

Method for calculating the longitudinal space-charge tune shift suitable for arbitrary bunch lengths

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The space-charge tune shift is a critical issue in circular proton, heavy-ion, and also low energy electron storage accelerators. Recently, some special electron rings with energies of several hundred MeV, nanometer-scale bunch lengths, and micrometer-scale bunch spacings have been proposed. In these novel rings, a significant longitudinal space-charge tune shift could be predicted by conventional methods. However, these methods may not be suitable when the bunches are extremely short. To address this issue, we derive a new set of formulas to calculate the longitudinal space-charge tune shift, which is applicable to arbitrary bunch lengths. These formulas incorporate the multibunch effects on the longitudinal space-charge tune shift, which have not been included in previous analyses of longitudinal space-charge tune shift calculations. The calculation results indicate that the tune shift predicted by our formulas is noticeably smaller than that obtained using the classical method. Moreover, our analysis demonstrates that the multibunch effect is not dominant in the longitudinal direction, even when the bunch spacing is extremely small.

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I. INTRODUCTION

Space-charge tune shifts play an important role in circular accelerators, and under unfavorable conditions, the tune shifts may result in serious beam loss by crossing some dangerous resonance lines [1–3]. Previous research has primarily focused on proton and heavy-ion rings rather than electron rings, as both the transverse and longitudinal space-charge tune shifts are relatively small in the latter.

Recently, novel electron rings with energies of several hundred MeV and nanometer-scale bunch length, such as the steady-state microbunching (SSMB) electron storage rings, have been proposed to generate coherent radiation at short wavelength [4–8]. According to classical formulas, the longitudinal tune shift is inversely proportional to the cube of the bunch length, indicating that SSMB rings may encounter significant challenges in managing this effect.

However, these conventional methods are not fully applicable when the bunch is extremely short, as they are derived under the assumption that $\gamma\sigma_z \gg \sigma_r$ (details of this assumption can be found in Appendix A), where γ is the Lorentz factor and σ_r and σ_z denote the rms bunch sizes in radial and longitudinal directions, respectively. In Fig. 1, we plot the parameters $\gamma\sigma_z$ and σ_r for some existing and planned rings [9–11]. It is evident that the data point corresponding to SSMB approaches the dashed line $\gamma\sigma_z = \sigma_r$, suggesting that the classical method may no longer hold validity for these novel rings with extremely short bunches.

Furthermore, laser modulation is usually used in place of rf modulation to generate such short bunches. The wavelength of typical laser modulation is on the micrometer scale, significantly smaller than that of rf modulation, resulting in an unusually small bunch spacing. This reduced spacing necessitates a careful evaluation of interactions between adjacent bunches, an effect that has not been considered in conventional analyses of longitudinal space-charge tune shifts.

In Ref. [12], we derived formulas for the transverse space-charge tune shift applicable to short bunch lengths, which serve as the theoretical foundation for this paper. Instead of the classical process presented in Appendix A, we calculate the electric field using the scalar potential in a comoving frame. The models chosen in our formulas are a

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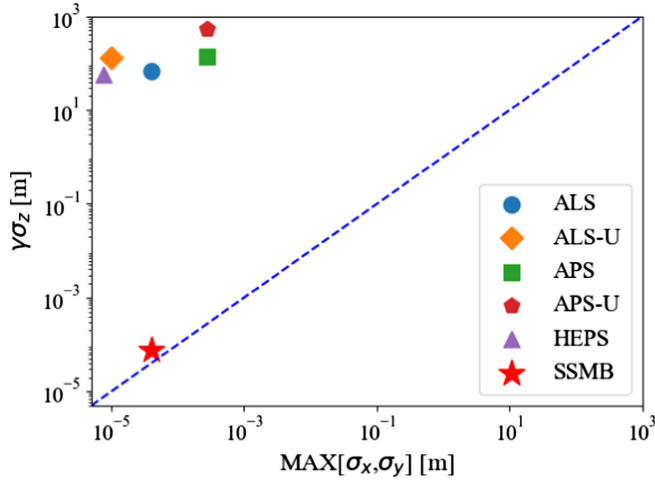


FIG. 1. The parameters $\gamma\sigma_z$ and σ_r for several existing and planned rings are plotted, with the maximum value between σ_x and σ_y selected as σ_r to ensure compliance with the criterion in both transverse directions. The blue dashed line represents the condition $\gamma\sigma_z = \sigma_r$, serving as the criterion for assessing the validity of the classical method.

3D Gaussian bunch and a boundary formed by two parallel plates. To satisfy the boundary condition, an infinite series of image sources is introduced, enabling convenient incorporation of the multibunch effect.

This paper is organized as follows: In Sec. II, the formulas suitable for arbitrary bunch lengths are derived, where image sources are introduced to satisfy the boundary condition. We also discuss the contribution of the multibunch effect in this section and show that it approaches zero in the uniform filling case. This is different from the transverse direction, where the multibunch effect becomes dominant under the short bunch condition [12]. In Sec. III, we present numerical calculations to validate our formulas. Additionally, we compute the longitudinal space-charge tune shifts for the HEPS storage ring and the SSMB ring. Conclusions and discussions are given in Sec. IV.

II. THEORETICAL DERIVATION FOR ARBITRARY BUNCH LENGTHS

As mentioned earlier, the classical method is not applicable to short bunches, as it is derived under the condition $\gamma\sigma_z \gg \sigma_{x,y}$. Therefore, in this section, we derive formulas that are valid for arbitrary bunch lengths. The formulations for the single-bunch and multibunch cases are presented in Secs. II A and II B, respectively. Both derivations originate from the same fundamental equations, with detailed steps provided in Appendix A:

$$\begin{aligned} \nu_z^2 - \nu_{z0}^2 &= -R^2 \Delta k \\ \Delta k &= -\frac{\partial}{\partial z} \left[\frac{\eta e E_z}{\beta^2 E_0} \right], \end{aligned} \quad (1)$$

where Δk is the linear perturbation induced by the space-charge force, $\beta = \sqrt{1 - \frac{1}{\gamma^2}}$, e is the elementary charge, E_0 is the energy of the reference electron, η is the phase slippage factor, R is the radius of the ring, ν_{z0} is the unperturbed longitudinal tune, and E_z is the longitudinal electric field. Now, the key is obtaining the longitudinal electric field E_z for arbitrary bunch length.

A. Single bunch

Let us start with the single-bunch case. A 3D Gaussian bunch model is chosen in our derivation, which is more general than the cylindrically symmetric transverse distribution discussed in Appendix A. The boundary is modeled as two parallel plates, allowing us to clearly illustrate the indirect effect, particularly those arising from adjacent bunches. To satisfy this boundary condition, we introduce an infinite series of image sources, as shown in Fig. 2.

To calculate the longitudinal electric field, we divide it into two components: the direct part, generated by the self-field in free space, and the indirect part, arising from image sources. This approach is commonly adopted in transverse space-charge analyses. Following this decomposition, the longitudinal tune shift can be expressed as

$$\Delta\nu_z = \Delta\nu_{z,\text{direct}} + \Delta\nu_{z,\text{indirect}}. \quad (2)$$

Consequently, we calculate the direct field generated by the bunch itself and the indirect field induced by the image sources separately. Both fields can be obtained from the scalar potential in the comoving frame [Eq. (A4) in Ref. [13]]:

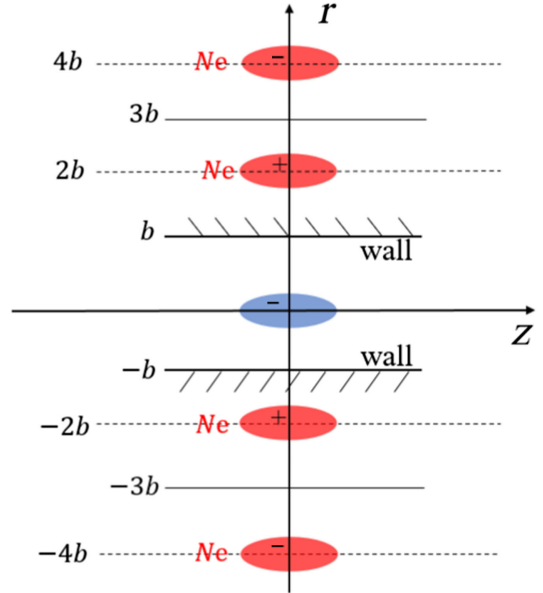


FIG. 2. An infinite series of image sources induced by the boundary, where b is the half-gap of the beam pipe, the bunch is shown as a blue ellipse, image sources are red, and the symbols “+” and “-” denote positive and negative charges, respectively.

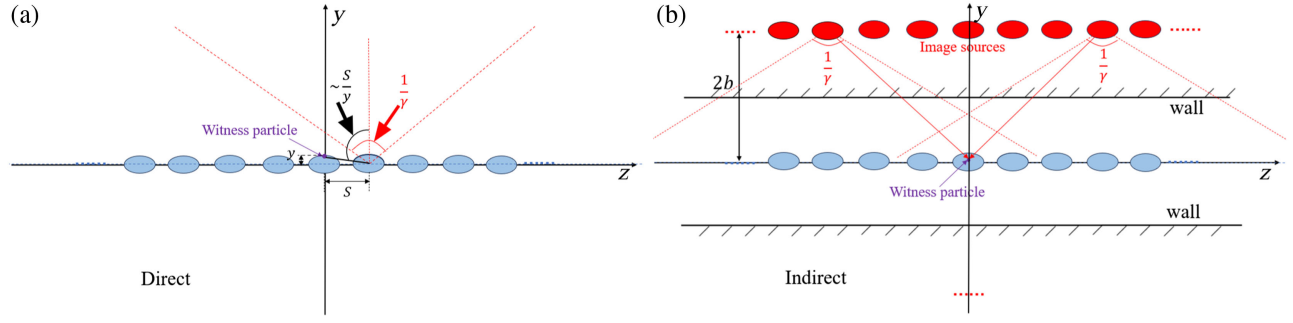


FIG. 3. Illustration of direct and indirect space-charge effects in multibunch scenarios. (a) and (b) represent the direct and indirect cases, respectively. In these two pictures, blue ellipses are bunches, red ellipses are image sources, purple dots represent the witness particle, and $(1/\gamma)$ is the opening angle of field.

$$\bar{\varphi} = \frac{Ne}{4\pi\epsilon_0\sqrt{\pi}} \int_0^\infty dq \frac{e^{-\frac{z^2}{2\sigma_x^2+q} - \frac{y^2}{2\sigma_y^2+q} - \frac{\bar{z}^2}{2\bar{\sigma}_z^2+q}}}{\sqrt{(2\sigma_x^2+q)(2\sigma_y^2+q)(2\bar{\sigma}_z^2+q)}}, \quad (3)$$

where ϵ_0 is the permittivity of vacuum, N is the number of electrons per bunch, $\bar{z} = \gamma z$ and $\bar{\sigma}_z = \gamma\sigma_z$. Through the relation that $\vec{E} = -\nabla\varphi$ and the Lorentz transformation $\bar{E}_z = E_z$, we obtain the longitudinal electric field in the laboratory frame. Then, since the primary concern is the maximum tune shift, with the corresponding electron located at the center of the bunch, we can assume that x , y , and z are all first-order small quantities. Under this assumption, Δk_{direct} and $\Delta k_{\text{indirect}}$ can be written as (the detailed derivation is presented in Appendix B)

$$\Delta k_{\text{direct}} = \frac{2N\eta r_e}{\beta^2\sqrt{\pi}} \int_0^\infty dq \frac{1}{\sqrt{(2\sigma_x^2+q)(2\sigma_y^2+q)(2\bar{\sigma}_z^2+q)^3}}, \quad (4)$$

$$\Delta k_{\text{indirect}} \approx \frac{4N\eta r_e}{\beta^2\sqrt{\pi}} \sum_{n=1}^\infty (-1)^n \int_0^\infty dq \frac{e^{-\frac{(2nb)^2}{q}}}{q\sqrt{(2\bar{\sigma}_z^2+q)^3}}, \quad (5)$$

where r_e is the classical radius of electron. In the derivation of Eq. (5), we assume that $\sigma_{x,y} \ll b$, which is always satisfied when the beam is stable.

For the short bunch case (i.e., $\bar{\sigma}_z \ll 2nb$) that we focus on, the term $(2\bar{\sigma}_z^2+q)$ can also be expressed as q , and the integral can be given as an analytical result¹:

¹Through transforming the integral variable q into $q^{-1/2}$ and the property:

$$\frac{\partial}{\partial a} \int_0^\infty x^n e^{-ax^2} = \int_0^\infty x^{n+2} e^{-ax^2} \quad (n = 0, 1, 2, \dots), \quad (6)$$

$$\int_0^\infty e^{-ax^2} = \frac{\sqrt{\pi}}{2\sqrt{a}}, \quad (7)$$

we can obtain Eq. (8), where the variable a is replaced by $2nb$.

$$\begin{aligned} \Delta k_{\text{indirect}}^{\text{short}} &= \frac{4N\eta r_e}{\beta^2\sqrt{\pi}} \sum_{n=1}^\infty (-1)^n \int_0^\infty dq \frac{e^{-\frac{(2nb)^2}{q}}}{q^{5/2}} \\ &= \frac{4N\eta r_e}{\beta^2\sqrt{\pi}} \sum_{n=1}^\infty (-1)^n \frac{\sqrt{\pi}}{2(2nb)^3} \\ &= -\frac{N\eta r_e}{4\beta^2 b^3} \eta_d(3), \end{aligned} \quad (8)$$

where the superscript “short” means this formula is only valid when the bunch is short ($\bar{\sigma}_z \ll 2nb$) and η_d is the Dirichlet-Eta function.²

B. Multibunches

We now consider the multibunch effect. First, we aim to demonstrate the necessity of evaluating the effect of adjacent bunches. Figure 3 illustrates the multibunch effect for both direct and indirect cases, considering a witness particle positioned at x, y, z with a bunch spacing of S between two adjacent bunches. For simplicity, only the vertical direction y is depicted, as the horizontal direction x shows no significant difference.

Using some typical SSMB parameters listed in Table I and assuming that y takes values on the order of tens of micrometers (comparable to the transverse size), we find that the value of S/y in Fig. 3(a) is significantly larger than the opening angle of $1/\gamma$, indicating the multibunch effect is not crucial for the direct space-charge tune shift, even when the bunch spacing is very small.

This conclusion does not hold for the indirect scenario as shown in Fig. 3(b), where the value of S/b is smaller than the opening angle $1/\gamma$ when the half-gap b is on the order of tens of millimeters. Therefore, it is necessary to evaluate the multibunch effect of indirect space charge in detail.

²The Dirichlet-Eta function can be expressed as

$$\eta_d(s) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^s}. \quad (9)$$

TABLE I. Some typical parameters of the SSMB ring.

Parameter, symbol	Unit	Values
Energy, E	MeV	400
Circumference, C ,	m	166
Bunch population, N		20,000
Phase slippage, η		1.13×10^{-6}
Bunch spacing, S	m	1×10^{-6}
Half-gap of pipes, b	m	0.01
Unperturbed longitudinal tune, ν_{z0}		0.087
Transverse rms size, σ_r	μm	40
Longitudinal rms length, σ_z	nm	100

The detailed derivations are presented in Appendix C, and the final formula for uniform filling is

$$\Delta k_{\text{indirect}}^{\text{multi}} = \frac{4N\eta r_e}{\beta^2} \sum_{p=-\infty}^{\infty} \sum_{n=1}^{\infty} (-1)^n \times \left[\frac{3\gamma^2 p^2 S^2}{2A(n, p)^{5/2}} - \frac{1}{2A(n, p)^{3/2}} \right], \quad (10)$$

where $A(n, p) = (2nb)^2 + (p\gamma S)^2$. By applying the parameters listed in Table I, with the phase slippage set to 0.001, to Eq. (10), we obtain Fig. 4, where the phase slippage acts merely as a coefficient outside the summation term, affecting the amplitude rather than the conclusion we aim to illustrate. As shown in Fig. 4(a), the Δk contributions from nearby and distant bunches have opposite signs, indicating a cancellation effect between them. This cancellation is illustrated in Fig. 4(b), where the total contribution gradually approaches zero as p increases.

For the longitudinal direction with nonuniform filling, the small bunch spacing (1 μm) results in a large number of bunches within a typical meter-scale bunch train. As shown in Fig. 4, the multibunch effect approaches zero within the first

few hundred bunches, indicating that the preceding analysis remains valid for the vast majority of bunches in the train.

In conclusion, for very small bunch spacing, the direct effect contributed by adjacent bunches remains negligible, while the indirect effect of a single nearby bunch becomes comparable in magnitude to the effect of the self-bunch (where “self-bunch” refers to the bunch containing the witness particle). However, the contributions from bunches at different positions are both positive and negative, causing the total multibunch effect to approach zero. Therefore, only the single-bunch case is considered in the following.

III. NUMERICAL CALCULATIONS

After completing the theoretical derivation, we now perform numerical calculations in this section using typical parameters for the SSMB and HEPS rings. Since the classical method assumes a symmetric transverse distribution, we set the bunch sizes in the x and y directions to be equal for a consistent comparison. To do this, we select the smaller size among the two directions for our calculations, which are 2.56 μm for the HEPS main ring and 40 μm for the SSMB ring, respectively. The unperturbed longitudinal tune of the SSMB ring is set at 0.087, while for the HEPS ring, it is calculated to be 0.00074 based on the parameters provided in Ref. [11]. All parameters used in the following calculations are listed in Tables I and II.

In Fig. 5, we scan σ_z and plot the ratio of Δk calculated using our formulas to that obtained from the classical method, with all other parameters set as the SSMB ring. In the long bunch range, where $\gamma\sigma_z \gg \sigma_{x,y}$, the ratio is approximately equal to 1, indicating our formulas agree with the classical ones. However, due to the fact that our method employs a two parallel plates model for the boundary, which differs from the cylindrical pipe model applied in the classical method, the ratio deviates slightly from 1. We think this small divergence is reasonable.

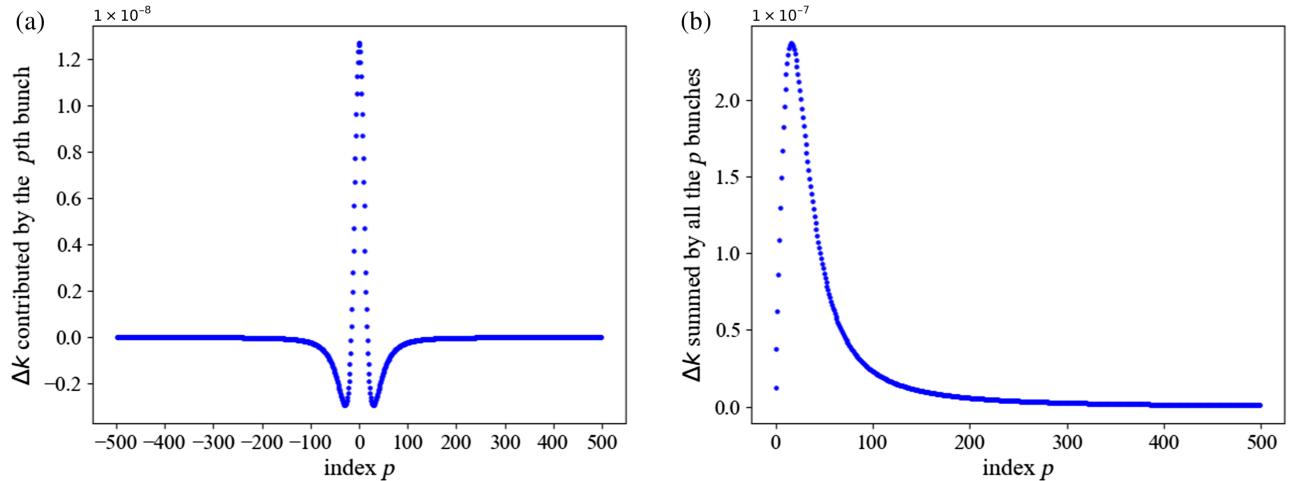


FIG. 4. (a) Δk contributed by p th bunch. (b) Δk summed by all the p bunches, where p represents the number of bunches considered both upstream and downstream.

TABLE II. Some typical parameters of the HEPS storage ring.

Parameter, symbol	Unit	Values
Energy, E	MeV	6000
Circumference, C	m	1360.4
Bunch population, N		$\sim 8.3 \times 10^9$
Phase slippage, η		1.56×10^{-5}
Unperturbed longitudinal tune, ν_{z0}		0.00074
Transverse rms size, σ_r	μm	2.56
Longitudinal rms length, σ_z	mm	5

Besides, as σ_z declines, the tune shift computed by the classical formula becomes larger than that obtained from our formulas, which meets our prediction in Sec. I, i.e., the tune shift will be inaccurate if using the classical formula when the bunch is very short. We can also see that the results given by the formula under the short bunch assumption [Eq. (8)] show excellent agreement with the complete formula [Eq. (5)] when σ_z is small.

Following the above analysis, we believe that the validity of our new formulas has been demonstrated. Using the value of Δk , we can figure out a longitudinal tune shift via Eq. (1). The computed results for a HEPS storage ring and an SSMB ring are summarized in Tables III and IV, where several cases with different half-gaps of the parallel plates are considered.

In Table III, no significant differences are observed between our results and the classical results in the HEPS ring, showing the validity of our method again. However,

TABLE III. The longitudinal tune shifts of a HEPS ring, with parameters presented in Table II and different half-gaps.

	$b = 5 \text{ mm}$	$b = 10 \text{ mm}$	$b = 20 \text{ mm}$
$\Delta\nu_z^{\text{classical}}$	3.42×10^{-7}	3.74×10^{-7}	4.05×10^{-7}
$\Delta\nu_z^{\text{our}}$	3.37×10^{-7}	3.68×10^{-7}	3.99×10^{-7}
$\Delta\nu_{z,\text{direct}}^{\text{our}}$	7.41×10^{-7}	7.41×10^{-7}	7.41×10^{-7}
$\Delta\nu_{z,\text{indirect}}^{\text{our}}$	-4.04×10^{-7}	-3.73×10^{-7}	-3.40×10^{-7}

TABLE IV. The longitudinal tune shifts of a typical SSMB ring, with parameters presented in Table I and different half-gaps.

	$b = 5 \text{ mm}$	$b = 10 \text{ mm}$	$b = 20 \text{ mm}$
$\Delta\nu_z^{\text{classical}}$	0.511	0.552	0.591
$\Delta\nu_z^{\text{our}}$	0.153	0.153	0.153
$\Delta\nu_{z,\text{direct}}^{\text{our}}$	0.153	0.153	0.153
$\Delta\nu_{z,\text{indirect}}^{\text{our}}$	-4.56×10^{-7}	-5.73×10^{-8}	-7.17×10^{-8}

all values are small, suggesting that the longitudinal tune shift is not a significant issue in typical electron rings.

In Table IV, it is evident that the classical formula overestimates the longitudinal tune shift in the SSMB ring. Additionally, the results given by the classical formula vary with the gap size, suggesting a significant influence of the boundary on $\Delta\nu_z$ when the bunch is very short. However, this assessment is inaccurate and the indirect tune shift caused by image sources is negligible compared to the direct tune shift at this time, as shown in the last two lines in Table IV. Finally, although the longitudinal tune shift obtained from our formulas is smaller than that computed using the classical formula, it remains significant relative to the unperturbed longitudinal tune, the influence of which warrants further investigation.

IV. CONCLUSIONS AND DISCUSSIONS

In this paper, we have derived new formulas to calculate the longitudinal space-charge tune shift. These formulas are applicable to arbitrary bunch lengths, extending the validity of previous approaches. Furthermore, we analyze the multibunch effect, demonstrating that its impact on both direct and indirect space-charge effects is negligible, even when the bunch spacing is extremely small.

Numerical calculations have been done using both classical formulas and our newly derived formulas. The results indicate that the longitudinal space-charge tune shifts computed by both methods agree with each other when the bunch length is relatively long. As the bunch length decreases, the tune shifts predicted by our formulas become smaller than those calculated using the classical formula. We also find that the influence of the boundary is negligible for very short bunches, which is different from the prediction of the classical method.

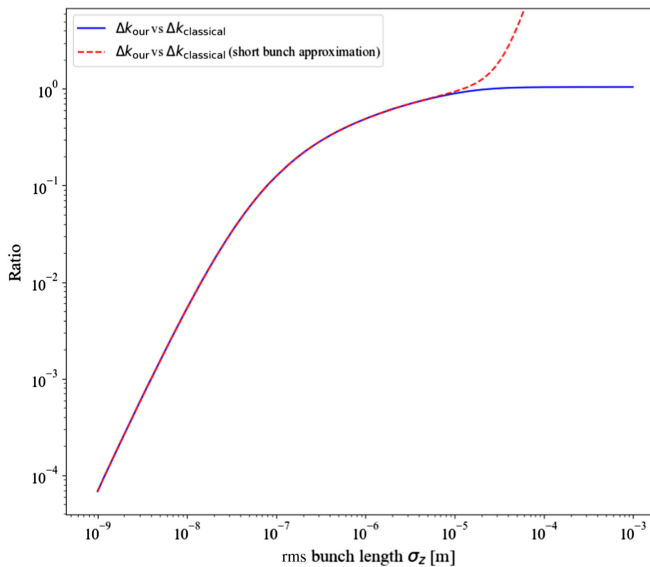


FIG. 5. The ratio of Δk computed by our formulas to that computed by the classical formula. In this picture, $\Delta k_{\text{classical}}$ is given by the classical formula [i.e., Eqs. (A4) and (A8)], while Δk_{our} means the sum of the direct and indirect Δk computed by our formulas [i.e., Eqs. (4) and (5)]. The “short bunch approximation” in the legend represents the indirect Δk computed by the simplified formula Eq. (8), which is valid when the bunch is short.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available at [14].

APPENDIX A

The following derivation is primarily based on Chapter 1.6 of Ref. [15], Chapter 21.2.6 of Ref. [16], and Chapter 3.8 of Ref. [17]. We begin with the equation of longitudinal motion including the space-charge force.:

$$\begin{aligned} z' &= -\eta\delta \\ \delta' &= \frac{1}{\eta} \left(\frac{\nu_{z0}}{R} \right)^2 z + \frac{eE_z}{\beta^2 E_0}, \end{aligned} \quad (\text{A1})$$

where $()'$ means $d()/ds$, $\beta = \sqrt{1 - \frac{1}{\gamma^2}}$, e is the elementary charge, E_0 is the energy of the reference electron, η is the phase slippage factor, R is the radius of the ring, ν_{z0} is the unperturbed longitudinal tune, and E_z is the longitudinal electric field. By linearizing the electric field and combining these two equations, we can obtain

$$z'' + \left(\frac{\nu_{z0}}{R} \right)^2 z = -\frac{\partial}{\partial z} \left[\frac{\eta e E_z}{\beta^2 E_0} \right] z. \quad (\text{A2})$$

Now, the longitudinal tune shift can be written as

$$\nu_z^2 - \nu_{z0}^2 = -R^2 \Delta k, \quad (\text{A3})$$

where we introduce the symbol Δk

$$\Delta k = -\frac{\partial}{\partial z} \left[\frac{\eta e E_z}{\beta^2 E_0} \right]. \quad (\text{A4})$$

The next step is computing the electric field E_z . In those textbooks, the bunch with cylindrically symmetric transverse distribution is chosen, and the boundary is typically assumed to be a conducting cylindrical pipe. By applying Faraday's law and the boundary condition that $E_z(b, z) = 0$, as shown in Fig. 6, we can obtain the equation:

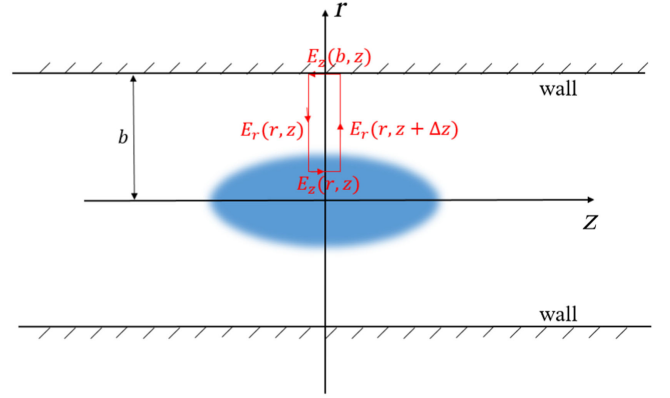


FIG. 6. Sketch of the chosen closed loop (red line) for a bunch with the cylindrically symmetric transverse distribution in the cylindrical coordinate system. In this picture, b is the half-gap of the beam pipe, while z and r represent the longitudinal and the radial directions, respectively.

$$\begin{aligned} E_z(r, z) \Delta z + \int_r^b E_r(r, z + \Delta z) dr - E_z(b, z) \Delta z \\ - \int_r^b E_r(r, z) dr = -\Delta z \frac{\partial}{\partial t} \int_r^b B_\phi(r) dr, \end{aligned} \quad (\text{A5})$$

where B_ϕ is the magnetic field perpendicular to the paper. By applying the following relations:

$$\begin{aligned} E_r(r, z + \Delta z) - E_r(r, z) &= \frac{\partial E_r(r, z)}{\partial z} \Delta z \\ z &= -\beta c t \\ B_\phi &= \frac{\beta}{c} E_r \end{aligned} \quad (\text{A6})$$

we can write the longitudinal electric field E_z through the radial electric field E_r :

$$E_z(r, z) = -\frac{1}{\gamma^2} \frac{\partial}{\partial z} \int_r^b E_r(r, z) dr. \quad (\text{A7})$$

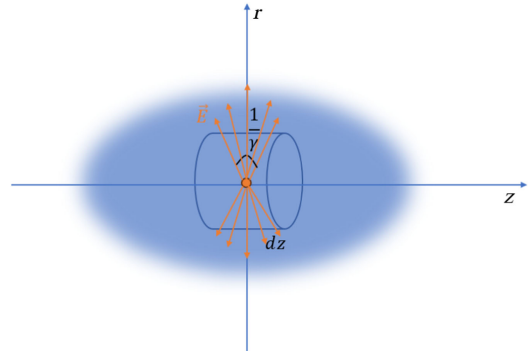


FIG. 7. Sketch of the electric field (orange arrows) generated by charges in a single bunch in the comoving frame (z, r) . The cylinder, with a radius r and an infinitesimal length dz , represents a closed surface inside a bunch.

In textbooks, the long bunch assumption that $\gamma\sigma_z \gg \sigma_r$ is usually used to obtain the radial electric field E_r . By applying Gauss's law to a cylinder inside a bunch with a radius r and an infinitesimal length dz , as shown in Fig. 7, we obtain:

$$E_{r,\text{classical}}(r, z) = \frac{1}{2\pi\epsilon_0} \frac{Ne}{\sqrt{2\pi}\sigma_z} e^{-\frac{z^2}{2\sigma_z^2}} \left[\frac{1 - e^{-\frac{r^2}{2\sigma_r^2}}}{r} \right] \quad (\text{A8})$$

ϵ_0 is the permittivity of vacuum, and N is the number of electrons per bunch. In this process, we suppose that $E_z \approx 0$, which is reasonable when $\gamma\sigma_z \gg \sigma_r$. Nevertheless, if the bunch length is extremely short, the flux of the two bottom surfaces will no longer be negligible, making the electric field given by Eq. (A8) larger than the actual value.

APPENDIX B

According to Eq. (3), the relation $\vec{E} = -\nabla\phi$ and the Lorentz transformation $\vec{E}_z = E_z$, we can obtain the

longitudinal electric field without assumption on bunch length:

$$E_z = \frac{Ne}{2\pi\epsilon_0\sqrt{\pi}} \int_0^\infty dq \frac{\gamma z e^{-\frac{x^2}{2\sigma_x^2+q} - \frac{y^2}{2\sigma_y^2+q} - \frac{z^2}{2\sigma_z^2+q}}}{\sqrt{(2\sigma_x^2+q)(2\sigma_y^2+q)(2\sigma_z^2+q)^3}}. \quad (\text{B1})$$

Since the particle located at the center of the bunch is what we are interested in, we can assume all location coordinates x , y , and z to be first-order small quantities. Under this assumption, all exceptional terms can be neglected. By combining the simplified electric field and Eq. (A4), we can obtain the direct perturbation Eq. (4).

For the indirect part, according to Fig. 2 and Eq. (B1), the longitudinal electric field contributed by image sources is

$$\begin{aligned} E_{z,\text{indirect}} &= 2[-E_z(y=2b) + E_z(y=4b) - E_z(y=6b) + E_z(y=8b) - \dots] \\ &= \frac{Ne}{\pi\epsilon_0\sqrt{\pi}} \sum_{n=1}^\infty (-1)^n \int_0^\infty dq \frac{\gamma z e^{-\frac{x^2}{2\sigma_x^2+q} - \frac{(2nb)^2}{2\sigma_y^2+q} - \frac{z^2}{2\sigma_z^2+q}}}{\sqrt{(2\sigma_x^2+q)(2\sigma_y^2+q)(2\sigma_z^2+q)^3}}, \end{aligned} \quad (\text{B2})$$

where the position coordinate y is replaced by $2nb$ in this equation. Holding the same assumption that x and z are still small and applying it to Eq. (A4), we obtain $\Delta k_{\text{indirect}}$:

$$\begin{aligned} \Delta k_{\text{indirect}} &= \frac{4N\eta r_e}{\beta^2\sqrt{\pi}} \sum_{n=1}^\infty (-1)^n \int_0^\infty dq \\ &\quad \times \frac{e^{-\frac{(2nb)^2}{2\sigma_y^2+q}}}{\sqrt{(2\sigma_x^2+q)(2\sigma_y^2+q)(2\sigma_z^2+q)^3}}. \end{aligned} \quad (\text{B3})$$

The formula for the indirect part is more complex than the direct equation provided in Eq. (4). However, for stable electron beams in the storage ring, the condition $\sigma_{x,y} \ll b$ is consistently satisfied. This indicates that the meaningful contributions to the integral arise only when $q \sim (2nb)^2$ or when $q > (2nb)^2$, at which point q is much bigger than $\sigma_{x,y}^2$. As a result, all terms $(2\sigma_{x,y}^2+q)$ can be simply written as q . Then, we obtain Eq. (5) in Sec. II A.

APPENDIX C

To include the multibunch effect, we replace z in Eq. (B2) with the sequence $-pS - z$, $(-p+1)S - z$, ..., $(p-1)S - z$, $pS - z$, and add them together. This approach assumes a uniform filling scenario. The indirect

electric field is

$$\begin{aligned} E_{z,\text{indirect}}^{\text{multi}} &= \frac{Ne}{\pi\epsilon_0\sqrt{\pi}} \sum_{p=-\infty}^\infty \sum_{n=1}^\infty (-1)^n \int_0^\infty dq \\ &\quad \times \frac{\gamma(pS - z) e^{-\frac{x^2}{2\sigma_x^2+q} - \frac{(2nb)^2}{2\sigma_y^2+q} - \frac{[\gamma(pS-z)]^2}{2\sigma_z^2+q}}}{\sqrt{(2\sigma_x^2+q)(2\sigma_y^2+q)(2\sigma_z^2+q)^3}}. \end{aligned} \quad (\text{C1})$$

Since the longitudinal position z is still small, the exponential term about z can be simplified to

$$e^{-\frac{(\gamma pS - z)^2}{2\sigma_z^2+q}} \approx e^{-\frac{(\gamma pS)^2}{2\sigma_z^2+q}} e^{-\frac{2\gamma^2 pS z}{2\sigma_z^2+q}} \approx e^{-\frac{(\gamma pS)^2}{2\sigma_z^2+q}} \left[1 + \frac{2\gamma^2 pS z}{2\sigma_z^2+q} \right]. \quad (\text{C2})$$

Holding the assumption that x is small and plugging Eq. (C2) into Eq. (C1), the numerator of the integral (expressed by ξ) in Eq. (C1) can be written as

$$\begin{aligned} \xi &\approx \gamma(pS - z) e^{-\frac{(2nb)^2}{2\sigma_y^2+q}} e^{-\frac{(\gamma pS)^2}{2\sigma_z^2+q}} \left[1 + \frac{2\gamma^2 pS z}{2\sigma_z^2+q} \right] \\ &\approx e^{-\frac{(2nb)^2}{2\sigma_y^2+q}} e^{-\frac{(\gamma pS)^2}{2\sigma_z^2+q}} \left[\gamma pS + \frac{2\gamma^3 p^2 S^2 z}{2\sigma_z^2+q} - \gamma z \right]. \end{aligned} \quad (\text{C3})$$

Combining Eqs. (C1), (C3), and the second equation of Eq. (A4), we can obtain the final result:

$$\Delta k_{\text{indirect}}^{\text{multi}} = \frac{4N\eta r_e}{\beta^2 \sqrt{\pi}} \sum_{p=-\infty}^{\infty} \sum_{n=1}^{\infty} (-1)^n \int_0^{\infty} dq \left[\frac{2\gamma^2 p^2 S^2}{2\bar{\sigma}_z^2 + q} - 1 \right] \times \frac{e^{\frac{(2nb)^2}{2\sigma_x^2 + q}} e^{\frac{(p\gamma S)^2}{2\bar{\sigma}_z^2 + q}}}{\sqrt{(2\sigma_x^2 + q)(2\sigma_y^2 + q)(2\bar{\sigma}_z^2 + q)^3}}. \quad (\text{C4})$$

Here, we want to define a “meaningful” multibunch scenario characterized by the following conditions: (i) $\sigma_{x,y} \ll b$, ensuring beam stability; (ii) $\gamma S \sim b$, indicating that the multibunch effect is influential; and (iii) $\sigma_z \ll S$, implying no significant overlap. Under these conditions, all terms $2\bar{\sigma}_z^2 + q$ and $2\sigma_{x,y}^2 + q$ can be written as q :

$$\Delta k_{\text{indirect}}^{\text{multi}} = \frac{4N\eta r_e}{\beta^2 \sqrt{\pi}} \sum_{p=-\infty}^{\infty} \sum_{n=1}^{\infty} (-1)^n \int_0^{\infty} dq \left[\frac{2\gamma^2 p^2 S^2}{q} - 1 \right] \frac{e^{\frac{(2nb)^2 + (p\gamma S)^2}{q}}}{q^{5/2}}. \quad (\text{C5})$$

Using Eqs. (6) and (7) but defining the variable a as $A(n, p)$, where $A(n, p) = (2nb)^2 + (p\gamma S)^2$, we obtain:

$$\begin{aligned} \Delta k_{\text{indirect}}^{\text{multi}} &= \frac{4N\eta r_e}{\beta^2 \sqrt{\pi}} \sum_{p=-\infty}^{\infty} \sum_{n=1}^{\infty} (-1)^n \left[4\gamma^2 p^2 S^2 \frac{3\sqrt{\pi}}{8A(n, p)^{5/2}} - \frac{\sqrt{\pi}}{2A(n, p)^{3/2}} \right] \\ &= \frac{4N\eta r_e}{\beta^2} \sum_{p=-\infty}^{\infty} \sum_{n=1}^{\infty} (-1)^n \left[\frac{3\gamma^2 p^2 S^2}{2A(n, p)^{5/2}} - \frac{1}{2A(n, p)^{3/2}} \right]. \end{aligned} \quad (\text{C6})$$

Remember that Eq. (C6) is only valid for the mentioned meaningful multibunch case.

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