

Extracting symplectic maps for space-charge dominated beams

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Symplecticity of transfer maps is important for reliable evaluation of space-charge dominated beams in accelerators. Unfortunately, most simulation codes that include collective effects, such as space charge, do not use canonical phase-space variables and therefore are not symplectic in the presence of electromagnetic fields. In this paper, we present a numerical method to extract local linear symplectic transfer maps using particle tracking simulation code IMPACT-T for space-charge dominated beams. We demonstrate this method for the photoinjector (113 MHz SRF gun) section of the Coherent electron Cooling (CeC) Proof of Principle (POP) experiment.

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I. INTRODUCTION

There is a need for rigorous analytical methods to study beam dynamics in accelerator lattices, which frequently involve solving the 3D Maxwell-Vlasov equation [1]. Symplecticity of the local linearized maps is critically important for finding analytically tractable solution (i.e., linear integral equation) for microscopic instabilities, which could be otherwise missed even by the most sophisticated codes. An example of this is the discovery of plasma cascade instability (PCI), which was first seen experimentally [2], but was not observed in IMPACT-T simulation data. The analytical treatment of PCI [3] subsequently allowed the IMPACT-T model to be updated and to correctly reproduce this phenomenon.

However, most of the commonly used particle codes, which include collective effects (CE), including popular particle-in-cell (PIC) codes such as IMPACT-T [4], WARP [5], ASTRA [6], and PARMELA [7], track mechanical momenta instead of canonical momenta and generate accurate but nonsymplectic transfer maps. Presence of external and/or internal fields, such as space-charge or other beam-induced electromagnetic (EM) fields, results in a difference between canonical ($H/c, \vec{P}$) and mechanical ($E/c, \vec{p}$) four-momenta,

$$H \equiv \mathbf{E} + e\varphi = \sqrt{m^2 c^4 + \left(\vec{P} - \frac{q}{c} \vec{A}\right)^2} + e\varphi$$

$$\vec{P} = \vec{p} + \frac{q}{c} \vec{A}, \quad \mathbf{E} = \gamma m c^2, \quad \vec{p} = \gamma m \mathbf{v}, \quad (1)$$

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where $\gamma = (1 - \vec{\beta}^2)^{-1/2}$ and $\vec{\beta} = \vec{v}/c$ are the usual relativistic factors and $(\varphi, \vec{A})$ is the four-potential of the EM field.

Other codes such as IMPACT-Z [8,9], XSuite [10], BMAD [11], and LEADS-3D [12] offer limited symplectic tracking. Specifically, we find that such codes do not preserve symplecticity for photoinjectors [13]. Additionally, obtaining symplecticity-preserving behavior in tracking codes can be computationally costly (such as for XSuite [10]). In any case, it is beyond the scope of this paper to benchmark the performance of all existing or proposed [14] symplecticity-preserving tracking codes. However, since most CE particle-tracking codes are nonsymplectic, and symplectic tracking codes usually have limited functionality for start-to-end simulations that include guns and injectors, we propose a general numerical postprocessing method to extract local linear symplectic maps (in the vicinity of a chosen phase-space trajectory) from any nonsymplectic particle tracking codes.

In Sec. II, we describe how to perform the transformation from mechanical to canonical phase-space coordinates in the presence of external EM fields and strong space-charge

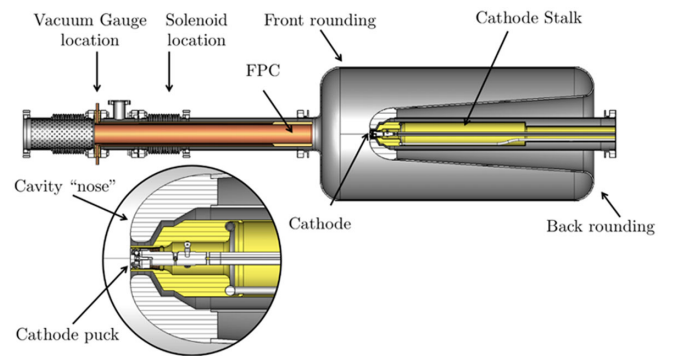


FIG. 1. Cross section of the 113 MHz SRF gun used in the CeC POP experiment [16].

forces. In Sec. III, we discuss the specific procedure required to extract nonsymplectic “mechanical” (or geometric) transfer maps from particle tracking data and subsequently adding EM potentials from rf gun, solenoid, and from space-charge forces to make them symplectic. In Sec. IV, we test this method by extracting symplectic transfer maps in the photoinjector section of the accelerator being used for the Coherent electron Cooling POP experiment at Brookhaven National Laboratory [15] (see Fig. 1). Finally, in Sec. V, we discuss tracking errors associated

with PIC codes and suggest an algorithm to remove these errors and obtain perfectly symplectic maps.

II. METHOD

For codes using time as the independent variable, the extraction of symplectic transfer maps requires knowledge of the geometric transfer map, M_g as well as the vector potential at both initial and final times [17]. Then, we can solve the rhs equation:

$$\{\vec{r}(t), \vec{p}(t)\} = M_g : \{\vec{r}(t_0), \vec{p}(t_0)\} \Leftrightarrow \left\{ \vec{r}(t), \vec{P}(t) - \frac{e}{c} \vec{A}[t, \vec{r}(t)] \right\} = M_g : \left\{ \vec{r}(t_0), \vec{P}(t_0) - \frac{e}{c} \vec{A}[t_0, \vec{r}(t_0)] \right\}. \quad (2)$$

A solution can be found by linearization of the map around the phase-space trajectories. Let us introduce six vectors to denote mechanical (Y) and canonical (X) phase-space coordinates:

$$Y = \begin{bmatrix} \vec{r} \\ \vec{p} \end{bmatrix} \Rightarrow Y(t) = \mathbf{M}_g : Y(t_0); \quad X = \begin{bmatrix} \vec{r} \\ \vec{P} \end{bmatrix} = Y + \frac{e}{c} \cdot \begin{bmatrix} 0 \\ \vec{A}^T(t, \vec{r}) \end{bmatrix} \Rightarrow X(t) = \mathbf{M}_s : X(t_0), \quad (3)$$

where $\mathbf{M}_s$ is the desirable, but unknown, symplectic transfer map. Let us pick one of the allowed space-phase trajectories (indicated by subscript o):

$$Y_o(t) \equiv \begin{bmatrix} \vec{r}_o(t) \\ \vec{p}_o(t) \end{bmatrix} = \mathbf{M}_g : Y_o(t_0); \quad \Rightarrow X_o(t) = Y_o(t) + \frac{e}{c} \cdot \begin{bmatrix} 0 \\ \vec{A}[t, \vec{r}_o(t)] \end{bmatrix}, \quad (4)$$

and expand map around it,

$$\begin{aligned} Y &= Y_o + \nu; \quad \nu = \begin{bmatrix} \vec{\rho} \\ \vec{\pi} \end{bmatrix}; \quad \nu(t) = \mathbb{M}_g \cdot \nu(t_0) + O(|\nu|^2); \quad \mathbb{M}_g = \left. \frac{\partial(\mathbf{M}_g : Y)}{\partial Y} \right|_{Y=Y_o(t)}; \\ X &= X_o + \xi; \quad \xi = \begin{bmatrix} \vec{\rho} \\ \vec{\Pi} \end{bmatrix}; \quad \xi(t) = \mathbb{M}_s \cdot \xi(t_0) + O(|\xi|^2); \quad \vec{\Pi} = \vec{\pi} + \mathbf{V}(t) \cdot \vec{\rho}; \quad \mathbf{V}(t) = \frac{e}{c} \cdot \left. \frac{\partial[\vec{A}(t, \vec{r})]}{\partial \vec{r}} \right|_{r=r_o(t)}, \end{aligned} \quad (5)$$

where the local 6×6 matrix $\mathbb{M}_g$ is defined by the geometric transfer map M_g and the chosen trajectory. The 3×3 matrix $\mathbf{V}$ is defined by the vector potential, and the symplectic 6×6 matrix, $\mathbb{M}_s$, still needs to be determined. For the rest of the paper, we will also neglect higher order terms identified as $O(\varepsilon^2)$, where ε is the expansion parameter.

We can rewrite Eq. (5) by identifying 3×3 blocks in the 6×6 matrices to arrive at the local symplectic transfer map, $\mathbb{M}_s$, using Alex Dragt's notation [18],

$$\begin{aligned} \mathbb{M}_g &= \begin{bmatrix} \mathbf{A}_g & \mathbf{B}_g \\ \mathbf{C}_g & \mathbf{D}_g \end{bmatrix}; \quad \mathbb{M}_s = \begin{bmatrix} \mathbf{A}_s & \mathbf{B}_s \\ \mathbf{C}_s & \mathbf{D}_s \end{bmatrix}; \quad \xi(t) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{V}(t) & \mathbf{I} \end{bmatrix} \cdot \nu(t); \\ \xi(t) &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{V}(t) & \mathbf{I} \end{bmatrix} \cdot \mathbb{M}_g \cdot \nu(t_0); \quad \nu(t_0) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{V}(t_0) & \mathbf{I} \end{bmatrix} \cdot \xi(t_0); \quad \xi(t) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{V}(t) & \mathbf{I} \end{bmatrix} \cdot \mathbb{M}_g \cdot \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{V}(t_0) & \mathbf{I} \end{bmatrix} \cdot \xi(t_0); \\ \mathbb{M}_s &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{V}(t) & \mathbf{I} \end{bmatrix} \cdot \mathbb{M}_g \cdot \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{V}(t_0) & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_g - \mathbf{B}_g \cdot \mathbf{V}(t_0) & \mathbf{B}_g \\ \mathbf{C}_g - \mathbf{D}_g \cdot \mathbf{V}(t_0) + \mathbf{V}(t) \cdot (\mathbf{A}_g - \mathbf{B}_g \cdot \mathbf{V}(t_0)) & \mathbf{D}_g + \mathbf{B}_g \cdot \mathbf{V}(t) \end{bmatrix}, \end{aligned} \quad (6)$$

where $\mathbf{I}$ and $\mathbf{0}$ are the unit and zero 3×3 block matrices, respectively. This concludes the extraction process of the locally symplectic transfer map, provided that the used tracking code provides full information about the vector potential [19], including that of the self-field (for example, space-charge forces).

III. EXTRACTING $\mathbb{M}_g$ AND ADDING EM FIELDS

Unfortunately, particle-tracking codes typically do not provide complete information about the vector potential, including self-field contributions, so the user must define the necessary components of the EM four-potential manually. In addition, the user needs to be aware of approximations used in the code, such as the type of the field expansions used. Otherwise, there will be additional errors caused by the difference between the potentials used for tracking and that for the extraction of the symplectic map.

It would be impossible to describe all possible cases encountered in various codes and for all accelerator elements in a single paper. Hence, we provide the specific example of how to use the well-benchmarked code, IMPACT-T [4] to extract the symplectic map for a space-charge dominated electron beam generated in an SRF gun and focused by a solenoid. Readers who wish to apply the method to an accelerator with additional elements or self-fields should consult the relevant expressions for the four-potentials.

First, we need to extract the geometric map, $\mathbb{M}_g$ using IMPACT-T tracking data. In order to do so, we track a set of particles in the near proximity of the selected reference particle. In our case, we choose the reference particle to be close to the location of the beam centroid. This is mainly because IMPACT-T outputs the 6D phase-space coordinates for all particles using BPMs at the time step where the beam centroid reaches a specified z position along the beam trajectory. This makes it convenient to calculate the vector potential contributions to each particle using the method described in Sec. III A. We use small deviations in the initial 6D phase-space coordinates of the tracked set of particles relative to the reference particle to ensure the linearity of the map given in Eq. (5). We optimized the size of the set and found that the best accuracy is achieved with a set sized $\sim 4\%$ of the rms values in each of six phase-space directions. This choice is specific to our accelerator as well as the choice of mesh size and time step in IMPACT-T (more details in Sec. V). Results can differ for other accelerators and tracking programs.

To find $\mathbb{M}_g$, let us first introduce the $6 \times n$ matrix, $\mathbf{Y} = [Y_1, Y_2, \dots, Y_n]$ where n is the number of tracked particles. Each six-vector, $Y_k = \begin{bmatrix} \vec{r}_k \\ \vec{p}_k \end{bmatrix}$, with $k = 1, 2, \dots, n$, corresponds to the 6D phase-space coordinates of the tracked particles. For $\mathbf{Y}_i$ and $\mathbf{Y}_f$ defined at the chosen initial and final times of the IMPACT-T simulation, respectively, we get using Eq. (3),

$$Y_{kf} = \mathbb{M}_g \cdot Y_{ki} \Rightarrow \mathbf{Y}_f = \mathbb{M}_g \cdot \mathbf{Y}_i, \quad (7)$$

where the subscripts i and f refer to the initial and final times, respectively. Now, we can proceed using the standard “method of least-squares,” which requires that we minimize the quantity,

$$\Phi = |\mathbf{Y}_f - \mathbb{M}_g \cdot \mathbf{Y}_i|^2. \quad (8)$$

The solution to this is given by

$$\begin{aligned} (\mathbf{Y}_f - \mathbb{M}_g \cdot \mathbf{Y}_i) \cdot \mathbf{Y}_i^T &= 0 \\ \Rightarrow \mathbb{M}_g &= (\mathbf{Y}_f \cdot \mathbf{Y}_i^T) \cdot (\mathbf{Y}_i \cdot \mathbf{Y}_i^T)^{-1}. \end{aligned} \quad (9)$$

Here, it is not necessary that an inverse of $\mathbf{Y}_i \cdot \mathbf{Y}_i^T$ exists. However, since $\mathbf{Y}_i \cdot \mathbf{Y}_i^T$ is a 6×6 matrix, there is a high probability for an inverse to exist if we have $n \gg 6$ particles. We find that even with ~ 10 particles, we are able to obtain a solution for $\mathbb{M}_g$ using this method. In our analysis, we typically use $n \sim 1000$. We would like to note that this linearization method can be used for both symplectic and nonsymplectic tracking codes.

As mentioned previously, IMPACT-T does not provide the necessary data about the EM four-potential, and we need to define it.

A. Vector potential due to external EM fields

Here, we describe the EM four-potential for the example of the 113-MHz superconducting rf (SRF) gun used in the CeC POP experiment [16], shown in Fig. 1, with lattice and machine parameters from an actual CeC POP experiment [20]. The definitions of the EM four-potential can be appropriately adapted to the lattice being studied.

The CeC gun system [16,20,21] comprises two main elements: the 1.25 MV SRF gun and the focusing solenoid. Figure 2 shows profiles of the on-axis accelerating field (E_z) in the gun and the longitudinal magnetic field (B_z) in the solenoid.

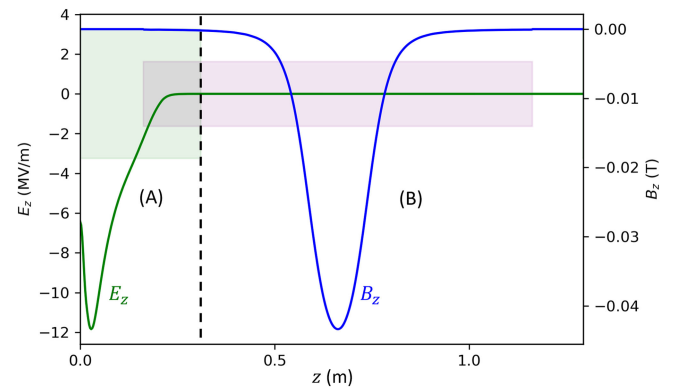


FIG. 2. On-axis accelerating E_z field (green curve) of the 113 MHz SRF gun and the B_z field (blue curve) in the gun solenoid. We break up the gun model in IMPACT-T into two regions (A) and (B). The green shaded rectangle corresponds to the E_z field extending from $z = 0.0$ to 0.3105 m. The pink shaded rectangle corresponds to the B_z field extending from $z = 0.162$ to 1.162 m.

1. SRF gun

The on-axis component of the rf accelerating E_z field in the gun [16,21] is shown in Fig. 2. z corresponds to the distance from the cathode. In order to simulate the rf E_z field in IMPACT-T, we use the Fourier series expansion,

$$E_z(z, t) = E_0 \left[\frac{a_0}{2} + \sum_{n=1} a_n \cos(k_n z) + b_n \sin(k_n z) \right] \times \cos(\omega t + \theta_0), \quad (10)$$

where E_0 is a scaling factor for the peak accelerating voltage of 1.25 MV [16]. a_0 , a_n , and b_n are Fourier coefficients obtained by fitting Eq. (10) to the on-axis rf E_z field data. We use $n = 100$ in the expansion in Eq. (10) for our simulations. θ_0 is the initial phase of the rf field. We also have

$$k_n = \frac{2\pi n}{L_{RF}}, \quad (11)$$

where the length of the rf E_z field is $L_{rf} = 0.3105$, and the angular frequency is $\omega = 2\pi/113$ MHz.

With the analytical expression for the E_z field given in Eq. (10), we are now able to calculate the exact analytical expression for the vector potential, $\vec{A}$ [Eq. (A20)] using Maxwell's equations as shown in Appendix A. Since we are interested in local linear maps, it is sufficient to use a truncated expression for the vector potential [22,23],

$$\begin{aligned} A_x &= \frac{1}{\omega} x \sum_{n=0}^{\infty} \frac{1}{2(n+1)} e'_n(z) r^{2n} \sin(\omega t + \theta), \\ A_y &= \frac{1}{\omega} y \sum_{n=0}^{\infty} \frac{1}{2(n+1)} e'_n(z) r^{2n} \sin(\omega t + \theta), \\ A_z &= -\frac{1}{\omega} \sum_{n=0}^{\infty} e_n(z) r^{2n} \sin(\omega t + \theta), \end{aligned} \quad (12)$$

where $r^2 = x^2 + y^2$ and

$$e_{n+1}(z) = -\frac{1}{4(n+1)^2} \left(e'_n(z) + \frac{\omega^2}{c^2} e_n(z) \right). \quad (13)$$

where $e_0(z) \equiv E_z(z)$ is the on-axis rf E_z field given in Eq. (10) without the time-dependent $\cos(\omega t + \theta_0)$ factor, and we only keep terms up to $n = 1$.

2. Gun solenoid

The $B_z(z)$ solenoid field can be expanded using Fourier series similar to Eq. (10),

$$B_z(z) = B_0 \left(\frac{b_0}{2} + \sum_{n=1} b_n \cos(k_n z) + c_n \sin(k_n z) \right). \quad (14)$$

For our simulations, we keep terms up to the $n = 39$ harmonic of the Fourier series expansion of the B_z field given in Eq. (14). In order to extract the expression for the vector potential associated with the gun solenoid field, we perform an expansion of the on-axis and off-axis contributions of Eq. (14) up to third order in derivatives of $B_z(z)$ with respect to z ,

$$\begin{aligned} B_x(z, r) &= -\frac{B'_z x}{2} + \frac{B'''_z x r^2}{16} + \mathcal{O}(B_z^{(5)}), \\ B_y(z, r) &= -\frac{B'_z y}{2} + \frac{B'''_z y r^2}{16} + \mathcal{O}(B_z^{(5)}), \\ B_z(z, r) &= B_z - \frac{B''_z r^2}{4} + \mathcal{O}(B_z^{(4)}), \end{aligned} \quad (15)$$

where $B_z^{(n)}$ is the n th derivative of B_z with respect to z . We adopt this approximation for the solenoidal $\mathbf{B}$ field to match the expressions for the magnetic field and vector potential used in IMPACT-T [23].

Then, using $\vec{B} = \nabla \times \vec{A}$ and keeping terms only up to second order in derivatives of B_z ,

$$A_x(z, r) = -\frac{B_z y}{2} + \frac{B''_z y r^2}{16}, \quad A_y(z, r) = \frac{B_z x}{2} - \frac{B''_z x r^2}{16}. \quad (16)$$

Since there is no current along the z direction in the gun solenoid, we have $A_z = 0$.

B. Correcting for space-charge effects

In IMPACT-T, the space-charge field is assumed to have the form of an electrostatic field in the comoving frame. In PIC tracking codes such as IMPACT-T, the space-charge forces are updated at each time step using the particles' position on the mesh grid. The particles are then pushed during tracking by adding the field contributions from the external fields (e.g., solenoid field) and the electrostatic field due to space charge. Such an approximation has been benchmarked and shown to perform well for various accelerator lattice models [24,25] including the CeC lattice [20].

For the quasistatic space-charge model in IMPACT-T, the electrostatic potential due to space-charge forces for a localized set of trajectories in a comoving frame can be approximated by

$$\vec{A}_c = 0, \quad \varphi_c(x_c, y_c, z_c) \neq 0, \quad (17)$$

where the subscript c denotes comoving frame. In the lab frame, this field is given by

TABLE I. “Symplecticity errors” ϵ_{sym} for geometric and canonical transfer maps for regions (A), (B), and (C) of the CeC photoinjector (gun) model. The $\epsilon_{\text{sym}}(\mathbb{M}_s)$ for symplectic transfer maps calculated using the symplectification algorithm is zero by construction [26,27].

Gun region	$\epsilon_{\text{sym}}(\mathbb{M}_{\text{ext}})$	$\epsilon_{\text{sym}}(\mathbb{M}_s)$
A	0.11	0.031
B	0.084	0.0018

$$\begin{aligned} x &= x_c, & y &= y_c, & z &= \gamma \cdot (z_c + vt_c) \\ A_{x,y} &= 0, & A_z &= \beta\varphi, \\ \varphi &= \gamma\varphi_c(x, y, z_c) = \gamma\varphi_c[x, y, \gamma \cdot (z - vt)]. \end{aligned} \quad (18)$$

For local linear maps extracted using IMPACT-T tracking data, the error in the space-charge vector potential due to such a “frozen-charge” approximation is expected to be much smaller than numerical PIC tracking errors as long as the space-charge distribution does not change very significantly. The numerical errors associated with the frozen-charge approximation are included in the symplecticity errors associated with the extracted transfer maps (reported in Table I for the CeC photoinjector). In the case that such errors due to frozen-charge approximation are too large, one can extract maps over smaller time steps or include explicit space-charge vector potential data from tracking code.

As indicated in Sec. II, we need expressions for vector potentials, including both external and internal EM fields, to define 3×3 matrix $\mathbf{V}$:

$$\mathbf{V} = \begin{bmatrix} \frac{\partial A_x(\vec{r}, t)}{\partial x} & \frac{\partial A_x(\vec{r}, t)}{\partial y} & \frac{\partial A_x(\vec{r}, t)}{\partial z} \\ \frac{\partial A_y(\vec{r}, t)}{\partial x} & \frac{\partial A_y(\vec{r}, t)}{\partial y} & \frac{\partial A_y(\vec{r}, t)}{\partial z} \\ \frac{\partial A_z(\vec{r}, t)}{\partial x} & \frac{\partial A_z(\vec{r}, t)}{\partial y} & \frac{\partial A_z(\vec{r}, t)}{\partial z} \end{bmatrix}_{\vec{r}=\vec{r}_0(t)} \quad (19)$$

at both initial (t_i) and final (t_f) times. Then, we can transform the extracted geometric matrix $\mathbb{M}_g$ into a symplectic one using Eq. (6),

$$\mathbb{M}_s = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{V}_f & \mathbf{I} \end{bmatrix} \cdot \mathbb{M}_g \cdot \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{V}_i & \mathbf{I} \end{bmatrix}. \quad (20)$$

Linearity of the EM field implies that the matrix $\mathbf{V}$ is a direct sum of the contributions from the external and internal components of the EM field,

$$\vec{A} = \vec{A}_{\text{gun}} + \vec{A}_{\text{sol}} + \vec{A}_{\text{sc}} \Rightarrow \mathbf{V} = \mathbf{V}_{\text{gun}} + \mathbf{V}_{\text{sol}} + \mathbf{V}_{\text{sc}}, \quad (21)$$

which in our case comprises three components: from the gun, the solenoid, and the space charge indicated by subscript sc. In the “eyes” of the tracking code, the EM potentials for the gun are defined only in region A (which includes the initial point of tracking), for the solenoid in a portion of region A and the entire region B (see Fig. 2), while space charge exists everywhere in the system.

As we have seen in Sec. III A, we have specific expressions for the gun and solenoid vector potentials; however, we only know the form of the vector potential due to space charge,

$$\mathbf{V}_{\text{sc}} = \frac{e}{c} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial A_z}{\partial x} & \frac{\partial A_z}{\partial y} & \frac{\partial A_z}{\partial z} \end{bmatrix}. \quad (22)$$

This means that there are three undetermined parameters in each of the initial (V_{isc}) and final (V_{fsc}) vector potentials due to space charge,

$$\mathbf{V}_{\text{fsc}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ m_1 & m_2 & m_3 \end{bmatrix} \quad \mathbf{V}_{\text{isc}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ m_4 & m_5 & m_6 \end{bmatrix} \quad (23)$$

and must be determined. Fortunately, the nature of extraction of the symplectic matrix from the geometric one allows us to apply the transformation given in Eq. (20) piecewise. This is a result of the multiplicativity relation

$$\begin{aligned} \begin{bmatrix} \mathbf{I} & 0 \\ \sum_k \mathbf{V}_k & \mathbf{I} \end{bmatrix} &\equiv \prod_k \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{V}_k & \mathbf{I} \end{bmatrix} \\ \Rightarrow \mathbb{M}_s &= \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{V}_{\text{fsc}} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{V}_{\text{fsol}} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{V}_{\text{fgun}} & \mathbf{I} \end{bmatrix} \cdot \mathbb{M}_g \cdot \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{V}_{\text{igun}} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{V}_{\text{isol}} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{V}_{\text{isc}} & \mathbf{I} \end{bmatrix}. \end{aligned} \quad (24)$$

This allows us to calculate the intermediate matrix ($\mathbb{M}_{\text{ext}}$),

$$\mathbb{M}_{\text{ext}} = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{V}_{\text{fsol}} + \mathbf{V}_{\text{fgun}} & \mathbf{I} \end{bmatrix} \cdot \mathbb{M}_g \cdot \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{V}_{\text{igun}} - \mathbf{V}_{\text{isol}} & \mathbf{I} \end{bmatrix}; \quad \mathbb{M}_s = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{V}_{\text{fsc}} & \mathbf{I} \end{bmatrix} \cdot \mathbb{M}_{\text{ext}} \cdot \begin{bmatrix} \mathbf{I} & 0 \\ -\mathbf{V}_{\text{isc}} & \mathbf{I} \end{bmatrix} \quad (25)$$

and use the symplectic condition,

$$\mathbb{M}_s^T \mathbb{S} \mathbb{M}_s = \mathbb{M}_s, \quad (26)$$

to determine the six unknown coefficients $m_1, \dots, m_6$. Equations (23) and (26) are an overdetermined system of 15 equations in 6 variables. Hence, the six coefficients can be found using the best fit estimate method to determine the final form of the extracted matrix $\mathbb{M}_s$.

IV. EXAMPLE: CEC PHOTOINJECTOR

There are two types of notation for six-component phase-space vectors used in accelerator physics. Conventional notation, frequently used in computer codes, has alternating space and momentum components, for example $X_c = (x, P_x, y, P_y, z, P_z)$ in the IMPACT-T code. The other notation is more convenient for theoretical description (also known as Alex Dragt's notation [18]) and has three space components

followed by three component of momenta. We use this notation in Secs. II and III: $X_d = (x, y, z, P_x, P_y, P_z)$.

Dragt's notation provided us with convenient definitions of the 3×3 block matrices [see Eq. (6)], the 3×3 matrices $\mathbf{V}$, and the connection between the geometric and symplectic matrices [see Eqs. (25)]. The antisymmetric matrix $\mathbf{S}$, which appears in the symplectic condition [Eq. (26)], also has a compact block matrix form,

$$\mathbf{S} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}. \quad (27)$$

Conventional notation would not allow us to use such compact matrix expressions.

Nevertheless, since the majority of the accelerator physics literature uses the conventional notation, we take the extra effort to provide the final results using conventional definitions of transport matrices, connected by the trivial but cumbersome relations of their coefficients,

$$\begin{aligned} X_c = \mathbb{N} \cdot X_d; \quad \mathbb{N} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \quad X_d = \mathbb{N}^{-1} \cdot X_c; \quad \mathbb{N}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \\ \mathbb{M}_c = \mathbb{N} \cdot \mathbb{M}_d \cdot \mathbb{N}; \quad \mathbb{M}_d = \mathbb{N}^{-1} \cdot \mathbb{M}_c \cdot \mathbb{N}^{-1}. \end{aligned} \quad (28)$$

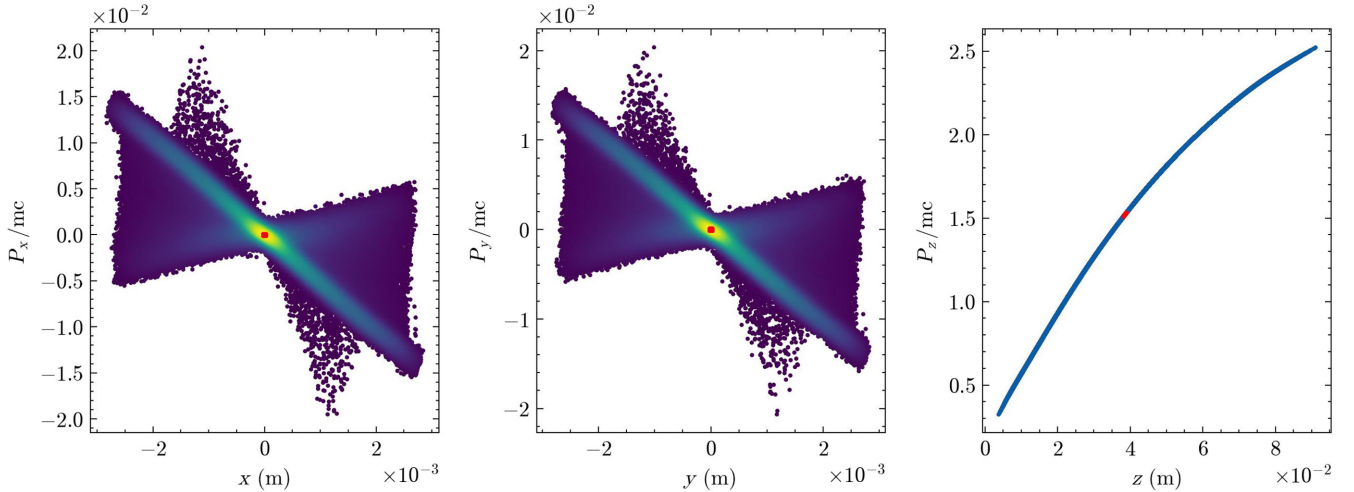


FIG. 3. Poincaré plot at the moment of time when beam centroid has reached $z = 0.04$ m. The color map indicates the density of particles with “yellower” regions indicating higher density and “bluer” regions indicating lower density. The red points correspond to a set of test particles used to extract the transfer map.

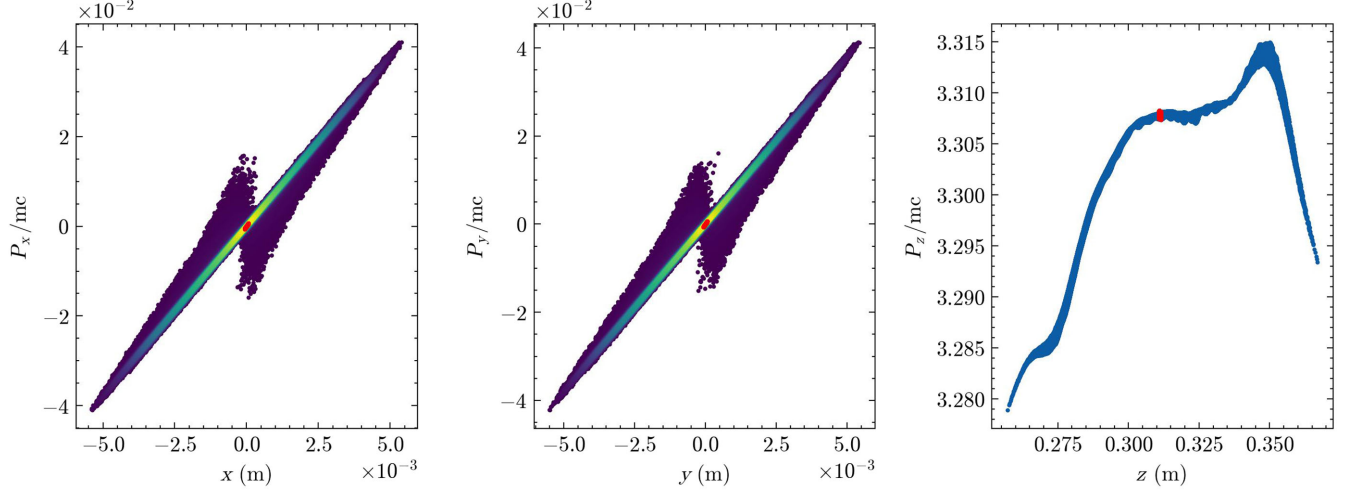


FIG. 4. Poincaré plot at the moment of time when beam centroid has reached $z = 0.31$ m. The beam centroid has not entered the region where the gun solenoid $\mathbf{B}$ field is defined. The red points correspond to the same test particles as in Fig. 3.

Simulation of beam dynamics in the CeC photoinjector is broken into two regions (shown in Fig. 2): (A) when the beam is fully loaded past the cathode (located at $z = 0$ m) to $z = 0.31$ m where the particles see the entire gun $\mathbf{E}$ field and a portion of the solenoid $\mathbf{B}$ field, (B) from $z = 0.31$ to 1.16 m where the particles see only the solenoid $\mathbf{B}$ field.

In region (A), the beam along with an initial set of test particles has been loaded from behind a photocathode (from negative z , photocathode at $z = 0$ m) using IMPACT-T's

cathode emission model [4]. We choose $\mathbf{Y}_i$ in Eq. (7) to be the 6D mechanical phase-space coordinates of the test particles after the entire beam is past the photocathode. Figure 3 shows a snapshot of the 6D phase-space coordinates $\mathbf{Y}_i = (x, p_x, y, p_y, z, p_z)$ of the CeC beam at a time step in the IMPACT-T simulation when the beam centroid has reached $z = 0.04$ m. Similarly, we extract the 6D phase-space coordinates at the final beam centroid position in region (A), $z = 0.31$ m (see Fig. 4).

Using the method described in Sec. III, we arrive at the following value for $\mathbb{M}_{\text{ext}}$,

$$\mathbb{M}_{\text{ext}(A)} = \begin{bmatrix} 1.857 & 0.1213 & 0.002294 & 9.483 \times 10^{-4} & 0.009718 & -4.292 \times 10^{-4} \\ 14.91 & 1.514 & 0.007074 & -0.01677 & 0.1616 & -0.007139 \\ -0.004358 & -1.937 \times 10^{-4} & 1.863 & 0.1215 & 0.002503 & -1.078 \times 10^{-4} \\ -0.04554 & -0.002192 & 15.02 & 1.516 & 0.04260 & -0.001846 \\ 4.180 \times 10^{-4} & 1.859 \times 10^{-5} & 8.918 \times 10^{-7} & 5.171 \times 10^{-5} & 0.8971 & 0.01091 \\ 0.2132 & 0.01076 & -0.01497 & 0.03070 & -19.82 & 0.8811 \end{bmatrix}, \quad (29)$$

where the subscript (A) denotes the region (A) to which the map corresponds.

As expected, because the space-charge vector potential still needs to be added, this matrix is not symplectic,

$$\mathbb{M}_{\text{ext}(A)}^T \mathbf{S} \mathbb{M}_{\text{ext}(A)} - \mathbf{S} = \begin{bmatrix} 0. & 0.002 & -0.02 & -0.018 & -0.044 & -0.009 \\ -0.002 & 0. & -0.001 & -0.001 & 0.005 & 0. \\ 0.002 & 0.001 & 0. & 0.001 & 0.055 & -0.002 \\ 0.018 & 0.001 & -0.001 & 0. & -0.027 & 0. \\ 0.044 & 0.005 & -0.055 & 0.027 & 0. & 0.007 \\ 0.009 & 0. & 0.002 & 0. & -0.007 & 0. \end{bmatrix} \neq 0; \quad \mathbf{S} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}. \quad (30)$$

It is useful to define “symplecticity error,” $\epsilon_{\text{sym}}(\mathbb{M})$ of a matrix $\mathbb{M}$ [28,29],

$$\epsilon_{\text{sym}}(\mathbb{M}) = \|\mathbb{M}^T \mathbf{S} \mathbb{M} - \mathbf{S}\|, \quad (31)$$

where $\|\mathbf{A}\| = \sqrt{\sum_{ij} |a_{ij}|^2}$ is the Frobenius norm of the matrix $\mathbf{A}$. For $\mathbb{M}_{\text{ext}(A)}$, Eq. (31) gives a symplecticity error of $\epsilon_{\text{sym}}(\mathbb{M}_{\text{ext}(A)}) = 0.11$ (also listed in Table I).

Next, it is important to account for the space-charge contribution to the vector potential. To do so, we first extract the best-fit coefficients m_i and then find the canonical transfer map $\mathbb{M}_{s(A)}$ [given in Eq. (B1)]. This improves the symplecticity error by an order of magnitude to $\epsilon_{\text{sym}}(\mathbb{M}_{s(A)}) = 3.1 \times 10^{-2}$.

Similarly, in region (B) with the focusing solenoid $\mathbf{B}$ field, we calculate symplectic transfer maps, $\mathbb{M}_{s(B)}$ [given in Eq. (B3)]. The “symplecticity errors” ϵ_{sym} for the geometric and canonical transfer maps are summarized in Table I.

The $\epsilon_{\text{sym}}(\mathbb{M}_s)$ values in Table I are $\mathcal{O}(10^{-2})$ after the vector potential contributions from external fields (accelerating SRF gun $\mathbf{E}$ field and focusing gun solenoid $\mathbf{B}$ field) and the strong space-charge forces in the high-charge (1.5 nC) CeC beam are included.

V. DISCUSSION

The extracted canonical transfer map, $\mathbb{M}_s$, inherits numerical errors from tracking. These errors (summarized in Table I) are $\mathcal{O}(10^{-2})$ after the algorithm explained in Sec. II is applied to the extracted mechanical maps from IMPACT-T. While this restores symplecticity significantly, the remaining symplecticity errors are still undesirable. These errors are a numerical effect arising due to the nonsymplectic preserving behavior of PIC codes.

We find that this remaining error is sensitive to the size of the distribution of test particles used to extract the matrix as shown in Fig. 5. We choose 4% variation for our calculations in Sec. IV since the numerical accuracy of particle tracking codes in capturing evolution of particles with very small variations in 6D coordinates is limited by mesh size, time step chosen for simulations. Using very small variations ($\sim 1\%$) yields only a minor improvement in ϵ_{sym} while significantly increasing the computational cost. On the other hand, too large of an initial variation in the 6D phase-space coordinates of the test particles leads to accumulation of PIC tracking errors, which in turn also results in larger symplecticity errors, as clearly seen in Fig. 5.

Additionally, we find that if we start with too small of an initial variation in the 6D coordinates of the test particles,

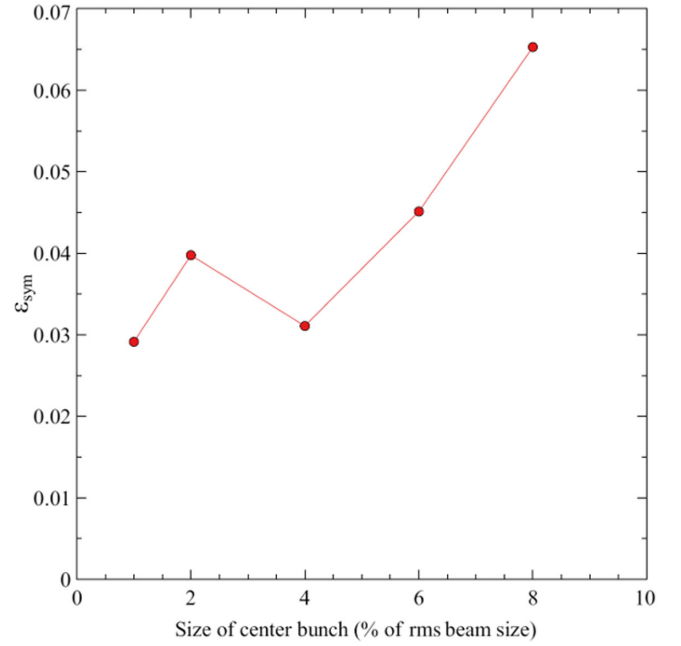


FIG. 5. Symplecticity error, ϵ_{sym} , for different sizes of variation in 6D coordinates of test particles used to extract transfer maps.

we observe “clumping” of particles. This is easily visible for variations which are $\lesssim \mathcal{O}(10^{-4}\%)$ of rms beam size as shown in Fig. 6. We see that the initially defined set of tracked test particles has no such clumping [Fig. 6(a)] and clumping occurs once the beam is loaded past the cathode. We also find a corresponding increase in symplecticity errors in such cases.

Increasing mesh size or reducing time step improves this situation but does not eliminate it even for large meshes such as $64 \times 64 \times 2048$ or small time steps of 10^{-2} ps. It should be noted that this choice of size of variations, mesh size, and time step was specific to the CeC accelerator model in IMPACT-T and may change depending the lattice and tracking code used.

Finally, we can use a “symplectification algorithm” [26,27] to eliminate the remaining tracking errors (listed in Table I) and obtain a symplectic map, $\mathbb{M}_{ps}$, that is numerically close to $\mathbb{M}_s$.

Applying this symplectification algorithm [26,27] on $\mathbb{M}_{s(A)}$, we obtain the symplectic matrix, $\mathbb{M}_{ps(A)}$ with $\epsilon_{\text{sym}}(\mathbb{M}_{ps(A)}) = 0$ (to numerical precision, $< 10^{-15}$),

$$\mathbb{M}_{ps(A)} = \begin{bmatrix} 1.855 & 0.1212 & 0.001004 & -4.335 \times 10^{-4} & 0.01125 & -1.889 \times 10^{-4} \\ 14.89 & 1.512 & -0.003303 & -0.007769 & 0.1743 & -0.003168 \\ 0.01221 & 2.990 \times 10^{-4} & 1.862 & 0.1214 & 0.001120 & -3.367 \times 10^{-4} \\ 0.08689 & 0.001374 & 15.01 & 1.515 & 0.03112 & -0.004531 \\ 0.004458 & 1.335 \times 10^{-4} & 0.003479 & 5.846 \times 10^{-5} & 0.8940 & 0.01087 \\ 0.07598 & 0.001658 & -0.03064 & 0.001048 & -19.76 & 0.8782 \end{bmatrix}. \quad (32)$$

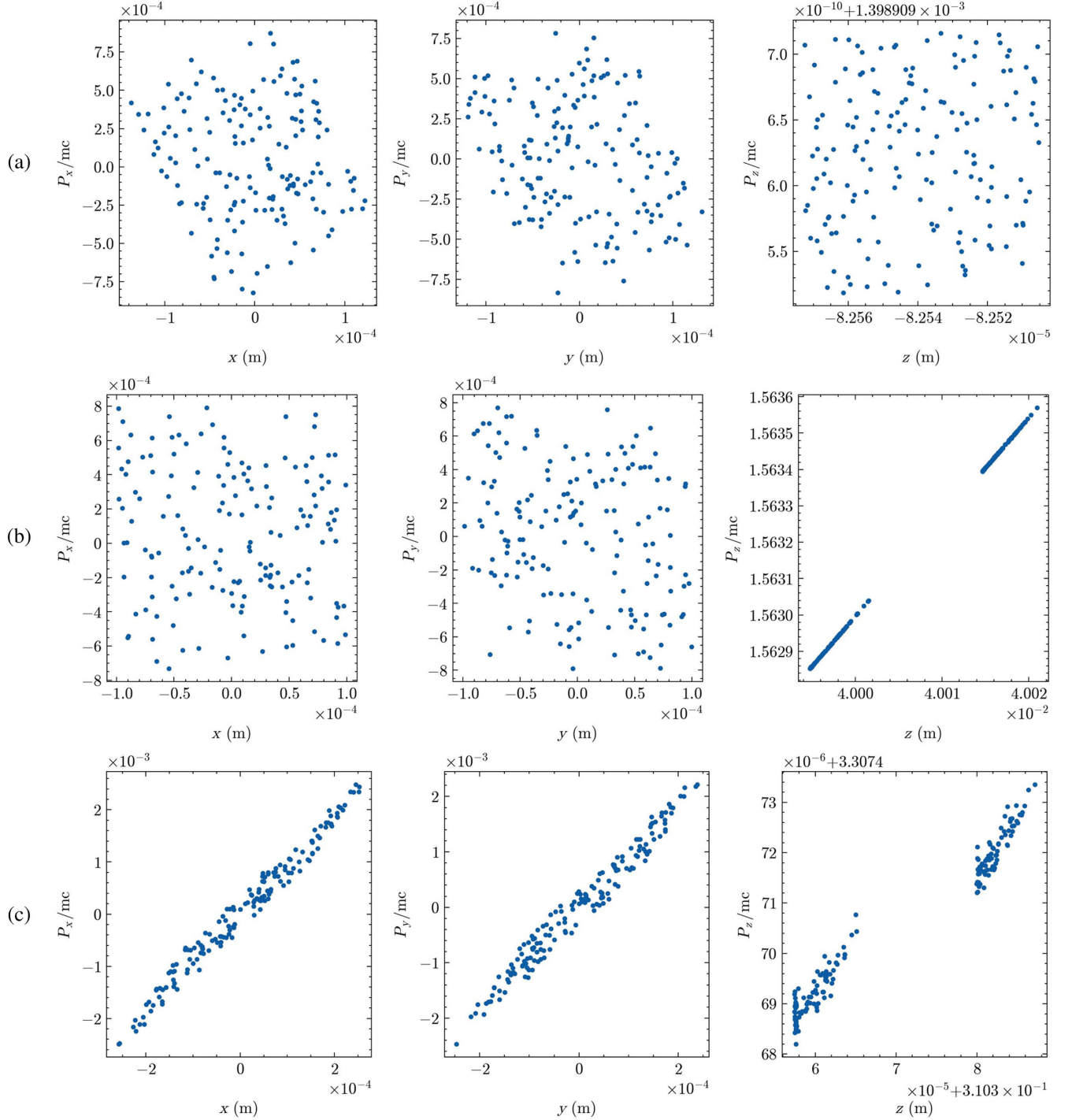


FIG. 6. Clumping of particles seen in IMPACT-T with very small variations in the initial 6D phase space of the test particles ($\lesssim \mathcal{O}(10^{-6})$ of rms beam size). These snapshots of the test particles are taken at (a) $t = 0$ when the particles are behind the cathode, (b) the start of region A when the beam centroid is located at $z = 0.04$ m, and (c) the end of region B when the beam centroid is located at $z = 1.16$ m.

The symplectic map for region B, $\mathbb{M}_{ps(B)}$ is given in Eq. (B4). A useful measure for how “close” the symplectic maps ($\mathbb{M}_{ps}$) are to those obtained using the method described in Sec. II ($\mathbb{M}_s$) is,

$$\Delta_s = \frac{\|\mathbb{M}_{ps} - \mathbb{M}_s\|}{\|\mathbb{M}_s\|}. \quad (33)$$

We find that $\Delta_s(A) = 0.006$ and $\Delta_s(B) = 0.0004$ for regions A and B of the photoinjector model (Fig. 2

respectively. Hence, the symplectification algorithm used only “tweaks” the canonical 6×6 map, which are obtained after adding vector potential contributions from external fields and space-charge forces, by 0.6% and 0.04% (combined for all 36 entries in the map) for regions A and B of the CeC photoinjector, respectively.

VI. CONCLUSION

We have presented a practical method to extract local linear symplectic transfer maps from particle tracking codes in the presence of external EM fields and strong space-charge forces. In the case where the particle-tracking code provides information regarding the EM four-potential, we can simply use the method described in Sec. II to obtain the symplectic map, $\mathbb{M}_s$, for any lattice.

However, most particle-tracking codes do not provide the necessary information for the EM four-potential including self-field forces such as space-charge forces. Hence, in Sec. III A, we describe the method to define the necessary expressions for the four-potential for an rf-based cavity and solenoid field using IMPACT-T tracking data. This allows us to calculate the contribution of external EM fields to the mechanical momenta. In Sec. III B, we present a method to include the vector potential contribution from space-charge forces present in charged beam dynamics. Hence, Sec. III describes a general method to compute the local linear canonical transfer maps ($\mathbb{M}_s$) from the geometric maps ($\mathbb{M}_g$).

In Sec. IV, we apply this method to the example of the CeC photoinjector model using IMPACT-T tracking data. We find that the canonical transfer maps have significantly reduced symplecticity errors, ϵ_{sym} (reduced to $\mathcal{O}(10^{-2})$, as shown in Table I). Finally, we leverage a theorem concerning symplectic matrices [26,27] to find the closest symplectic matrix, $\mathbb{M}_{ps}$ (with $\epsilon_{\text{sym}} < 10^{-15}$). The matrices $\mathbb{M}_{ps}$ are $< 0.6\%$ different from the canonical map, $\mathbb{M}_s$ for the CeC photoinjector example.

ACKNOWLEDGMENTS

We would like to thank K. Shih and Y. Jing for useful discussions regarding the CeC simulation using IMPACT-T. We would also like to acknowledge the continued support provided by J. Qiang with IMPACT-T. This research also used computer resources at the National Energy Research Scientific Computing Center.

DATA AVAILABILITY

Some of the data that support the findings of this article are openly available [30], embargo periods may apply. The data used in this study was simulated using IMPACT-T, an open-source simulation software available at [23] (github repository link). No experimental or external datasets were used.

APPENDIX A: DERIVATION OF $\vec{A}$ FOR ON-AXIS RF E FIELD

We start with the on-axis expression for the rf E_z field in Eq. (10). For such a periodic EM field, the vector potential and electric field are related according to

$$\vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t} \vec{A}(\vec{r}, t). \quad (\text{A1})$$

Also, for an on-axis axis-symmetric electric field, the time dependence comes from the factor $\cos(\omega t + \theta_0)$. It is therefore easy to verify that the on-axis vector potential in the z direction is related to the on-axis electric field by

$$A_z(z) = -\frac{1}{\omega} E_z(z), \quad (\text{A2})$$

which is also axially symmetric. Using the Maxwell equations in vacuum ($\rho = 0$) with appropriate gauge ($\varphi = 0$) (in SI units), we have

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 0; & \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, \\ \vec{\nabla} \cdot \vec{B} &= 0; & \vec{\nabla} \times \vec{B} &= \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}. \end{aligned} \quad (\text{A3})$$

Using Eq. (A3), we find that

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) &= -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) \\ \Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \Delta \vec{E} &= -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}. \end{aligned} \quad (\text{A4})$$

Substituting Eq. (10) and $\vec{\nabla} \cdot \vec{E} = 0$ [Eq. (A3)] into Eq. (A4), we get

$$\Delta \vec{E} + \left(\frac{\omega}{c}\right)^2 \vec{E} = 0. \quad (\text{A5})$$

This implies due to Eq. (A1),

$$\Delta \vec{A} + \left(\frac{\omega}{c}\right)^2 \vec{A} = 0. \quad (\text{A6})$$

In cylindrical coordinates, the d'Alembertian operator is given by

$$\Delta = \frac{\partial^2}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right). \quad (\text{A7})$$

Applying the d'Alembertian operator on axis-symmetric electric field and vector potential ($\partial/\partial\theta$ term goes to zero), we get

$$\frac{\partial^2}{\partial z^2} A_z(z, r) + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} A_z(z, r) \right) + \left(\frac{\omega}{c} \right)^2 A_z(z, r) = 0. \quad (\text{A8})$$

From Eqs. (A2) and (10), it is easy to show

$$\frac{\partial^2}{\partial z^2} A_z(z, r) = k_n^2 A_z(z, r). \quad (\text{A9})$$

To find a solution, we expand $A_z(z, r)$ as a Fourier series similar to Eq. (10),

$$A_z(z, r) = \frac{a_0(r)}{2} + \sum_{n=1} a_{n1}(r) \cos(k_n z) + a_{n2}(r) \sin(k_n z), \quad (\text{A10})$$

where a_0 , a_{n1} , and a_{n2} are undetermined coefficients. Substituting this back into Eq. (A8) and expanding terms, we find that the undetermined coefficients in Eq. (A10) must satisfy the relation,

$$\begin{aligned} & \sum_{n=1} \left(r \frac{\partial^2 a_{n1}(r)}{\partial r^2} + \frac{\partial a_{n1}(r)}{\partial r} + r \chi a_{n1}(r) \right) \cos(k_n z) \\ & + \sum_{n=1} \left(r \frac{\partial^2 a_{n2}(r)}{\partial r^2} + \frac{\partial a_{n2}(r)}{\partial r} + r \chi a_{n1}(r) \right) \sin(k_n z) \\ & + r \frac{\partial^2 a_0(r)}{\partial r^2} + \frac{\partial a_0(r)}{\partial r} + r k_0^2 a_0(r) \\ & = 0, \end{aligned} \quad (\text{A11})$$

where $\chi = k_0^2 - k_n^2$. Equation (A11) is equivalent to the set of conditions,

$$r \frac{\partial^2 a_0(r)}{\partial r^2} + \frac{\partial a_0(r)}{\partial r} + r k_0^2 a_0(r) = 0 \quad (\text{A12})$$

and

$$r \frac{\partial^2 a_n(r)}{\partial r^2} + \frac{\partial a_n(r)}{\partial r} + r \chi a_n(r) = 0, \quad (\text{A13})$$

where the second equation actually represents $2n$ equations which can be explicitly written by replacing $a_n(r)$ by each of the coefficients $a_{n1}(r)$ and $a_{n2}(r)$.

Now, we notice that Bessel functions of order ν are solutions of the equation,

$$x^2 \frac{\partial^2}{\partial x^2} f_\nu + x \frac{\partial}{\partial x} f_\nu + (x^2 - \nu^2) f_\nu = 0. \quad (\text{A14})$$

Substituting $r = \lambda x$ into Eq. (A12) and equating to zero gives

$$\begin{aligned} & \lambda x \frac{\partial^2 a_0(r)}{\lambda^2 \partial x^2} + \frac{\partial a_0(r)}{\lambda \partial x} + \lambda x k_0^2 a_0(r) = 0 \\ & \Rightarrow x^2 \frac{\partial^2 a_0(r)}{\partial x^2} + x \frac{\partial a_0(r)}{\partial x} + x^2 \lambda^2 k_0^2 a_0(r) = 0. \end{aligned} \quad (\text{A15})$$

We notice that if we set $x = k_0 r$ and $\nu = 0$ in Eq. (A14), then a solution to Eq. (A12) is simply given by

$$a_0(r) = A_0 \cdot J_0(k_0 r), \quad (\text{A16})$$

where J_0 is the ordinary Bessel function of the first kind of order zero. Similarly, a solution to each coefficient $a_{n1}(r)$ and $a_{n2}(r)$ is given by

$$a_n(r) = A_n \cdot B_0(\kappa_n r), \quad (\text{A17})$$

where $\kappa_n = \sqrt{|k_0^2 - k_n^2|}$ and,

$$B_k(\kappa_n r) = \begin{cases} J_k(\kappa_n r), & |k_n| < k_0 \\ I_k(\kappa_n r), & |k_n| > k_0 \end{cases}, \quad (\text{A18})$$

where J_k and I_k are the ordinary and modified Bessel functions of order k , respectively.

From Eq. (A2), we know that the on-axis ($r = 0$) vector potential, A_z is given by Eq. (A10) with,

$$\begin{aligned} a_0(r=0) &= A_0 = -E_0 \frac{e_0}{\omega}, \\ a_{n1}(r=0) &= A_{n1} = -E_0 \frac{e_{n1}}{\omega}, \\ a_{n2}(r=0) &= A_{n2} = -E_0 \frac{e_{n2}}{\omega}, \end{aligned} \quad (\text{A19})$$

where we have used $B_0(0) = 1$. The coefficients e_0 , e_{n1} , and e_{n2} correspond to those in the Fourier expansion of the on-axis electric field given in Eq. (10). Substituting Eq. (A19) into Eqs. (A16), (A17), and then into Eq. (A10) gives us the longitudinal component of the vector potential

$$A_z(z, r, t) = -\frac{E_0}{\omega} \sin(\omega t + \theta_0) \times \left[\frac{e_0}{2} J_0(k_0 r) + \sum_{n=1} \{e_{n1} \cos(k_n z) + e_{n2} \sin(k_n z)\} B_0(\kappa_n r) \right]. \quad (\text{A20})$$

To obtain the transverse component of the axis-symmetric vector potential, $\vec{A} = \hat{z}A_z + \hat{r}A_r$, we use $\vec{\nabla} \cdot \vec{A} = 0$, which gives

$$\frac{\partial A_z}{\partial z} + \frac{1}{r} \frac{\partial}{\partial r}(rA_r) = 0 \Rightarrow \frac{1}{r} \frac{\partial}{\partial r}(rA_r) = -\frac{\partial A_z}{\partial z}. \quad (\text{A21})$$

Using the expression for $A_z(z, r)$ from Eq. (A20) and ignoring the oscillating term $\sin(\omega t + \theta_0)$ for now, we get

$$\frac{1}{r} \frac{\partial}{\partial r}(rA_r) = \frac{E_0}{\omega} \times \left(\sum_{n=1} k_n B_0(\kappa_n r) \{-e_{n1} \sin(k_n z) + e_{n2} \cos(k_n z)\} \right). \quad (\text{A22})$$

Substituting $x = \kappa_n r$, Eq. (A22) simplifies to

$$\frac{1}{x} \frac{\partial}{\partial x}[xA_r(x)] = \frac{E_0}{\omega} \times \left[\sum_{n=1} \frac{k_n}{\kappa_n} \{-e_{n1} \sin(k_n z) + e_{n2} \cos(k_n z)\} B_0(x) \right]. \quad (\text{A23})$$

Ordinary and modified Bessel functions satisfy the property

$$\left(\frac{1}{x} \frac{\partial}{\partial x} \right)^n \left[x^\nu B_\nu(x) \right] = x^{\nu-n} B_{\nu-n}(x) \Rightarrow \frac{1}{x} \frac{\partial}{\partial x} \left[x B_1(x) \right] = B_0(x). \quad (\text{A24})$$

Applying this to Eq. (A23), we get the transverse component of the vector potential

$$A_r(z, r, t) = \frac{E_0}{\omega} \sin(\omega t + \theta_0) \times \left(\sum_{n=1} \frac{k_n}{\kappa_n} \{-e_{n1} \sin(k_n z) + e_{n2} \cos(k_n z)\} B_1(\kappa_n r) \right). \quad (\text{A25})$$

APPENDIX B: TRANSFER MAPS FOR CEC PHOTO-INJECTOR MODEL

The geometric and symplectic transfer maps, $\mathbf{M}_{g(A)}$ and $\mathbf{M}_{s(A)}$ for gun region A (shown in Fig. 2) are given in Eqs. (29) and (32), respectively. The canonical transfer map, $\mathbf{M}_{s(A)}$, for the tracked “center bunch” in this region is given in Eq. (B1). The transfer maps in field region B is calculated using the method described in Sec. II are given in Eqs. (B2)–(B4). The symplecticity errors, ϵ_{sym} of these transfer maps are reported in Table I.

$$\mathbb{M}_{s(A)} = \begin{bmatrix} 1.857 & 0.1213 & 0.002101 & -9.483 \times 10^{-4} & 0.009719 & -4.292 \times 10^{-4} \\ 14.91 & 1.514 & 0.003872 & -0.01677 & 0.1616 & -0.007138 \\ -0.004365 & -1.937 \times 10^{-4} & 1.863 & 0.1215 & 0.002504 & -1.078 \times 10^{-4} \\ -0.04565 & -0.002192 & 15.02 & 1.516 & 0.04261 & -0.001846 \\ 0.001052 & 1.859 \times 10^{-5} & 0.004893 & 5.171 \times 10^{-5} & 0.8970 & 0.01091 \\ 0.1497 & 0.003248 & -0.06124 & 0.001994 & -19.83 & 0.8812 \end{bmatrix} \quad (\text{B1})$$

$$\mathbb{M}_{\text{ext}(B)} = \begin{bmatrix} 0.1866 & 0.1319 & 0.1581 & 0.1109 & 3.992 \times 10^{-5} & 6.029 \times 10^{-5} \\ -3.938 & 0.3547 & -3.315 & 0.2988 & -1.294 \times 10^{-4} & 7.621 \times 10^{-5} \\ -0.1582 & -0.1109 & 0.1890 & 0.1317 & 1.693 \times 10^{-4} & -1.439 \times 10^{-5} \\ 3.316 & -0.2986 & -3.938 & 0.3548 & 9.103 \times 10^{-4} & -4.155 \times 10^{-5} \\ -0.001323 & 1.591 \times 10^{-4} & -3.191 \times 10^{-4} & 3.403 \times 10^{-5} & 0.9973 & 0.02156 \\ -0.05911 & 0.007434 & -0.007407 & 4.688 \times 10^{-4} & -0.1342 & 1.000 \end{bmatrix}, \quad (\text{B2})$$

$$\mathbb{M}_{s(B)} = \begin{bmatrix} 0.1866 & 0.1319 & 0.1581 & 0.1109 & 3.991 \times 10^{-5} & 6.029 \times 10^{-5} \\ -3.938 & 0.3547 & -3.315 & 0.2988 & -1.294 \times 10^{-4} & 7.621 \times 10^{-5} \\ -0.1582 & -0.1109 & 0.1890 & 0.1317 & 1.693 \times 10^{-4} & -1.439 \times 10^{-5} \\ 3.316 & -0.2986 & -3.938 & 0.3548 & 9.103 \times 10^{-4} & -4.155 \times 10^{-5} \\ 1.675 \times 10^{-4} & 1.591 \times 10^{-4} & -1.259 \times 10^{-4} & 3.403 \times 10^{-5} & 0.9973 & 0.02156 \\ -5.966 \times 10^{-4} & -8.399 \times 10^{-5} & 9.701 \times 10^{-4} & 2.999 \times 10^{-5} & -0.1342 & 1.000 \end{bmatrix}, \quad (\text{B3})$$

$$\mathbb{M}_{ps(B)} = \begin{bmatrix} 0.1865 & 0.1319 & 0.1580 & 0.1109 & 3.782 \times 10^{-5} & 4.4078 \times 10^{-5} \\ -3.937 & 0.3546 & -3.314 & 0.2987 & -9.504 \times 10^{-5} & -3.439 \times 10^{-4} \\ -0.1581 & -0.1109 & 0.1890 & 0.1316 & 1.709 \times 10^{-4} & 2.887 \times 10^{-6} \\ 3.315 & -0.2985 & -3.938 & 0.3547 & 8.834 \times 10^{-4} & 2.332 \times 10^{-4} \\ -7.514 \times 10^{-5} & 8.414 \times 10^{-5} & -1.263 \times 10^{-4} & 2.230 \times 10^{-5} & 0.9970 & 0.02155 \\ -5.666 \times 10^{-4} & -8.444 \times 10^{-5} & 9.700 \times 10^{-4} & 3.091 \times 10^{-5} & -0.1342 & 1.000 \end{bmatrix}. \quad (\text{B4})$$

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