

Threshold studies of coherent synchrotron radiation induced microwave instability beyond adiabatic approximation

J. H. Bian¹, X. J. Deng^{2,*}, W. H. Huang^{1,†} and C. X. Tang¹

¹*Department of Engineering Physics, Tsinghua University, Beijing 100084, China*

²*Institute for Advanced Study, Tsinghua University, Beijing 100084, China*



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In typical studies of collective effects in a storage ring, an important assumption is that the longitudinal bunch length remains constant over one turn, which is referred to as the adiabatic approximation. This approximation is reasonable for conventional storage rings. However, for extremely short bunch length storage rings, such as steady-state microbunching storage ring, the global phase slippage is very small, and the local or partial phase slippage can significantly influence the longitudinal dynamics of the bunch. Consequently, the bunch length can vary greatly around the ring, and the adiabatic approximation no longer holds. In this work, we investigate the microwave instability caused by the coherent synchrotron radiation, which represents one of the dominant collective effects in short bunch storage rings, beyond the adiabatic approximation. We present a general formula for the instability threshold considering the variation of longitudinal optics around the ring and offer valuable guidance for the design of extremely short bunch storage rings. Our results show that depending on the distribution of the partial phase slippage, the instability threshold of a real lattice can be increased by a factor of 2 or even more compared to the classical prediction.

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I. INTRODUCTION

Accelerator-based synchrotron radiation light sources are of paramount significance for scientific research. Currently, a prominent research direction focuses on generating short electron bunches or microbunches in synchrotron radiation rings, which can produce high-power and high-flux coherent radiation, thereby providing novel opportunities for industrial applications such as extreme ultraviolet lithography. The steady-state microbunching (SSMB) represents an innovative accelerator-based light source scheme that holds the potential to deliver high-average-power, narrow-band coherent radiation across the wavelength spectrum from THz to soft x-ray. Related research is actively underway, demonstrating promising progress [1–21], particularly evident in recent proof-of-principle experiments [22,23].

To obtain short bunches in storage rings, a well-established approach is to design a quasi-isochronous magnetic lattice with a very small global phase slippage factor η . This

approach is based on the classical “zero-current” bunch length scaling law $\sigma_z \propto \sqrt{|\eta|}$, given by Sands [24]. However, when the global phase slippage is sufficiently small, the effects of the local or partial phase slippage begin to gain prominence and the classical scaling law is no longer applicable. In this scenario, single-particle dynamics become more intricate and have been investigated by Deng *et al.* [9] utilizing longitudinal Courant-Snyder formalism.

The collective effects of bunches, when considering the partial phase slippage, constitute another critical issue that has not yet been addressed. Typical studies of collective effects assume that the bunch length remains constant around the ring over one turn, which is also known as the adiabatic approximation. However, when considering the partial phase slippage, the bunch length can vary significantly around the ring within one turn and the adiabatic approximation no longer holds. This notable difference also makes the collective effects more complex. The influence of the partial phase slippage typically becomes significant in extremely short bunch storage rings, and the coherent synchrotron radiation (CSR) emerges as one of the dominant collective effects in such rings. Therefore, in this work, we investigate the impact of the CSR wake on beam dynamics beyond the adiabatic approximation. The method presented however applies also to other wakefields in general.

The impact of the CSR wake in conventional short bunch storage rings has been extensively studied, and the

*Contact author: dengxiujie@mail.tsinghua.edu.cn

†Contact author: huangwh@mail.tsinghua.edu.cn

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threshold of microwave instability (MWI) induced by CSR has been provided through various methods [25,26]. In this work, we have developed a particle tracking code that can account for the variation of bunch length around the ring. We have conducted simulations of the MWI induced by CSR under various conditions considering the partial phase slippage and based on it presented a general formula for the instability threshold. This formula can be used for a quick assessment of the CSR induced MWI threshold in a storage ring where the partial phase slippage plays a dominant role. Our results can also provide valuable insights for the design of the magnetic lattice of extremely short bunch storage ring aimed at achieving higher instability threshold.

In the following section, we provide a brief overview of longitudinal single-particle dynamics considering the partial phase slippage. In Sec. III, we describe the particle tracking code we developed and its benchmark. Section IV outlines the design of our simulations, while Sec. V gives the simulation results and related discussions. Finally, we conclude with remarks and outlook.

II. LONGITUDINAL COURANT-SNYDER FORMALISM

In this section, we describe longitudinal single-particle dynamics while considering partial phase slippage based on the longitudinal Courant-Snyder formalism presented in Ref. [9]. Here, we focus exclusively on longitudinal dynamics and the particle state vector $X = (z, \delta)^T$ is used, where z and δ are the longitudinal position and relative energy deviation with respect to the reference particle, respectively, and ^{T} means transpose. We treat the case

where there is only one radiofrequency (rf) cavity located at a dispersion-free location in the storage ring. Then at a specific position s_j , the ring can be divided into three parts, with their transfer matrices given by

$$\begin{aligned} T(s_{\text{rf}}, s_j) &= \begin{pmatrix} 1 & -\tilde{\eta}(s_{\text{rf}}, s_j)C_0 \\ 0 & 1 \end{pmatrix}, \\ T(s_{\text{rf}}, s_{\text{rf}}) &= \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}, \\ T(s_j, s_{\text{rf}}) &= \begin{pmatrix} 1 & -\tilde{\eta}(s_j, s_{\text{rf}})C_0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \quad (1)$$

where

$$\tilde{\eta}(s_2, s_1) = \frac{1}{C_0} \int_{s_1}^{s_2} \left(\frac{D_x(s)}{\rho(s)} - \frac{1}{\gamma^2} \right) ds \quad (2)$$

is the partial phase slippage from s_1 to s_2 , and $\tilde{\eta}(s_{\text{rf}}, s_j) + \tilde{\eta}(s_j, s_{\text{rf}}) = \eta$ represents the global phase slippage, C_0 is the circumference of the ring, D_x is the horizontal dispersion, ρ is the bending radius of the trajectory, γ is the Lorentz factor, s is the path length along the reference orbit, $h = eV_{\text{rf}}k_{\text{rf}} \cos \phi_s / E_0$ denotes the rf acceleration gradient, in which e is the elementary charge, V_{rf} is the rf voltage, $k_{\text{rf}} = 2\pi/\lambda_{\text{rf}}$ is the rf wave number, ϕ_s is the synchronous phase, and E_0 is the energy of the reference particle. In the analysis, the rf cavity is assumed to be of zero length. The one-turn map at s_j is then

$$\begin{aligned} M(s_j) &= T(s_j, s_{\text{rf}})T(s_{\text{rf}}, s_{\text{rf}})T(s_{\text{rf}}, s_j) \\ &= \begin{pmatrix} 1 - \tilde{\eta}(s_j, s_{\text{rf}})hC_0 & -\eta C_0 + \tilde{\eta}(s_j, s_{\text{rf}})\tilde{\eta}(s_{\text{rf}}, s_j)hC_0^2 \\ h & 1 - \tilde{\eta}(s_{\text{rf}}, s_j)hC_0 \end{pmatrix}. \end{aligned} \quad (3)$$

Following the analysis of Courant and Snyder [27], we can parametrize $M(s_j)$ as

$$M(s_j) = \begin{pmatrix} \cos \Phi_z + \alpha_z(s_j) \sin \Phi_z & \beta_z(s_j) \sin \Phi_z \\ -\gamma_z(s_j) \sin \Phi_z & \cos \Phi_z - \alpha_z(s_j) \sin \Phi_z \end{pmatrix}, \quad (4)$$

where Φ_z is the longitudinal phase advance per turn. Thus,

$$\beta_z = \frac{-\eta C_0 + \tilde{\eta}(s_j, s_{\text{rf}})\tilde{\eta}(s_{\text{rf}}, s_j)hC_0^2}{\sin \Phi_z}, \quad (5)$$

and the bunch length at s_j is given by $\sigma_z(s_j) = \sqrt{\epsilon_z \beta_z(s_j)}$, where ϵ_z is the longitudinal emittance.

The first term in the numerator of Eq. (5) represents the global phase slippage, while the second term describes the influence of the partial phase slippage. The analysis of Sands [24] neglects the second term, therefore β_z is a

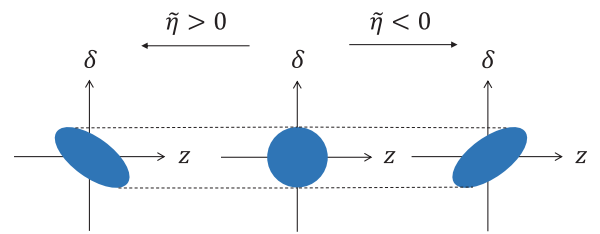


FIG. 1. Physical picture to explain how the partial phase slippage operates. The ellipse is the beam distribution in longitudinal phase space.

constant value around the ring. Consequently, the bunch length also remains constant within one turn, which holds for most currently operational storage rings. However, when the global phase slippage is extremely small, the impact of the partial phase slippage becomes significant. Therefore, the bunch length can vary significantly within one turn.

Figure 1 shows the physical mechanism about how the partial phase slippage operates. It is important to note that, for a given ϵ_z and β_z and thus bunch length, the beam orientation in the longitudinal phase space can have two distinct configurations, which complicates the analyses of collective effects, as will be discussed further.

III. CODE AND BENCHMARK

To describe the variation of bunch length along the ring, we have developed a particle tracking code. The code currently focuses on longitudinal dynamics, with the potential to be extended to include transverse-longitudinal coupling dynamics in the future. For the issue we are concerned with, the longitudinal MWI caused by CSR, we only need to consider two types of components: an rf cavity and dipole magnets.

In the code, when the bunch passes through the rf, the particle gets an energy kick depending on its longitudinal position coordinate according to

$$\begin{aligned} z_f &= z_i, \\ \delta_f &= \delta_i + h z_i, \end{aligned} \quad (6)$$

where i and f represent the particle state just before and after the component, respectively. We have linearized the sinusoidal waveform of rf acceleration, since typically the bunch length is much shorter than the rf wavelength. When the bunch passes through the dipoles, they are need to be sliced to describe the impact of the variation of the bunch length. In each slice, the evolution of the phase space coordinates is given by

$$\begin{aligned} z_f &= z_i - \tilde{\eta}_{\text{slice}} C_0 \delta_i, \\ \delta_f &= \delta_i + \delta_{\text{CSR}}, \end{aligned} \quad (7)$$

where $\tilde{\eta}_{\text{slice}}$ is the partial phase slippage of each slice and δ_{CSR} describes the energy change caused by CSR. The dipoles are sliced into a sufficient number of segments to ensure that the simulation results converge.

It can be seen that the simulation of linear dynamics in the code is conventional, which utilizes the method of transfer matrices. The core of the code is to simulate the CSR effects in dipole magnets. Referencing Borland [28], the rate of energy change for the particle with a unit charge in the bunch due to steady-state CSR wake in free space can be expressed as

TABLE I. Parameters used in the benchmark simulation.

Parameter	Value	Description
C_0	100 m	Ring circumference
E_0	800 MeV	Beam energy
η	8×10^{-5}	Global phase slippage
ρ	1.27 m	Bending radius
σ_z	1×10^{-4} m	Bunch length
σ_δ	5×10^{-4}	Natural energy spread
h	0.2 m^{-1}	rf accelerator gradient

$$\frac{dE(z)}{ds} = -\frac{1}{2\pi\epsilon_0(3\rho^2)^{1/3}} \int_{-\infty}^z \frac{\lambda'}{(z-z')^{1/3}} dz', \quad (8)$$

where E is the energy of the particle, ϵ_0 is the vacuum permittivity, ρ is the bending radius of the dipole magnet, λ' denotes the derivative of the density distribution with respect to the longitudinal position coordinate, z and z' represent the longitudinal position coordinates of the test particle and the source particle, respectively. Computation of δ_{CSR} in the code is performed as follows: (1) longitudinal position coordinates of the particles are binned; (2) the density histogram is smoothed using a Savitzky-Golay filter; (3) the same filter is used to take the derivative of the smoothed density distribution; (4) the rate of energy change for each bin is calculated using Eq. (8), where simple sums are used for the integral required; (5) the rate of energy change for each particle is the same as the rate for the bin it occupies. Note that in our simulations, 10^6 macroparticles, 3000 bins, and a first-order Savitzky-Golay smoothing filter with a half-width of 5 bins are used to overcome the effects of noise.

We have conducted a test that simulates the MWI caused by CSR in a traditional storage ring to validate the reliability of our code. The case has been extensively studied [25,26] and the instability threshold formula, considering only free space steady-state CSR, is given by

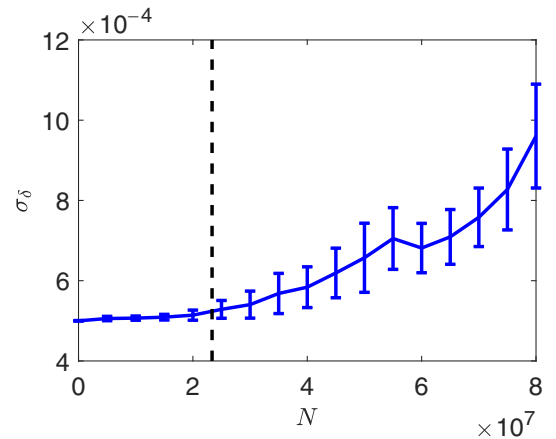


FIG. 2. Energy spread as a function of bunch population. The black dashed line represents the threshold predicted by Eq. (9).

$$N_{\text{th}} = \frac{\pi \nu_s \gamma \sigma_\delta \sigma_z^{4/3}}{r_e \rho^{1/3}}, \quad (9)$$

where N represents the number of particles per bunch, ν_s is the synchrotron tune, σ_δ is the natural energy spread, and r_e is the classical electron radius.

In the test, we assume that the effect of the CSR wake is localized in a single spot, calculated only once per turn. The assumption is reasonable in traditional storage rings, as the bunch length remains constant within one turn and the synchrotron tune is typically very small, thus the longitudinal relative positions of the particles in the bunch approximately remain unchanged within one turn. The lattice consists of only one rf and one dipole magnet of length $2\pi\rho$, and the magnet does not need to be sliced in the simulation.

The specific parameters used for the test are shown in Table I, with values selected based on current achievable technology. The simulation results are illustrated in Fig. 2, where the threshold intensity is taken as the intensity at which the energy spread increases by 5% compared to zero-current energy spread. It can be observed that our simulation results align well with the predictions made by Eq. (9).

IV. SIMULATION DESIGN

The main difference in longitudinal beam dynamics when considering the partial phase slippage is that the bunch length can vary significantly within one turn. The longitudinal optical function of an actual lattice may be very complex, so we have established a simplified toy model to study the issue we are concerned with. The specific lattice used in our study is shown in Fig. 3. The entire storage ring consists of only one rf cavity and four dipole magnets with the same bending radius ρ and the same length $L = \pi\rho/2$. The dispersion function of each dipole is artificially assigned to adjust the evolution of the partial phase slippage. It is important to note that the dispersion function may not conform to real physical laws. However, we believe that this model can be used to

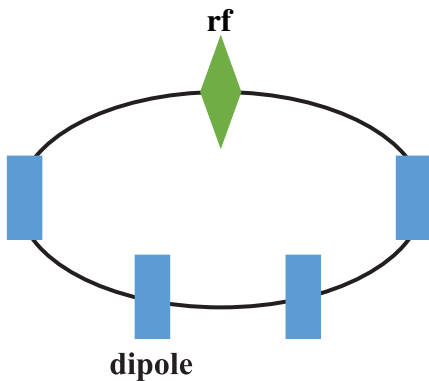


FIG. 3. Layout of the simulation lattice.

investigate the issue we concern. The specific parameters selection for the simulation lattice is also based on Table I.

As mentioned at the end of Sec. II, one β_z can correspond to two kinds of distributions of the bunch in phase space. Therefore, even if the evolution of the bunch length is the same within one turn, the phase space evolution of the bunch can have different possibilities due to the influence of the sign of the partial phase slippage. In our study, we exclusively consider three categories of cases, referred to as case 0, case A, and case B. Case 0 represents the traditional storage ring, where the effect of the partial phase slippage is negligible. Case A and case B are two extreme cases where the partial phase slippage is in effect. In case A, the sign of the partial phase slippage from rf is half positive and half negative within the dipoles, where steady-state CSR wake has an effect. While in case B, the sign remains the same in all four dipoles. The energy spread and the global phase slippage for the three kinds of cases are set to be the same to

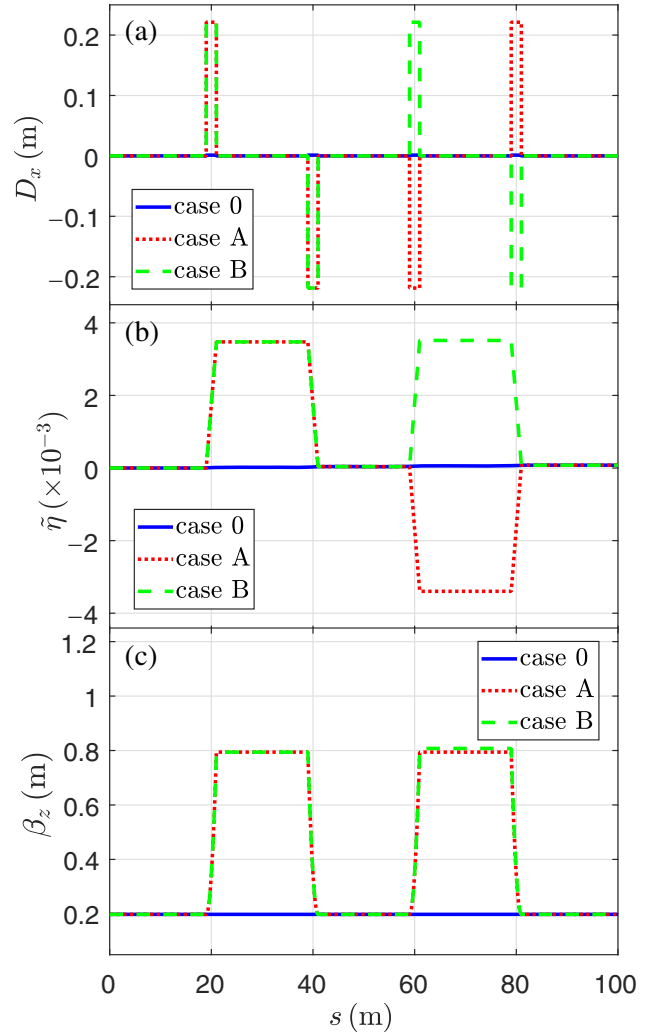


FIG. 4. Evolution of (a) D_x , (b) $\tilde{\eta}$, and (c) β_z around the ring in different cases. The dispersion values outside the dipoles are set to 0, as they do not contribute to the phase slippage.

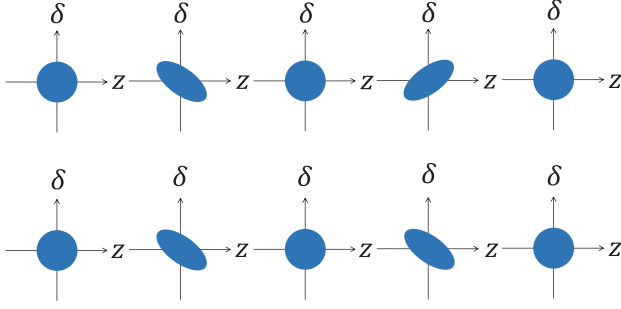


FIG. 5. Evolution of the phase space in case A (top) and case B (bottom).

ensure that the minimum bunch length is identical in each case, facilitating comparison.

In case 0, the dispersion function of each dipole is set to a constant value of $\eta C_0/2\pi$ to ensure that the global phase slippage equals η . Note that in our work, the particles are close to being ultrarelativistic, and the contribution of the second term in Eq. (2) to the phase slippage is negligible. The dispersion function, the partial phase slippage and the longitudinal beta function evolution around the ring are illustrated by the blue lines in Fig. 4. We can observe that the beta function remains approximately constant along the ring, as we anticipated.

For case A and case B, the method to alter the evolution of the partial phase slippage is to adjust the dispersion value of each dipole based on case 0. In case A, a positive constant dispersion value is added to the first and fourth dipoles, while an equal negative dispersion value is applied to the second and third dipoles. This ensures that the global phase slippage remains unchanged. By varying the magnitude of the added dispersion value, the maximum bunch length to which the beam can be stretched within one turn can be adjusted. The red dotted lines in Fig. 4 illustrate the optical functions when the maximum bunch length is twice the minimum bunch length in case A. The trend of the

evolution of the optical functions is similar for other stretch factors.

In case B, a positive constant dispersion value is added to the first and third dipoles, while an equal negative dispersion value is applied to the second and fourth dipoles based on case 0. The green dashed lines in Fig. 4 illustrate the optical functions in case B, also using a stretch factor of 2 as an example. We can observe that the evolution of the longitudinal beta function is nearly the same in case A and case B for the same stretch factor, and the beam experiences two stretches and contractions. The difference lies in the fact that in case A, the evolution directions of the phase space of the bunch during the two stretching processes are opposite, while in case B, the evolution directions are the same. The schematic diagram is shown in Fig. 5.

V. SIMULATION RESULTS AND DISCUSSIONS

The simulation results are shown in Fig. 6, illustrating the relationship between the energy spread of the bunch and the bunch population in different cases. Only case 0 and the scenarios with a stretch factor of 2 and 4 in case A and case B are depicted. We still use a 5% increase in energy spread relative to the zero-current energy spread as the marker for the onset of instability.

From the simulation results, we can observe that the instability threshold in case 0 is consistent with the benchmark test result in Sec. III, which indicates that the adiabatic approximation is reasonable for traditional storage rings. For case A and case B, two intuitive conclusions can be drawn. First, the greater the stretch factor of the bunch length, the higher the instability threshold for both case A and case B. This aligns with the physical intuition, as a longer bunch length weakens the effect of the wake. The other conclusion is that, compared to case B, case A has a higher instability threshold for the same stretch factor. We believe this is because, compared to case B, some particles may experience head-tail exchanges in case A. This process partially cancels

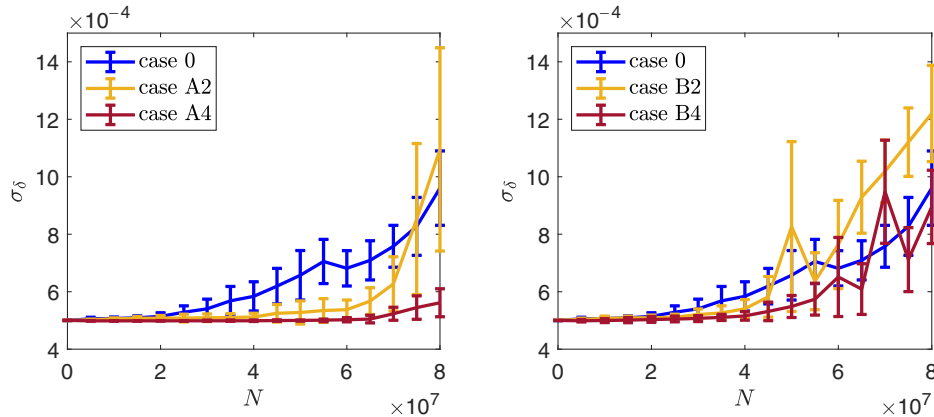


FIG. 6. Energy spread as a function of bunch population in case 0, case A, and case B. Case A2 represents the scenario with a stretch factor of 2 in case A, and the meanings of other legends are similar.

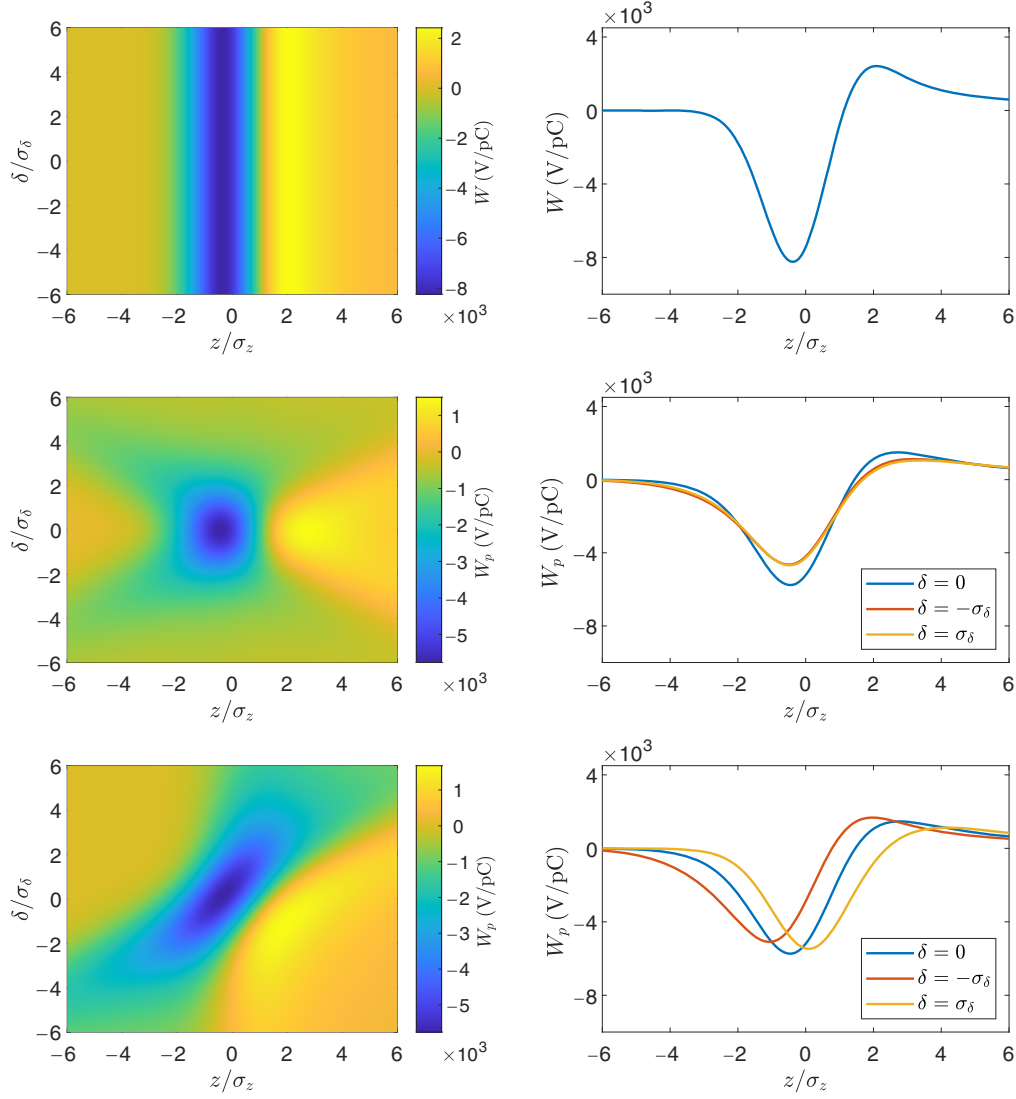


FIG. 7. Bunch wake of a Gaussian bunch over one turn in phase space and at various specific relative energy deviations in case 0 (top), case A (middle), and case B (bottom).

out the impact of the wake, thereby increasing the instability threshold.

To conduct a more in-depth analysis, we consider the bunch wake of a Gaussian bunch over one turn, while neglecting the potential well distortion. For a conventional storage ring, the longitudinal relative positions of the particles in a bunch remain approximately unchanged within one turn. Therefore, the bunch wake over one turn can be expressed as

$$W(z) = \frac{2^{5/6} \rho^{1/3}}{3^{7/3} \epsilon_0 \sigma_z^{4/3}} \left[\frac{\sqrt{\pi} q_1 F_1\left(\frac{7}{6}, \frac{3}{2}, -\frac{q^2}{2}\right)}{\sqrt{3} \Gamma\left(\frac{5}{3}\right)} - \frac{2^{5/6} \Gamma\left(\frac{2}{3}\right) {}_1F_1\left(\frac{2}{3}, \frac{1}{2}, -\frac{q^2}{2}\right)}{\Gamma\left(\frac{7}{3}\right)} \right], \quad (10)$$

where $q = z/\sigma_z$, which depends solely on the longitudinal position coordinate of the particle, and ${}_1F_1$ is a confluent hypergeometric function of the first kind [29]. Figure 7 (top) is a schematic diagram that illustrates the wake in phase space and at various specific relative energy deviations. It can be seen that the wake is independent of the relative energy deviation, as discussed. The standard deviation of the relative energy deviation change caused by the CSR wake over one turn, referred to as kick divergence, can be calculated as

$$K = \frac{4\pi\epsilon_0 N r_e}{\gamma} \sqrt{\int (W(z) - \bar{W})^2 f(z) dz}, \quad (11)$$

where $\bar{W} = \int W(z) f(z) dz$ and f represents the distribution of the bunch. We can derive that

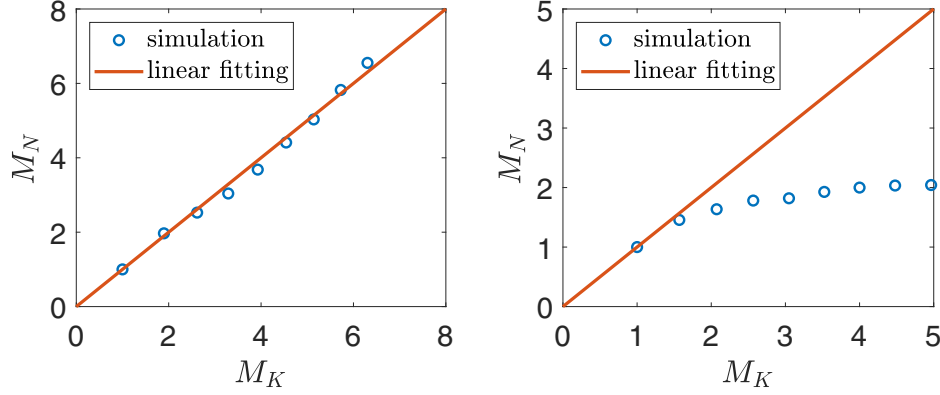


FIG. 8. Relationship between M_N and M_K in case A (left) and case B (right).

$$K = 1.545 \frac{Nr_e \rho^{1/3}}{\gamma \sigma_z^{4/3}}, \quad (12)$$

which is directly proportional to $\rho^{1/3}/\sigma_z^{4/3}$, while the instability threshold equation (9) is inversely proportional to it. ρ and σ_z are two parameters that are directly related to the strength of the CSR wake. Thus, there is an inverse relationship between the kick divergence and the threshold when other parameters are constant.

For the cases that considering the partial phase slippage, the Gaussian bunch wake over one turn is more complex, which is dependent on both the longitudinal position and the relative energy deviation of the particles. It can be expressed as

$$W_p(z, \delta) = \int_{\text{dipole}} \frac{W(z - \tilde{\eta}(s)C_0\delta)}{2\pi\rho} ds, \quad (13)$$

where it is important to note that $\sigma_z(s)$ implicitly included in W also varies with s . The integration is performed in the dipoles, as steady-state CSR wake has an effect only in them. Figure 7 (middle and bottom) illustrates the bunch wake in phase space and at various relative energy deviations in case A and case B with a stretch factor of 2, respectively. It can be observed that the overall magnitude of the wake decreases compared to case 0, which is due to the stretch of the bunch length. The wakes experienced by the particles with the same longitudinal position but different relative energy deviations are different as a result of the partial phase slippage. In case A, the wake in phase space is almost symmetric with respect to the position coordinate axis, which means that for particles with equal magnitudes of relative energy deviation but opposite signs, the evolution of the wake in relation to the position coordinate is nearly the same. An intuitive physical explanation is as follows. Among particles with the same longitudinal position coordinate, the changes in the position coordinate of the particles with opposite relative energy deviations within one turn due to the impact of

the partial phase slippage are completely opposite. In case A, the evolution of the bunch length in the first and last two dipoles is the same, while the changes in the position coordinate of the particles with nonzero relative energy deviation are totally contrary. Some particles in the bunch undergo head-tail exchanges. Therefore, particles with the same longitudinal position and opposite relative energy deviations experience the same bunch wake within one turn. However, in case B, the change in the position coordinate of the particles with nonzero relative energy deviation in the first and last two dipoles is the same. Thus, the bunch wake in phase space is no longer symmetric and appears to undergo a rotation.

In case A, although the maximum amplitude of the wake decreases as the absolute value of the relative energy deviation increases, the evolution trend of the wake with respect to the position coordinate remains almost the same for a given relative energy deviation as it does when there is no relative energy deviation. Therefore, we speculate that the inverse relationship observed in traditional storage rings still holds, meaning the threshold is inversely proportional to the kick divergence. The kick divergence considering the partial phase slippage can be expressed as

$$K_p = \frac{4\pi\epsilon_0 Nr_e}{\gamma} \sqrt{\iint (W_p(z, \delta) - \bar{W}_p)^2 f(z, \delta) dz d\delta}, \quad (14)$$

where $\bar{W}_p = \iint [W_p(z, \delta)] f(z, \delta) dz d\delta$. Through the results of our simulations, we find that this inverse proportionality still approximately holds true. Figure 8 (left) illustrates the relationship between the multiplicative increase in the threshold $M_N = \tilde{N}_{\text{th}}/N_{\text{th}}$ and the multiplicative decrease in the kick divergence $M_K = K/K_p$ relative to case 0 with different stretch factors in case A, where M_K rises with the increase of the stretch factor. It can be seen that the two are approximately linearly related. However, in case B, when calculating the same data, the two no longer conform to a linear relationship, as shown in Fig. 8 (right). The deviation

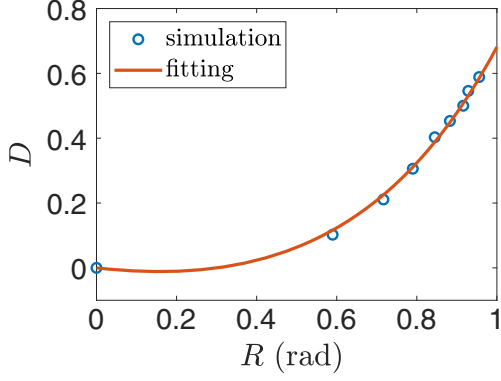


FIG. 9. Relative deviation coefficient D as a function of phase space rotation angle R .

from linearity becomes more pronounced, as the stretch factor increases. This is because, in case B, the evolution of the wake with respect to the position coordinate is no longer the same for different relative energy deviations. From the perspective of phase space, the wake appears to undergo a rotation, and as the stretch factor increases, the rotation angle becomes larger.

We use the rotation angle R , which has units of rad, as a variable to fit the relationship between the relative deviation $D = (M_K - M_N)/M_K$ and R , as shown in Fig. 9. The two can be well fitted using a double exponential function minus a constant, expressed as

$$D = 0.89e^{-0.44R} + 0.11e^{2.31R} - 1. \quad (15)$$

It is observed that there is no rotation in case A, thus, the relative deviation is zero.

To summarize the above results, a general instability threshold formula taking into account the impact of the partial phase slippage can be proposed as

$$\widetilde{N}_{\text{th}} = (1 - D)M_K N_{\text{th}}. \quad (16)$$

In addition to the cases described above, we have also tested the applicability of our formula for other more complex cases, and found that it can still predict the threshold effectively.

Here, we would like to present an example of applying our formula to a real lattice of the Metrology Light Source (MLS) [30,31] of the Physikalisch-Technische Bundesanstalt in Berlin. The lattice consists of one rf cavity and eight dipole magnets and Fig. 10 illustrates its optical functions. This specific lattice optics is slightly modified based on the one used in a recent experiment validating the impact of partial phase slippage on the equilibrium energy spread [21]. The global phase slippage of this lattice is small while the partial phase slippage is large. As can be seen, the longitudinal beta function evolves significantly around the ring, indicating that the partial phase slippage may have a notable impact on the MWI threshold. Other relevant parameters are given in

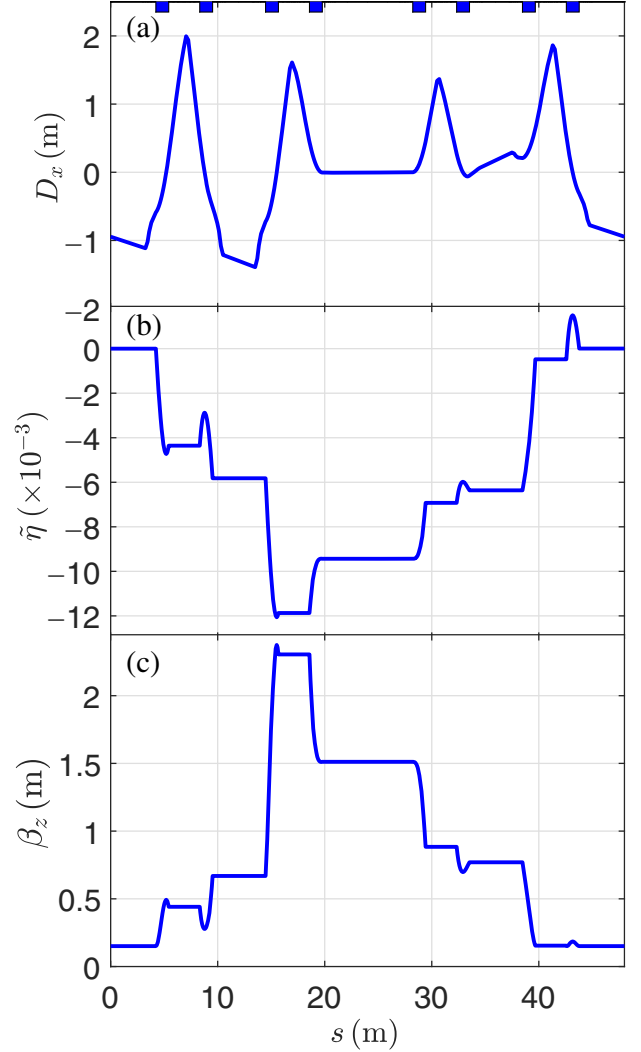


FIG. 10. Evolution of (a) D_x , (b) $\tilde{\eta}$, and (c) β_z around the ring in the test case. The dipole magnets are shown at the top as blue rectangles. Each dipole has a length of 1.2 m.

Table II and we can directly calculate $N_{\text{th}} = 2.88 \times 10^6$ without considering the partial phase slippage. The bunch wake over one turn in phase space is shown in Fig. 11 and we can get $M_K = 2.76$, $R = 0.77$, and thus $D = 0.28$. Finally, we can obtain the instability threshold $\widetilde{N}_{\text{th}} = 5.67 \times 10^6$.

TABLE II. Parameters used in the test case.

Parameter	Value	Description
C_0	48 m	Ring circumference
E_0	630 MeV	Beam energy
V	600 kV	rf voltage
η	5×10^{-6}	Global phase slippage
ρ	1.27 m	Bending radius
σ_z	1.7×10^{-4} m	Bunch length
σ_δ	1.1×10^{-3}	Energy spread
h	0.01 m^{-1}	rf accelerator gradient

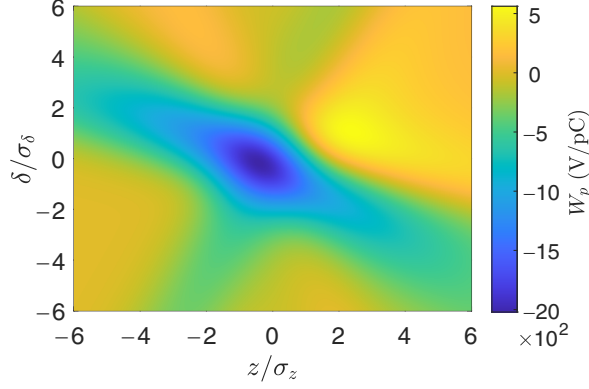


FIG. 11. Bunch wake over one turn in phase space.

The simulation results are shown in Fig. 12 and it can be seen that the threshold aligns well with our prediction.

The general ability of Eq. (16) in predicting the instability threshold is based on the condition that the particle revolution period is much shorter than the time for instability to grow. Thus, the prescribed motion of particles in one turn is not much affected by the instability, and we can lump the effect of wake on the bunch each turn for threshold assessment. Compared to the classical treatment, the generalization in our work is that now the one-turn-lumped wake depends not only on the relative longitudinal position of particles, but also on their relative energy deviations. To capture the main feature of this lumped wake, two newly defined parameters, kick divergence K_p and rotation angle R , are thus introduced. The parameter K_p is used to quantify the different energy kicks received by different particles, which is the origin of the instability, while the parameter R is used to describe the two-dimensional dependence of the lumped wake. For any given lattice, once the bunch wake over one turn in phase space is

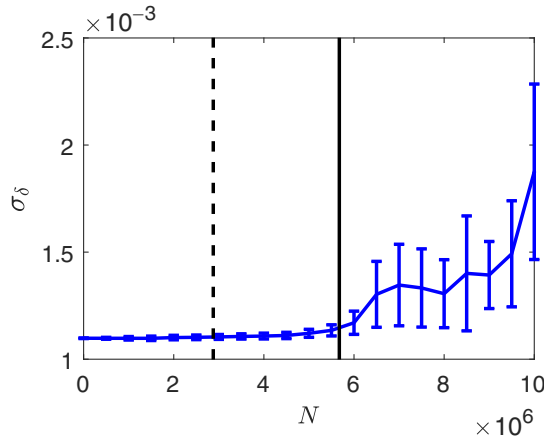


FIG. 12. Energy spread as a function of bunch population. The black dashed line represents the classical prediction while the black solid line represents the threshold predicted by our new formula.

obtained through integration, the threshold can be determined with the assistance of these two new parameters.

In practical applications, we first calculate the energy spread and utilize the global phase slippage to achieve the shortest bunch length within one turn of the storage ring. Then, we substitute the energy spread and the shortest bunch length into the formula to get the kick divergence K and instability threshold N_{th} without considering the partial phase slippage. Next, we determine the kick divergence K_p and the rotation angle R of bunch wake over one turn in phase space considering the partial phase slippage through numerical calculations based on Eqs. (13) and (14). Finally, we calculate $M_K = K/K_p$ and D using Eq. (15), and substitute all relevant parameters into Eq. (16) to obtain the instability threshold $\tilde{N}_{th}$ under general conditions. Note that compared to the time-consuming particle tracking, our method can quickly estimate the threshold of CSR induced MWI in a storage ring where the partial phase slippage is in effect.

Our results can also guide the lattice design of extremely short bunch storage rings to achieve higher instability threshold. From Eq. (16), it can be seen that there are two optimization directions for adjusting the optical functions when N_{th} is determined. One is to allow the bunch to experience greater stretching, thereby increasing M_K . The other is to introduce more head-tail exchanges among the particles in the bunch to reduce the rotation angle R of the bunch wake in phase space, thereby decreasing D . It is important to note that the partial phase slippage also affects single-particle dynamics, and both the longitudinal emittance and the energy spread are related to it. Therefore, the final design requires a comprehensive consideration.

VI. CONCLUSIONS

In this paper, we present a threshold formula for the MWI caused by CSR considering the variation of the bunch length within one turn due to the partial phase slippage. This general formula can quickly assess the instability threshold compared to the time-consuming particle tracking and can also guide the optical design of storage rings with extremely short bunch lengths to achieve a higher instability threshold. Our results show that depending on the distribution of the partial phase slippage, the instability threshold can be increased by a factor of $(1 - D)M_K$ as compared to the prediction based on the adiabatic approximation, where D represents the relative deviation due to the rotation of wake in phase space, and M_K is the multiplicative decrease in the kick divergence. Although our work focuses solely on free space CSR wake, the physical effects of the partial phase slippage are similar for other forms of wake. Therefore, our conclusions have more general implications.

It is important to note that this work only addresses one-dimensional longitudinal dynamics. In practice, when the

bunch is extremely short, the transverse-longitudinal coupling can also affect the bunch length significantly. A more accurate bunch length formula should be $\sigma_z = \sqrt{\epsilon_z \beta_z + \epsilon_x H_x}$, where ϵ_x is the horizontal emittance and H_x is the horizontal chromatic function. The effects of transverse-longitudinal coupling on collective effects are currently under investigation.

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DATA AVAILABILITY

The data supporting this study's findings are available from the authors upon reasonable request.

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