

Simple model for electron beam dynamics in laser modulators and radiation characteristics of steady-state microbunching storage ring

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(Received 12 December 2024; accepted 2 July 2025; published 24 July 2025)

Recent studies have actively explored the feasibility of a novel storage ring light source based on the steady-state microbunching (SSMB) mechanism. Many interesting investigations have been limited to single-particle or pure-optics effects. In SSMB storage rings, laser modulators, consisting of an undulator and a copropagating laser beam, are utilized to manipulate the longitudinal bunch length at the radiator position. When the electron bunch traverses the modulator undulator, it emits coherent undulator radiation around the resonant wavelength with respect to the modulation laser wavelength. To increase the intensity of the modulation laser, an enhancement cavity can be used. Within the enhancement cavity, the modulation laser field is confined in a closed path, allowing the intracavity power to be accumulated and enhanced. Within this cavity, not only is there an accumulated modulation laser from external injection, but also coherent undulator radiations—both of which influence the dynamics of the electron beam on subsequent turns. In this paper, we develop a simple model to analyze the electron beam dynamics in laser modulator cavities, including the effects of finite loss due to mirrors in the cavity and the externally injected laser power required to compensate for this energy dissipation, as well as the dynamical effects of coherent undulator radiation accumulated in the cavity over multiple turns. This model also allows the estimation of the radiation characteristics in the radiator of the SSMB storage ring. Our method combines the beamline transfer matrices with a low-gain free-electron-laser oscillator model. The advantage of this approach lies in its ability to provide quick estimates for the aforementioned dynamical effects. We then discuss the feasibility of three recently proposed scenarios, namely, the amplifier SSMB, the frequency-beating SSMB, and the harmonic SSMB, in the presence of the aforementioned dynamical effects. The analysis indicates that all three scenarios remain feasible under the current preliminary design parameters.

DOI: [10.1103/kwxj-4bff](https://doi.org/10.1103/kwxj-4bff)

I. INTRODUCTION

Accelerator-based light sources have played a significant role in various fields of scientific research and industrial applications for the past three decades. Currently, accelerator-based light sources can be classified into three categories: single-pass linear-accelerator (linac)-driven high-gain free-electron lasers (FEL), storage-ring-based synchrotron light sources, and energy-recovery-linac (ERL)-driven light sources that combine the features of both the above through few-pass energy boost and recovery.

Generally speaking, linacs provide the most excellent beam quality, and linac-based FEL light sources deliver high peak brightness. In storage ring accelerators, the equilibrium beam quality is determined by the radiation damping and quantum excitation. With the cyclic nature of beam circulation, synchrotron light sources provide exceptionally high average brightness. Due to their few-pass nature, ERL-based particle beams have not yet reached equilibrium and can achieve beam quality comparable to linacs; however, unlike linacs, ERL-based coherent light sources offer high average brightness. These three conventional types of electron accelerator-based light sources use radio-frequency (rf) or microwave fields to compensate for electron beam energy losses due to radiation and other factors, while simultaneously providing longitudinal beam focusing.

For storage ring light sources, a novel mechanism called the steady-state microbunching (SSMB) has been proposed in recent years [1]. Following the initial proposal of SSMB, considerable research efforts have been devoted

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to exploring different implementation schemes of this mechanism (see, for example, Refs. [2–21]). Unlike traditional storage rings, a distinctive feature of this mechanism is the replacement of the aforementioned two functions of the rf cavity: energy compensation can be achieved through a conventional rf cavity or an induction accelerator, while longitudinal focusing is implemented via a laser modulator. The laser modulator is realized by copropagating a laser beam with the electron beam along a common axis within an undulator. To reuse the laser beam and accumulate sufficient intensity while relaxing laser requirements, we can confine the laser beam in a closed path and configure the laser modulator as an optical cavity, commonly referred to as a laser modulator cavity or enhancement cavity. Notably, while the laser modulator functions similarly to an rf cavity, the wavelength has been reduced by 5 to 6 orders of magnitude, implying that the longitudinal phase-space bucket width in the SSMB storage ring will correspondingly decrease by 5 to 6 orders of magnitude, i.e., from about 0.5 m down to μm . Consequently, the minimum electron bunch length is similarly reduced. When the bunch length becomes shorter than the target radiation wavelength, these electron bunches become electron *microbunches*, producing coherent radiation. This radiation will manifest as high-power coherent radiation approaching continuous-wave (cw) operation in the storage ring configuration. Currently, the laser modulator is a crucial component that must provide sufficient modulation strength or longitudinal focusing to effectively confine the microbunch train within the mini phase-space buckets. Given that the state-of-the-art laser cavity technology can store $\lesssim 1$ MW average laser power [22,23], considerable effort [6] has been devoted to designing schemes that reduce the dependence on laser power to achieve practical SSMB designs. Here, we remind that a potential source of confusion may lie in the distinction between the stored power in the laser modulator cavity and the input or injection power to the laser modulator cavity. The latter is typically much lower than 1 MW, and in many cases, an injection power of around 100 W or above may lead to unstable operation in the enhancement cavity. Excessive injection power may impart increased amplitude and phase noise, degraded beam quality, or mismatched spatial modes. These issues can hinder phase locking and mode matching into the cavity and consequently limit the achievable cavity performance. Given current laser enhancement cavity technology, intracavity powers well above 1 MW can induce unwanted thermal effects on the mirrors. These include thermally induced surface deformation and changes in the radius of curvature, leading to optical mode deformation and deterioration in mode matching, eventually compromising the stability of the enhancement cavity [22,23].

From both engineering and beam physics perspectives, the influence of laser modulators on beam dynamics

represents a new and significant branch of the studies. The existing literature for the studies of electron dynamics in the laser modulators mainly focuses on single-particle effects, e.g., Refs. [6–9,11–14]. The collective dynamics occurred in the laser modulators, as the next step in evaluating SSMB feasibility, has only recently been studied using a wake-impedance model for the case of a single laser modulator [15–17,24].

According to the recent developments [19], SSMB implementation schemes can be categorized into three scenarios: the amplifier SSMB, the frequency-beating SSMB, and the harmonic SSMB. The first scenario targets the radiation wavelength comparable to that of the modulation laser. Based on the present high-power laser cavity technology, we assume that the laser modulator operates at an infrared (IR) wavelength of 1064 nm. For target wavelengths longer than the laser modulator wavelength, say the terahertz (THz) regime, the wavelength beating techniques can be employed, as demonstrated in the second scenario (see more discussions later in this paper). More challenging is the generation of coherent high-average-power radiation at wavelengths shorter than the modulation laser wavelength, which has numerous potential applications, including the coherent ultraviolet (UV) 200 nm light sources for material surface processing or biomedical applications or coherent 13.5 nm light sources for the extreme ultraviolet (EUV) lithography in semiconductor core demands. This is the third scenario, known as the harmonic SSMB, which involves further compression of the microbunches within the mini phase-space bucket.

In this paper, we develop a simple model for analyzing the electron beam dynamics in laser modulator cavities of an SSMB storage ring for the above three scenarios. This includes physical effects within the laser modulator cavity due to energy loss from cavity mirrors, external laser power injection for energy dissipation compensation, and the turn-by-turn dynamical effects from coherent undulator radiation generated by the traversing electron beams. This model allows to estimate the radiation characteristics in the radiator of the SSMB storage ring. Our method combines the transfer matrix approach, commonly used in accelerator physics, with the low-gain FEL oscillator model. The advantage of this approach is its capability to quickly estimate the aforementioned dynamical effects.

This paper is organized as follows: In Sec. II A, we describe the electron beam dynamics in a laser modulator cavity by deriving the transfer matrix of the laser modulator. The effect of the modulation laser on the beam is characterized by a lumped parameter called the energy chirp strength. This energy chirp strength parameter is then used to construct a transfer matrix. When considering the coherent radiation generated by the electron beam passing through the modulator undulator, which can be confined and accumulated turn by turn in the laser cavity, the process resembles a low-gain FEL oscillator. This dynamic effect is described in Sec. II B. To establish an SSMB storage ring,

we must incorporate additional components and physical effects beyond the modulator undulator, including the full storage ring transfer matrix, radiation damping, quantum excitation (both in the storage ring and modulator undulator), and the equilibrium energy spread. In Sec. II C, summarizing these relevant effects and combining the beam sigma matrix commonly used in accelerator physics [25], we can track various bunch characteristics during each revolution. In Sec. III, we apply our developed model to analyze the beam dynamics and radiation characteristics of all three SSMB scenarios in the presence of the aforementioned dynamical effects. Finally, we summarize this work in Sec. IV.

II. THEORETICAL MODEL

A. Transfer matrix of the laser modulator

In this subsection, our objective is to construct a transfer matrix for the laser modulator. Here, we primarily consider the longitudinal effects of the laser modulator on the electron beam, while approximating the transverse effect as a free-space drift. Since it is placed at zero phase, the laser modulator's longitudinal effect on the electron beam mainly manifests as providing an energy chirp, or longitudinal focusing, similar to the action of a quadrupole in the transverse dimension. The energy chirp is related to the energy gain experienced by the resonant particle after passing through the modulator undulator, defined as

$$h = k_L \frac{\Delta\mathcal{E}}{\gamma mc^2}, \quad (1)$$

where $k_L = 2\pi/\lambda_L$ is the wave number of the modulation laser, γ is the Lorentz relativistic factor of the electron energy, m is the electron rest mass, and c is the speed of light in vacuum. In the above expression, the energy difference $\Delta\mathcal{E}$ received by an electron after passing through the modulator undulator can be evaluated by integrating along the undulator axis, i.e.,

$$\Delta\mathcal{E} = \int_{-L_u/2}^{L_u/2} \frac{d\mathcal{E}}{ds} ds, \quad (2)$$

where the rate of energy change $d\mathcal{E}/ds$ can be derived by the Lorentz force equation. Skipping the detailed derivation, which can be found in Appendix A, we summarize the results here. For the on-resonance case, the energy chirp strength of the laser modulator can be expressed as

$$h = \frac{eK_u[\text{JJ}]}{\gamma^2 mc^2} \sqrt{\frac{4P_L Z_0 z_R}{\lambda_L}} \tan^{-1} \left(\frac{L_u}{2z_R} \right) k_L, \quad (3)$$

where $e > 0$ is the unit charge, K_u is the undulator parameter with $K_u \approx 0.934 B_0[\text{T}] \lambda_u[\text{cm}]$ with B_0 the peak magnetic field and λ_u the undulator period, $L_u = N_u \lambda_u$ is

the undulator length with N_u the number of undulator periods. Here z_R is the Rayleigh length, $Z_0 \approx 377 \Omega$ is the free-space impedance, λ_L is the modulation laser wavelength, and P_L is the modulation laser power. The coupling coefficient of the planar undulator is defined as

$$[\text{JJ}] = J_0 \left(\frac{K_u^2}{4 + 2K_u^2} \right) - J_1 \left(\frac{K_u^2}{4 + 2K_u^2} \right), \quad (4)$$

where J_0, J_1 are, respectively, the zeroth and first order Bessel functions of the first kind.

Having obtained the chirp strength received by the electron through the modulator undulator, we can construct a longitudinal transfer matrix for the longitudinal phase-space coordinates $[z \delta]^T$, with z the longitudinal coordinate and δ the relative energy deviation in the bunch, as [6]

$$\mathbf{M}_{\text{LM}} = \begin{pmatrix} \cos(\sqrt{-R_{56,\text{LM}}h}) & \frac{R_{56,\text{LM}} \sin(\sqrt{-R_{56,\text{LM}}h})}{\sqrt{-R_{56,\text{LM}}h}} \\ \frac{h \sin(\sqrt{-R_{56,\text{LM}}h})}{\sqrt{-R_{56,\text{LM}}h}} & \cos(\sqrt{-R_{56,\text{LM}}h}) \end{pmatrix}, \quad (5)$$

where the longitudinal momentum compaction function of the laser modulation is

$$R_{56,\text{LM}} = \frac{1 + K_u^2/2}{\gamma^2} L_u. \quad (6)$$

Here, we note that this symplectic matrix is derived under the plane wave approximation, i.e., the laser field experienced by the electron is roughly uniform along the undulator. When written in the form that includes both the transverse and longitudinal phase-space coordinates $[y y' z \delta]^T$, with y and y' the transverse vertical coordinate and vertical divergence in the bunch, we have

$$\mathbf{M}_{\text{LM}} = \begin{pmatrix} 1 & N_u \lambda_u & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\sqrt{-R_{56,\text{LM}}h}) & \frac{R_{56,\text{LM}} \sin(\sqrt{-R_{56,\text{LM}}h})}{\sqrt{-R_{56,\text{LM}}h}} \\ 0 & 0 & \frac{h \sin(\sqrt{-R_{56,\text{LM}}h})}{\sqrt{-R_{56,\text{LM}}h}} & \cos(\sqrt{-R_{56,\text{LM}}h}) \end{pmatrix}. \quad (7)$$

It is worth noting that the transverse phase-space coordinates $[y, y']$ are adopted in this context because the forthcoming discussion on the third scenario involves coupling of the transverse emittance in the y direction to the longitudinal z dimension to achieve bunch compression.

B. Coherent radiation in a laser modulator cavity

To further enhance the modulation laser power, the laser beam is confined in a closed path and the laser modulator is configured as an optical cavity, which is called an enhancement cavity or laser modulator cavity. In this cavity, in

addition to the externally injected laser field, an undulator radiation field is also generated each time the electron beam passes through the modulator undulator. The dominant undulator radiation fields, with a wavelength that fulfills the resonance condition $\lambda_L = \frac{\lambda_u}{2\gamma^2} (1 + \frac{K_u^2}{2})$, will be confined within the cavity. This process is similar to a low-gain FEL oscillator. We will utilize a well-known low-gain FEL oscillator model [26,27] to describe this dynamic process, while combining results from the previous section to present a model that includes the coherent undulator radiation effects (or FEL-like effects). It should be noted that Dattoli's model [26,27], developed through fitting numerous simulation results, has been recognized as reliable and is widely applied by the FEL community in the design analysis of FEL oscillators or storage ring FELs.

First, in the laser cavity, the evolution of the laser power per turn is subject to cavity mirror loss while potentially receiving gain from FEL-like processes. The former is reflected in the cavity's total power reflectivity $R \leq 1$, where the equality indicates no mirror loss (ideal case). The latter is described by the gain G_n . From the n th turn to the $(n+1)$ th turn, the modulation laser power P_L evolves according to

$$P_{L,n+1} = R(1 + G_n)P_{L,n}, \quad (8)$$

where the gain function G_n is given by [26]

$$G_n = G_{M,n} \frac{1 - e^{-\beta X_n}}{\beta X_n}, \quad \beta = \frac{\pi}{2} (1.0145), \quad X_n = \frac{P_{L,n}}{P_{\text{sat}}}, \quad (9)$$

with

$$\begin{aligned} a_1 &= 3.0, & b_1 &= 209.11, & c_1 &= 2.0, & d_1 &= 5.7212, & e_1 &= 9.8361 \\ a_2 &= 40.0, & b_2 &= 58.01, & c_2 &= 5.2427, & d_2 &= 78.9575, & e_2 &= 142.589 \\ a_3 &= 11461, & b_3 &= 128.60, & c_3 &= 2.9317, & d_3 &= 9.0363, & e_3 &= 13.369. \end{aligned} \quad (14)$$

According to Ref. [26], these fitting formulas produce results with a relative error of 1.7% compared to actual simulation results, valid within $|\kappa| < 10$, which is sufficient for our purposes. In Eq. (9), the effective saturation power is defined as

$$P_{\text{sat}}[\text{W}] = 6.93 \times 10^{12} \left(\frac{\gamma}{N_u} \right)^4 \frac{\pi(w_0[\text{m}])^2}{(\lambda_u[\text{cm}] \cdot K_u \cdot [\text{JJ}])^2}, \quad (15)$$

with w_0 the laser beam waist size.

We will iterate Eq. (8) to evaluate the turn-by-turn evolution of the laser power stored in the modulator cavity.

$$G_{M,n} = \frac{e^{-0.132P(g_0)\mu_{\epsilon,n}^2}}{1 + 1.6Q(g_0)\mu_{\epsilon,n}^2} \sum_{j=1}^3 g_0^j g_j(\kappa). \quad (10)$$

Here the small-signal gain coefficient g_0 can be written as

$$g_0 = 8\pi \frac{I_P}{I_A} \frac{K_u^2 [\text{JJ}]^2}{\left(1 + \frac{K_u^2}{2}\right)^2} \frac{\gamma N_u^3 \lambda_L^2}{\mathcal{A}_\perp}, \quad (11)$$

with the peak bunch current I_P , the Alfvén current $I_A \approx 17045$ A, the transverse effective area $\mathcal{A}_\perp = 2\pi(\sigma_x^2 + \frac{\lambda_L}{4\pi} z_R)$, and σ_x being the transverse electron beam size. The remaining quantities in Eq. (10) are [26]

$$\begin{aligned} P(g_0) &= 1 - 3.57 \times 10^{-3} g_0 + 6.83 \times 10^{-2} g_0^2 \\ &\quad - 4.79 \times 10^{-3} g_0^3, \\ Q(g_0) &= 1 + 0.142 g_0 - 0.011 g_0^2 + 1.8 \times 10^{-4} g_0^3. \end{aligned} \quad (12)$$

These fitting coefficients reflect the effects induced by the beam finite energy spread σ_δ , where $\mu_{\epsilon,n} = 4N_u \sigma_{\delta,n}$, valid for $\mu_{\epsilon,n} \leq 1$. For the low-gain FEL interaction, in the sum expression of Eq. (10), we have for the detuning parameter $\kappa = 2\pi N_u \frac{\omega_u - \omega}{\omega_L}$ [26]

$$\begin{aligned} g_1(\kappa) &\approx \frac{\pi}{a_1} \exp\left(-\frac{\kappa^2}{b_1}\right) \sin\left(\frac{\kappa}{c_1}\right) \cos\left(\frac{\kappa}{d_1}\right) \cos\left(\frac{\kappa}{e_1}\right) \\ g_2(\kappa) &\approx \frac{\pi^2}{a_2} \exp\left(-\frac{\kappa^2}{b_2}\right) \cos\left(\frac{\kappa}{c_2}\right) \cos\left(\frac{\kappa^2}{d_2}\right) \cos\left(\frac{\kappa^2}{e_2}\right), \\ g_3(\kappa) &\approx \frac{\pi^3}{a_3} \exp\left(-\frac{\kappa^2}{b_3}\right) \sin\left(\frac{\kappa}{c_3}\right) \cos\left(\frac{\kappa}{d_3}\right) \cos\left(\frac{\kappa}{e_3}\right) \end{aligned} \quad (13)$$

where

On each turn, when the electron beam passes through the modulator undulator, the emitted coherent undulator radiation slightly increases the modulation laser power, which can be written as

$$\begin{aligned} P_{L,n} &= P_{L,n}^{\text{ext}} + \frac{\pi}{\epsilon_0 c} N_u \left(\frac{K_u^2}{4 + 2K_u^2} \right) \\ &\quad \times [\text{JJ}]^2 F F_\perp(S) |b|^2 I_P^2 \left(\frac{\sin \frac{\kappa}{2}}{\frac{\kappa}{2}} \right)^2, \end{aligned} \quad (16)$$

where $P_{L,n}^{\text{ext}}$ is the laser power already existing in the cavity, $|b|$ is the bunching factor of the electron beam (to be defined later), I_P is the peak bunch current, and the last sinc

TABLE I. Initial beam, laser modulator, radiator, and storage ring parameters for the first example: IR SSMB.

Name	Value	Unit
Beam energy	250	MeV
Peak bunch current	0.55	A
rms energy spread	2.76×10^{-4}	
rms bunch length	175	nm
rms beam size	100	μm
Laser modulation wavelength	1064	nm
Laser modulation power	24	kW
Rayleigh length	0.33	m
Nominal chirp strength	500	m^{-1}
Modulator undulator period	5	cm
Modulator undulator length	1	m
Modulator undulator parameter	4.29	
Total reflectivity of the cavity mirrors	0.99963	
Radiator undulator period	5	cm
Radiator undulator length	2	m
Radiator undulator parameter	4.29	
Storage ring circumference	50	m
Storage ring slippage factor	5×10^{-6}	
Synchrotron tune	0.056	
Radiation damping time	10	ms

function characterizes the detuning effect. The second term in the equation represents the additional increment [18], where the electron beam is assumed to be a transverse round beam with the transverse form factor

$$FF_{\perp}(S) = \frac{2}{\pi} \left[\tan^{-1} \left(\frac{1}{2S} \right) + S \ln \left(\frac{(2S)^2}{(2S)^2 + 1} \right) \right], \quad (17)$$

where $S = \frac{\gamma \omega_L}{L_u}$.

Before ending this subsection, we discuss the validity of the above model in our case. Here, we assume that the microbunch is located near the center of the phase-space bucket. We neglect the phase offset between the bucket center and the emitted radiation field, i.e., we presume that the radiation field and the microbunch are at the same phase. This is justified if we consider the interaction is low gain [28]. For the high-gain interaction, this assumption becomes violated. To ensure this assumption holds in our case, let us estimate the FEL gain length as follows. Taking the parameters from Table I that we will demonstrate later as an example, for 250 MeV, $I_P = 0.55$ A, $K_u = 4.29$, $\lambda_L = 1064$ nm, $\sigma_x \approx 100$ μm , the Pierce parameter

$$\rho_{\text{FEL}} = \left[\frac{1}{8\pi} \frac{I_P}{I_A} \left(\frac{K_u [JJ]}{1 + \frac{K_u^2}{2}} \right)^2 \frac{\gamma \lambda_L^2}{2\pi \sigma_x^2} \right]^{1/3} \approx 1 \times 10^{-3}. \quad (18)$$

The 1D power gain length $L_{G0} = \frac{\lambda_u}{4\pi\sqrt{3}\rho_{\text{FEL}}} \approx 2.3$ m, longer than the length of the modulator undulator (1 m). In general, within the first two gain lengths (4.6 m), this

assumption is valid. The gain length can be longer when considering 3D effects [29]. Therefore, the validity of the model in our case is justified.

From the above two subsections, we have now developed the theoretical model of the laser modulator cavity. When considering the FEL-like effects in the laser modulator, the modulation laser power varies turn by turn, and this $P_{L,n}$ will lead to different chirp strength h_n on each turn, thus the numerical values of the transfer matrix [Eqs. (5) and (7)] of the laser modulator will also vary slightly turn by turn. These will affect the electron beam dynamics in laser modulators and radiation characteristics in the radiator of an SSMB storage ring. In the next subsection, we will describe the remaining parts of the SSMB storage ring using common transfer matrices, while considering the radiation damping and quantum excitation effects.

C. One-turn matrix and other relevant effects

For symmetry considerations, the SSMB radiator is typically installed at a position symmetric to the laser modulator cavity. Consequently, we divide the storage ring in half; the longitudinal transfer matrix for half of the ring can be written as

$$\mathbf{R}_{1/2} = \begin{pmatrix} \cos \pi \nu_s & \beta_{zs} \sin \pi \nu_s \\ -\frac{1}{\beta_{zs}} \sin \pi \nu_s & \cos \pi \nu_s \end{pmatrix}, \quad (19)$$

with β_{zs} being the longitudinal beta function and ν_s being the synchrotron tune. For longitudinal weak focusing, we have [6,11]

$$\nu_s \approx \frac{\sqrt{h\eta_{\text{tot}}C_0}}{2\pi} \ll 1 \quad (20)$$

and

$$\beta_{zs} \approx \sqrt{\frac{\eta_{\text{tot}}C_0}{h}}. \quad (21)$$

Here, η_{tot} is the storage ring slippage factor and C_0 is the storage ring circumference. When including both the transverse and longitudinal motion, the transfer matrix for half of the storage ring can be written as

$$\mathbf{R}_{1/2} = \begin{pmatrix} \cos \pi \nu_y & \beta_y \sin \pi \nu_y & 0 & 0 \\ -\frac{1}{\beta_y} \sin \pi \nu_y & \cos \pi \nu_y & 0 & 0 \\ 0 & 0 & \cos \pi \nu_s & \beta_{zs} \sin \pi \nu_s \\ 0 & 0 & -\frac{1}{\beta_{zs}} \sin \pi \nu_s & \cos \pi \nu_s \end{pmatrix}, \quad (22)$$

with β_y the transverse vertical betatron function and ν_y being the vertical betatron tune.

Now we can describe the one-turn matrix $\mathcal{O}$ that includes both the laser modulator and the remaining parts to form the

complete storage ring. In the subsequent section, we will demonstrate three SSMB scenarios, for which we need to construct three corresponding one-turn matrices $\mathcal{O}$. With the one-turn matrix established, in our model, we use the beam sigma matrix to describe the second-moment characteristics of the beam. For the 2D case, the longitudinal beam sigma matrix is

$$\Sigma = \begin{pmatrix} \sigma_z^2 & \sigma_{z\delta} \\ \sigma_{z\delta} & \sigma_\delta^2 \end{pmatrix}. \quad (23)$$

For the 4D case, including the transverse and longitudinal beam sigma matrix, we have

$$\Sigma = \begin{pmatrix} \sigma_y^2 & \sigma_{yy'} & \sigma_{yz} & \sigma_{y\delta} \\ \sigma_{yy'} & \sigma_{y'}^2 & \sigma_{y'z} & \sigma_{y'\delta} \\ \sigma_{yz} & \sigma_{y'z} & \sigma_z^2 & \sigma_{z\delta} \\ \sigma_{y\delta} & \sigma_{y'\delta} & \sigma_{z\delta} & \sigma_\delta^2 \end{pmatrix}. \quad (24)$$

Turn-by-turn evolution of the beam sigma matrix can be formulated as [25]

$$\Sigma_{n+1} = \mathcal{O}_n \Sigma_n \mathcal{O}_n^T. \quad (25)$$

Here, we add the subscript n to the one-turn matrix, writing it as $\mathcal{O}_n$, to reflect the changes when FEL-like effects are considered in the laser modulator cavity. To initiate the iteration, the initial parameters for turn-by-turn tracking will be calculated from single-particle effects. Without considering the aforementioned dynamical effects in the laser modulator cavity, the equilibrium energy spread can be expressed as

$$\sigma_{\delta 0} = \sqrt{\frac{C_q \gamma^2}{J_s \rho}}, \quad (26)$$

where ρ is the average radius of the storage ring bending magnets, J_s is the longitudinal damping partition number (assumed to be 2), $C_q = 3.832 \times 10^{-13}$ m. The equilibrium bunch length is $\sigma_{z0} = \sigma_{\delta 0} \beta_{zs}$. We assume the initial $\sigma_{z\delta} = 0$. During turn-by-turn tracking, the radiation damping effects can be implemented through the following approach

$$\mathbf{R}_{1/2}(z, \delta) = \mathbf{R}_{1/2}(z, \delta) \times e^{-1/2N_{\text{RD}}}, \quad (27)$$

for the longitudinal dimension, or

$$\mathbf{R}_{1/2}(y, y') = \mathbf{R}_{1/2}(y, y') \times e^{-1/4N_{\text{RD}}}, \quad (28)$$

for the transverse dimension. Here $N_{\text{RD}} = \tau_z / T_{\text{rev}}$ is the longitudinal damping time expressed in terms of the number of revolution turns. For simplicity, we assume that the transverse radiation damping time is twice the longitudinal radiation damping time. In addition to the radiation

damping, the quantum excitation contributes to the longitudinal incoherent energy spread and the transverse emittance growth. On each turn, these effects can be estimated as

$$\Delta\sigma_\delta^2 = \frac{\sigma_{\delta 0}^2}{N_{\text{RD}}/4}, \quad \Delta\epsilon_y = \frac{\epsilon_{y0}}{N_{\text{RD}}}, \quad (29)$$

where $\sigma_{\delta 0}$ and ϵ_{y0} represent the equilibrium energy spread and the emittance, respectively. According to Eq. (29), the quantum excitation updates the beam sigma matrix turn by turn as

$$\Sigma_n \rightarrow \Sigma_n + \Delta\Sigma, \quad (30)$$

with

$$\Delta\Sigma = \begin{pmatrix} \Delta\epsilon_y \beta_y & -\Delta\epsilon_y \alpha_y & 0 & 0 \\ -\Delta\epsilon_y \alpha_y & \Delta\epsilon_y \gamma_y & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta\sigma_\delta^2 \end{pmatrix}. \quad (31)$$

In addition to the radiation damping and quantum excitation occurring in a storage ring, the quantum diffusion causes an increase in the longitudinal incoherent energy spread when the electron beam passes through a modulator undulator. On each turn, this increase is given by [30]

$$\sigma_{\delta,n}^2 = \sigma_{\delta,0}^2 + \Delta\sigma_{\delta u}^2, \quad (32)$$

with

$$\Delta\sigma_{\delta u}^2 \approx r_e \lambda_C \gamma^2 \left(\frac{k_u K_u}{\sqrt{2}} \right)^3 L_u, \quad (33)$$

where $r_e \approx 2.818 \times 10^{-15}$ m is the classical electron radius, and $\lambda_C \approx 2.426 \times 10^{-12}$ m is the Compton wavelength of the electron. We comment that this increase, $\Delta\sigma_{\delta u}^2$, is approximately 2 orders of magnitude smaller than the quantum excitation contribution from the storage ring [Eq. (29)]. Compared to other effects, this contribution can be small, but we still include it to reflect the FEL-like heating effect.

III. EXAMPLES

In the previous section, we developed a theoretical model, which will now be applied in this section to analyze three SSMB scenarios. The first example is an amplifier SSMB, the simplest configuration, targeting a radiation wavelength similar to that of the laser modulator. The second example targets a radiation wavelength longer than that of the laser modulator, known as the frequency-beating SSMB. In the third example, we consider coherent radiation with a wavelength shorter than that of the laser modulator, referred to as the harmonic SSMB. For all three examples, we assume the laser modulator operates in the infrared (IR) region at 1064 nm.

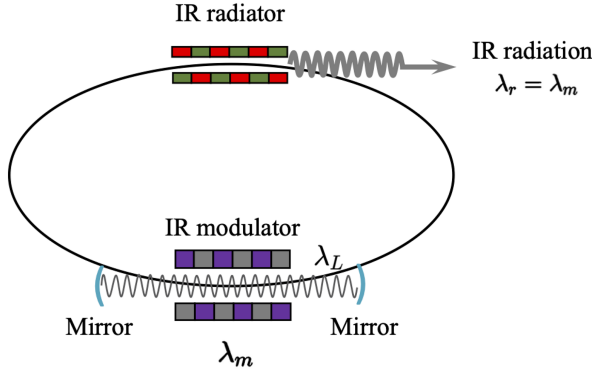


FIG. 1. Schematic layout of the amplifier SSMB. The electron bunch undergoes energy modulation or longitudinal focusing in the IR modulator, followed by coherent radiation emission at the same wavelength as the modulator in the downstream IR radiator. After radiation emission in the radiator, the electron beam returns to the IR modulator, which provides energy modulation or longitudinal focusing to maintain the longitudinal bunch length around the storage ring.

A. Amplifier scenario: IR SSMB

In the first example, as shown in Fig. 1, the electron bunch is modulated or longitudinally focused in an IR laser modulator cavity, maintaining the microbunch length at approximately $\sigma_{z0} = 175$ nm per turn. When the microbunch passes through the radiator, it generates coherent IR radiation. In this case, the radiation wavelength is equal to the modulation laser wavelength.

For this scenario, considering symmetry, starting from the radiator, the one-turn matrix can be formulated as

$$\mathcal{O} = \mathbf{R}_{1/2} \mathbf{M}_{\text{LM}} \mathbf{R}_{1/2}. \quad (34)$$

Here, the radiator is simply treated as a drift with its $R_{56,\text{rad}}$ counted as part of the storage ring slippage factor. Once the initial microbunch parameters are specified, we can combine Eq. (34) and use Eq. (25) to calculate the bunch characteristics at the modulator or radiator turn by turn. The bunch properties at other locations in the storage ring can be obtained in a similar manner. Of particular interest is the bunching factor at the radiator or the modulator, defined as

$$|b| \equiv \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(z, \delta) e^{-i(\frac{2\pi}{\lambda_L})z} dz d\delta \right|, \quad (35)$$

where we assume the longitudinal phase-space distribution is Gaussian, i.e.,

$$\begin{aligned} f(z, \delta) &= \frac{1}{2\pi\sqrt{\det \Sigma}} e^{-\frac{1}{2}\mathbf{Z}^T \Sigma^{-1} \mathbf{Z}} \\ &= \frac{1}{2\pi\sqrt{\sigma_z^2 \sigma_\delta^2 - \sigma_{z\delta}^2}} e^{-\frac{1}{2\det \Sigma_z}(z^2 \sigma_\delta^2 + \delta^2 \sigma_z^2 - 2z\delta \sigma_{z\delta})}, \end{aligned} \quad (36)$$

where $\det$ denotes the determinant operator and $\mathbf{Z} = [z \ \delta]^T$. Then the bunching factor can be analytically integrated

$$|b| = e^{-\frac{1}{2}(\frac{2\pi}{\lambda_L})^2 \sigma_z^2}, \quad (37)$$

where σ_z can be extracted from the beam sigma matrix calculated for any given turn. In addition to the bunching factor, we are also concerned with the output radiation power at the radiator. According to Ref. [18], the radiation power can be expressed as

$$P_{\text{peak}} = \frac{\pi}{\epsilon_0 c} N_{u,\text{rad}} \left(\frac{K_{u,\text{rad}}^2}{4 + 2K_{u,\text{rad}}^2} \right) [\text{JJ}]^2 FF_{\perp}(S) |b|^2 I_P^2, \quad (38)$$

where $[\text{JJ}]$ is given by Eq. (4), and $FF_{\perp}$ is defined in Eq. (17). The absence of the detuning term [cf. Eq. (16)] is because we have assumed that the radiation is on resonance in the radiator. In practical units, the above expression becomes

$$P_{\text{peak}} [\text{kW}] = 1.183 N_{u,\text{rad}} \left(\frac{K_{u,\text{rad}}^2}{4 + 2K_{u,\text{rad}}^2} \right) [\text{JJ}]^2 FF_{\perp}(S) |b|^2 I_P^2 [\text{A}]. \quad (39)$$

This formula will be used later to calculate the output radiation power at the radiator.

Table I summarizes the initial beam, laser modulator, radiator, and storage ring parameters for the first example. We use the parameters in Table I to construct the one-turn matrix and calculate the turn-by-turn evolution of the beam sigma matrix. According to Eq. (6), the $R_{56,\text{LM}}$ of the laser modulator is 42.56 μm . Under ideal conditions, i.e., without cavity loss, the equilibrium bunch length provided in Table I is approximately 175 nm, and the relative energy spread is 2.76×10^{-4} . These values serve as the initial conditions in Figs. 2 and 3. The turn-by-turn tracking results are shown in Figs. 2 and 3. Figure 2 illustrates the evolution of the microbunch's bunch length, energy spread, and bunching factor in the laser modulator cavity, along with the modulation laser power stored in the cavity as a function of the turn number under different physical conditions. In the case of single-particle effects, where the radiation damping and quantum excitation are considered but the FEL-like interaction in the modulator is ignored, when cavity loss is taken into account but without external laser injection to compensate for the loss, the modulation laser power decreases turn by turn because of power loss in cavity mirrors, as expected. As a result, the modulation strength weakens, and the longitudinal focusing of the microbunch is gradually lost, eventually leading to bunch lengthening. This process is reflected in the results shown in Fig. 2, represented by the dashed lines. As the bunch length in the laser modulator increases, the bunch

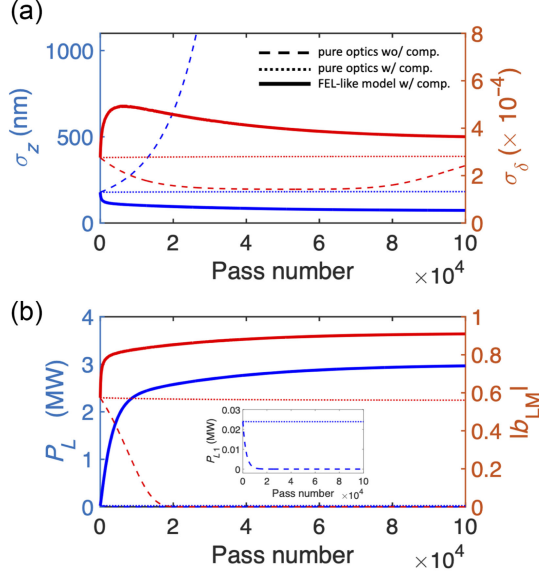


FIG. 2. Turn-by-turn evolution of the microbunch beam parameters σ_z , σ_δ (a), $|b_{LM}|$ and the laser power P_L (b) in the laser modulator for the case of an amplifier SSMB. The radiation damping time is set to approximately 10 ms, corresponding to 6×10^4 turns. This is a dual y-axis plot, where the blue lines correspond to the left y-axis and the red lines correspond to the right y-axis. The dashed lines account for radiation damping and quantum excitation effects, without external laser power injection to compensate for the cavity loss. The dotted lines additionally include laser power compensation. The solid lines represent simulations using an FEL-like model for the laser modulator, incorporating the coherent radiation power generated by the electron beam passing through the modulator undulator and the turn-by-turn accumulation of dynamic effects. The inset shows a close-up of a selected region in the bottom plot.

length at the radiator also increases, and the microbunch is no longer able to produce sufficient coherent radiation at the radiator. This is demonstrated by the dashed lines in Fig. 3. Without the quantum diffusion effect in the modulator undulator, the energy spread of the bunch decreases initially as the bunch lengthens but then returns to its equilibrium value due to radiation damping and quantum excitation. Here, the radiation damping time is set to approximately 10 ms, corresponding to 6×10^4 turns. Since the design of the SSMB storage ring is not yet fully finalized, this value is chosen as a reasonable yet somewhat arbitrary setting.

According to our numerical setup, the total cavity reflectivity is 0.99963, also of somewhat arbitrary choice, corresponding to a power loss of approximately 8.9 W per turn. Still ignoring the FEL-like effects in the modulator, we now externally inject 8.9 W of the laser power per turn to compensate for the cavity loss. With this setup, the laser modulator is able to operate stably, as shown by the dotted lines in Figs. 2 and 3. As can be seen, the bunch length in the laser modulator is maintained at approximately 178 nm,

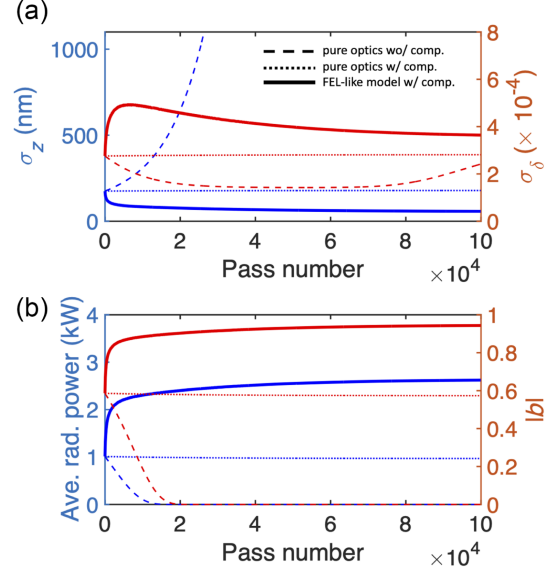


FIG. 3. Turn-by-turn evolution of the microbunch beam parameters σ_z , σ_δ (a), the average radiation output power and $|b_{LM}|$ (b) in the radiator for the case of an amplifier SSMB. The simulation parameters and line conventions are maintained as described in Fig. 2 (see also Table I).

the energy spread remains at 2.8×10^{-4} , and the laser power in the modulator is stabilized at 23.9 kW, all consistent with the expected design values. Additionally, as shown in Fig. 3, the microbunch length and energy spread at the radiator remain comparable to those in the modulator. The output radiation power is approximately 1 kW, and the bunching factor is about 0.58, both matching the expected performance. Here, we emphasize that the relevant amplification process in this amplifier scenario does not involve the 24 kW stored in the laser modulator cavity and the 1 kW radiation power generated in the radiator. Instead, an appropriate comparison should be between the injection laser power of 8.9 W and the radiation power of 1 kW.

Now, after considering the coherent radiation in the laser modulator cavity and FEL-like gain effects introduced in Sec. II B within the modulator, a more complete physical picture of the laser modulator emerges. The results are shown as the solid lines in Figs. 2 and 3. Several interesting observations can be made. First, in the laser modulator, the radiation generated by the microbunch passing through the undulator significantly amplifies the stored laser power in the cavity, increasing from 24 kW to 3 MW, as shown by the blue solid line in the bottom panel of Fig. 2. This increase in laser power provides stronger modulation or longitudinal focusing. Consequently, the microbunch length in the laser modulator shortens from 178 nm (under single-particle effects only) to approximately 74 nm. This is accompanied by an increase in energy spread, rising from 2.8×10^{-4} . After including radiation damping, the energy spread eventually approaches 3.6×10^{-4} . Correspondingly,

the microbunch length at the radiator also shortens, while the bunching factor increases from 0.58 to 0.95. The emitted radiation power rises from 1 to 2.6 kW. Here, we comment that the present laser cavity technology can already support stored powers of approximately ≥ 700 kW [22,23] using an optical enhancement cavity. We believe that achieving an average storage power of approximately 1 MW in the foreseeable future appears to be a realistic and achievable goal. We also recognize that the actual gain of an optical enhancement cavity will decrease when the stored laser power approaches several hundred kilowatts [22,23], ultimately leading to saturation at a certain power level. This power-dependent behavior of the cavity gain is neglected in our analysis. From Figs. 2 and 3, it is evident that the FEL-like effects in the laser modulator are promising: they enhance the longitudinal focusing in the modulator and increase the output radiation power at the radiator. Of course, the cost of this better performance is the need for additional energy compensation for the electron beam, such as through an rf cavity or induction linac in the storage ring. This aspect is beyond the scope of this paper and will not be discussed here.

The results in Figs. 2 and 3 assume that the laser modulator operates on resonance, that is, the modulation laser wavelength λ_L exactly matches the undulator resonance wavelength λ_m . If detuning exists, i.e., $\lambda_m \neq \lambda_L$ [see also Eqs. (A7) and (A8)], for example, by fine-tuning the magnetic field strength of the modulator undulator (adjusting K_u), the modulation strength of the laser modulator on the microbunch will vary, leading to change in the output radiation power at the radiator. Over-modulation might even occur. Figure 4 shows the stored laser power in the modulator and the radiation power generated at the radiator as a function of the detuning parameter $\kappa = 2\pi N_u \frac{\lambda_m - \lambda_L}{\lambda_L}$. It can be observed that, when $\kappa > 0$, particularly for $\kappa > 0.5$, excessive laser power in the modulator overmodulates the microbunch. This results in an unsuitable bunch length at the downstream radiator, preventing it from emitting coherent radiation at the intended wavelength. When $\kappa > 5$, as the FEL-like gain diminishes, the system stabilizes. In the unstable region of $0.5 < \kappa < 5$, tracking results indicate that the microbunch characteristics no longer evolve stably with the turn number but instead exhibit significant fluctuations. From Fig. 4, we see that slight tuning toward $\kappa > 0$ can improve the gain, but the system tends to become unstable. Tuning toward $\kappa < 0$ reduces the gain, but the laser modulator remains in a stable region.

In this subsection, we have demonstrated the amplifier scenario of IR SSMB, considering cavity loss, external laser injection to compensate for the loss, and the effects of coherent radiation in the laser modulator cavity as the electron beam passes through the modulator undulator. In the next subsections, we will continue to analyze other scenarios using the developed theoretical model.

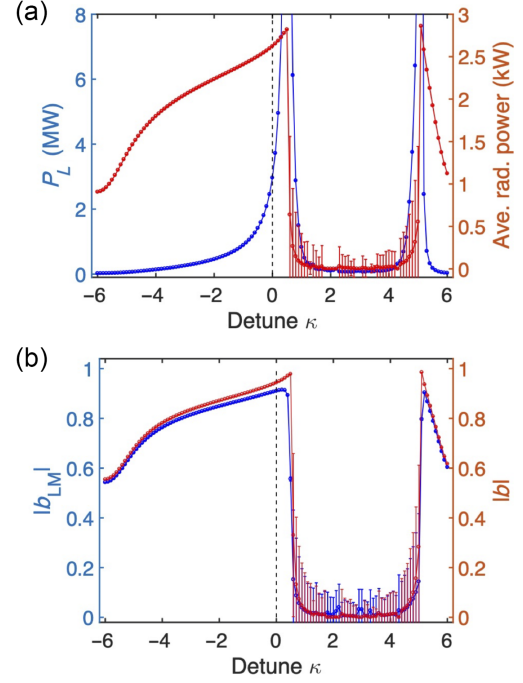


FIG. 4. (a) Detune dependence of the laser modulation power (blue), output radiation power (red), and (b) the bunching factors at the modulator (blue), and radiator (red). All the simulation parameters except for the detune parameter κ are the same as those in Figs. 2 and 3. To quantify the stability of the tracking results, error bars are calculated using the standard deviation of the relevant quantities over the final 1000 turns of simulation.

B. Frequency-beating scenario: THz SSMB

To generate coherent radiation with a wavelength longer than the modulation laser wavelength, one can install two back-to-back modulator undulators with slightly different λ_u at the modulator (see Fig. 5). These two modulator undulators then have slightly different resonant frequencies, both within their resonance bandwidths, covering the modulation laser. The frequency difference between the two resonances corresponds to the target radiation frequency. After traversing the modulator, the electron beam distribution will develop a beat-frequency component at the target wavelength, producing coherent radiation at that wavelength. For detailed discussions on the beat-wave envelope bucket from two modulator undulators, we refer interested readers to Appendix B.

To elaborate the idea, we consider two undulators at the modulator with different periods, denoted as $\lambda_{u1} = \lambda_{u0} - \Delta\lambda_u$ and $\lambda_{u2} = \lambda_{u0} + \Delta\lambda_u$, where λ_{u0} is the averaged undulator period and define

$$\Delta = \frac{\Delta\lambda_u}{\lambda_{u0}}. \quad (40)$$

Then the resonant wavelength at the two undulators becomes $\lambda_{m1} = \lambda_L(1 - \Delta)$ and $\lambda_{m2} = \lambda_L(1 + \Delta)$. Moreover, we let

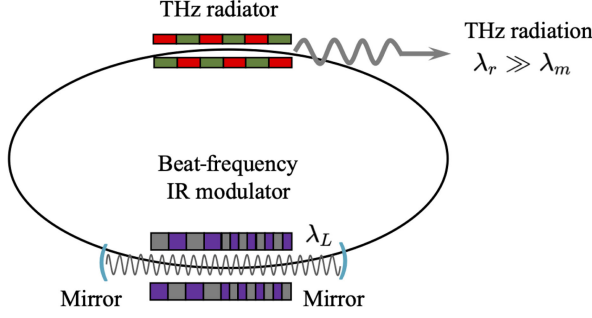


FIG. 5. Schematic layout of the frequency-beating SSMB. The microbunches in the storage ring receive energy modulation or longitudinal focusing from the beat-frequency IR modulator, consisting of two back-to-back modulator undulators with slightly different undulator periods. Then the microbunches with frequency-beating component will emit coherent radiation at the beating frequency or wavelength in the downstream radiator. After radiation emission in the radiator, the electron beam returns to the modulator, which provides energy modulation or longitudinal focusing to maintain the longitudinal bunch length around the storage ring.

the length of the first modulator undulator be $L_{u1} = N_{u1}\lambda_{u1}$ and the length of the second modulator undulator be $L_{u2} = N_{u2}\lambda_{u2}$. Here, we note that using two undulators with different K_u values is also possible, but in this example, we assume different undulator periods and the same K_u . Based on an analysis similar to that in Sec. II A, the energy deviation imparted to electrons after passing through the two modulator undulators is given by

$$\Delta\mathcal{E} = \int_{L_{u1}} \left. \frac{d\mathcal{E}}{ds} \right|_{\epsilon = -\Delta\lambda_u/\lambda_{u0}} ds + \int_{L_{u2}} \left. \frac{d\mathcal{E}}{ds} \right|_{\epsilon = +\Delta\lambda_u/\lambda_{u0}} ds, \quad (41)$$

with $\epsilon = (\lambda_m - \lambda_L)/\lambda_L$ the fractional difference between λ_m and λ_L .

The chirp strength can be evaluated by $h = k_L \frac{\Delta\mathcal{E}}{\gamma mc^2}$. As will be shown in the subsequent analysis, this chirp strength depends on Δ and simultaneously determines the target radiation frequency of the downstream radiator, written as $h = h(\Delta; f_{\text{rad}})$. When considering the coherent radiation effects (i.e., FEL-like effects) in the modulator cavity, as analyzed in Sec. II B, we have

$$P_{L,n} = P_{L,n}^{\text{ext}} + \delta P^{(n)}, \quad (42)$$

where

$$\begin{aligned} \delta P^{(n)} \approx & \frac{\pi}{\epsilon_0 c} \chi_{\text{JJ}}^2 F F_{\perp}(S) |b|^2 I_p^2 \\ & \times \left[N_{u1} \left(\frac{\sin(\frac{\kappa}{2} - \pi N_{u1} \Delta)}{\frac{\kappa}{2} - \pi N_{u1} \Delta} \right)^2 \right. \\ & \left. + N_{u2} \left(\frac{\sin(\frac{\kappa}{2} + \pi N_{u2} \Delta)}{\frac{\kappa}{2} + \pi N_{u2} \Delta} \right)^2 \right]. \end{aligned} \quad (43)$$

The radiation wavelength at the radiator is determined by the difference between the resonant frequencies of the two distinct $\lambda_{u1,u2}$ undulators at the laser modulator, corresponding to the beating frequency. The beating period forms an envelope bucket (see Appendix B), comprising several mini phase-space buckets. The radiation wavelength at the radiator is given by

$$\begin{aligned} \lambda_{\text{THz}} &= \frac{c}{\Delta f_m} = \frac{2\pi c}{\omega_{m1} - \omega_{m2}} \\ &= \frac{\lambda_{m1}\lambda_{m2}}{\lambda_{m2} - \lambda_{m1}} \approx \frac{\lambda_L^2}{\lambda_L \cdot 2\Delta} = \frac{\lambda_L}{2\Delta}, \end{aligned} \quad (44)$$

where Δ is defined in Eq. (40). Once the target radiation wavelength is specified, assuming λ_L is known, this equation can be used to guide the design of the modulator undulators, leading to

$$\Delta = \frac{\Delta\lambda_u}{\lambda_{u0}} \approx \frac{\lambda_L f_{\text{THz}}}{2c} \approx 1.67 \times 10^{-3} \lambda_L [\mu\text{m}] f_{\text{rad}} [\text{THz}]. \quad (45)$$

Given the desired radiation frequency at the radiator, the difference in undulator periods $\Delta\lambda_u$ for the two back-to-back modulator undulators can be calculated. To clarify, we consider a numerical example with $\lambda_L = 1064$ nm. Assuming the average λ_{u0} corresponds to the resonance wavelength of 1064 nm, this ensures effective interaction between the electron beam and the modulation laser. Referring to Eq. (41), the chirp strength depends on this undulator period difference Δ . For the modulation laser power of 24 kW, as specified in Table II, the chirp strength corresponding to the desired radiation frequencies from 0.5 to 11 THz is shown in Fig. 6. For lower frequencies (longer wavelengths), the required Δ is smaller, indicating a minor difference between the two back-to-back modulator undulators. As expected, two more similar undulators allow the modulation laser to interact more effectively with the microbunch, resulting in a larger chirp strength.

In addition to the energy chirp strength in the modulator, we also need to decide on the undulator period $\lambda_{u,\text{rad}}$ and undulator parameter $K_{u,\text{rad}}$ of the radiator. The choice is not unique but can be selected within a range, depending on the magnetic field of the radiator undulator. To find $\lambda_{u,\text{rad}}$ and $K_{u,\text{rad}}$, we recall that Eq. (45) provides the constraint on Δ for the two modulator undulators based on the desired radiation frequency. To determine $\lambda_{u,\text{rad}}$ and $K_{u,\text{rad}}$, the resonance wavelength of the radiator undulator should satisfy

$$\lambda_{\text{THz}} = \frac{\lambda_{u,\text{rad}}}{2\gamma^2} \left(1 + \frac{K_{u,\text{rad}}^2}{2} \right). \quad (46)$$

In the meanwhile, from Eq. (45) and the resonance condition $\lambda_L = \frac{\lambda_{u0}}{2\gamma^2} (1 + \frac{K_u^2}{2})$, we have

TABLE II. Initial beam, laser modulator, radiator, and storage ring parameters for the second example: THz SSMB.

Name	Value	Unit
Beam energy	250	MeV
Peak bunch current	0.55	A
rms energy spread	2.76×10^{-4}	
rms bunch length	175	μm
rms beam size	100	μm
Laser modulation wavelength	1064	nm
Laser modulation power	24	kW
Rayleigh length	0.33	m
Nominal chirp strength	500	m^{-1}
Modulator undulator period $\lambda_{u1}, \lambda_{u2}$	4.955, 5.045	cm ($\Delta = 0.009$)
Number of modulator undulator periods N_{u1}, N_{u2}	10	
Modulator undulator parameter	4.29	
Total reflectivity of the cavity mirrors	0.99963	
Resonant radiation frequency	5	THz
Radiator undulator period	7.36	cm
Number of radiator undulator periods	40	
Radiator undulator parameter	27.66	
Storage ring circumference	50	m
Storage ring slippage factor	5×10^{-6}	
Synchrotron tune	0.055	
Radiation damping time	10	ms

$$\lambda_{\text{THz}} = \frac{\lambda_L}{2\Delta} = \frac{\lambda_{u0}}{4\gamma^2\Delta} \left(1 + \frac{K_u^2}{2}\right). \quad (47)$$

The above Eqs. (46) and (47) provide the following constraint on $\lambda_{u,\text{rad}}$ and $K_{u,\text{rad}}$

$$\frac{\lambda_{u0}}{2\Delta \cdot \lambda_{u,\text{rad}}} = \frac{1 + \frac{K_{u,\text{rad}}^2}{2}}{1 + \frac{K_u^2}{2}}. \quad (48)$$

Provided that $K_{u,\text{rad}} \approx 0.934 B_{0,\text{rad}}[\text{T}] \lambda_{u,\text{rad}}[\text{cm}]$, we solve Eq. (48) for the $\lambda_{u,\text{rad}}$ in terms of the λ_{u0} and K_u of the laser modulator to obtain

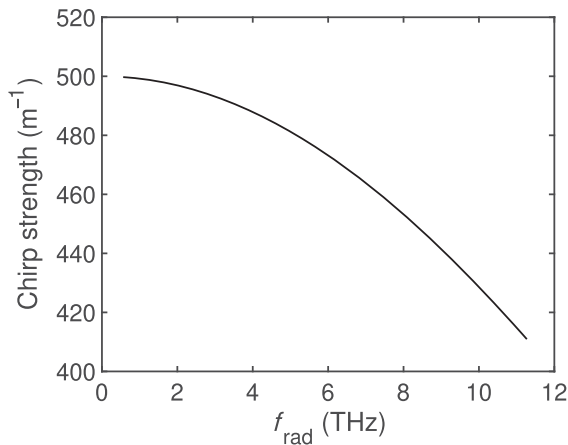


FIG. 6. Dependence of the chirp strength h as a function of the resonant frequency in the radiator.

$$\lambda_{u,\text{rad}} = \frac{4 \cdot 6^{2/3} B_{0,\text{rad}}^2 k^2 \Delta^2 - 6^{1/3} \Upsilon^{2/3}}{6 B_{0,\text{rad}}^2 k^2 \Delta \Upsilon^{1/3}}, \quad (49)$$

with $k = 0.934$ and

$$\begin{aligned} \Upsilon = & -9 B_{0,\text{rad}}^4 k^4 (2 + K_u^2) \Delta^2 \lambda_{u0} \\ & + \sqrt{3} \sqrt{B_{0,\text{rad}}^6 k^6 \Delta^4 (128 \Delta^2 + 27 B_{0,\text{rad}}^2 k^2 (2 + K_u^2)^2 \lambda_{u0}^2)}. \end{aligned} \quad (50)$$

Substituting the value of $\lambda_{u,\text{rad}}$ obtained from Eq. (49) into $K_{u,\text{rad}} \approx 0.934 B_{0,\text{rad}}[\text{T}] \lambda_{u,\text{rad}}[\text{cm}]$, we can determine the radiator undulator parameter. Figure 7 shows the functional dependence of $\lambda_{u,\text{rad}}$ and $K_{u,\text{rad}}$ on the peak undulator magnetic field, assuming that the modulation laser wavelength is 1064 nm. After selecting an appropriate $B_{0,\text{rad}}$, the corresponding set of $\lambda_{u,\text{rad}}$ and $K_{u,\text{rad}}$ gives the required parameters for the radiator undulator.

Now we have completed the setup for the second example. The one-turn matrix for this example is the same as that in the first example [Eq. (34)], except that evaluation of the chirp strength is based on the above Eqs. (41)–(43). Turn-by-turn evolution of the microbunch second moments is governed by Eq. (25). Table II summarizes the initial beam, laser modulator, radiator, and storage ring parameters based on the 1064 nm modulation laser and the laser modulation power of 24 kW, where we have somewhat arbitrarily set the radiator frequency to 5 THz. These parameters are not optimized but only for

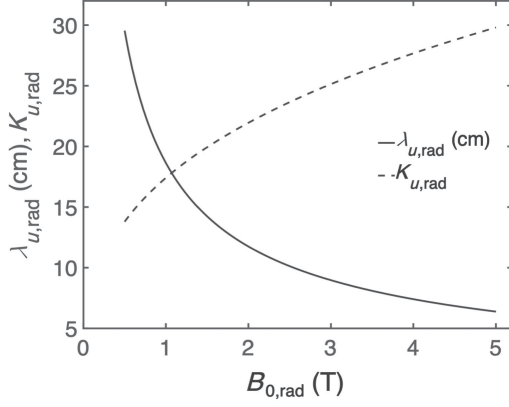


FIG. 7. Dependence of the radiator undulator period $\lambda_{u,\text{rad}}$ and the radiator undulator parameter $K_{u,\text{rad}}$ as a function of the peak undulator magnet field $B_{0,\text{rad}}$ in the radiator. Here, we assume that the target radiation frequency is 5 THz and the laser modulator operates at 1064 nm.

demonstration purposes. The corresponding undulator periods for the two modulator undulators are 4.955 and 5.045 cm, respectively, with the same number of periods $N_{u1} = N_{u2} = 10$. For the radiator, we choose the corresponding radiator parameters as $\lambda_{r,\text{rad}} = 7.36$ cm and $K_{u,\text{rad}} = 27.66(B_{0,\text{rad}} \approx 4 \text{ T})$, with the number of periods $N_{u,\text{rad}} = 40$. The choice of $B_{0,\text{rad}} \approx 4 \text{ T}$ indicates that a superconducting undulator will be required. We will not consider other derivative effects as they are beyond the scope of this paper.

Similar to the previous amplifier-SSMB example, we plot the evolution of microbunch length, energy spread, the modulation laser power, and radiation power at both the modulator and radiator locations as a function of the number of revolution turns, as shown in Figs. 8 (modulator) and 9 (radiator). Using the same legend, the dashed lines represent the case that considers the single-particle effects (the radiation damping and quantum excitation) and cavity loss, but without external laser compensation. The dotted lines, on the other hand, show the case where the external laser injection is added to compensate for cavity loss. The total reflectivity loss is set in the same way as in the first example. Comparing Fig. 8 with Fig. 2, we see that, in the modulator, the microbunches and the modulation laser exhibit very similar characteristics to those in the first example. This is perhaps not surprising, as for this second example, we assumed the same external laser parameters, and the two back-to-back undulators are not significantly different from the single modulator undulator in the first example.

The radiation and beam characteristics at the radiator are also similar to the amplifier case; see Figs. 3 and 9. For the case without external laser injection to compensate for the cavity loss, the radiation power at the radiator quickly drops from 1.68 kW. The power level of 1.68 kW can be calculated using Eq. (39). The external laser injection

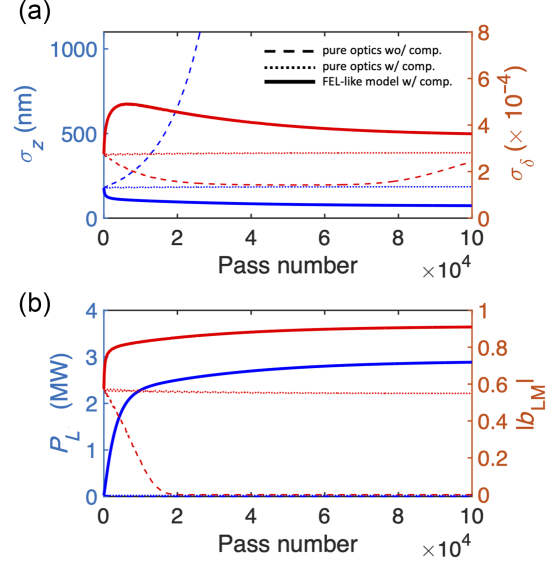


FIG. 8. Turn-by-turn evolution of the microbunch beam parameters σ_z, σ_δ (a), $|b_{\text{LM}}|$ and the laser power P_L (b) in the laser modulator for the case of a frequency-beating SSMB. The radiation damping time is set to approximately 10 ms, corresponding to 6×10^4 turns. In the dual y-axis plot, the blue lines correspond to the left y axis and the red lines correspond to the right y axis. The dashed lines account for radiation damping and quantum excitation effects, without external laser power injection to compensate for the cavity loss. The dotted lines additionally include laser power compensation. The solid lines represent simulations using an FEL-like model for the laser modulator, incorporating the coherent radiation power generated by the electron beam passing through the modulator undulator and the turn-by-turn accumulation of dynamic effects.

compensation is crucial if we wish to maintain the required modulation or longitudinal focusing. If we consider the coherent radiation generated by the electron beam as it passes through the undulator in the modulator and accumulates turn by turn, the modulation enhancement effect discussed in the first example will occur. Referring to Figs. 8(b) and 9(b), it implies that after considering the FEL-like effects, the electron beam in both the modulator and radiator receives better longitudinal focusing, and the radiation output power at the radiator increases up to 3 kW.

C. Harmonic scenario: UV SSMB

In the previous two subsections, we have discussed two scenarios: one where the amplifier SSMB generates coherent radiation at the same wavelength as the modulation laser, and another based on the frequency beating to produce longer wavelengths, such as in the THz regime. From Eqs. (21) and (26), we can see that, given the bunch length in the storage ring, $\sigma_{z0} = \sigma_{\delta 0} \beta_{zS} \propto 1/\sqrt{h}$, a shorter bunch length imposes more stringent requirements on the modulation laser [31]. Specifically, from Eq. (3),

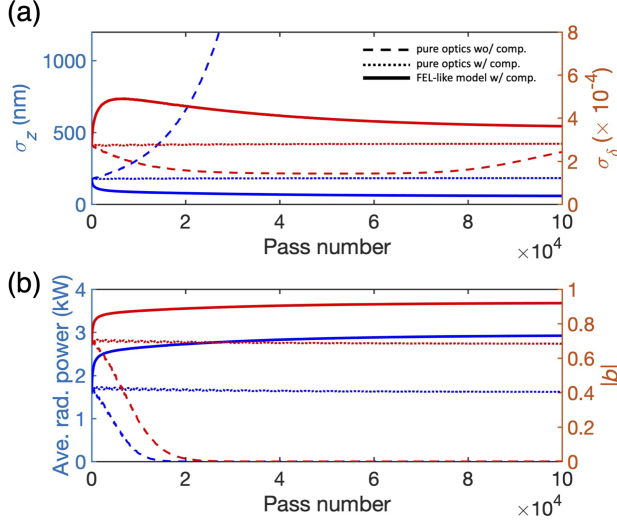


FIG. 9. Turn-by-turn evolution of the microbunch beam parameters σ_z , σ_δ (a), $|b|$ and the average radiation output power (b) in the radiator for the case of a frequency-beating SSMB. The simulation parameters and line conventions are maintained as described in Fig. 8 (see also Table II).

$\sigma_{z0} \propto P_L^{-1/4}$, meaning that to produce shorter-wavelength coherent radiation, a shorter microbunch length is required, which in turn necessitates a significant increase in modulation laser power. For the 1064 nm modulation laser wavelength, our first amplifier-SSMB example required 24 kW. To produce 200 nm UV coherent radiation, which is about one-fifth of the wavelength of 1064 nm, the required modulation laser power would increase by a factor of $5^4 = 625$, resulting in approximately 20 MW. This level of high-average-power laser is already far beyond the current state-of-the-art laser technology. Therefore, to alleviate the

requirements for the modulation laser, in the third example, we demonstrate a harmonic SSMB scenario based on the transverse-longitudinal coupling (TLC) scheme [12,32]. Here, we do not intend to introduce the details of this TLC scheme. For further discussions on this approach, we refer the readers to Refs. [12,13].

The TLC SSMB scheme is illustrated in Fig. 10, also called the generalized longitudinal strong focusing scheme. It consists of three laser modulators, one of which is installed in the main ring to provide weak longitudinal focusing in the storage ring. The remaining two laser modulators, with the radiator placed in between, play a crucial role in the TLC scheme. The top row in Fig. 10 shows the two laser modulators surrounding the radiator, including the vertical bending magnets entering the first modulator and leaving the second modulator, along with two transport lines resembling vertical doglegs. This setup is referred to as the insertion section, which ideally should be transparent with respect to the remaining storage ring, meaning that from the transfer matrix perspective, the insertion section behaves as an identity matrix. The concept of the TLC scheme is to utilize the vertical bending magnets to introduce the longitudinal-vertical (y, y', z, δ) coupling, thereby coupling the small emittance in the y direction to the longitudinal z direction. Since the vertical geometric emittance ϵ_y^G is very small, achieving sufficient modulation is relatively easier, similar to the angular dispersion induced microbunching scheme recently proposed in the FEL community [33].

To analyze this situation, we perform an analysis in the 4D phase space $[y, y', z, \delta]^T$. For simplicity, we assume that outside the insertion section, the remaining storage ring is a (x, s) -planar storage ring and neglect the xy coupling. Since this study primarily concerns the impact of the laser

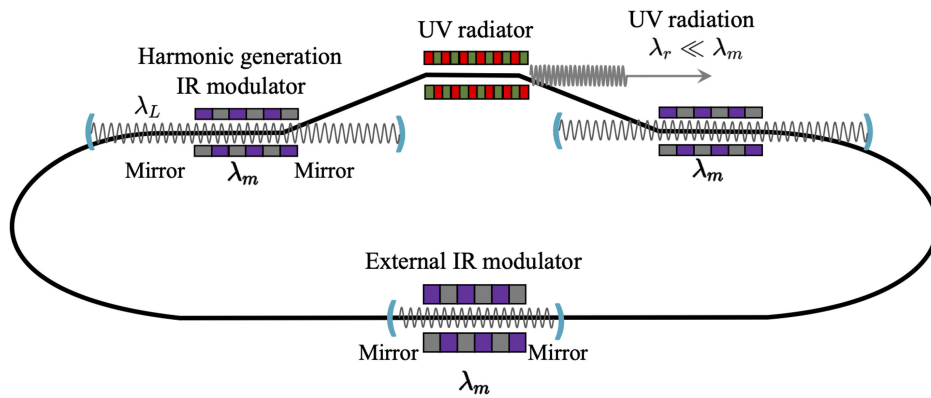


FIG. 10. Schematic layout of the harmonic SSMB based on the transverse-longitudinal coupling (TLC) scheme. In the figure, two harmonic-generation IR modulators sandwiching the UV radiator are used to implement energy modulation or longitudinal compression specifically to the TLC scheme. This transport line segment is also called the insertion section. Bending in the insertion section, including entering the first laser modulator and leaving the second laser modulator and the two dogleglike structures, is in the transverse vertical y dimension. Another modulator, designated as the external IR modulator, provides longitudinal focusing of the electron beam to maintain the bunch length in the storage ring. The ideal TLC scheme design assumes that the insertion section represents an identity matrix, making it transparent to the storage ring.

modulators, we concentrate on the vertical-longitudinal 4D dynamics. Before the electron beam enters the first modulator (or the upstream modulator), it passes through a vertical bending magnet, whose transfer matrix can be written as

$$\mathbf{M}_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & R_{46} \\ -R_{46} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (51)$$

where R_{46} will be constrained by the TLC scheme, and the corresponding value will be provided below. Here, we comment that the upper left 2×2 block matrix for y and y' does not necessarily have to be an identity matrix; we desire the overall transfer matrix of the insertion section to be an identity matrix. Considering the symmetry, when the electron beam exits the second modulator (or the downstream modulator), the transfer matrix of the vertical bending magnet must satisfy

$$\mathbf{M}_B^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -R_{46} \\ R_{46} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (52)$$

To further couple the transverse vertical dimension to the longitudinal dimension, after the electron beam exits the first modulator and before entering the radiator, it will pass through a dogleg with the corresponding transfer matrix

$$\mathbf{M}_{DL} = \begin{pmatrix} 1 & 0 & 0 & R_{54} \\ 0 & 1 & 0 & 0 \\ 0 & R_{54} & 1 & R_{56} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (53)$$

where the R_{54} and R_{56} will be constrained by the TLC scheme, and their values will be provided below. Furthermore, on leaving the radiator, before entering the second modulator, there is a symmetric inverse dogleg, whose transfer matrix is

$$\mathbf{M}_{DL}^{-1} = \begin{pmatrix} 1 & 0 & 0 & -R_{54} \\ 0 & 1 & 0 & 0 \\ 0 & -R_{54} & 1 & -R_{56} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (54)$$

For the second or the downstream modulator, the inverse of the laser modulator transfer matrix can be derived from Eq. (7). For simplicity, we only write the longitudinal part here

$$\mathbf{M}_{LM}^{-1} = \begin{pmatrix} \cos(\sqrt{-R_{56,LM}h}) & -\frac{R_{56,LM} \sin(\sqrt{-R_{56,LM}h})}{\sqrt{-R_{56,LM}h}} \\ -\frac{h \sin(\sqrt{-R_{56,LM}h})}{\sqrt{-R_{56,LM}h}} & \cos(\sqrt{-R_{56,LM}h}) \end{pmatrix}. \quad (55)$$

Clearly, the whole insertion section $(\mathbf{M}_B^{-1} \mathbf{M}_{LM}^{-1} \mathbf{M}_{DL}^{-1}) \times (\mathbf{M}_{DL} \mathbf{M}_{LM} \mathbf{M}_B) = \mathcal{I}$. Now we shall summarize the constraints based on the TLC scheme. According to the analysis in Refs. [12,13], the dispersion invariant for the two modulators can be approximately written as

$$\mathcal{H}_y(\text{Mod}) \approx \frac{\sigma_z^2(\text{Mod})}{\epsilon_y^G}, \quad (56)$$

where ϵ_y^G is the geometric emittance in the y direction, $\sigma_z(\text{Mod})$ is the bunch length at the entrance of the first modulator. For the TLC scheme-based SSMB insertion section design, the required R_{46} for the vertical bending magnet is [12,32]

$$R_{46} \approx \sqrt{\frac{\mathcal{H}_y(\text{Mod})}{\beta_y}}, \quad (57)$$

where β_y is the vertical Courant-Snyder function at the entrance of the first modulator and is treated as a given parameter. Additionally, the TLC scheme also requires that R_{56} and R_{54} in the dogleg be determined by the following constraints [12,32]

$$R_{56} = -\frac{1}{h}, \quad R_{54} = -\frac{R_{56}}{R_{46}}. \quad (58)$$

Now, we have completed all the preparatory work for the third example. To formulate the one-turn matrix, we start from the middle of the storage ring, i.e., the external IR laser modulator. We do not explicitly model the external IR modulator but assume that the longitudinal focusing is hidden in β_{zs} [see Eq. (21)]. The transfer matrix for the first half of the insertion section is $\mathcal{T}_{1/2}^{(1)} = \mathbf{M}_{DL} \mathbf{M}_{LM} \mathbf{M}_B$. The second half of the insertion section is the reverse order of the first, i.e., $\mathcal{T}_{1/2}^{(2)} = [\mathcal{T}_{1/2}^{(1)}]^{-1}$, such that $\mathcal{T}_{1/2}^{(2)} \mathcal{T}_{1/2}^{(1)} = \mathcal{I}$. This identity exactly holds for the pure optics case, i.e., neglecting radiation damping, quantum excitation, FEL-like effects, and power loss in cavity mirrors. When including laser modulator dynamic effects, the one-turn matrix can be written as

$$\mathcal{O}_n = \mathbf{R}_{1/2} \mathcal{T}_{1/2,n}^{(2)} \mathcal{T}_{1/2,n}^{(1)} \mathbf{R}_{1/2}. \quad (59)$$

Here, the subscript n is added in the one-turn matrix and the transfer matrix of the insertion section.

TABLE III. Initial beam, laser modulator, radiator, and storage ring parameters for the third example: UV harmonic SSMB.

Name	Value	Unit
Beam energy	360	MeV
Peak bunch current	1	A
rms energy spread	3.1×10^{-4}	
rms bunch length in the storage ring	500	μm
rms beam size	176	μm
Laser modulation wavelength	1064	nm
Laser modulation power	0.33	MW
Rayleigh length	0.65	m
Nominal chirp strength	1000	m^{-1}
Modulator undulator period	15	cm
Modulator undulator length	1.95	m
Modulator undulator parameter	3.35	
Vertical geometric emittance	10	pm
Vertical betatron function	10	m
Vertical betatron tune	0.63	
Vertical dispersion invariant	25	mm
Total reflectivity of the cavity mirrors	0.99963	
Resonant radiation wavelength	200	nm
Radiator undulator period	3	cm
Radiator undulator length	1.5	m
Radiator undulator parameter	3.35	
Storage ring circumference	50	m
Storage ring slippage factor	1×10^{-6}	
Synchrotron tune	0.005	
Radiation damping time	10	ms
Longitudinal betatron function	1.6	mm

Turn-by-turn tracking equation is based on Eq. (25), given the above one-turn matrix. Table III provides the preliminary design parameters for the case of a 1064 nm modulation laser wavelength, with a laser modulation power of 0.33 MW, targeting at 200 nm UV coherent radiation [34]. From Table III, the $R_{56,LM}$ of the laser modulator is 26 μm , $\mathcal{H}_y(\text{Mod}) \approx 0.025$ m, and the R_{46} of the vertical bending magnet is 0.05. To achieve the microbunch length 20 nm at the radiator, the required chirp strength in the laser modulator will be 1000 m^{-1} and the corresponding R_{56} and R_{54} in the dogleg, according to Eq. (58), are -1 mm and 0.02 m, respectively. In what follows, we discuss the feasibility of such a harmonic SSMB based on the TLC scheme, considering the aforementioned dynamical effects, using the theoretical model developed in this paper.

Figures 11 and 12 show the microbunch length, energy spread, the modulation laser power, and the output radiation characteristics at the first modulator and at the radiator, respectively, for three different setups of the physical effects. When considering the single-particle effects (the radiation damping and quantum excitation) and cavity loss without external compensation, it can be seen that the laser in the modulator quickly decays to zero, and naturally, no stable 200 nm coherent radiation is

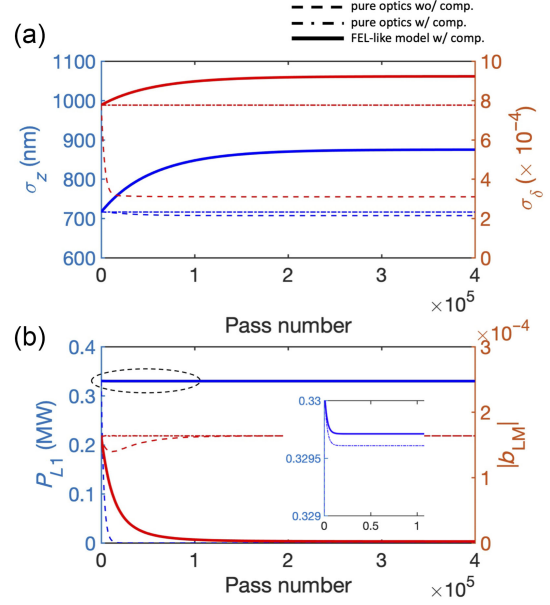


FIG. 11. Turn-by-turn evolution of the microbunch beam parameters σ_z, σ_δ (a), $|b_{LM}|$ and the laser power P_L (b) in the laser modulator for the case of a harmonic SSMB. The radiation damping time is set to approximately 10 ms, corresponding to 6×10^4 turns. In the dual y axis plot, the blue lines correspond to the left y axis and the red lines correspond to the right y axis. The dashed lines account for radiation damping and quantum excitation effects, without external laser power injection to compensate for the cavity loss. The dashed-dotted lines additionally include laser power compensation. The solid lines represent simulations using an FEL-like model for the laser modulator, incorporating the coherent radiation power generated by the electron beam passing through the modulator undulator and the turn-by-turn accumulation of dynamic effects. In the bottom plot, the blue dashed-dotted line almost overlaps with the blue solid line (see the inset).

produced from the radiator. It is important to note that the bunch length does not stretch indefinitely, as this is prevented by the laser modulator outside the insertion section (in the remaining storage ring) to maintain weak longitudinal focusing. When the external laser injection is used to compensate for cavity loss, the scheme works as expected. According to our numerical settings, the total cavity reflectivity is $R \approx 0.99963$, with a corresponding power loss per turn $0.33 \text{ MW} \cdot (1 - R) \approx 122 \text{ W}$ per cavity. When setting the compensation laser power to 122 W, we observe that the dynamical system maintains the equilibrium, as expected. Interestingly, when considering FEL-like effects in the laser modulator cavity, we see at the first glance that the results differ from the previous two examples, as shown by the solid lines in Figs. 11 and 12. The fact that the modulation laser power does not increase due to FEL-like effects is not surprising, because the bunching factors in the laser modulators are small [see the right panel of Fig. 11(b)]. Therefore, one might not expect that the bunch shortening happens. Moreover, the

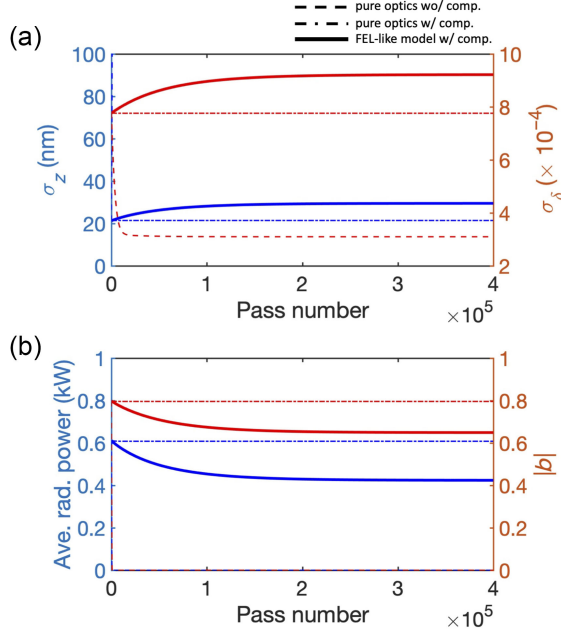


FIG. 12. Turn-by-turn evolution of the microbunch beam parameters σ_z , σ_δ (a), $|b|$ and the average radiation output power (b) in the radiator for the case of a harmonic SSMB. The simulation parameters and line conventions are maintained as described in Fig. 11 (see also Table III).

microbunch begins to lengthen and the energy spread increases, because of the incoherent energy spread from quantum diffusion in the modulator undulator. This then leads to microbunch lengthening or bunching factor reduction in the radiator and a slight decrease of the radiation power from 0.6 to 0.4 kW.

The harmonic SSMB scenario based on the TLC scheme has relatively stringent requirements for the emittance in the y direction within the modulator, compared with requirements for other physical quantities. Figure 13 shows the vertical beam size and vertical emittance of the bunch at the first modulator under different physical conditions. It can be seen that when proper external laser injection compensation is applied, the vertical beam size and emittance of the bunch are maintained at the design values. When there is no compensation for cavity loss, the vertical beam size initially experiences a mismatch due to differences in modulation strength. However, when radiation damping is considered, the mismatch can be recovered to the design value after a few turns. When FEL-like effects are taken into account, stronger mismatches occur due to the increasing modulation strength, combined with quantum diffusion occurring in the undulator [see Eq. (33)], which ultimately results in an increase in the vertical beam size and emittance. From Fig. 12, we believe that the increase in the vertical beam size and emittance of the bunch remains within an acceptable range and does not significantly affect the expected design values.

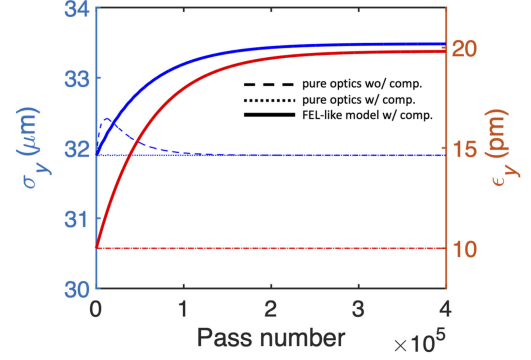


FIG. 13. Turn-by-turn evolution of the vertical size of the vertical emittance of the microbunch beam in the first laser modulator. In the dual y axis plot, the blue lines correspond to the left y axis and the red lines correspond to the right y axis. The dashed lines account for radiation damping and quantum excitation effects without external laser power injection to compensate the cavity loss. The dotted lines additionally include laser power compensation. The solid lines represent simulations using an FEL-like model for the laser modulator. The simulation parameters are the same as those used in Fig. 11. Note that the red dashed line and the red dotted line almost overlay each other.

IV. SUMMARY

In this paper, we have developed a simple model for analyzing electron beam dynamics and radiation characteristics in a laser modulator cavity, incorporating effects of the finite losses due to cavity mirrors, of the external laser power injection for energy dissipation compensation, and of the turn-by-turn accumulation of dynamical effects from coherent undulator radiation generated by the traversing electron beams. Our method combines the transfer matrix approach with the low-gain FEL oscillator model. As applications of our developed model, we discuss the feasibility of three proposed scenarios: the amplifier IR SSMB, the frequency-beating THz SSMB, and the harmonic UV SSMB. The analysis indicates that, under current preliminary design parameters and in the presence of the aforementioned dynamical effects, all three scenarios remain feasible. Compared to the wake-impedance macro-particle model [15–17], the method developed in this paper offers the advantage of quick estimation of the beam collective effects, thereby enabling efficient assessment of the preliminary design parameters.

ACKNOWLEDGMENTS

The author C.-Y. T. is supported by the Fundamental Research Funds for the Central Universities (HUST) under Project No. 2021GCRC006 and National Natural Science Foundation of China under Project No. 12275094. He also acknowledges P. D. Tsai for help in the preparation of this manuscript. The author X. J. D. is supported by the National

Natural Science Foundation of China (NSFC Grants No. 12035010 and No. 12342501), the National Key Research and Development Program of China (Grant No. 2022YFA1603401) and the Tsinghua University Dushi Program.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available because no suitable repository exists for hosting data in this field of study. The data are available from the authors upon reasonable request.

APPENDIX A: ENERGY CHIRP STRENGTH INTRODUCED BY THE LASER MODULATOR

Here, we derive the energy chirp strength induced by the laser modulator. To begin with, along the undulator axis s , we assume that the transverse electric field of the modulation laser can be expressed as $E_x = \text{Re}(\tilde{E}_x e^{-i\omega t})$, where, by solving the paraxial wave equation [35,36], we can obtain the approximate expression of the transverse electric field $\tilde{E}_x$

$$\tilde{E}_x \approx \frac{E_0}{1 + i \frac{s}{z_R}} e^{iks}, \quad (\text{A1})$$

where E_0 is the transverse electric field amplitude, the Rayleigh length $z_R = \pi w_0^2 / \lambda_L$, w_0 is the laser beam waist, and λ_L is the central wavelength of the modulation laser. The central frequency of the laser is $\omega_L = k_L c = 2\pi c / \lambda_L$. In the above expression, the wave front curvature term is neglected, assuming that the laser beam waist is relatively larger than the electron beam radius. In other words, the laser field experienced by the electron is roughly uniform along the undulator. The normalized transverse velocity of the electron in the modulator undulator can be written as

$$\beta_x = \frac{v_x}{c} = -\frac{K_u}{\gamma} \sin k_u s \quad (\text{A2})$$

and the normalized longitudinal velocity after averaging over the wiggling motion is

$$\bar{\beta}_z = \frac{\bar{v}_z}{c} = 1 - \frac{1}{2\gamma^2} \left(1 + \frac{K_u^2}{2} \right), \quad (\text{A3})$$

where γ is the Lorentz relativistic factor of the electron energy, $k_u = 2\pi / \lambda_u$ with λ_u being the undulator period. Here, the undulator parameter $K_u \approx 0.934 B_0 [\text{T}] \lambda_u [\text{cm}]$, with B_0 the peak magnetic field. Here we assume a planar undulator. Integrating Eq. (A3) over the undulator coordinate s , we obtain

$$s = s_0 + \bar{\beta}_z c t + \frac{K_u^2}{8\gamma^2 k_u} \sin 2k_u \bar{\beta}_z c t. \quad (\text{A4})$$

Here, we assume that the undulator center is located at $s_0 = 0$. As can be observed, the electron's longitudinal motion exhibits a longitudinal modulation due to its wiggling motion, known as the figure-8 motion [29]. Such a characteristic figure-8 motion will be reflected in the following [JJ] factor (to be defined below). The energy change per unit length experienced by a traversing electron after passing through the modulator undulator due to the laser interaction can be described by the Lorentz force equation. After some mathematical derivation [29,36], we obtain

$$\begin{aligned} \frac{d\mathcal{E}}{ds} &\approx (-e)\beta_x E_x = \frac{ecK_u E_0}{\gamma} \sin(k_u s) \text{Re} \left(\frac{e^{iks-i\omega t}}{1 + i \frac{s}{z_R}} \right) \\ &\approx \frac{eK_u [\text{JJ}]}{2\gamma} E_0 \text{Re} \left(\frac{e^{iks_0 + iek_u s}}{1 + i \frac{s}{z_R}} \right), \end{aligned} \quad (\text{A5})$$

where

$$[\text{JJ}] = J_0 \left(\frac{K_u^2}{4 + 2K_u^2} \right) - J_1 \left(\frac{K_u^2}{4 + 2K_u^2} \right), \quad (\text{A6})$$

with J_0, J_1 the zeroth and first order Bessel function of the first kind. Here $[\text{JJ}] < 1$ characterizes the coupling reduction when electrons interact with the laser field in a planar undulator. Since the laser wavelength λ_L may not exactly match the resonant wavelength of the modulator undulator λ_m , considering their possible difference, i.e., detuning, we can express the undulator resonant wavelength as

$$\lambda_m = \lambda_L (1 + \epsilon) = \frac{\lambda_u}{2\gamma^2} \left(1 + \frac{K_u^2}{2} \right) (1 + \epsilon), \quad (\text{A7})$$

where $\epsilon = (\lambda_m - \lambda_L) / \lambda_L = (\omega_L - \omega) / \omega_L$ is the detuning parameter. For convenience in subsequent discussions, we define the normalized detuning parameter

$$\kappa = 2\pi N_u \epsilon. \quad (\text{A8})$$

The energy change received by an electron after passing through the modulator undulator, by integrating along the undulator axis, is

$$\Delta\mathcal{E} = \int_{-L_u/2}^{L_u/2} \frac{d\mathcal{E}}{ds} ds. \quad (\text{A9})$$

Then around the zero-crossing phase, we have the corresponding chirp strength defined as

$$h = k_L \frac{\Delta\mathcal{E}}{\gamma m c^2}, \quad (\text{A10})$$

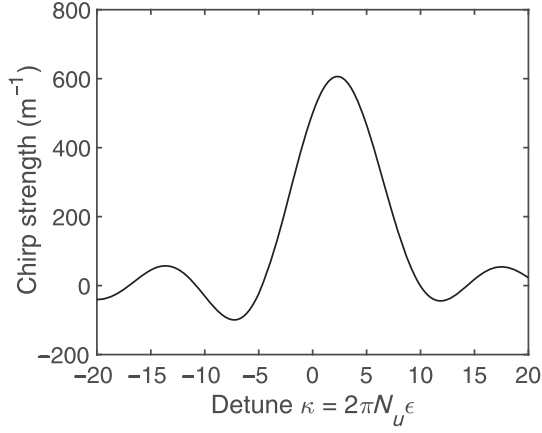


FIG. 14. Dependence of the chirp strength h as a function of the detuning parameter $\kappa = 2\pi N_u \epsilon$. Here, we assumed a beam energy of 250 MeV, an undulator length of $L_u = 1$ m ($N_u = 20$, $\lambda_u = 5$ cm), an undulator parameter $K_u \approx 4.29$, a modulation laser power of 24 kW, a central wavelength of $\lambda_L = 1064$ nm, and a Rayleigh length of $z_R = 0.33$ m (see Table I).

with m the electron rest mass and c the speed of light in vacuum.

It may be appropriate to substitute some numerical values here to obtain more physical insights. The calculated results are from the relevant parameters in Table I (the first example in Sec. III). Figure 14 shows the dependence of the chirp strength on the detuning parameter. For the case with zero detuning, the chirp strength is about 500 m^{-1} , corresponding to the nominal case in the first example. Furthermore, we observe a nonsymmetric chirp strength with respect to the detuning, similar to the low-gain FEL detuning curve. However, this is slightly different from the typical low-gain FEL curve (Madey theorem [37]), because we have only accounted for the zero-phase region, where the microbunch locates near the phase-space bucket center, instead of a coasting beam occupying the phase from 0 to 2π for the conventional FEL case. In our simple model, we lump the effects of the modulation laser field and the undulator radiation field on the electron beam into this chirp strength parameter, which will enter the transfer matrix of the laser modulator.

For the zero detuning case, the energy change can be expressed analytically with

$$\Delta\mathcal{E} = \frac{eK_u[\text{JJ}]}{\gamma} E_0 z_R \tan^{-1}\left(\frac{L_u}{2z_R}\right) \cos ks_0 \quad (\text{A11})$$

and the corresponding chirp strength

$$h = \frac{eK_u[\text{JJ}]}{\gamma^2 mc^2} \sqrt{\frac{4P_L Z_0 z_R}{\lambda_L}} \tan^{-1}\left(\frac{L_u}{2z_R}\right) k_L, \quad (\text{A12})$$

where the laser power can be approximately expressed as

$$P_L \approx \frac{\pi w_0^2}{4Z_0} E_0^2. \quad (\text{A13})$$

APPENDIX B: BEAT-WAVE ENVELOPE BUCKET FROM TWO MODULATOR UNDULATORS

In Sec. III B, we mentioned that two back-to-back modulator undulators with slightly different undulator periods generate radiation wavelengths slightly different from the modulation laser wavelength. The resulting beating component forms a beat-wave envelope bucket. To illustrate this, we write down the equations of motion for electrons in the two-frequency undulator [38]

$$\begin{aligned} \frac{d\psi_1}{ds} &= 2k_{u1}\delta \\ \frac{d\psi_2}{ds} &= 2k_{u2}\delta \approx 2(1 - 2\Delta)\delta, \end{aligned} \quad (\text{B1})$$

where the undulator wave numbers $k_{u1,u2} = 2\pi/\lambda_{u1,u2}$, δ represents the energy deviation, and $\psi_{1,2}$ are the phase differences of the electron motion relative to the external laser field in the two undulators

$$\psi_{1,2} \approx (k_{u1,u2} + k_L)z - \omega_L t, \quad (\text{B2})$$

the difference of which, $\Delta\psi = \psi_1 - \psi_2$, can be written as

$$\frac{\Delta\psi}{2} \approx \frac{2\pi\Delta\lambda_u}{\lambda_{u0}^2} z = k_{u0} \left(\frac{\Delta\lambda_u}{\lambda_{u0}} \right) z \approx \bar{\psi} \Delta, \quad (\text{B3})$$

where $\Delta = \Delta\lambda_u/\lambda_{u0}$ and $\bar{\psi} = (\psi_1 + \psi_2)/2$ is the average phase. From the above two equations of Eq. (B1), we have the equation for $\Delta\psi$

$$\frac{d\Delta\psi}{ds} = \frac{d\psi_1}{ds} - \frac{d\psi_2}{ds} = 4k_{u0}\Delta \cdot \delta \ll 1, \quad (\text{B4})$$

and the equation for the averaged phase $\bar{\psi}$

$$\frac{d\bar{\psi}}{ds} = \frac{1}{2} \left(\frac{d\psi_1}{ds} + \frac{d\psi_2}{ds} \right) \approx 2k_{u0}\delta, \quad (\text{B5})$$

and the energy equation [38]

$$\begin{aligned} \frac{d\delta}{ds} &= -\frac{1}{2} (\Omega_1^2 \sin \psi_1 + \Omega_2^2 \sin \psi_2) \\ &\approx -\Omega^2 \sin\left(\frac{\psi_1 + \psi_2}{2}\right) \cos\left(\frac{\psi_1 - \psi_2}{2}\right) \\ &\approx -\Omega^2 \sin \bar{\psi} \cos\left(\frac{\Delta\psi}{2}\right) = -\Omega^2 \sin \bar{\psi} \cos(\bar{\psi} \Delta), \end{aligned} \quad (\text{B6})$$

where

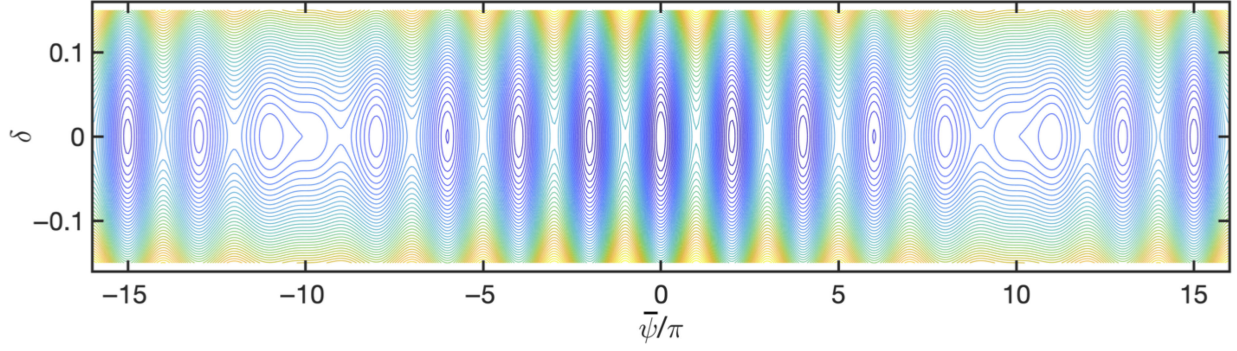


FIG. 15. Illustration of the longitudinal phase-space microbunch buckets in the laser modulators consisting of two back-to-back modulator undulators with slightly different undulator periods. Here, we assume the relative deviation of the undulator periods $\Delta = 0.05$, $\Omega^2 = 0.1$.

$$\Omega^2 = \frac{eK_u[\text{JJ}]E_0}{2\gamma^2 mc^2}. \quad (\text{B7})$$

So far, we have obtained the equations of motion expressed in terms of the averaged phase $\bar{\psi}$ and energy deviation δ , and the corresponding Hamiltonian can be derived

$$\mathcal{H}(\bar{\psi}, \delta; \Delta) = k_{u0}\delta^2 - \Omega^2 \left(\frac{\cos \bar{\psi} \Delta \cdot \cos \bar{\psi} + \Delta \sin \bar{\psi} \Delta \cdot \sin \bar{\psi}}{1 - \Delta^2} \right), \quad (\text{B8})$$

from which we can explore the phase-space structure of the beating period. Figure 15 illustrates the phase-space torus based on Eq. (B8). For illustration, we choose the relative deviation of the two undulator periods $\Delta = 0.05$, and the modulation laser strength $\Omega^2 = 0.1$. It is clearly seen that most of the individual (with a few exceptions on two sides) phase-space buckets have widths determined by the external laser wavelength, and several adjacent buckets are clustered, surrounded by an invisible (large) envelope, forming the beat-wave bucket. Upon further reflection, it becomes evident that the width of the beat-wave bucket corresponds to the target radiation wavelength. More quantitatively, the width of the beat-wave bucket is the beating period $= \pi/\Delta$, within which approximately a number of $\frac{1}{2\Delta} - 2$ microbunches are contained inside the beat-wave bucket. The factor of -2 represents the elimination of two microbunches at the edges of the beat-wave bucket.

In practice, as we demonstrate in Sec. III B, the beat-wave bucket will contain many more (for example, about 56, in our second example) buckets with widths of $\approx \lambda_L$. It may not be easy to accurately calculate the radiation power generated by multiple microbunches within this beat-wave bucket as they pass through the radiator undulator. For a rough estimate, however, we can still use Eq. (39), but the bunching factor needs some modification. First, we make a straightforward assumption, starting from the equilibrium

state of the storage ring without considering the electron-radiation interaction in the laser modulator. In this situation, the electron energy distribution follows a Gaussian distribution, $f(\bar{\psi}, \delta) \propto e^{-\frac{\delta^2}{2\sigma_\delta^2}}$.

Therefore, combining with the Hamiltonian in Eq. (B8), the longitudinal phase-space distribution of the electron beam within the beat-wave bucket can be expressed as

$$f(\bar{\psi}, \delta) = \frac{C_1}{\sqrt{2\pi}\sigma_\delta} e^{-\frac{\mathcal{H}}{2k_{u0}\sigma_\delta^2}}. \quad (\text{B9})$$

By integrating over the energy deviation δ , the corresponding bunch density distribution can be written as

$$\rho(\bar{\psi}) = C_1 \exp \left[\frac{\Omega^2}{2k_{u0}\sigma_\delta^2} \times \left(\frac{\cos \bar{\psi} \Delta \cdot \Delta \cos \bar{\psi} + \Delta \sin \bar{\psi} \Delta \cdot \sin \bar{\psi}}{1 - \Delta^2} \right) \right], \quad (\text{B10})$$

where the coefficient C_1 will be determined from the normalization of the charged density function, which accounts for how many microbuckets are included in the beat-wave bucket, i.e.,

$$\int_{-\frac{2\pi}{4\Delta}}^{\frac{2\pi}{4\Delta}} \rho(\bar{\psi}) d\bar{\psi} = \frac{1}{2\Delta} - 2. \quad (\text{B11})$$

The corresponding bunching factor at radiator (inside the beat-wave bucket) will be

$$b = \int_{-\pi}^{\pi} \rho(\varphi) e^{-i\varphi} d\varphi, \quad (\text{B12})$$

where $\varphi = 2\bar{\psi}\Delta$. Here, through Eq. (B11), given Δ , we first calculate the normalization coefficient C_1 , and then compute the bunching factor using Eq. (B12). Once the bunching factor is obtained, we can substitute it into Eq. (39) to estimate the radiation power at the radiator.

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