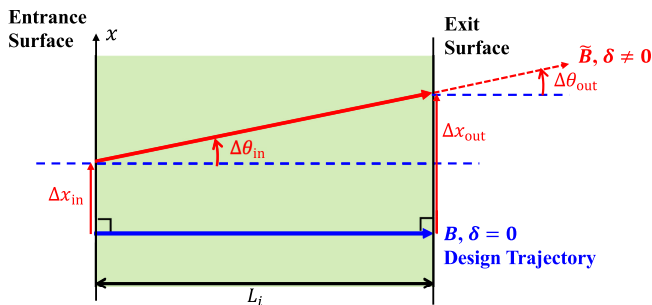


FIG. 9. Beam trajectories through the drift spaces D3 to D8.

where

$$\theta = -\sin^{-1} \left(\sin \Delta \theta_{\text{in}} - \frac{\rho_0 \sin \theta_0}{\tilde{\rho}_0 (1 + \delta)} \right) + \Delta \theta_{\text{in}}.$$

I. Mapping of D9

The maps for D9 are obtained by changing the sign of θ_0 in the case of D1.

$$\Delta l = \frac{L_9 + \Delta x_{\text{in}} \sin \theta_0}{\cos \Delta \theta_{\text{in}}} - L_9, \quad (35)$$

$$\Delta x_{\text{out}} = \frac{L_9 \sin \Delta \theta_{\text{in}} + \Delta x_{\text{in}} \cos(\theta_0 - \Delta \theta_{\text{in}})}{\cos \Delta \theta_{\text{in}}}, \quad (36)$$

$$\Delta \theta_{\text{out}} = \Delta \theta_{\text{in}}. \quad (37)$$

J. Mapping of D10

The maps for D10 are obtained by changing the sign of θ_0 in the case of D2.

$$\Delta l = \frac{L_{10} \cos \theta_0 + \Delta x_{\text{in}} \sin \theta_0}{\cos(\Delta \theta_{\text{in}} + \theta_0)} - L_{10}, \quad (38)$$

$$\Delta x_{\text{out}} = \frac{L_{10} \sin \Delta \theta_{\text{in}} + \Delta x_{\text{in}} \cos \Delta \theta_{\text{in}}}{\cos(\Delta \theta_{\text{in}} + \theta_0)}, \quad (39)$$

$$\Delta \theta_{\text{out}} = \Delta \theta_{\text{in}}. \quad (40)$$

K. Mapping of B4

The maps for B4 are obtained by changing the sign of ρ_0 , θ_0 , and $\tilde{\rho}_0$ in the case of B2.

$$\Delta l = \tilde{\rho}_0 (1 + \delta) \theta - \rho_0 \theta_0, \quad (41)$$

$$\Delta x_{\text{out}} = \Delta x_{\text{in}} + \rho_0 \sin \theta_0 \tan \left(\theta_0 + \Delta \theta_{\text{in}} - \frac{\theta}{2} \right) - \rho_0 (1 - \cos \theta_0), \quad (42)$$

$$\Delta \theta_{\text{out}} = \Delta \theta_{\text{in}} - \theta + \theta_0, \quad (43)$$

where

$$\theta = -\sin^{-1} \left(\sin(\theta_0 + \Delta \theta_{\text{in}}) - \frac{\rho_0 \sin \theta_0}{\tilde{\rho}_0 (1 + \delta)} \right) + \theta_0 + \Delta \theta_{\text{in}}.$$

L. Numerical results

Figure 10 shows R_{56} and T_{566} obtained as the coefficients of δ and δ^2 in Δl , respectively, by applying the above maps from the entrance of B1 to the exit of B4 over

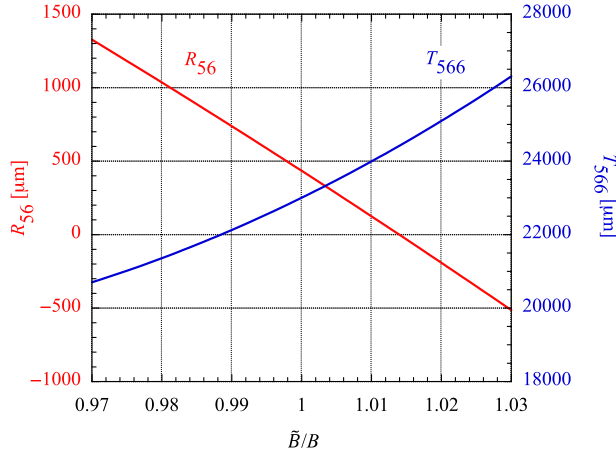


FIG. 10. Variation of R_{56} and T_{566} with the change in $\tilde{B}$ calculated by using the developed sequential procedure.

the entire two-DBA dogleg. The R_{56} and T_{566} values obtained in the case of $\tilde{B}/B = 1$ are $434 \mu\text{m}$ and $2.30 \times 10^4 \mu\text{m}$, respectively, which agree well with the doubled R_{56} and T_{566} of one DBA section, as discussed in Sec. II. We see that the parameter R_{56} becomes zero around $\tilde{B}/B = 1.014$ and in the region where $\tilde{B}$ is smaller than this zero-crossing point, R_{56} becomes positive. Figure 11 shows the energy-dependent pathlength variations calculated from R_{56} and T_{566} in Fig. 10 using $\tilde{B}$ as a parameter. In the figure, the pathlength variations calculated from doubled R_{56} and T_{566} of one DBA section (the first DBA section) are also shown by the filled circles. We see that the two agree well and doubling R_{56} and T_{566} of one DBA section is a good model to represent the energy-dependent pathlength variation for the entire two-DBA dogleg in a relative energy deviation range from -1 to $+1\%$. We have cross checked these results by using the ELEGANT code.

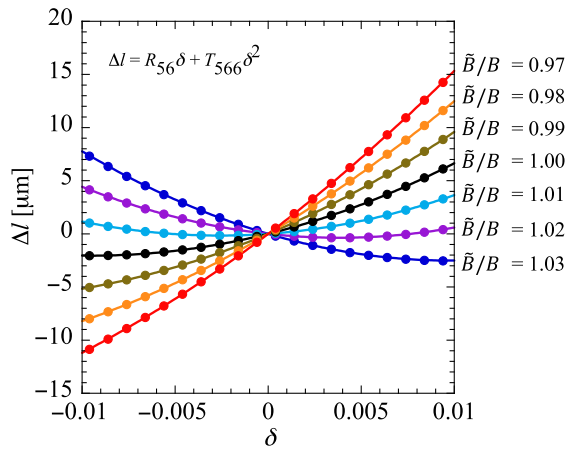


FIG. 11. Energy-dependent pathlength variations calculated from the developed sequential procedure using $\tilde{B}$ as a parameter. The filled circles represent values calculated from doubled R_{56} and T_{566} of one DBA section.

V. THE NONLINEAR EFFECTS OF THE TWO-DBA DOGLEG

Before discussing the nonlinear effects, we first summarize the key features of the dogleg shown in Fig. 1: (i) the Twiss parameters are mirror-symmetric with respect to the center of the dogleg section, (ii) the linear energy dispersion has point symmetry, (iii) the horizontal phase advance of the transport channel connecting the two DBA structures is set to π , and (iv) the horizontal phase advance between the two Q1s is set to 2π . These are needed for suppressing the CSR effects [26].

To control R_{56} and T_{566} of the two-DBA dogleg, only the DBA bending angle needs to be changed. Due to the above features, the beam trajectory and linear energy dispersion downstream of the dogleg remain unchanged, as shown in Fig. 12. However, this simple scheme cannot confine nonlinear energy dispersions within the dogleg. When the magnetic field of the bending magnets is changed, the quadrupole Q1 in the arc can return the beam trajectory to the original position and angle at the exit of the DBA section, as shown in Fig. 12(a). However, when Q1 operates under the off-axis condition, it introduces additional linear energy dispersion, as shown in Fig. 12(b). This extra linear dispersion does not close within the DBA section and leaks into the beam transport section bridging

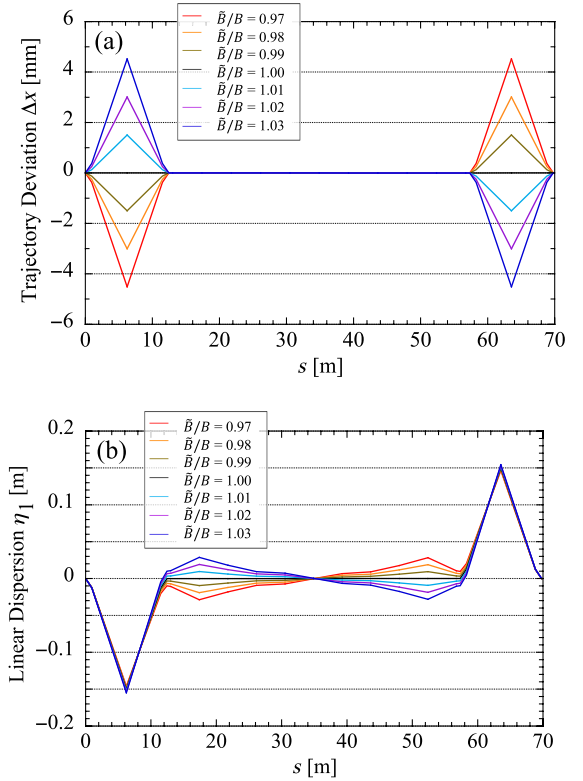


FIG. 12. Deviation of beam trajectory (a) and the linear dispersion (b) in the two-DBA dogleg calculated from the developed sequential procedure using $\tilde{B}$ as a parameter.

the two DBA sections. This linear dispersion leakage then generates the nonlinear energy dispersion by coupling with the quadrupole magnets at the transport section. The leaked linear dispersion is closed at the exit of dogleg due to an optimum horizontal phase difference of 2π between the two Q1 magnets. The leakage of higher order transverse dispersion under the given dogleg condition is evaluated up to any required order of δ as variations in the horizontal position Δx and angle $\Delta\theta$ at the exit of the dogleg. This can be obtained in the same way as R_{56} and T_{566} , using the exact mapping of Δx , $\Delta\theta$, and Δl described in Sec. IV. Figure 13 shows the leaked nonlinear energy dispersions up to the fourth order of δ in changing the bending magnet field from -3 to $+3\%$. Considering that the absolute value of δ is ~ 0.01 at most, we found that the leakage effect of the second-order energy dispersion is dominant compared to those of the third- and fourth-order dispersions.

Let us estimate how much these nonlinear energy dispersions deteriorate emittance of the electron beam. A horizontal position and angle of the i th electron at the exit of the dogleg are divided into the contributions from a horizontal betatron oscillation and horizontal energy dispersions as

$$x_i = x_{\beta,i} + x_{\delta,i}, \quad x'_i = x'_{\beta,i} + x'_{\delta,i},$$

$$x_{\delta,i} = \sum_{n \geq 2} \eta_{x,n} \delta_i^n, \quad x'_{\delta,i} = \sum_{n \geq 2} \eta'_{x,n} \delta_i^n, \quad (44)$$

where the suffixes β and δ represent the contributions from the betatron oscillation and energy dispersion, respectively; the suffix i represents the i th electron in the bunch; the parameters $\eta_{x,n}$ and $\eta'_{x,n}$ are the n th order nonlinear horizontal energy dispersion and its gradient, respectively. Note that in Eq. (44), the contribution from linear dispersion ($n = 1$) is excluded from the summation, since it is closed at the end of the dogleg due to a horizontal phase advance of 2π between the DBA Q1 quadrupoles.

Assuming Gaussian distributions with no correlation between the betatron oscillation and the energy spread, the effective horizontal emittance $\varepsilon_{x,\text{eff}}$, including contributions from energy dispersion, is defined as [33]

$$\varepsilon_{x,\text{eff}}^2 = \{ \langle x_{\beta,i}^2 \rangle + \langle (\Delta x_{\delta,i})^2 \rangle \} \{ \langle x_{\beta,i}'^2 \rangle + \langle (\Delta x_{\delta,i}')^2 \rangle \}$$

$$- \{ \langle x_{\beta,i} x_{\beta,i}' \rangle + \langle (\Delta x_{\delta,i}) (\Delta x_{\delta,i}') \rangle \}^2$$

$$= \varepsilon_x^2 + \varepsilon_x \{ \beta_x \langle (\Delta x_{\delta,i}')^2 \rangle + \gamma_x \langle (\Delta x_{\delta,i})^2 \rangle \}$$

$$+ 2\alpha_x \langle (\Delta x_{\delta,i}) (\Delta x_{\delta,i}') \rangle$$

$$+ \langle (\Delta x_{\delta,i})^2 \rangle \langle (\Delta x_{\delta,i}')^2 \rangle - \langle (\Delta x_{\delta,i}) (\Delta x_{\delta,i}') \rangle^2, \quad (45)$$

with

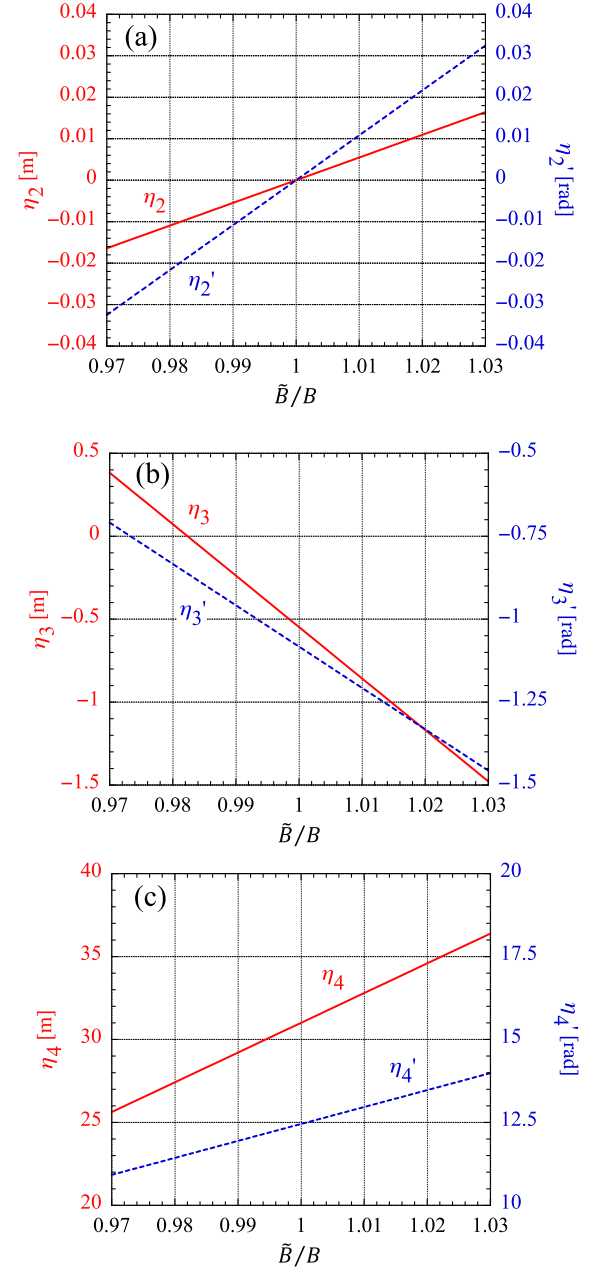


FIG. 13. Leakage of the nonlinear energy dispersions from the SACLA BL2 two-DBA dogleg up to the fourth order of δ in changing the bending magnet field within $\pm 3\%$. The (a)–(c) are the second-order, third-order, and fourth-order nonlinear dispersions, respectively, calculated at the end of the dogleg.

$$\langle x_{\beta,i}^2 \rangle = \beta_x \varepsilon_x, \quad \langle x_{\beta,i}'^2 \rangle = \gamma_x \varepsilon_x, \quad \langle x_{\beta,i} x_{\beta,i}' \rangle = -\alpha_x \varepsilon_x,$$

$$\varepsilon_x = \sqrt{\langle x_{\beta,i}^2 \rangle \langle x_{\beta,i}'^2 \rangle - \langle x_{\beta,i} x_{\beta,i}' \rangle^2},$$

$$\Delta x_{\delta,i} \equiv x_{\delta,i} - \langle x_{\delta,i} \rangle, \quad \Delta x_{\delta,i}' \equiv x_{\delta,i}' - \langle x_{\delta,i}' \rangle,$$

where $\langle P \rangle$ denotes ensemble average of P over the number of electrons in the bunch and ε_x is the electron beam horizontal emittance of the horizontal betatron oscillation;

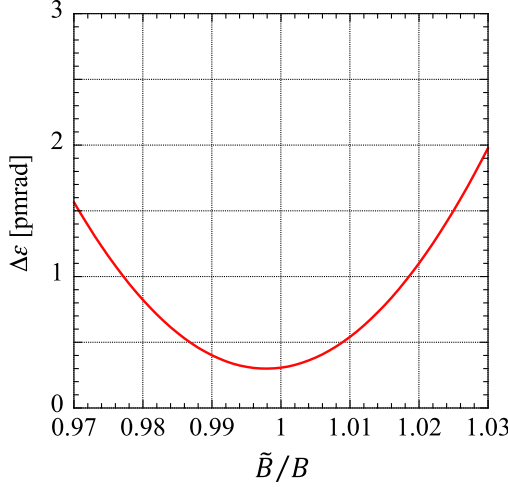


FIG. 14. Emittance increment generated by the leaked nonlinear energy dispersions up to the fourth order of δ in tuning the energy-dependent pathlength variation. Relative normalized energy deviation σ_δ is set to 0.005 and the bending magnet field is changed from -3 to $+3\%$.

γ_x , α_x , and β_x are the horizontal Courant-Snyder parameters [34] at the exit of the dogleg.

The effective horizontal emittance $\epsilon_{x,\text{eff}}$ is calculated by using Eqs. (44) and (45) and the nonlinear horizontal dispersions up to fourth order of δ shown in Fig. 13. The energy-dependent horizontal trajectory variations are given by Eq. (44). Then, by using Eq. (45), the trajectory variations are ensemble-averaged over a huge number of electrons in a bunch, assuming that there is no correlation between the betatron oscillation (or the natural emittance) and the energy spread distribution. Consequently, $\epsilon_{x,\text{eff}}$ is calculated with the horizontal Courant-Snyder parameters [34] and assumed rms energy spread σ_δ . Since we have assumed a Gaussian distribution for the energy spread, the average of δ_i^n is evaluated as

$$\langle \delta_i^n \rangle = \begin{cases} (n-1)!! \sigma_\delta^n & n = \text{even} \\ 0 & n = \text{odd} \end{cases},$$

and we can calculate the emittance increment downstream from the dogleg defined as $\Delta\epsilon \equiv \epsilon_{x,\text{eff}} - \epsilon_x$. The origin of this $\Delta\epsilon$ is the leaked nonlinear energy dispersions, and in Fig. 14, we show calculated $\Delta\epsilon$ when the field of bending magnets is changed from -3 to $+3\%$, assuming $\sqrt{\langle \delta_i^2 \rangle} = \sigma_\delta$ is 0.005.

The maximum emittance increment due to the nonlinear energy dispersions up to the fourth order is estimated to be about 2 pm rad for a bending magnet field change of $\pm 3\%$. Emittance degradation is not significant compared to the electron beam emittance of SACLA, which is 60 pm rad or 1 mm mrad in a normalized value at 8 GeV. The impact would be more pronounced for smaller electron beam emittance, e.g., around 30 pm rad.

One possible countermeasure to address this issue is to adjust the strength of Q1 in the dispersive arc as to cancel out the perturbation by the bending magnet field change and close the linear dispersion within the DBA section. By this adjustment, the linear dispersion leakage at the beam transport line—which is the source causing the higher-order nonlinear dispersion leakage at the exit—can be suppressed. This countermeasure reduces $\Delta\epsilon$ generated by the leaked nonlinear energy dispersions up to the fourth order down to a negligibly small level of about 0.5 pm rad even in the case where the bending magnet field is changed by 3%. However, this adjustment causes distortion of the beam trajectory of the beam transport line between the first and second DBA sections in the dogleg. The trajectory distortion can close within the dogleg by setting the horizontal betatron phase advance between the two Q1s in the first and second DBA sections to an even multiple of π . The magnitude of this distortion at the beam transport line is less than 1 mm at maximum for a bending field change from -3 to $+3\%$, which is small compared to the typical vacuum chamber aperture of several tens of millimeters. This trajectory distortion only generates much smaller linear dispersion by coupling with the quadrupole magnets compared with that at the case where the linear dispersion is directly leaked into the beam transport line.

The CSR suppression condition requires a horizontal betatron phase advance of π between the first and second DBA sections, and as shown in [26] the actual permissible range is 0.8π to 1.3π at SACLA BL2. This requirement does not conflict with the condition for closing the trajectory distortion in the dogleg when the phase advance between the two Q1s is set to 2π . Therefore, it is possible to simultaneously perform both the precise adjustment of the energy-dependent pathlength variation and the suppression of emittance degradation by CSR.

VI. SUMMARY

Analytical expressions of R_{56} and T_{566} are derived under the on-axis condition for an ordinary two-DBA dogleg configuration to investigate the bunch compression using a self-modulated reverse energy chirp. A method for directly evaluating R_{56} and T_{566} is also proposed for the off-axis condition with an arbitrary bending angle of DBA. Applying the proposed method to the two-DBA dogleg of SACLA BL2 confirmed that (1) doubling R_{56} and T_{566} of one DBA section is a good model to represent the energy-dependent pathlength variation for the entire two-DBA dogleg, and (2) closing the linear energy dispersion within each DBA section by adjusting the quadrupole strength in the dispersive arc is effective for suppressing the beam emittance growth downstream from the dogleg caused by nonlinear energy dispersion leakage.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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