

Quantitative evaluation of a longitudinal rms emittance decrease using nonlinear space charge force in an intense deuteron linac

J. Hyun^{1,*} and A. Mizuno^{1,2}

¹*National Institutes for Quantum Science and Technology (QST),*

2-166 Omotedate, Obuchi, Rokkasho, Kamikita, Aomori 039-3212, Japan

²*Japan Synchrotron Radiation Research Institute (JASRI), 1-1-1 Kouto, Sayo, Hyogo 679-5198, Japan*



(Received 7 May 2025; accepted 4 September 2025; published 22 September 2025)

Radio frequency (rf) bunchers are often utilized to control a longitudinal bunch length in low- and medium-energy transport lines. However, nonlinear rf fields in the bunchers deform a longitudinal phase space distribution, which causes emittance growth leading to beam quality degradation and unexpected beam losses. This paper describes a decrease in the longitudinal rms emittance in a high-intensity deuteron linac, where the longitudinal phase space deformed by a nonlinear rf field in the buncher is self-linearized by a longitudinal nonlinear space charge force. To quantitatively explain this phenomenon, we develop two formulas to express deformations of phase space and forces acting on the beam. We perform particle tracking simulations with the TraceWin code using a 125 mA, 5 MeV deuteron beam at the Linear IFMIF Prototype Accelerator (LIPAc) and validate the two formulas by applying them to LIPAc. The quantitative evaluations of the deformations with the formulas lead to a deeper understanding of beam physics and are also expected to be useful in mitigating phase space deformation.

DOI: [10.1103/kx6r-1d8r](https://doi.org/10.1103/kx6r-1d8r)

I. INTRODUCTION

A root mean square (rms) beam emittance, which represents an rms area of an ellipse in phase space and beam quality, is conserved over a beamline as long as linear forces act on a beam. Nonlinear forces originating from focusing magnets, rf cavities, and space charge (SC) effects cause emittance growth, resulting in beam losses and quality degradation that lead to unexpected operation conditions. Once the rms emittance worsens, it is challenging to reduce it. Therefore, it is essential to suppress emittance degradation to perform beam operations without limiting the machine performance.

A longitudinal space charge (LSC) force F_z is one of the main sources to cause emittance growth. When a bunch length is longer than the diameter of a beam pipe, it can analytically be written as $F_z(r, z) \propto -\frac{g}{\gamma^2} \frac{\partial \lambda(z)}{\partial z}$, where λ is the longitudinal line charge density (charge per length), γ is the Lorentz factor of a beam, and g is the geometric factor depending on radii of a transverse beam size and a beam pipe [1–3]. When the longitudinal distribution is a

uniform, the LSC force does not contribute to the longitudinal plane of the beam due to $\partial \lambda(z)/\partial z = 0$. In a Gaussian distribution, particles near the bunch center experience a linear LSC force, while those at the bunch head and tail are decelerated and accelerated due to $-\partial \lambda(z)/\partial z$, respectively. Therefore, the longitudinal phase space (LPS) distribution gradually deforms into an S-shape as the beam travels a beamline, which causes the emittance growth [4]. In the transverse plane, the emittance growth by SC forces is known well, and the method to mitigate rms emittance growth caused by SC force in a proton linac is proposed in [5].

rf bunchers are employed to control a bunch length using the velocity differences of particles in electron and hadron accelerators. It is well known that a nonlinear rf field component in the buncher deforms the LPS distribution into an inverse S-shape. The deformed beam distribution limits the minimum bunch length and a transverse beam size in a dispersive region when maximally focused. To address this, some approaches to mitigate the deformed LPS have been proposed and applied, especially in electron linacs [6,7].

We found that when an LPS is deformed into an inverse S-shaped distribution in a buncher, it is compensated by an LSC force with an S-shaped distribution due to the opposing curvatures. This is expected to lead to a decrease in the rms emittance. Similar phenomena, such as transverse emittance reductions in electron injectors, have been reported in [8]. This phenomenon shows that the

*Contact author: hyun.jibong@qst.go.jp

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radial phase space deformed to an inverse S-shape by the image charge effect at a cathode is rapidly linearized by the transverse S-shaped SC force. Additionally, the deformations of the phase space and the transverse SC force acting on the beam were qualitatively evaluated by introducing two physical quantities defined as mean emittance and mean force nonlinearity in [9]. However, longitudinal emittance decreases via self-linearization have not been reported. Additionally, the two expressions are only applicable to the radial phase space. Therefore, we developed the formulas to qualitatively evaluate the deformations of the phase space and the internal and external forces acting on a beam in the Cartesian coordinate system.

To validate the LPS linearization, we apply the two developed formulas to a 5 MeV beamline for the intermediate commissioning stage called Phase B+ of the Linear IFMIF prototype Accelerator (LIPAc) [10]. The Phase B+ configuration of LIPAc can accelerate a 125 mA deuteron beam with an energy of up to 5 MeV. The LIPAc beamline consists of a low-energy beam transport line (LEBT) with an ion source, an RFQ with a length of 9.8 m, a medium-energy beam transport line (MEBT) with bunchers, a MEL (MEBT extended line), a high-energy beam transport line (HEBT) with a bending magnet, and a beam dump. Each section after the RFQ mainly comprises quadrupole, dipole magnets, and beam monitors to guide the beam to the beam dump and has no emittance compensation devices. Using TraceWin [11], we perform the particle tracking simulations assuming a single bunch to obtain longitudinal phase space distributions along the beamline.

In this paper, we explain the mechanism of the longitudinal emittance decrease after the buncher and quantitatively evaluate the nonlinearities of the LPS and the SC force using the mean emittance and the mean force nonlinearity defined in the Cartesian coordinate system. First, we review the rms emittance and then derive a formula of the normalized mean emittance in the longitudinal and transverse planes in Sec. II. In Sec. III, we derive a formula of the mean force nonlinearity of the LSC force. We summarize the characteristics of the mean emittance and the mean force nonlinearity in Sec. IV. We present the LIPAc beamline, the simulated transverse and longitudinal beam sizes, and transverse rms emittances along the beamline in Secs. V and VI. To calculate distributions of the space charge force and the rf field force in a buncher acting on a beam, we show the analytical formulas in Sec. VII. In Secs. VIII and IX, we show the evolution of the normalized rms emittance, the normalized mean emittance, the MFN and explain the mechanism of the longitudinal rms emittance evolution. The parameter dependence of the minimum longitudinal rms emittance is shown in Sec. X, and conclusions are presented in Sec. XI.

II. NORMALIZED RMS EMITTANCE AND MEAN EMITTANCE

We first review an rms emittance and then explain a mean emittance that can express deformation of a phase space distribution of a beam.

A. rms emittance

The longitudinal rms emittance can be written as

$$\epsilon_{z,z'} = \sqrt{\langle z^2 \rangle \langle z'^2 \rangle - \langle z \cdot z' \rangle^2},$$

or

$$\epsilon_{z,p} = \sqrt{\langle z^2 \rangle \left\langle \left(\frac{\Delta P}{P_0} \right)^2 \right\rangle - \left\langle z \cdot \frac{\Delta P}{P_0} \right\rangle^2}, \quad (1)$$

where $z(=s-s_0)$ is the difference of the longitudinal particle position s with respect to the synchronous particle s_0 on the design trajectory, z' is the particle divergence for the design trajectory, P_0 is the central momentum, and ΔP is the momentum difference. The bracket notation $\langle \cdot \rangle$ denotes the average over the number of all particles, and $\langle \cdot^2 \rangle$ stands for the second moment. For example, it can be calculated as $\langle z^2 \rangle = \langle (z - \langle z \rangle)^2 \rangle$. Additionally, z' is expressed as

$$z' = \frac{dz}{ds} = \frac{dz}{dt} \frac{dt}{ds} = \frac{\Delta v_z}{v_0} = \frac{\Delta \beta}{\beta} = \frac{1}{\gamma^2} \frac{\Delta P}{P_0}, \quad (2)$$

where v_0 is the velocity of the synchronous particle, γ is the Lorentz factor, and $\beta = v_0/c$. Using Eq. (2), we can relate $\epsilon_{z,z'}$ to $\epsilon_{z,p}$ as $\epsilon_{z,z'} = (1/\gamma^2)\epsilon_{z,p}$. The normalized rms emittance in the longitudinal plane can be written as

$$\epsilon_{n,z} = \beta \gamma \epsilon_{z,p} = \beta \gamma^3 \epsilon_{z,z'}. \quad (3)$$

In the transverse plane, the normalized rms emittance $\epsilon_{n,\perp}$ is given by

$$\epsilon_{n,\perp} = \beta \gamma \epsilon_{\perp,\perp'}, \quad (4)$$

where $\perp = (x, y)$, $\perp' = (x', y')$, and $\epsilon_{\perp,\perp'}$ is the transverse unnormalized rms emittance.

Here, we consider a beam consisting of n particles. The square of the rms emittance ϵ_u where $u = x, y, z$ is

$$\begin{aligned} \epsilon_u^2 &= \langle u^2 \rangle \langle u'^2 \rangle - \langle u \cdot u' \rangle^2 \\ &= \left(\frac{1}{n} \sum_{i=1}^n u^2 \right) \left(\frac{1}{n} \sum_{i=1}^n u'^2 \right) - \left(\frac{1}{n} \sum_{i=1}^n uu' \right)^2. \end{aligned} \quad (5)$$

Equation (5) can be calculated as

$$\begin{aligned} n^2 \epsilon_u^2 &= (u_1^2 + u_2^2 + u_3^2 + \cdots + u_n^2)(u_1'^2 + u_2'^2 + u_3'^2 + \cdots + u_n'^2) - (u_1 u_1' + u_2 u_2' + \cdots + u_n u_n')^2 \\ &= (u_1 u_2' - u_2 u_1')^2 + (u_1 u_3' - u_3 u_1')^2 + \cdots + (u_{n-1} u_n' - u_n u_{n-1}')^2. \end{aligned} \quad (6)$$

Each term $u_i u_j' - u_j u_i' (= \vec{i} \times \vec{j})$ means a parallelogram area spanned by two vectors pointing from the origin to two particle positions i, j . Since the number of the parallelograms in the phase space is ${}_n C_2 = \frac{n(n-1)}{2} (\approx \frac{n^2}{2})$, the rms emittance defined with each parallelogram area $W_{k,u}$ is

$$\epsilon_u = \sqrt{\frac{\sum_{k=1}^{{}_n C_2} W_{k,u}^2}{n^2}} = \sqrt{\frac{\sum_{k=1}^{n^2/2} W_{k,u}^2}{2(n^2/2)}} = \sqrt{\frac{\langle W_u^2 \rangle}{2}}. \quad (7)$$

Each parallelogram area $W_{k,u}$ can be positive, zero, or negative, while the rms emittance is independent of the sign of each parallelogram area $W_{k,u}$ due to the square of $W_{k,u}$.

B. Mean emittance

To introduce the mean emittance into the Cartesian coordinate system, which can characterize different types of phase space deformation, we determine the sign of $W_{k,u}$ corresponding to three types of phase space deformation: linear shape, S-shape, and inverse S-shape. For simplicity, we first consider any two particles i, j in a phase space with a line distribution and then extend to a particle ensemble. We select the vector closer to the origin between two vectors $\vec{i}, \vec{j}$ as a reference vector. Figure 1 illustrates the relationship between $W_{k,u}$ and each type of phase space deformation. As shown in Fig. 1(a), a linear phase space

distribution gives the parallelogram area $W_{ij,u} = 0$ due to the cross product of the two vectors. When particle j is below (or above) the extended line of the reference vector as shown in Figs. 1(b) and 1(c), the phase space distribution deforms as an S-shape (or an inverse S-shape). The S-shape in the phase space is the typical distribution when a beam is subjected to a nonlinear space charge force. Here, we define the sign of $W_{ij,u}$ below so that $W_{ij,u} > 0$ when the deformation is the S-shape corresponding to a curvature of an SC force acting on a beam

$$W_{ij,u} = 0: \text{linear shape}, \quad (8)$$

$$W_{ij,u} > 0: \text{S-shape}, \quad (9)$$

$$W_{ij,u} < 0: \text{inverse S-shape}. \quad (10)$$

The parallelogram area taking into account the signs can be given as

$$W_{ij} = (u_i u_j' - u_j u_i') \cdot \text{sgn}(|u_i| - |u_j|) \cdot \text{sgn}(u_i u_j) \quad (11)$$

$$\begin{aligned} &= (u_i u_j' - u_j u_i') \cdot \frac{|u_i| - |u_j|}{||u_i| - |u_j||} \cdot \frac{u_i u_j}{|u_i u_j|} \\ &(i = 1, 2, \dots, n-1, j = i+1, \dots, n). \end{aligned} \quad (12)$$

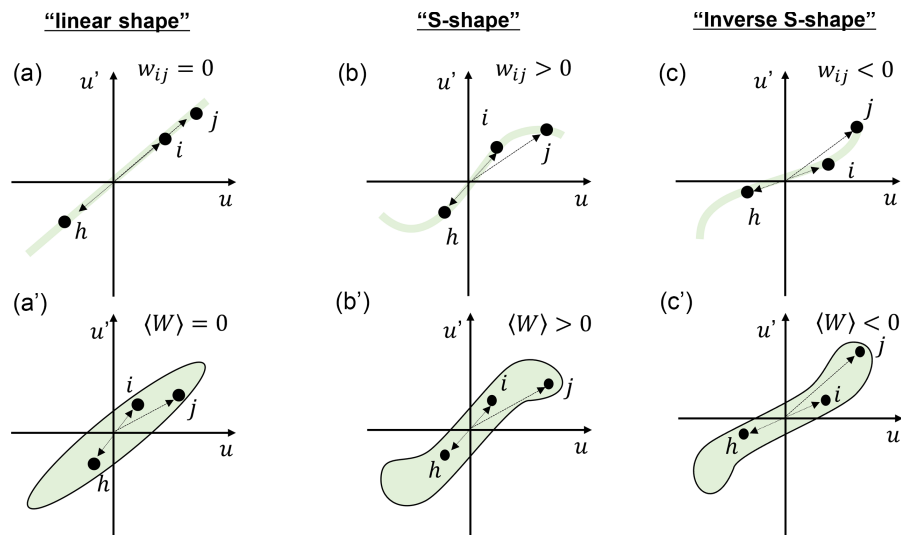


FIG. 1. The signs of a mean emittance corresponding to three deformation cases in phase space distributions. For instance, (a) and (a') are linear shapes (ideal case), (b) and (b') are S-shapes, and (c) and (c') are inverse S-shapes. Here, $\vec{i}$ is chosen as a reference vector.

The sign functions are defined as

$$\text{sgn}(u) = \begin{cases} \frac{u}{|u|} & (u \neq 0) \\ 1 & (u = 0). \end{cases} \quad (13)$$

The sign functions are introduced to ensure that the sign of $W_{ij,u}$ is consistent with that of $W_{hj,u}$ when two particles i and j are located in the first quadrant and another particle h is in the third quadrant as shown in Figs. 1(a)–1(c). When extending to n particles as shown in Figs. 1(a')–1(c'), we take an average of $W_{ij,u}$ over the number of particles. The relationship between the signs of each parallelogram area and the phase space deformations is found to be the same as Eqs. (8)–(10) for the case of the line distributions.

$$\langle W_u \rangle = 0: \text{linear shape}, \quad (14)$$

$$\langle W_u \rangle > 0: \text{S-shape}, \quad (15)$$

$$\langle W_u \rangle < 0: \text{inverse S-shape}. \quad (16)$$

Here, we introduce a mean emittance $\epsilon_{\text{mean},u}$ as a function of $\langle W_u \rangle$ that characterizes both a beam emittance and a phase space deformation. It is defined as

$$\epsilon_{\text{mean},u} = \alpha \langle W_u \rangle, \quad (17)$$

where α is a proportional constant chosen such that $\epsilon_{\text{mean},u}$ coincides with the rms emittance described in Eq. (7) when the beam has a Gaussian distribution at each $W_{ij,u} \geq 0$. Then, the two moments $\langle W_u \rangle$ and $\langle W_u^2 \rangle$ are given by

$$\begin{aligned} \langle W_u \rangle &= \frac{\int_0^\infty W_u f(W_u) dW_u}{\int_0^\infty f(W_u) dW_u}, \\ \langle W_u^2 \rangle &= \frac{\int_0^\infty W_u^2 f(W_u) dW_u}{\int_0^\infty f(W_u) dW_u}. \end{aligned} \quad (18)$$

Here, $f(W_u)$ for a Gaussian distribution is

$$\begin{aligned} f(W_u) &= \frac{1}{\sqrt{2\pi}\sigma_{W_u}} \int_0^\infty \exp\left(-\frac{W_u^2}{2\sigma_{W_u}^2}\right) dW_u, \\ \left(\Rightarrow \int_0^\infty f(W_u) dW_u = \frac{1}{2}\right), \end{aligned} \quad (19)$$

where σ_{W_u} stands for the rms width. Two moments in Eq. (18) are

$$\langle W_u \rangle = \frac{\frac{1}{\sqrt{2\pi}}\sigma_{W_u}}{\frac{1}{2}} = \sqrt{\frac{2}{\pi}}\sigma_{W_u}, \quad \langle W_u^2 \rangle = \frac{\frac{\sigma_{W_u}^2}{2}}{\frac{1}{2}} = \sigma_{W_u}^2, \quad (20)$$

where we used the Gaussian integral formulas below

$$\int_0^\infty W_u \exp\left(-\frac{W_u^2}{2\sigma_{W_u}^2}\right) dW_u = \frac{2\sigma_{W_u}^2}{2} = \sigma_{W_u}^2, \quad (21)$$

$$\begin{aligned} \int_0^\infty W_u^2 \exp\left(-\frac{W_u^2}{2\sigma_{W_u}^2}\right) dW_u &= \frac{1}{4} \sqrt{\frac{\pi}{(2\sigma_{W_u}^2)^{-3}}} \\ &= \frac{\sqrt{2\pi}\sigma_{W_u}^3}{2}. \end{aligned} \quad (22)$$

Equation (20) gives the relation between $\langle W_u \rangle$ and $\langle W_u^2 \rangle$ as

$$\langle W_u \rangle = \sqrt{\frac{2}{\pi}} \sqrt{\langle W_u^2 \rangle}. \quad (23)$$

From Eqs. (7) and (17), the constant α is given as

$$\alpha \sqrt{\frac{2}{\pi}} \sqrt{\langle W_u^2 \rangle} = \frac{\sqrt{\langle W_u^2 \rangle}}{\sqrt{2}} \Rightarrow \alpha = \frac{\sqrt{\pi}}{2}. \quad (24)$$

Therefore, the mean emittance can be expressed as

$$\epsilon_{\text{mean},u} = \frac{\sqrt{\pi}}{2} \langle W_u \rangle. \quad (25)$$

The normalized mean emittances in the longitudinal and transverse planes can be also obtained with $\epsilon_{n,z} = \beta\gamma^3 \epsilon_z$ in Eq. (3) and with Eq. (4)

$$\begin{aligned} \epsilon_{n,\text{mean},z} &= \beta\gamma^3 \frac{\sqrt{\pi}}{2} \langle W_z \rangle = \frac{\sqrt{\pi}}{2} \langle \beta\gamma^3 W_z \rangle \\ &= \frac{\sqrt{\pi}}{2} \langle W_{n,z} \rangle, \end{aligned} \quad (26)$$

$$\epsilon_{n,\text{mean},\perp} = \frac{\sqrt{\pi}}{2} \langle \beta\gamma W_\perp \rangle = \frac{\sqrt{\pi}}{2} \langle W_{n,\perp} \rangle, \quad (27)$$

where $W_{n,z} = \beta\gamma^3 W_z$ and $W_{n,\perp} = \beta\gamma W_\perp$ are used.

III. MEAN FORCE NONLINEARITY

Here, we present the mean force nonlinearity (MFN) that can express the deformation of the force distribution acting on a beam. We consider a change in the parallelogram area in the normalized phase space caused by longitudinal force while the particles travel a distance Δs along the beam propagation direction. When we focus on two particles i, j out of an ensemble of n particles, the rate of change of $W_{n,ij,z}$ is

$$\begin{aligned} \frac{dW_{n,ij,z}}{ds} &= (\beta\gamma^3 z'_j \cdot J_{ij}) \frac{dz_i}{ds} + z_i \frac{d}{ds} (\beta\gamma^3 z'_j \cdot J_{ij}) \\ &\quad - (\beta\gamma^3 z'_i \cdot J_{ij}) \frac{dz_j}{ds} - z_j \frac{d}{ds} (\beta\gamma^3 z'_i \cdot J_{ij}), \end{aligned} \quad (28)$$

where β and γ stand for the mean values of the ensemble of the particles and

$$J_{ij} = \text{sgn}(|z_i| - |z_j|) \cdot \text{sgn}(z_i z_j). \quad (29)$$

$\frac{dz}{ds} = z'$ reduces Eq. (28) to

$$\frac{dW_{n,ij,z}}{ds} = z_i \frac{d}{ds} (\beta \gamma^3 z'_j \cdot J_{ij}) - z_j \frac{d}{ds} (\beta \gamma^3 z'_i \cdot J_{ij}). \quad (30)$$

Using Eq. (2), the differential in the first term gives

$$\begin{aligned} \frac{d}{ds} (\beta \gamma^3 z'_j \cdot J_{ij}) &= \frac{d}{ds} \left[\beta \gamma^3 \left(\frac{1}{\gamma^2} \frac{\Delta P_j}{P_0} \right) \cdot J_{ij} \right] \\ &= \frac{d}{ds} \left[\beta \gamma^3 \left(\frac{1}{\gamma^2} \frac{\Delta P_j}{m_0 c \beta \gamma} \right) \cdot J_{ij} \right] \\ &= \frac{J_{ij}}{m_0 c} \frac{d \Delta P_j}{ds} = \frac{J_{ij}}{m_0 \beta c^2} \frac{d \Delta P_j}{dt} \\ &= \frac{J_{ij}}{m_0 \beta c^2} F_{z_j}, \end{aligned} \quad (31)$$

where $\frac{dz}{ds} \frac{d}{dz} \text{sgn}(z) = 0$ is used and F_{z_j} is the longitudinal force acting on a particle j . Thus, Eq. (28) results in

$$\begin{aligned} \frac{dW_{n,ij,z}}{ds} &= z_i \left(\frac{F_{z_j} \cdot J_{ij}}{m_0 \beta c^2} \right) - z_j \left(\frac{F_{z_i} \cdot J_{ij}}{m_0 \beta c^2} \right) \\ &= \frac{(z_i F_{z_j} - z_j F_{z_i}) \cdot J_{ij}}{m_0 \beta c^2}. \end{aligned} \quad (32)$$

Equation (32) means that the change in the area formed by i and j particles in the z - $\beta \gamma^3 z'$ phase space is equal to the area formed by the particle positions and forces acting on each particle in the z - f_z distribution. The numerator in Eq. (32) becomes the same expression as Eq. (12) when u'_j is replaced with F_{z_j} . The parallelogram area $W_{F_z,ij}$ can be written as

$$\begin{aligned} W_{F_z,ij} &= (z_i F_{z_j} - z_j F_{z_i}) \cdot \text{sgn}(|z_i| - |z_j|) \cdot \text{sgn}(z_i z_j) \\ &= (z_i F_{z_j} - z_j F_{z_i}) \cdot \frac{|z_i| - |z_j|}{||z_i| - |z_j||} \cdot \frac{z_i z_j}{|z_i z_j|} \\ &(i = 1, 2, \dots, n-1, j = i+1, \dots, n). \end{aligned} \quad (33)$$

Equation (33) shows the deformation of a force distribution acting on the beam. The relation between the deformation and the sign of the parallelogram area is the same as those shown in Eqs. (8)–(10). Taking the average of the change in the normalized mean emittance over the number of particles n , Eq. (32) becomes

$$\begin{aligned} \left\langle \frac{dW_{n,z}}{ds} \right\rangle &= \frac{1}{m_0 \beta c^2} \frac{\sum^n (z_i F_{z_j} - z_j F_{z_i}) \cdot J_{ij}}{n} \\ &= \frac{1}{m_0 \beta c^2} \langle W_{F_z} \rangle. \end{aligned} \quad (34)$$

By differentiating the longitudinal mean emittance $\epsilon_{n,\text{mean},z}$ in Eq. (26) with s , we have

$$\epsilon_{n,\text{mean},z} = \frac{\sqrt{\pi}}{2} \langle W_{n,z} \rangle \Rightarrow \frac{d\epsilon_{n,\text{mean},z}}{ds} = \frac{\sqrt{\pi}}{2} \left\langle \frac{dW_{n,z}}{ds} \right\rangle. \quad (35)$$

Substituting Eq. (34) into Eq. (35), the mean force nonlinearity (MFN) in the longitudinal plane χ_z [rad] can be defined as

$$\chi_z = \frac{d\epsilon_{n,\text{mean},z}}{ds} = \frac{\sqrt{\pi}}{2m_0 \beta c^2} \langle W_{F_z} \rangle. \quad (36)$$

Similarly, the transverse MFN $\chi_\perp$ can be derived by replacing $\beta \gamma^3 \rightarrow \beta \gamma$ and $(z, z') \rightarrow (\perp, \perp')$ into Eq. (28):

$$\chi_\perp = \frac{d\epsilon_{n,\text{mean},\perp}}{ds} = \frac{\sqrt{\pi}}{2m_0 \beta c^2} \langle W_{F_\perp} \rangle. \quad (37)$$

The mean force nonlinearity (MFN) χ_u is obtained from the change in the normalized mean emittance along the beam-line and expresses the deformation of the force distribution (u - f_u) acting on the beam.

IV. CHARACTERISTICS OF NORMALIZED MEAN EMITTANCE AND MFN

Here, we briefly summarize the characteristics of the normalized mean emittance $\epsilon_{n,\text{mean},u}$ and the MFN χ_z . They indicate the deformations of a normalized phase space distribution and a force distribution. Their signs tell us whether the distributions are linear, S-shaped, or inverse S-shaped. Table I summarizes the signs of the normalized mean emittance and the MFN corresponding to the distribution shapes. When they are positive (negative), the distributions have an S-shape (an inverse S-shape). According to [9], the normalized rms emittance in the radial plane decreases when $\epsilon_{n,\text{mean},u}$ and χ_u have opposite signs. This decrease occurs because the force curve works to counteract the deformation of the radial phase space. Conversely, when their signs are identical, the deformation of the curve is more pronounced, leading to an increase in the rms emittance. Table II shows the correlation between a change in the rms emittance and the signs of $\epsilon_{n,\text{mean},u}$ and χ_z .

TABLE I. The signs of $\epsilon_{n,\text{mean},u}$ and χ_u corresponding to the distribution shapes.

Phase space and force distributions	$\epsilon_{n,\text{mean},u}$ and χ_u
Linear shape	$\epsilon_{n,\text{mean},u} \cdot \chi_u = 0$
S-shape	$\epsilon_{n,\text{mean},u} \cdot \chi_u > 0$
Inverse S-shape	$\epsilon_{n,\text{mean},u} \cdot \chi_u < 0$

TABLE II. The relation between a change in the rms emittance and the signs of $\epsilon_{n,\text{mean},u}$ and χ_u . When the signs of $\epsilon_{n,\text{mean},u}$ and χ_u are identical (different), the rms emittance increases (decreases).

Change in rms emittance	$\epsilon_{n,\text{mean},u}$ and χ_u
Increase ($\nearrow$)	$\epsilon_{n,\text{mean},u} \cdot \chi_u > 0$
Decrease ($\searrow$)	$\epsilon_{n,\text{mean},u} \cdot \chi_u < 0$

The relation between the normalized rms and mean emittances is defined so that they are agreed when a particle distribution has a Gaussian shape and each $W_{ij} \geq 0$. For a beam with a line distribution in the u - u' plane, the rms emittance is equal to the absolute value of the normalized mean emittance. As a simple case, considering a linear line distribution as shown in Fig. 1(a), we have $\epsilon_{n,\text{mean},u} = \epsilon_{n,\text{rms}} = 0$. However, for an ellipse distribution with finite area as shown in Fig. 1(a'), we have $\epsilon_{n,\text{rms}} \neq 0$ while $\epsilon_{n,\text{mean},u} = 0$ because the positive and negative areas formed by the particles become mixed. Moreover, if the phase particle distribution does not conform to a Gaussian distribution, the absolute mean emittance differs from the normalized rms emittance.

The MFN is derived from a force acting on particles, as discussed in Sec. III. When several types of forces such as rf field force f_{rf} , space charge force f_{sc} , and magnetic force f_{mag} contribute to the particles, the total MFN can be obtained by substituting $F_{z_j} = f_{rf,z_j} + f_{sc,z_j} + f_{\text{mag},z_j}$ into Eq. (32). Finally, we have

$$\begin{aligned} \chi_u &= \frac{\sqrt{\pi}}{2m_0\beta c^2} [\langle W_{rf,u} \rangle + \langle W_{sc,u} \rangle + \langle W_{\text{mag},u} \rangle] \\ &= \chi_{rf,u} + \chi_{sc,u} + \chi_{\text{mag},u}, \end{aligned} \quad (38)$$

where $\chi_{rf,u}$, $\chi_{sc,u}$, and $\chi_{\text{mag},u}$ stand for the MFNs for the rf field force, the space charge force, and the magnetic force acting on the particles, respectively. The total MFN is obtained by summing each MFN of the forces acting on the particles. Furthermore, Eqs. (36) and (37) show that the MFN is derived from a change in the normalized mean emittance along the beamline. Thus, we can identify whether the total MFN is positive, negative, or zero by examining the evolution of the normalized mean emittance.

V. LINEAR IFMIF PROTOTYPE ACCELERATOR

This section describes the Linear IFMIF Prototype Accelerator (LIPAc) to investigate the self-linearization of a deformed LPS distribution and verify the theory mentioned above. The International Fusion Materials Irradiation Facility (IFMIF) is an accelerator-based neutron source designed to study materials for future fusion reactors [12]. It consists of two parallel accelerators capable of accelerating 125 mA deuteron beams to 40 MeV and a liquid lithium target. The facility enables the production of intense 14 MeV neutrons by simultaneously irradiating the lithium target with the deuteron beams. The LIPAc is designed and developed to demonstrate an acceleration of the low-energy section of IFMIF up to 9 MeV while maintaining particle losses below 1 W/m across the beamline in continuous wave mode. The design of LIPAc is shown at the top of Fig. 2. The construction and beam commissioning of LIPAc have been conducted in several phases in a collaboration between Japanese and European Union institutes in Japan [10]. Currently, all components except for the SRF linac have been installed in the beamline. The place to install the SRF linac is a temporary beam transport line with quadrupole magnets named MEL (MEBT extension line). The configuration of the LIPAc from the latter half of the RFQ to the beam dump without the SRF linac is illustrated at the bottom of Fig. 2. The beam commissioning was performed

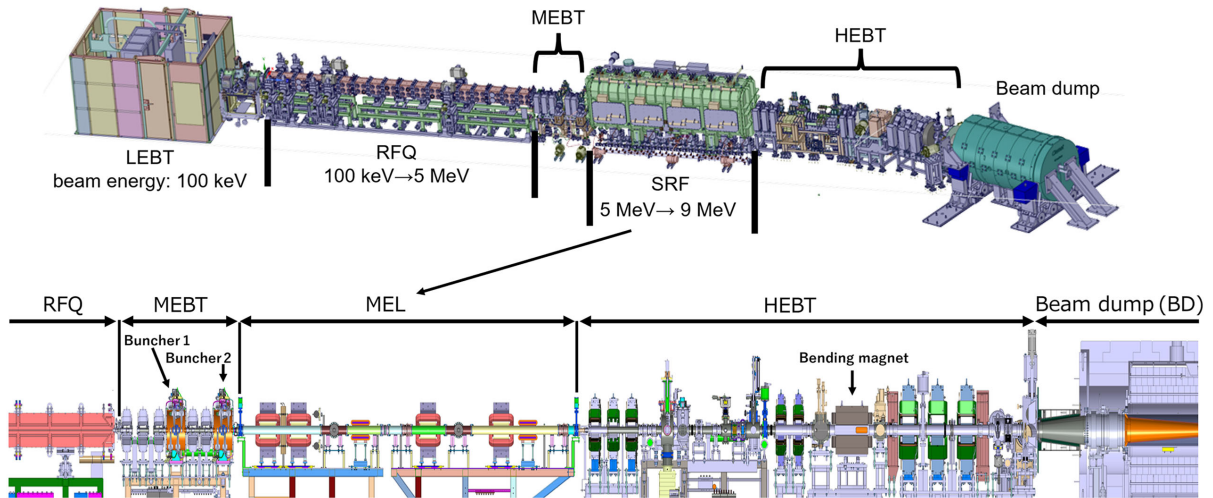


FIG. 2. The designed configuration of LIPAc. The top figure shows the final configuration of LIPAc. The bottom figure shows the beamline without the SRF linac from the latter half of the RFQ to beam dump used to validate the developed theory.

to characterize beam quality and parameters including all beam diagnostics toward the beam operation with the SRF linac [13–17]. We performed the particle tracking simulations for the configuration without the SRF linac to validate the developed theory described above.

A deuteron beam with a current of 125 mA generated by an ECR ion source [18,19] is accelerated to 5 MeV in the RFQ with a length of 9.8 m operating at a frequency of 175 MHz [20]. The beam is then transported to the MEBT section with quadrupole magnets arranged as a doublet and a triplet and two rf bunchers to focus the beam longitudinally [21]. The two bunchers, also operating at 175 MHz, have an identical structure with five cell cavities [22,23]. MEL following the MEBT section has two sets of a quadrupole doublet and two steering magnets. When the SRF linac is installed into the beamline, the beam can be accelerated from 5 to 9 MeV [24]. The HEBT section downstream of MEL is composed of a quadrupole doublet, two quadrupole triplets, a bending magnet with a bending angle of 20° , and the beam dump [25]. Table III summarizes the parameters of the LIPAc. The normalized beam emittances and beam rms sizes at the exit of the RFQ are $(\epsilon_x, \epsilon_y, \epsilon_z) = (0.20, 0.20, 0.46) \pi \text{ mm mrad}$ and $(\sigma_x, \sigma_y, \sigma_z) = (1.5, 1.4, 4.1) \text{ mm}$. The beam dump requires transverse beam size and divergence to be $(\sigma_x, \sigma_y) \sim (40, 40) \text{ mm}$ and $(\sigma'_x, \sigma'_y) \sim (16, 16) \text{ mrad}$ to avoid thermal damage to the beam dump [25]. Buncher 1 is used to control the bunch length so that the SRF linac can accelerate all particles, and therefore, buncher 1 is always set to 0 kV in the particle tracking simulations.

VI. TRANSVERSE AND LONGITUDINAL BEAM SIZES AND NORMALIZED TRANSVERSE RMS EMITTANCES ALONG BEAMLINE

We performed particle tracking simulations for the beamline without the SRF linac assuming a single bunch beam with 5 MeV/125 mA from the exit of RFQ to the

TABLE III. Relevant machine and beam parameters of the LIPAc.

Ion species	Deuteron
Beam energy	5 MeV (9 MeV with SRF linac)
Lorentz factor γ and β	1.003 and 0.073
rf frequency	175 MHz
rf wavelength $\beta\lambda$	125.06 mm
Beam current	125 mA
Nominal bunch charge	0.724 nC
Nominal buncher voltage	(buncher 1, buncher 2) = (0, 170) kV
Normalized rms emittance at the RFQ exit	$(\epsilon_x, \epsilon_y, \epsilon_z) = (0.20, 0.20, 0.46) \pi \text{ mm-mrad}$
rms beam size at the RFQ exit	$(\sigma_x, \sigma_y, \sigma_z) = (1.5, 1.4, 4.1) \text{ mm}$
Transverse rms beam size at the beam dump	$(\sigma_x, \sigma'_x) = (\sigma_y, \sigma'_y) = (40 \text{ mm}, 16 \text{ mrad})$
Bending angle	20°

beam dump with the TraceWin code. TraceWin calculates space charge effects using Particle-In-Cell Numerical Integration between cubes method, which has been benchmarked against a variety of simulation codes to validate its accuracy and reliability [26–28].

Figure 3 shows the transverse rms beam sizes, the bunch lengths, the normalized transverse rms emittances, the momentum spreads for different buncher 2 voltages. As shown in Fig. 3(c), the bunch length is minimized at the beam dump entrance when the buncher 2 voltage is set to 170 kV. The quadrupole setting applied in the simulations was optimized at a 170 kV buncher voltage so that particles are not lost over the beamline and the transverse rms beam sizes and divergences also meet the beam dump requirement to avoid thermal damage [25]. Here, we refer to the quadrupole setting at a buncher 2 voltage of 170 kV as a nominal setting. A higher buncher 2 voltage strongly compresses the bunch length, followed by rapid elongation, while no notable differences are observed in the transverse beam sizes among different buncher settings as shown in Figs. 3(a) and 3(b).

Figures 3(d) and 3(e) show the evolution of the normalized horizontal and vertical rms emittances for different buncher 2 voltages. The horizontal rms emittances remain approximately constant along the beamline until the bending magnet. Inside the bending magnet, however, the horizontal rms emittance increases due to the effects of dispersion and momentum spread [Fig. 3(f)]. In the vertical plane, the rms emittances for all buncher voltages except for 210 kV show little change along the beamline. At a buncher 2 voltage of 210 kV, a slight increase in the vertical rms emittance is observed around $s = 5 \text{ m}$. This increase is attributed to the strong beam focus in both the vertical and longitudinal planes.

In the simulations above, we employed 7 grids in the transverse (x, y) directions and 32 grids in the longitudinal direction (z) covering a range of $\pm 3.5 \times \sigma_{x,y,z}$ and a step size of 1 mm. The simulations were performed on an 11th Gen Intel(R) Core(TM) i7 @ 2.5 GHz, 8 cores. We used 10^5 particles, which were confirmed to have negligible impact on the simulation results. The dependence of the simulation results on the number of particles is discussed in Appendix A. The computation time from the RFQ exit to the beam dump was approximately 10 min.

VII. FORCES ACTING ON PARTICLES

In Secs. VIII and IX, we show the evolution of the normalized longitudinal rms emittance along the beamline and explain the mechanism of the behavior using the normalized mean emittance $\epsilon_{n,\text{mean},z}$ and the MFN χ_z . Standard tracking codes such as TraceWin do not directly provide the two physical parameters. Equations (12) and (26) allow us to calculate the normalized mean emittance using the beam position (z, z') in the longitudinal phase space generated by these codes. On the other hand, the

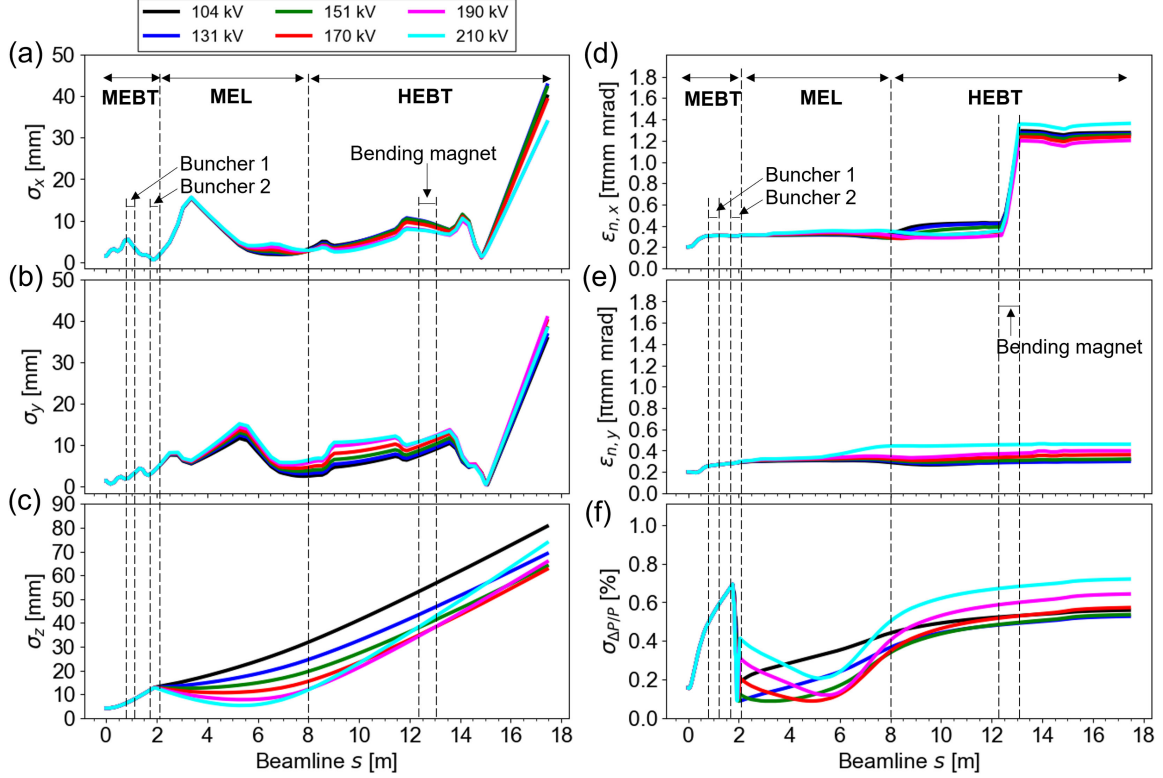


FIG. 3. Evolution of transverse and longitudinal rms beam sizes, normalized transverse emittances, and rms momentum spreads from the RFQ exit to the beam dump for different buncher 2 voltages. Simulations were performed using a 5 MeV/125 mA deuteron beam. (a), (b), and (c) show the rms beam sizes in the horizontal, vertical, and longitudinal planes, respectively. (d) and (e) show the normalized rms emittances in the horizontal and vertical planes. (f) shows the rms momentum spread.

MFN requires obtaining all force distributions ($z' - f_z$) acting on the particles. In the following discussion, we need distributions of space charge force and rf field force in buncher 2. Here, we show the analytical formulas for calculating the two force distributions using particle positions (x, y, z) provided by the tracking codes.

A. Space charge force

We derive the LSC force $\mathbf{F}_{z,sc}$ acting on particles in a single bunch with the Liénard-Wiechert potentials [29,30]. First, we consider that a particle with a charge of q travels with a constant velocity of $\mathbf{v}_0 = (0, 0, v_z)$ along a trajectory s . When the particle at point (x_0, y_0, z_0) radiates an electromagnetic field, $\mathbf{E}$ and $\mathbf{B}$ observed at time t and point $\mathbf{r} = (x, y, z)$ can be written as

$$\mathbf{E}(\mathbf{r}, t) = -\text{grad}\phi(\mathbf{r}, t) - \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t}, \quad (39)$$

$$\mathbf{B}(\mathbf{r}, t) = \text{rot} \mathbf{A}(\mathbf{r}, t), \quad (40)$$

where ϕ is the electric potential and $\mathbf{A}$ is the vector potential. Using the charge and current densities for a point charge at $s(t)$

$$\rho(\mathbf{r}, t) = q\delta^3[\mathbf{r} - \mathbf{s}(t)], \quad \mathbf{J}(\mathbf{r}, t) = q\mathbf{v}_0\delta^3[\mathbf{r} - \mathbf{s}(t)], \quad (41)$$

the electric and vector potentials are expressed as

$$\begin{aligned} \phi(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{r}_s, t')}{|\mathbf{r} - \mathbf{r}_s|} d^3\mathbf{r}_s \\ &= \frac{q}{4\pi\epsilon_0} \int_V \frac{\delta^3[\mathbf{r}_s - \mathbf{s}(t')]}{|\mathbf{r} - \mathbf{r}_s|} d^3\mathbf{r}_s, \end{aligned} \quad (42)$$

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}_s, t')}{|\mathbf{r} - \mathbf{r}_s|} d^3\mathbf{r}_s \\ &= \frac{\mu_0 \mathbf{v}_0}{4\pi} \int_V \frac{\delta^3[\mathbf{r}_s - \mathbf{s}(t')]}{|\mathbf{r} - \mathbf{r}_s|} d^3\mathbf{r}_s, \end{aligned} \quad (43)$$

where $t' = t - \frac{|\mathbf{r} - \mathbf{r}_s|}{c}$ is the retarded time. Substituting Eqs. (42) and (43) into Eqs. (39) and (40), $\mathbf{E}$ and $\mathbf{B}$ observed at time t and point $\mathbf{r} = (x, y, z)$ can be calculated as

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0\gamma^2} \times \frac{(x - x_0, y - y_0, z - z_0)}{R}, \quad (44)$$

$$\mathbf{B} = \frac{qv_z}{4\pi\epsilon_0\gamma^2 c^2} \times \frac{[-(y - y_0), x - x_0, 0]}{R}, \quad (45)$$

with

$$R = \left[(z - v_z t)^2 + \frac{1}{\gamma^2} [(x - x_0)^2 + (y - y_0)^2] \right]^{3/2}. \quad (46)$$

We consider a space charge force $F_{i,sc}$ acting on i th particle in a single bunch with a total charge of Q consisting of N particles. Since the space charge force can be obtained by $\mathbf{F} = q(\mathbf{E} + \mathbf{v}_0 \times \mathbf{B})$, $F_{i,sc}$ acting on i th particle is

$$F_{x_i,sc} = q(E_{x_i} - v_{z_i}B_{y_i}) = \frac{qQ}{4\pi\epsilon_0 N} \sum_{j \neq i}^N \frac{x_i - x_j}{\gamma_j [\gamma_j^2(z_i - z_j)^2 + [(x_i - x_j)^2 + (y_i - y_j)^2]]^{3/2}}, \quad (47)$$

$$F_{y_i,sc} = q(E_{y_i} + v_{z_i}B_{x_i}) = \frac{qQ}{4\pi\epsilon_0 N} \sum_{j \neq i}^N \frac{y_i - y_j}{\gamma_j [\gamma_j^2(z_i - z_j)^2 + [(x_i - x_j)^2 + (y_i - y_j)^2]]^{3/2}}, \quad (48)$$

$$F_{z_i,sc} = qE(z_i) = \frac{qQ}{4\pi\epsilon_0 N} \sum_{j \neq i}^N \frac{\gamma_j(z_i - z_j)}{[\gamma_j^2(z_i - z_j)^2 + [(x_i - x_j)^2 + (y_i - y_j)^2]]^{3/2}}, \quad (49)$$

where we assumed that the number of particles in the bunch is enough large to hold $(N - 1) \approx N$.

In the derivation of the formulas above, we assumed the electromagnetic field of a charged beam moving in free space. However, in reality, the beam travels through a vacuum chamber, whose geometry and material properties affect the electromagnetic field of the beam. Accurate calculation of the LSC force requires incorporating the boundary conditions into the derivation above, making the derivation more complex and beyond the scope of this paper. When the radius of a vacuum chamber b is larger than a bunch length l_z , the screening effect of the chamber wall on the LSC force is small [31,32]. In LIPAc, the beam pipe typically has a radius of 50 mm. As shown in Fig. 3(c), since the condition $b > \sigma_z$ is satisfied along most of the

beamline, except in the latter half of the HEBT section under the nominal setting, Eqs. (47)–(49) are applicable to the LIPAc beamline.

B. rf field force in a buncher

A buncher is a device to compress the bunch length making use of the velocity differences of particles as they travel through a drift space downstream of the buncher. Inside the buncher, an electromagnetic field induced by rf decelerates high-energy particles at the bunch head while accelerating the low-energy particles at the tail.

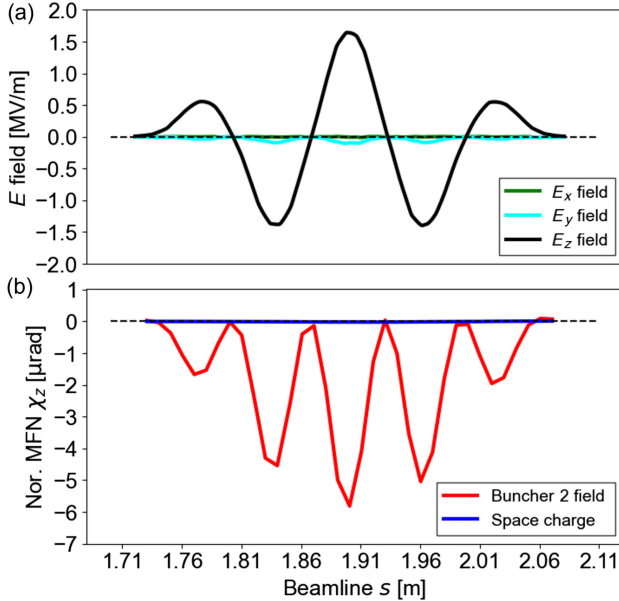


FIG. 4. Electromagnetic fields and longitudinal MFN in buncher 2 at a buncher voltage of 170 kV. (a) Electric field of the rf. (b) Longitudinal MFNs of the rf field force (red) and the space charge force (blue).

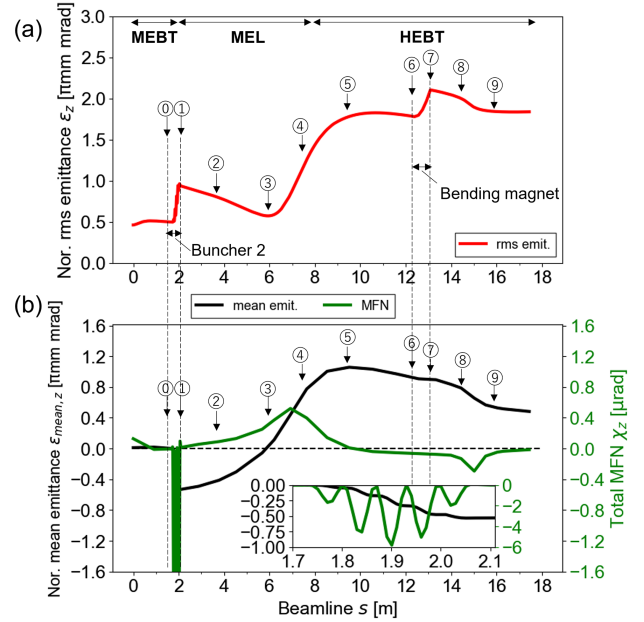


FIG. 5. (a) Normalized longitudinal rms emittance at a buncher 2 voltage of 170 kV simulated by TraceWin. (b) Normalized longitudinal mean emittance (black) and the total MFN (green) computed using Eqs. (26) and (36). Inset shows the normalized mean emittance and the total MFN inside buncher 2. A 5 MeV/125 mA deuteron beam was used in the simulation.

The longitudinal rf field force that the beam experiences in the buncher can be written as

$$f_{rf,z} = qE_z \cos(\omega t - \pi/2) + q(v_x B_y - v_y B_x) \sin(\omega t - \pi/2), \quad (50)$$

where E_z, B_x, B_y are the electromagnetic force elements of the rf field, $\omega = 2\pi f$ is the angular frequency, and v_x, v_y are the velocities of the particles. Figure 4(a) shows E inside buncher 2 when the buncher voltage is set to 170 kV. Since v_x, v_y are significantly small, Eq. (50) can be reduced to

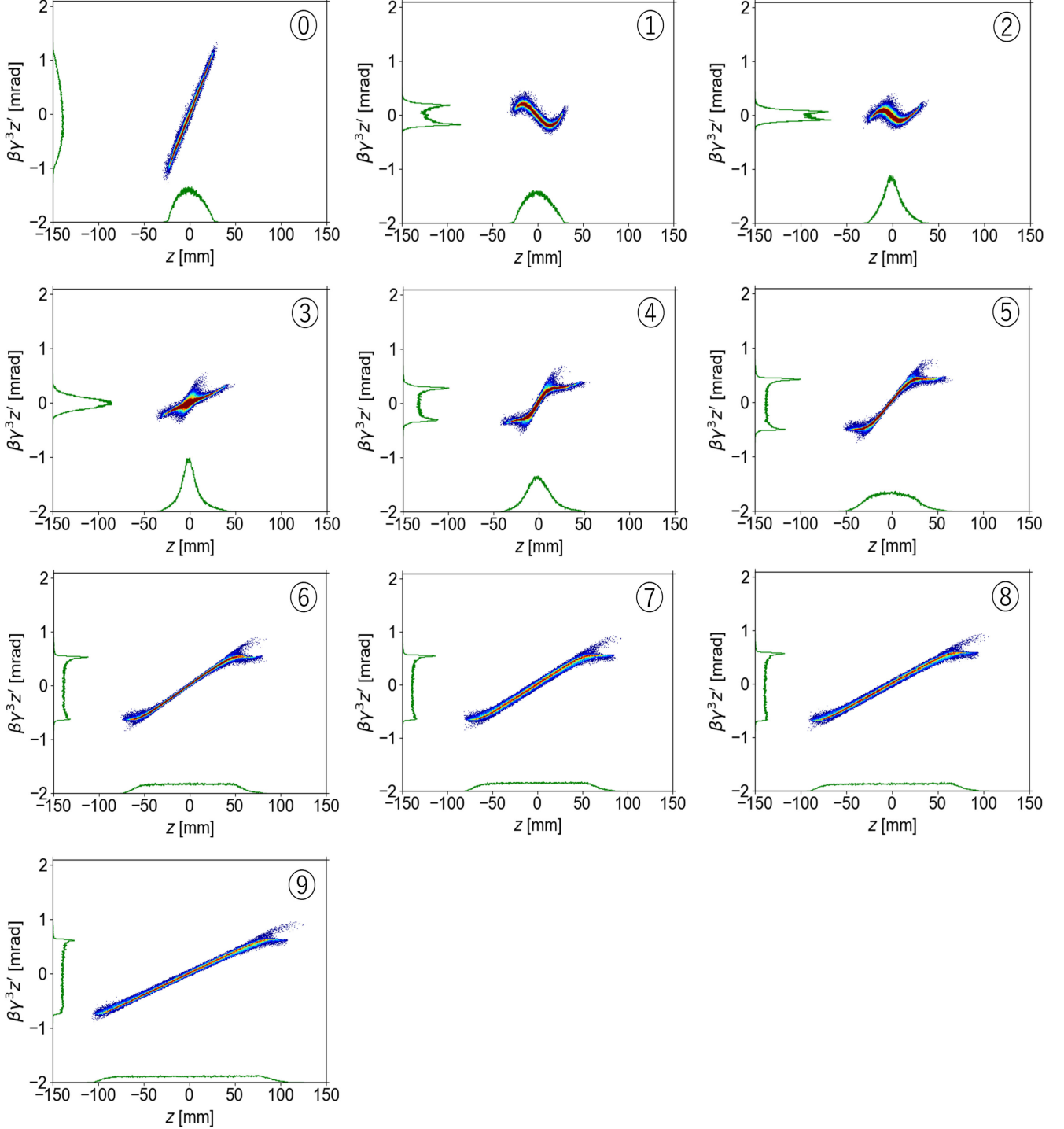


FIG. 6. Normalized longitudinal phase space (LPS) distributions at different locations at a buncher 2 voltage of 170 kV obtained by TraceWin. The numbers ①–⑨ correspond to those in Fig. 5. A 5 MeV/125 mA deuteron beam was used in the simulation.

$$f_{rf,z} \sim qE_z(z) \sin \left[\frac{2\pi f(z \pm 3\sigma_z)}{c\beta} \right]. \quad (51)$$

For a 125-mA deuteron beam, the rms bunch length becomes $\sigma_z \sim 11.5$ mm at the buncher 2 entrance as shown in Fig. 3(c), and the total bunch length ($= \pm 3\sigma_z$) exceeds the half-wavelength of an rf of 62.53 mm. From the longitudinal rf force given in Eq. (51), the sine curvature of the rf field significantly deforms the LPS distribution. To quantitatively evaluate the deformation of the rf field force distribution, we calculate the MFN χ_z of the rf field force at a buncher voltage of 170 kV using Eqs. (33), (36), and (50) and show the result in Fig. 4(b). The buncher MFN χ_z fluctuates between zero and negative values synchronizing the electric field E_z of the rf, which indicates that the longitudinal nonlinear rf force deforms the LPS into an inverse S-shaped distribution.

VIII. EVOLUTION OF THE NORMALIZED LONGITUDINAL RMS EMITTANCE, MEAN EMITTANCES, AND LONGITUDINAL MFN

Figure 5(a) shows the normalized longitudinal rms emittance at a buncher 2 voltage of 170 kV simulated by TraceWin. It first increases in buncher 2, then gradually decreases in the MEL section, reaching a minimum value close to that at the entrance of buncher 2. It subsequently increases between $s = 6$ m and $s = 10$ m, followed by a slight decrease up to the entrance of the bending magnet. After the bending magnet, it significantly decreases again and finally becomes constant.

To examine the detail of the normalized rms emittance evolution, we present the normalized LPS ($z\text{-}\beta\gamma^3 z'$) and the LSC force distributions ($z\text{-}f_z$) computed at different beam-line locations in Figs. 6 and 7. The numbers at different locations in Figs. 6 and 7 correspond to those in Fig. 5. The normalized LPS distributions were simulated by TraceWin, and LSC force distributions were calculated by substituting particle positions (x, y, z) at each location provided by TraceWin into Eq. (49). In Fig. 7, a small number of particles exhibiting slightly divergent values of f_z [N] can be observed. This divergence occurs because the interparticle distance becomes excessively small, causing the denominator in Eq. (49) to decrease significantly. Due to the limitation on the number of particles used in the

calculation, such divergence arises in the calculation of particle-particle interactions. However, since the mean emittance and MFN are primarily determined by the distribution shape of the core region, the influence of halo particles is negligible.

To investigate the deformations of the two distributions shown in Figs. 6 and 7, we computed the normalized mean emittance $\epsilon_{n,\text{mean},z}$ and the MFN χ_z with Eqs. (12), (26), (33), and (36). Figure 5(b) shows the evolution of the calculated normalized longitudinal mean emittance and total MFN. This total MFN is the sum of the MFNs of the rf field force in buncher 2 and the LSC force.

As a benchmark, we confirmed that the longitudinal bunch length, rms emittance, and mean emittance shown in Figs. 3(c) and 5 are consistent with the results obtained using a tracking code different from the one employing the PIC method. The detailed comparison is presented in Appendix B.

Here, we verify whether the calculated normalized mean emittance and MFN are correctly related to the deformations of the normalized LPS ($z\text{-}\beta\gamma^3 z'$) and the LSC force ($z\text{-}f_z$) distributions as summarized in Table I. We confirm the relation between the normalized LPS distribution and mean emittance, followed by the force distribution and the MFN.

As shown by ① in Figs. 5(b) and 6, the normalized LPS has an elliptical distribution from the RFQ exit to the buncher 2 entrance. In this region, the corresponding normalized mean emittance is 0π mm mrad. At the buncher 2 exit, the normalized LPS has an inverse S-shaped distribution, as depicted in Fig. 6 ①. This deformation is gradually reduced and nearly eliminated at $s \sim 6$ m, as shown in Figs. 6 (①–③). In this region, the mean emittance varies from -0.5 to 0π mm mrad, as indicated in Fig. 5(b) (①–③).

For $6\text{ m} < s < 17.5$ m, the normalized LPSs basically have S-shaped distributions, and the corresponding mean emittances have positive values, as shown in Figs. 5(b) and 6 (④–⑨). However, starting from $s \sim 10$ m, the normalized LPS distribution gradually transforms into a uniform distribution, resulting in a decrease in the mean emittance, as seen in Figs. 5(b) and 6 (⑤–⑨). At the entrance of the bending magnet [Fig. 6 (⑥)], the S-shaped curvatures are observed in both the bunch head and tail. Eventually, the normalized LPS approaches a uniform

TABLE IV. The relation between the normalized rms emittance change and the signs of the normalized mean emittance $\epsilon_{n,\text{mean},z}$ and the MFN χ_z in the longitudinal plane along the beamline. The signs and the normalized rms emittance change are extracted from Fig. 5. $\rightarrow$, $\nearrow$, and $\searrow$ stand for constant, increase, and decrease, respectively.

s (m)	$0 < s < 1.8$	$1.8 < s < 2$	$2 < s < 6$	$6 < s < 10$	$10 < s < 16$	$s > 16$
$\text{sgn}(\epsilon_{n,\text{mean},z})$	0	-1	-1	+1	+1	+1
$\text{sgn}(\chi_z)$	~ 0	-1	1	+1	-1	~ 0
Change in rms emittance	$\rightarrow$	$\nearrow$	$\searrow$	$\nearrow$	$\searrow$	$\rightarrow$

distribution, as shown in Fig. 6 (⊙). Nevertheless, the normalized mean emittance remains significantly different from $0\pi\text{ mm mrad}$ [⊙ in Fig. 5(b)] because of the slight deformations in the head and tail of the bunch. We confirmed that $\epsilon_{n,\text{mean},z} = 0\pi\text{ mm mrad}$ within the range

$-50\text{ mm} \leq z \leq 50\text{ mm}$ where the normalized LPS is uniformly distributed. These observations demonstrate that the deformation of the normalized LPS distributions is correctly related to the normalized mean emittance as summarized in Table I.

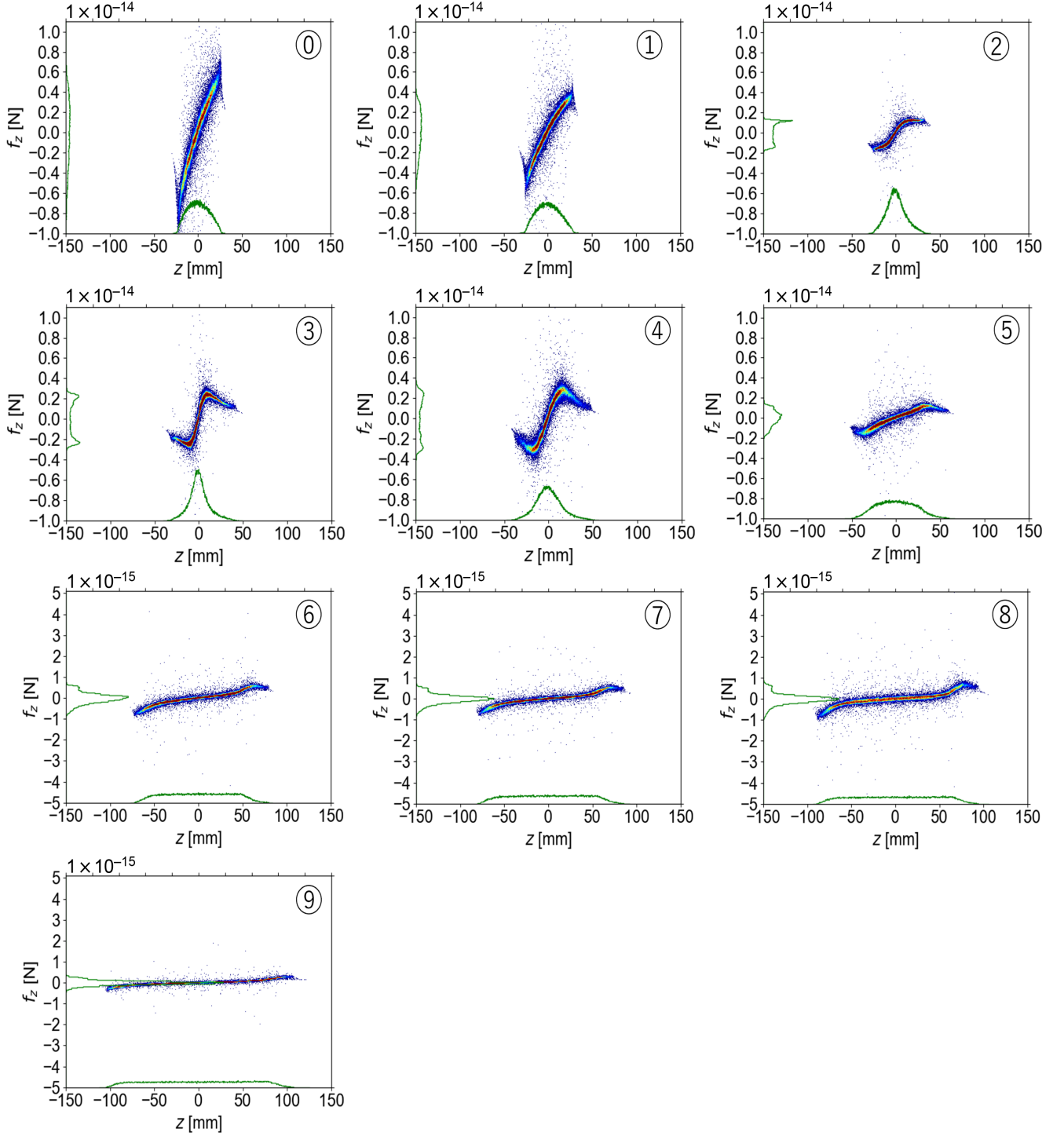


FIG. 7. Longitudinal space charge (LSC) force distributions acting on particles at different locations calculated with Eq. (49). The numbers (⊙–⑨) correspond to those in Fig. 5. The vertical scale ranges for ⑥–⑨ are different from those for ①–⑤ in order to clearly show the deformations at the edges on the horizontal axis.

Next, we examine the relationship between the deformation of the LSC force distribution (z - f_z) and the sign of the MFN χ_z . The LSC force distributions are linear at the entrance and exit of buncher 2 as shown in Fig. 7 (⊙ and ⊙), and the corresponding MFNs are 0 μ rad at ⊙ and ⊙ in Fig. 5(b). The blue line in Fig. 4(c) plots the MFN of the LSC force distribution inside buncher 2, where the MFN remains at 0 μ rad. As the bunch length shortens downstream of buncher 2, the LSC force distribution is gradually deformed into an S-shape [Fig. 7 (⊙–⊙)], and the MFN increases from 0 to 0.5 μ rad at $s = 7$ m, as shown in Fig. 5(b). Subsequently, as the bunch length increases, the MFN returns to 0 μ rad at $s \sim 9$ m. We can confirm that the MFNs indicate zero and positive values in this region where the LSC force distributions are linear and S-shaped. For $s > 10$ m, the inverse S-shaped deformations in the LSC force distributions are slightly observed at the bunch head and tail as illustrated in Fig. 7 (⊙–⊙), and the MFNs show negative values. Starting from $s \sim 14$ m, the MFN starts decreasing, reaching -0.4 μ rad, as shown in Fig. 5(c). This decrease is due to the strong LSC force induced by the tight focusing of the transverse beam size at the beam dump, which was necessary to meet the beam dump requirement, see Figs. 3(a) and 3(b). For $s > 16$ m, the MFN finally returns to 0 μ rad, as the longitudinal distribution approaches a uniform one where the SC force does not act on the particles [Fig. 7 (⊙)]. Based on these results, the deformations of the LSC force distributions can be quantitatively evaluated using the MFN.

We examine the relationship between the normalized rms and mean emittances in the longitudinal plane. In Sec. II B, we defined the coefficient α that relates the mean emittance to the rms emittance such that they agree when the z distribution is Gaussian and each $W_{ij} \geq 0$. As shown in Figs. 5(a) and 5(b), the absolute value of the mean emittance does not match the rms emittance over the beamline. This discrepancy arises because the positive and negative regions formed by the particles in the normalized LPS are mixed, and the longitudinal beam distribution is also non-Gaussian after the bending magnet.

IX. MECHANISM OF THE LONGITUDINAL RMS EMITTANCE EVOLUTION

We describe the mechanism of the longitudinal rms emittance evolution at a 170 kV buncher voltage using the normalized mean emittance and the MFN. As we summarized the conditions for the rms emittance reduction in the radial plane in Table II, the rms emittance decreases (increases) when $\epsilon_{n,\text{mean}}$ and χ have opposite (identical) signs. To confirm the signs of $\epsilon_{n,\text{mean},z}$ and χ_z in the longitudinal plane, Table IV presents their signs and the rms emittance change along the beamline extracted from Figs. 5(a) and 5(b).

From the RFQ exit to the buncher 2 entrance ($0 \text{ m} < s < 1.8 \text{ m}$), where the MFN χ_z is almost zero,

the normalized longitudinal rms emittance is conserved because the beam experiences only linear SC forces. While the rms emittance remains constant, the condition $\text{sgn}(\epsilon_{n,\text{mean},z}) \cdot \text{sgn}(\chi_z) = 0$ is satisfied, as shown in Table IV.

Inside buncher 2 ($1.8 \text{ m} < s < 2 \text{ m}$), the nonlinear rf field force ($\chi_{z,\text{rf}} \leq 0$) significantly deforms the linear normalized LPS distribution into an inverse S-shape: $\text{sgn}(\epsilon_{n,\text{mean},z}): 0 \rightarrow -1$, which causes the rms emittance growth in buncher 2. During the increase in the normalized rms emittance, $\text{sgn}(\epsilon_{n,\text{mean},z}) \cdot \text{sgn}(\chi_z) > 0$ holds, see Table IV.

After the beam passes through buncher 2 ($2 \text{ m} < s < 6 \text{ m}$), the LSC force increases as the bunch length is compressed, forming an S-shaped distribution: $\text{sgn}(\chi_{z,\text{sc}}): 0 \rightarrow +1$. Simultaneously, the S-shaped LSC force gradually counteracts the inverse S-shaped normalized LPS distribution: $\text{sgn}(\epsilon_{n,\text{mean},z}): -1 \rightarrow 0$, which results in a decrease in the longitudinal rms emittance. Then, the rms emittance reaches its minimum value when the normalized LPS deformation is maximally reduced. In this region, the relation between the signs satisfies $\text{sgn}(\epsilon_{n,\text{mean},z}) \cdot \text{sgn}(\chi_z) < 0$, as shown in Table IV.

For $6 \text{ m} < s < 10 \text{ m}$, the once mitigated normalized LPS distribution is again deformed into an S-shape since the strong LSC force with the S-shape distribution keeps

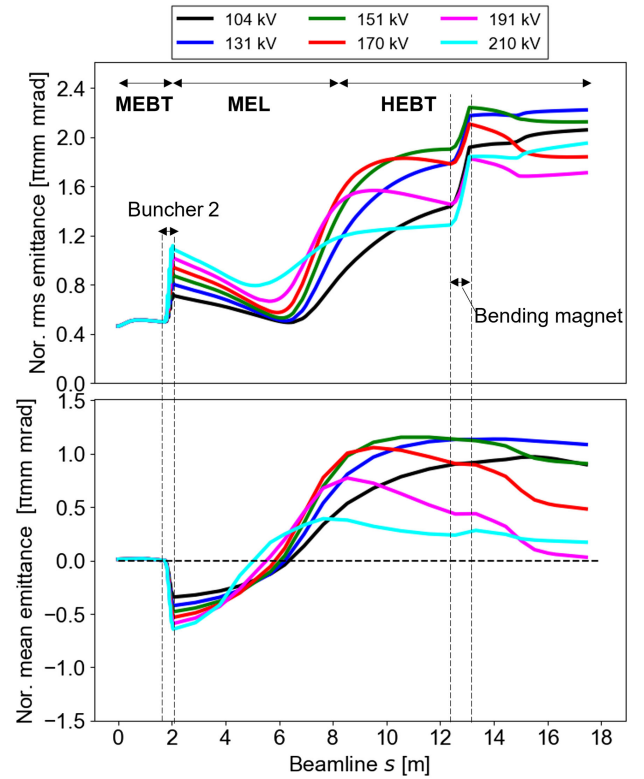


FIG. 8. Evolution of longitudinal rms (top) and mean (bottom) emittances at different buncher 2 voltages. A 5 MeV/125 mA deuteron beam was used in the simulations.

acting on the beam. Consequently, the rms emittance increases again. During this rms emittance growth, $\text{sgn}(\epsilon_{n,\text{mean},z}) \cdot \text{sgn}(\chi_z) > 0$. After $s \sim 7$ m, the MFN values gradually decrease as the bunch length increases, as shown in Fig. 3(c), and the normalized LPS approaches a uniform distribution.

For $10 \text{ m} < s < 16 \text{ m}$, the normalized LPS has the S-shaped distributions at the head and tail of the bunch ($\epsilon_{n,\text{mean},z} \geq 0$), while the LSC force shows the inverse S-shaped distribution at the head and tail of the bunch ($\chi_z < 0$). Therefore, the LSC mitigates the deformation of the normalized LPS distribution at the head and tail of the

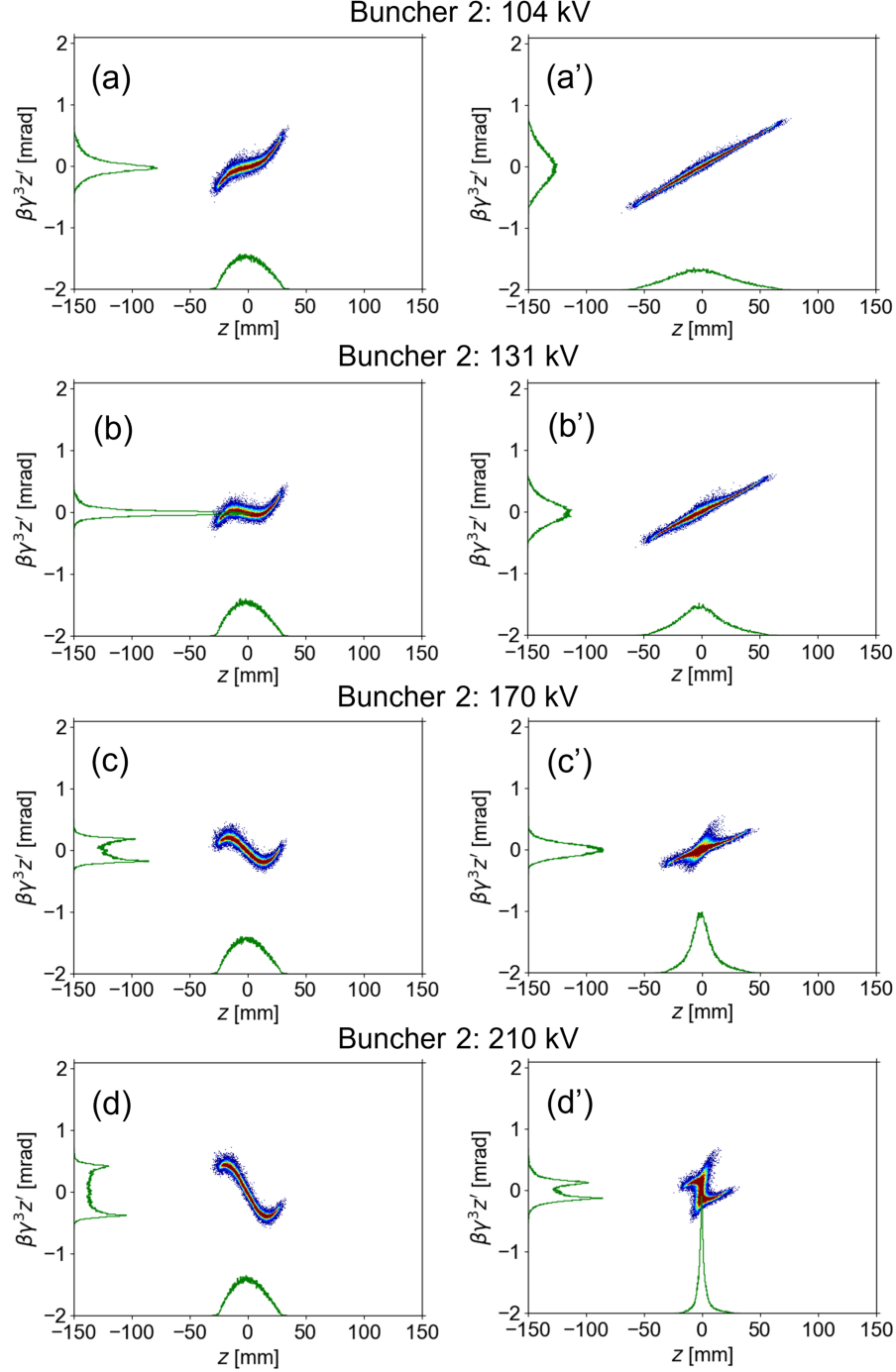


FIG. 9. Normalized LPS distributions at the buncher 2 exit and the minimal rms emittance position for different buncher 2 voltages. (a)–(d) and (a')–(d') show the LPS distributions at the buncher 2 exit and the minimum rms emittance location, respectively. (a) and (a') 104 kV, (b) and (b') 131 kV, (c) and (c') 170 kV, (d) and (d') 210 kV. A 5 MeV/125 mA deuteron beam was used in the simulations.

bunch, leading to a decrease in the rms emittance again. However, since the amount of the deformation of the normalized LPS is small, the decrease in emittance is also slight. During the decrease in rms emittance, $\text{sgn}(\epsilon_{n,\text{mean},z}) \cdot \text{sgn}(\chi_z) < 0$ holds. Then, the normalized LPS reaches a uniform distribution around $s = 16$ m, and the LSC force no longer acts on the beam: $\text{sgn}(\epsilon_{n,\text{mean},z}) \cdot \text{sgn}(\chi_z) = 0$. For $s > 16$ m, the rms emittance remains constant unless nonlinear forces act on the beam.

X. PARAMETER DEPENDENCE ON MINIMUM RMS EMITTANCE

We carried out particle simulations from the RFQ exit to the beam dump to examine the parameter dependence of the normalized LPS deformation and rms emittance in the longitudinal plane. Figure 8 shows the evolution of the normalized rms and mean emittances in the longitudinal plane for different buncher 2 voltages. As the buncher voltage increases, the normalized mean emittance has a large negative value at the buncher 2 exit since the rf field force significantly deforms the normalized LPS distribution. Additionally, the distance at which the rms emittance reaches the minimum value is close to the buncher 2 exit due to strong focusing in the longitudinal plane. Importantly, the minimum rms emittance for a high buncher voltage does not return to the value at the buncher 2 entrance because the LSC force does not completely compensate for the deformation. Figure 9 shows the normalized LPS distributions at the buncher 2 exit and at each minimum rms emittance location for different buncher voltages. When the buncher voltage is set to 210 kV, the normalized LPS at the minimum rms emittance shows a star-shaped distribution since the large normalized LPS deformation at the bunch head and tail caused by the rf field force is not completely mitigated, whereas the normalized LPS at a buncher voltage of 104 kV shows a linear distribution since the deformation is canceled. Thus, the minimum rms emittance at 210 kV is larger than those at other lower buncher voltages. However, after reaching the minimum rms emittance, the rms emittances at high buncher voltages are not higher than those at lower buncher voltages because the normalized LPS soon transforms into a uniform distribution.

To see the impact of the SC force on the normalized LPS distribution deformed by the rf field in buncher 2, we performed particle tracking simulations from the buncher 2 entrance with different beam currents. As the initial parameters at the buncher 2 entrance, we used the parameters simulated with a 125 mA beam current from the RFQ exit to the buncher 2 entrance. Figure 10 shows the normalized rms emittance in the longitudinal plane for different beam currents from buncher 2 to the beam dump. The buncher 2 voltage is set to 170 kV for all cases. At the buncher 2 exit, all normalized rms emittances are almost the same because the SC force inside buncher 2 is significantly

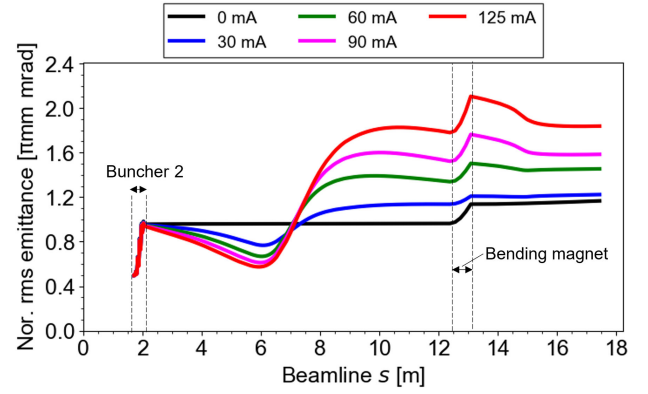


FIG. 10. Normalized longitudinal rms emittance evolution for different beam currents at a buncher 2 voltage of 170 kV. The starting point is the buncher 2 entrance.

weak compared with the rf field force, as mentioned in Sec. VIII. When the beam current is low, the minimum rms emittance does not return to the value at the buncher 2 entrance since the LSC force does not completely cancel the deformation. When the beam current is zero, the increased rms emittance is conserved along the beamline except in the dispersive region because nonlinear internal and external forces do not act on the beam. As the beam current increases, the minimum rms emittance after buncher 2 is close to the value at the buncher 2 entrance. The rms emittance decrease is particularly noticeable in high-current deuteron linacs, such as LIPAc. However, after reaching the minimum rms emittance, the rms emittance increases significantly due to the strong LSC force. When the beam current is low, the increase in the rms emittance is low after the minimum rms emittance due to the weak LSC force.

XI. CONCLUSIONS

We showed that a longitudinal rms emittance decreased and increased along a beamline consisting of rf bunchers and magnets in an intense deuteron linac. To quantitatively evaluate the mechanism, we developed the normalized mean emittance $\epsilon_{n,\text{mean}}$ and mean force nonlinearity χ applicable in the Cartesian coordinate system, which can express the deformations of the normalized transverse and longitudinal phase spaces and force distributions. When the signs of $\epsilon_{n,\text{mean}}$ and χ are positive or negative, respectively, the phase space and the force acting on the particles have S-shaped or inverse S-shaped distributions. Since general particle tracking codes cannot output SC force distributions, we derived the transverse and longitudinal SC force formulas acting on the particles from the Liénard-Wiechert potentials to calculate them using each particle position in a bunch. We applied the two physical parameters to the Linear IFMIF Prototype Accelerator (LIPAc) with a 5 MeV/125 mA beam to reveal the mechanism of the rms emittance decrease.

The rms emittance increased by a nonlinear force in an rf buncher is gradually decreased by the LSC force as the beam travels along the beamline. This is because the normalized LPS deformed into an inverse S-shape by the nonlinear rf field in a buncher is mitigated to be a linear distribution by the LSC force with an S-shape distribution. The minimum rms emittance downstream of the buncher, which depends on the deformations of the LSC force and the normalized LPS at the buncher exit, reaches close to that at the buncher entrance. We confirmed a decrease (increase) in the longitudinal rms emittance when $\epsilon_{m,\text{mean}}$ and χ have opposite (identical) signs. We also verified that the signs of the two physical parameters are consistent with the deformations of the phase spaces and force distributions at each beamline location.

The theory developed in this study requires experimental validation in the future. However, precise measurement of longitudinal emittance in high-current hadron beams is extremely challenging [33], and validation of the theory requires measurements at multiple locations. These are important issues to be addressed going forward. On the other hand, we have confirmed that the transverse beam size measured using a 5 MeV/110 mA deuteron beam agrees well with the simulation results [34]. This agreement suggests that the longitudinal particle distribution is also consistent, since any discrepancy in the longitudinal distribution would affect the transverse beam size. Therefore, this supports the occurrence of longitudinal emittance evolution as described in the paper.

Although the two physical parameters are useful for interpreting the deformations of phase space and force distributions, their quantitative evaluation requires detailed particle tracking simulations. This may limit their practical use in experimental environments. Nevertheless, in simulation-based design and optimization, these parameters provide valuable insights into beam dynamics and contribute to improved linac design and tuning. These developed formulas enable quantitative evaluations of phase space and force distribution deformations, thereby deepening our understanding of beam dynamics. Furthermore, the concept of reducing longitudinal emittance without using special devices may be applicable to linac designs aiming for low longitudinal emittance.

ACKNOWLEDGMENTS

We would like to thank Dr. Ken Takayama (KEK) for useful discussions on the simulation results. Part of the computations were carried out using the JFRS-1 supercomputer system at Computational Simulation Centre of International Fusion Energy Research Centre (IFERC-CSC) in Rokkasho Fusion Institute of QST (Aomori, Japan). This work was undertaken under the Broader Approach Agreement between the European Atomic Energy Community and the Government of Japan.

The views and opinions expressed herein do not necessarily state or reflect those of the Parties to this Agreement.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

APPENDIX A: DEPENDENCE OF NORMALIZED LONGITUDINAL RMS AND MEAN EMITTANCES ON PARTICLE NUMBER

We describe the dependence of the normalized rms and mean emittances in the longitudinal plane on the number of particles. Figure 11 shows the normalized longitudinal rms and mean emittances along the beamline, simulated using 10^2 (0.1k), 10^3 (1k), 10^4 (10k), and 10^5 (100k) particles. The rms emittance was simulated using TraceWin, and the mean emittance was subsequently calculated using Eqs. (12) and (26). The beam current and the buncher 2 voltage were set to 125 mA and 170 kV, respectively. The results indicate that when the number of particles exceeds 10k, further increases in particle count have a negligible impact on the calculated emittance values. Therefore, 100k particles were adopted for the particle tracking simulations.

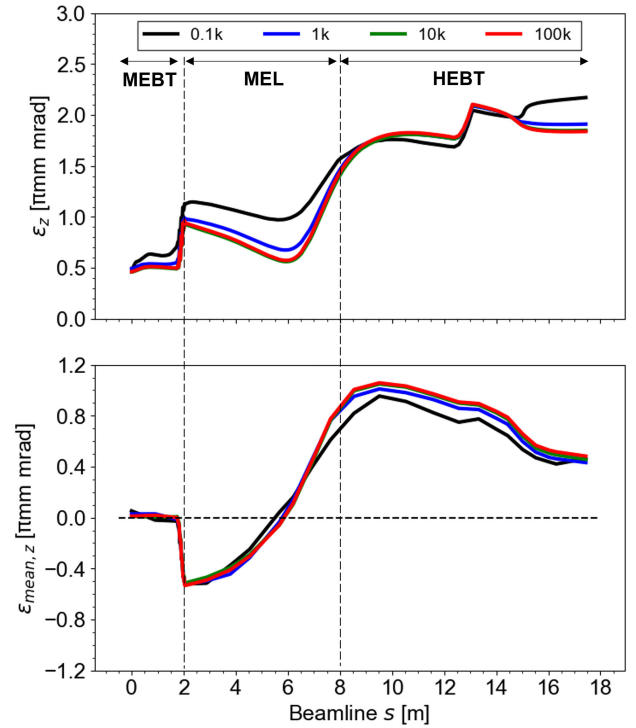


FIG. 11. Normalized longitudinal rms and mean emittances simulated along the beamline for 0.1k, 1k, 10k, and 100k. k stands for thousand.

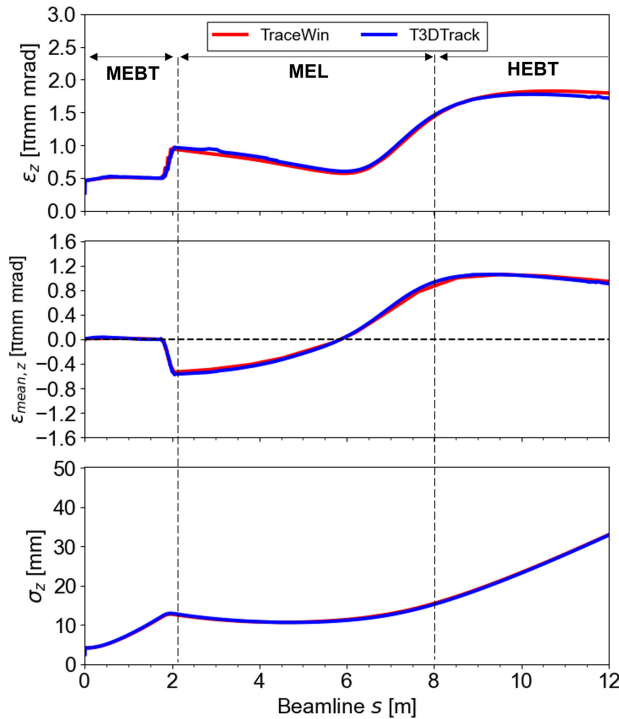


FIG. 12. Normalized rms emittance (top), normalized mean emittance (middle), and rms bunch length (bottom) in the longitudinal plane simulated using TraceWin and T3DTrack.

APPENDIX B: BENCHMARKING OF LONGITUDINAL BEAM PARAMETERS WITH DIFFERENT TRACKING CODES

We compare the rms bunch length, the normalized rms, and mean emittances obtained from simulations using TraceWin and a true three-dimensional tracking code (T3DTrack). T3DTrack calculates interactions between each macroparticle directly [35] and does not employ the PIC method. Since T3DTrack is not capable of modeling the bending magnet effect, the simulations were performed from the RFQ exit to the entrance of the bending magnet. Figure 12 shows comparisons of the simulated bunch length and longitudinal rms and mean emittances using TraceWin and T3DTrack. The simulations employed 100 k particles for TraceWin and approximately 87 k for T3DTrack. The results show good agreement, indicating that both simulation approaches yield consistent values for the bunch length, as well as for the rms and mean emittances.

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