

Modeling of arbitrarily shaped resistive objects with a finite-difference time-domain electromagnetic field solver

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An implementation of surface-impedance boundary conditions in finite-difference time-domain electromagnetic field solvers was described by Oh and Schutt-Aine. This algorithm allows for modeling of the effects of objects with finite conductivity which lie along grid-conformal surfaces. Here we describe the extension of the algorithm to allow for modeling of arbitrarily shaped objects. This is accomplished by combining the model of Oh and Schutt-Aine with the locally conformal method for modeling of perfectly conducting objects developed by Dey and Mittra. The resulting hybrid algorithm yields a second-order accurate implicit field solution, in which the standard orthogonal grid is deformed only locally at the conductor-vacuum interface, which is modeled by partially filled grid cells. Field solutions inside the conductor are not sought and the approximate boundary condition is used to set the nonzero tangential electric fields along interfaces. This method obviates the need to resolve skin depths on the grid. We briefly review the Oh and Schutt-Aine algorithm and describe its implementation in the Dey-Mittra partial-cell treatment. The use of the model is then demonstrated by performing several test problems involving objects with finite conductivity in both relatively simple and more complex geometries.

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I. INTRODUCTION

Particle-in-cell (PIC) codes have been used extensively in designing and developing high-current pulsed-power accelerators such as the Z-machine at Sandia National Laboratories [1] and continue to be used in the development of next generation accelerators [2–5]. In these devices, high-current pulses, on the order of tens of MAs, are delivered to the load by a system of magnetically insulated transmission lines (MITLs). To model the behavior of the MITLs, it is necessary to include such nonideal effects [6] as the presence of anode and cathode plasmas, neutrals liberated from electrode surfaces, etc. Many of these effects can be treated effectively by PIC codes with the inclusion of the requisite surface-physics models, see e.g., Ref. [7]. Ohmic heating due to the finite conductivity of the MITL electrodes has also been recognized as an important loss mechanism [8,9]. In order to allow this effect to be accurately captured in PIC simulations, it is desirable to add this resistive loss mechanism to the surface-physics

models already employed in PIC simulations, and, ideally, this capability should be applicable to arbitrarily shaped conductors.

PIC codes generally employ the finite-difference time-domain (FDTD) method to temporally advance electromagnetic fields in the simulation domain. These codes also generally make use of an orthogonal grid in curvilinear coordinates, in which complex structures are approximated on the grid by the “stair-stepping” approximation. This can introduce errors [10] and spurious solutions [11]. Moreover, conducting objects are generally considered to be perfect electrical conductors (PECs) for which tangential electric fields are set to zero at conductor boundaries. This is, however, only an approximation, even in highly conductive metals. Ideally, in the case of objects with finite conductivity, the proper behavior at the object surfaces is obtained by including the propagation of fields throughout the object. This becomes numerically problematic in FDTD codes for “good” conductors, due to the need to spatially resolve the skin depth, $\propto (\omega\sigma)^{-1/2}$, where σ is the conductivity and ω is a characteristic angular frequency of the problem [12]. This shortcoming can be alleviated by the use of an approximate boundary condition in which the nonzero tangential electric fields at conducting boundaries are treated as a small perturbation of the PEC boundary condition. Such boundary conditions have been used widely in scattering, propagation, and waveguide analyses [13].

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In standard FDTD electromagnetic solvers, objects with an arbitrary finite conductivity can be modeled by adding an Ohm's law current density,

$$\vec{J} = \sigma \vec{E}, \quad (1)$$

to Ampere's law in the field solution. A region of the simulation grid can be assigned a finite conductivity and permittivity value, ϵ . In the absence of dielectric or magnetic losses, Ampere's law in that region can be written as

$$\epsilon \partial \vec{E} / \partial t = \nabla \times \vec{H} - \sigma \vec{E}. \quad (2)$$

Maxwell's equations are then solved on the simulation grid throughout the conducting region. In the following, we refer to this approach as either the bulk or volume conductivity model. Inspection of Eq. (2) shows that a simple FDTD approach to solving this equation requires $\sigma \Delta t / \epsilon$ to be small. Similarly, the spatial grid must resolve the skin depth of the material for all characteristic frequencies. For finite, but large, conductivities, these constraints may become prohibitive.

In the alternate approach discussed above, high-conductivity regions of the simulation space are treated by the application of a surface-impedance boundary condition (SIBC) on the vacuum-conductor interface. In this case, field solutions inside the conductor are not sought, and the approximate boundary condition is used to set the nonzero tangential electric fields along the interface. This method clearly assumes conducting objects that are large compared to the conducting skin depths, and it will break down for skin depths on the order of the object dimensions.

The surface-impedance concept was first introduced by Leontovich [14] and considered by many other authors [13,15–20]. An efficient implementation of the use of the frequency-domain SIBC in a time-domain code with an orthogonal grid was described by Oh and Schutt-Aine [21]. In their paper, a relatively simple recursive convolution algorithm was developed to set tangential electric fields along conducting surfaces. This algorithm was limited, however, to applications in which the vacuum-conductor interfaces are confined to grid-conformal boundaries. We have extended the SIBC algorithm to allow for arbitrarily shaped conducting objects.

In order to mitigate the effect of the stair-stepping approximation, a simple but accurate three-dimensional model for analysis of curved perfectly conducting objects was described by Dey and Mittra [11,22], in which the orthogonal grid is modified only locally in cells which are partially filled by the perfect conductor. In this paper, we describe the extension of the Dey-Mittra algorithm to allow for its use with finite-conducting, rather than perfect-conducting, objects, by making use of the recursive SIBC treatment of Oh and Schutt-Aine. This generalized algorithm has been implemented into the hybrid particle-in-cell code CHICAGO [23], and simulations have been

performed for test problems including both relatively simple and more complex geometries in both 2D-cylindrical and 3D-Cartesian coordinates. Results are compared to both theoretical results and highly resolved simulations making use of a bulk conductivity model, as in Eq. (2).

This paper is organized as follows: In Sec. II, we briefly review the original Oh and Schutt-Aine algorithm in which the frequency-domain surface-impedance boundary condition is converted to a convolution integral in the time domain. By fitting the normalized surface impedance to a sum of simple first-order rational functions, Oh and Schutt-Aine derive an implicit boundary condition, suitable for use at vacuum-conductor interfaces that lie along grid-conformal boundaries in FDTD codes. The fitting of the frequency-dependent surface-impedance function is considered in Sec. III. A new more accurate fitting is also introduced in this section. In Sec. IV, we describe how the SIBC can be extended for use with the locally conformal partial-cell treatment of Dey-Mittra to allow for modeling of arbitrarily shaped finite-conducting objects. The use of the resulting fully implicit second-order field advance, as implemented into CHICAGO, is then demonstrated in Sec. V in a series of test problems. Conclusions and suggestions for future work are presented in Sec. VI.

II. REVIEW OF SURFACE IMPEDANCE BOUNDARY CONDITION MODEL OF OH AND SCHUTT-AINE

In this section, we briefly review the original SIBC model developed by Oh and Schutt-Aine [21]. To illustrate the use of the model, we consider a lossy dielectric half-space as shown in Fig. 1. Medium 1, shown on the left, is assumed to be nonconducting: $\sigma_1 = 0$ and is characterized by a permittivity and permeability of ϵ_1 and μ_1 , while medium 2, on the right, is characterized by a nonzero but finite conductivity, σ_2 , ϵ_2 , and μ_2 . In the nonconducting medium exist the arbitrary fields $\vec{E}$ and $\vec{H}$.

The boundary condition at the interface between medium 1 and 2, in the frequency domain, is then written as

$$\vec{E}_t(\omega) = Z_c(\omega)[\hat{n} \times \vec{H}_t(\omega)], \quad (3)$$

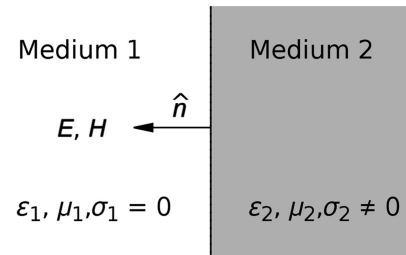


FIG. 1. Schematic of lossy dielectric half-space. Adapted from Ref. [21].

where $\hat{n}$ is the surface normal, and $\vec{E}_t$ and $\vec{H}_t$ are the tangential components of the electric and magnetic fields at the interface (i.e., $\vec{E}_t = \hat{n} \times \vec{E}$). The surface impedance is given by

$$Z_c(\omega) = \eta_2 \sqrt{\frac{-i\omega\epsilon_2/\sigma_2}{1 - i\omega\epsilon_2/\sigma_2}}, \quad (4)$$

where

$$\eta_2 = \sqrt{\mu_2/\epsilon_2}. \quad (5)$$

Note that, as in Ref. [21], MKS units are used (i.e., σ/ϵ has units of inverse time). The boundary condition in Eq. (3) can, in principle, be extended to arbitrarily shaped resistive objects in the limit that the both the object dimensions and radius of curvature are large compared to the skin depth.

Making the substitutions $-i\omega \rightarrow s$ and $\sigma_2/\epsilon_2 \rightarrow a$, Eq. (3) may be rewritten as

$$Z_c(s) = \eta_2 \sqrt{\frac{s/a}{1 + s/a}}. \quad (6)$$

Defining $s' = s/a$, a normalized impedance function may be written as

$$Z_N(s') = Z_c(s)/\eta_2 = \sqrt{\frac{s'}{1 + s'}}. \quad (7)$$

In the time domain, the tangential electric and magnetic fields are obtained by the inverse Fourier transform of Eq. (3) which yields the convolution integral

$$\vec{E}_t(t) = Z_c(t) * [\hat{n} \times \vec{H}_t(t)], \quad (8)$$

where $Z_c(t)$ is given by the inverse Laplace transform of $Z_c(s)$.

To perform the convolution integral in Eq. (8), Oh and Schutt-Aine assume that the normalized impedance function may be written as a sum of rational fractions,

$$Z_N(s') = 1 - \sum_{i=1}^L \frac{C_i}{\omega_i + s'}, \quad (9)$$

where L is the integer number of terms in the sum, and C_i and ω_i are fitting constants. Performing the convolution integral with this impedance model in the discrete time domain [$\vec{E}_t(t) \simeq \vec{E}_t(n\Delta t) \equiv \vec{E}_t^n$] yields the recursive result

$$\vec{E}_t^n = \eta_2 [\hat{n} \times \vec{H}_t^n] - \sum_{i=1}^L \vec{A}_i^n, \quad (10)$$

$$\vec{A}_i^n = p_{i1} [\hat{n} \times \vec{H}_t^n] + p_{i2} [\hat{n} \times \vec{H}_t^{n-1}] + p_{i3} \vec{A}_i^{n-1}, \quad (11)$$

where

$$\begin{aligned} p_{i1} &= \eta_2 \frac{C_i}{\omega_i} [1 + (e^{-a\omega_i\Delta t} - 1)/(a\omega_i\Delta t)], \\ p_{i2} &= \eta_2 \frac{C_i}{\omega_i} \{1/(a\omega_i\Delta t) + e^{-a\omega_i\Delta t} [1 + 1/(a\omega_i\Delta t)]\}, \\ p_{i3} &= e^{-a\omega_i\Delta t}. \end{aligned} \quad (12)$$

In the Oh and Schutt-Aine treatment, tangential electric fields on conductor edges are set by Eq. (10), which make use of the $\vec{A}_i$ vectors obtained from Eq. (11). Maxwell's equations are then solved with this effective boundary condition.

III. FITTING OF NORMALIZED IMPEDANCE FUNCTION

Use of the SIBC algorithm depends on fitting the normalized impedance to the ansatz of Eq. (9). Oh and Schutt-Aine have performed nonlinear curve fitting to fit Eq. (7) to Eq. (9) for $L = 6, 7$, and 8 . The constants C_i and ω_i have been tabulated in Ref. [21]. Progressively better fits can be obtained by going to larger values of L . To demonstrate this, we have performed the curve fitting procedure for $L = 9$. For reference, the resulting fitted values for C_i , and ω_i , in this case, are given in the Appendix. We have plotted the relative error, $|(Z_N(s') - Z_{\text{fit}}(s'))/Z_N(s')|$, for all four fits in Fig. 2 over a range of s' from 10^{-6} to 10^2 . Over this entire range, the $L = 9$ fit is accurate to better than a percent and is about an order of magnitude better than the fits of Oh and Schutt-Aine. For $s' < 10^{-6}$ the fits for all orders of L up to 9 begin to break down.

There is no reason, in principle, that L cannot be increased to values greater than 9 . However, it can be shown that, regardless of L , Eq. (9) will deviate from Eq. (7) for

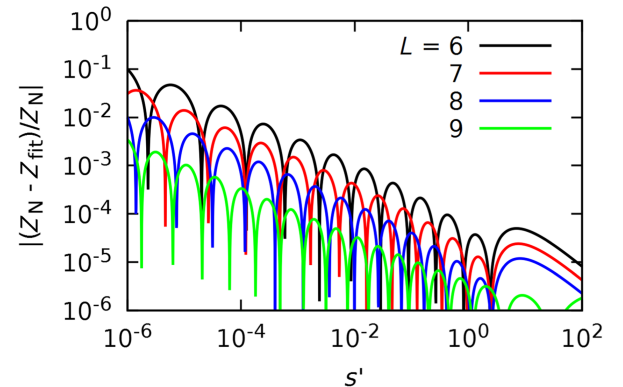


FIG. 2. Relative error of nonlinear fitting of surface impedance [Eq. (7)] to a sum of L rational functions [Eq. (9)]. Result are shown for $L = 6, 7, 8$, and 9 . The error is seen to drop considerably as L is increased. All fits, however, break down at small enough values of s' .

small enough values of s' . To lowest order in s' , Eq. (9) may be written as

$$Z_N(s') \simeq \left(1 - \sum_{i=1}^L C_i/\omega_i\right) + \left(\sum_{i=1}^L C_i/\omega_i^2\right)s', \quad (13)$$

which asymptotes either to a constant value, or linearly in s' , if the sum of the C_i/ω_i coefficients is identically one, whereas the actual surface impedance will asymptote to zero as $s'^{1/2}$. Since $|s'| = \omega\epsilon/\sigma$, this sets a minimum frequency (or maximum conductivity) for which the fit is accurate to a desired tolerance. The use of the SIBC algorithm then has two separate requirements. As already mentioned, for a given σ , the frequency must be large enough so that the skin depth, $\propto (\omega\sigma)^{-1/2}$, is small compared to relevant length scales. But must also be large enough so that the rational function fitting procedure is valid. It is important that the user of the algorithm be aware of both of these restrictions. For all the sample problems considered below, we have assured the skin depths are relatively small and that $\omega\epsilon/\sigma > 10^{-6}$.

We can demonstrate the accuracy of the fits by considering a normally incident TEM wave on a lossy dielectric half-space, as shown in Fig. 1, where $\epsilon_1 = \epsilon_2 = \epsilon_o$ and $\mu_1 = \mu_2 = \mu_o$. The analytic reflection coefficient is given by

$$\Gamma = \frac{Z_C(\omega) - Z_o}{Z_C(\omega) + Z_o}, \quad (14)$$

where $Z_o = \sqrt{\mu_o/\epsilon_o}$ is the impedance of free space. The exact answer is obtained from Eq. (4). Using the fitted approximation of Eq. (9) yields

$$\Gamma_{\text{fit}} = -\left(\sum_{i=1}^L \frac{aC_i}{a\omega_i - i\omega}\right) \left(2 - \sum_{i=1}^L \frac{aC_i}{a\omega_i - i\omega}\right)^{-1}. \quad (15)$$

The magnitude of the reflection coefficients is shown in Fig. 3 over a frequency range from 0 to 10 GHz for conductivities of 0.2 S/m (top) and 200 S/m (bottom). The exact results are shown as black lines. The results showing the order 6 and order 9 fits are shown as black and red dots, respectively. At 0.2 S/m, there is a good agreement between the exact results and both the $L = 6$ and $L = 9$ fits. At the low conductivity value, the order 6 result is in good agreement with the exact result, but, at the high-conductivity value, it is less accurate than the order 9 result over the frequency range shown. In our implementation of the algorithm, the user may choose L as 6, 7, 8, or 9. The value of 9 is the default and is used throughout in the test problems discussed below.

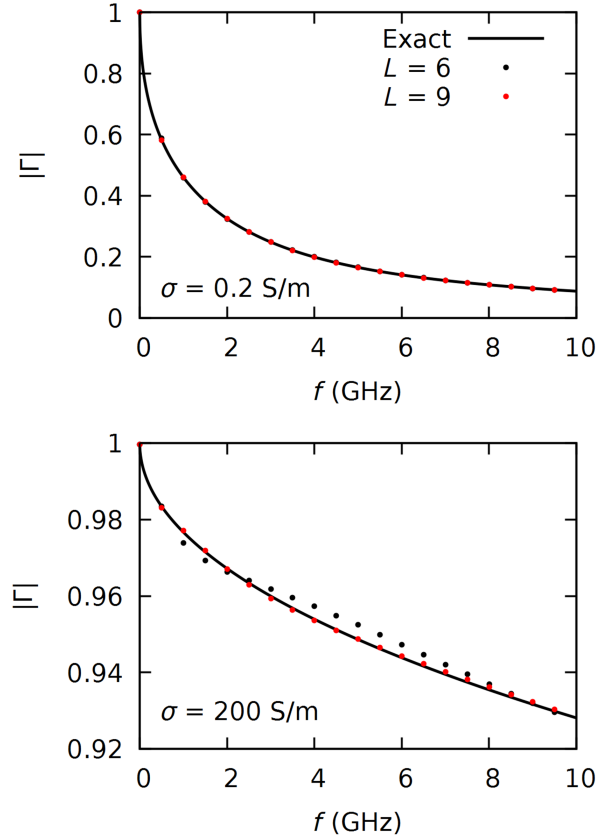


FIG. 3. Exact-analytic and fitted-approximate spectral-reflection coefficients of a lossy dielectric half-space. At higher σ values, the $L = 9$ fit is found to be more accurate than fits at lower values of L .

IV. EXTENSION OF SIBC MODEL TO ARBITRARILY-SHAPED CONDUCTORS MODELED BY PARTIALLY FILLED CELLS

The original SIBC algorithm of Oh and Schutt-Aine was formulated to be applied along grid-conformal boundaries, in which tangential electric fields are set by tangential magnetic fields components located one half cell into the vacuum side of the boundary. In this section, we describe the extension of the SIBC concept to allow for its use with the locally conformal partial-cell treatment of Dey and Mittra [11,22]. The global second-order spatial accuracy of the standard FDTD field solution is dropped to first order by the “step-stepping” approximation of non-grid-conformal conducting surfaces [24]. The Dey and Mittra algorithm has been shown to restore global second-order spatial accuracy to the field solution [25]. This algorithm, previously implemented into CHICAGO [26], was intended for use in modeling arbitrarily shaped perfectly conducting objects, in which the orthogonal grid is modified only locally at interfaces, in order to accommodate for the curvature of the object. In the following, we describe how the original Dey-Mittra algorithm for perfect conductors can be combined with the Oh and Schutt-Aine

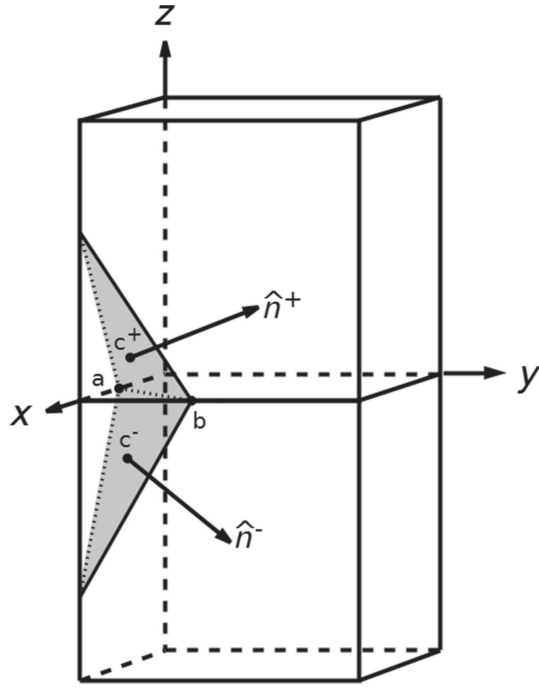


FIG. 4. Two adjacent grid cells in 3D-Cartesian coordinates. The approximate surface of a conducting object is shown in each cell in gray.

algorithm to account for arbitrarily shaped objects with finite conductivity.

We describe the implementation of the algorithm in 3D-Cartesian coordinates, but the extension to curvilinear coordinates is straightforward. A schematic drawing of two adjacent grid cells is shown in Fig. 4, in which both the upper and lower cells are intersected by a conducting object. The approximate surface between conducting and nonconducting regions in each cell is shown as a gray surface. Assuming that the local radius of curvature of the object in each cell is much greater than the cell dimensions, the conducting object surface can be approximated by a plane defined by the surface center, c and a normal vector, $\hat{n}$. By convention, the normal vector points outward from the conducting material into the adjacent nonconducting medium. The $+$ and $-$ superscripts in the figure are used to denote the surfaces in the upper and lower cells, respectively. In the large radius-of-curvature approximation, $\hat{n}^+ \simeq \hat{n}^-$. The figure shows a rather large difference in the direction between the upper and lower cell normal vectors, but this is for illustrative purposes only. In this same approximation, the intersection of the object surface and the cell faces may be approximated by straight lines. For example, in the $z = 0$ plane, the interface between conducting and nonconducting cells is given by the line a from b . This is true for both the upper and lower cells.

The upper cell in Fig. 4 is denoted by the cell indices i, j, k , and lower cell by $i, j, k - 1$. The shared interface at $z = 0$ between the upper and lower cells is shown in Fig. 5.

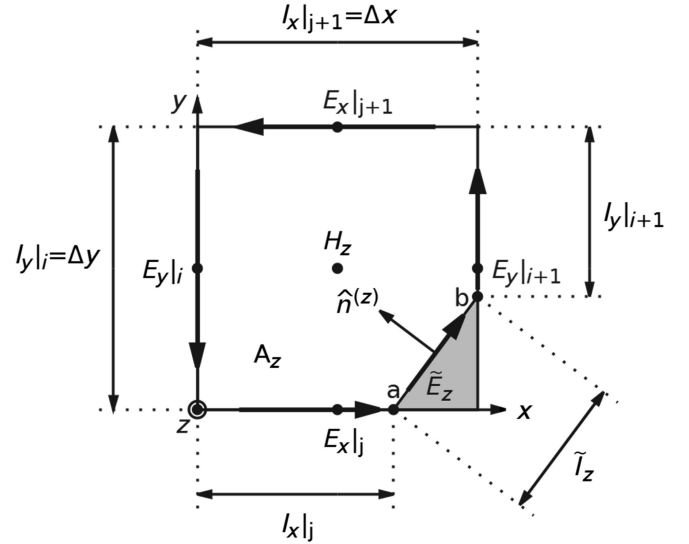


FIG. 5. Plot of the cell face shared by the two cells in Fig. 4 in the $z = 0$ plane.

The conducting region in this plane is shown in gray, and the interface between conducting and nonconducting regions is represented again by the line from a to b . For this cell face, designated by the indices i and j , we have shown with dots the positions of the magnetic and electric field components on the standard Yee cell [27]. The H_z component is located in the center of the cell face, and, by convention, “belongs” to the upper cell (i, j, k). The E_y component on the left belongs to the i th cell, while the component on the right belongs to the adjacent $i + 1$ cell.

Applying Stokes’s theorem to Faraday’s law along the path denoted by the thick arrows in Fig. 5, we obtain

$$\mu A_z \partial H_z / \partial t = -[E_y l_{y|i+1} - E_y l_{y|i} - E_x l_{x|j+1} + E_x l_{x|j}] - \tilde{E}_z \tilde{l}_z, \quad (16)$$

where the l values give the lengths along the cell-face edge segments of the path integral. Since there is no intersection with a conducting medium along the left and top, $l_{x|j+1}$ and $l_{y|i}$ are given by Δx and Δy , respectively. As in Ref. [11], H_z is considered to have a constant value over the nonconducting surface area denoted A_z , and the electric field values are assumed to be constant along the entire path lengths along the cell-face edges. The contribution of the path integral, along the interface from a to b , is denoted by the product $\tilde{E}_z \tilde{l}_z$, where $\tilde{E}_z$ is the component of the electric field along the path length $\tilde{l}_z$. This field is assumed to be constant. In this context, the subscripts applied to both $\tilde{E}_z$ and $\tilde{l}_z$ are used to designate only that they are associated with the calculation of H_z .

If $\hat{n}^{(z)}$ denotes the outward unit vector normal in the plane to the interface along the line from a and b , then

$$\tilde{E}_z = E_x n_y^{(z)} - E_y n_x^{(z)}. \quad (17)$$

We note that Eq. (17) is merely a geometric relation, where E_x and E_y represent (as yet undetermined) constant electric field components along the path and are not to be associated with any of the electric field components shown as dots in the figure. Again assuming planar interface surfaces of conducting objects with grid cells (as in Fig. 4), one can demonstrate that

$$\begin{aligned} n_x^+ &= n_x^- \equiv n_x, \\ n_y^+ &= n_y^- \equiv n_y, \end{aligned} \quad (18)$$

from which it follows that

$$\hat{n}^{(z)} = c_z(n_x \hat{x} + n_y \hat{y}), \quad (19)$$

where

$$c_z = 1/\sqrt{n_x^2 + n_y^2}. \quad (20)$$

In the upper cell of Fig. 4, for which $\vec{E}_t = \hat{n}^+ \times \vec{E}$, simple geometric arguments show that

$$\tilde{E}_z = -c_z E_{tz}. \quad (21)$$

Making use of Eq. (21), a time-centered second-order accurate leapfrog implementation of Eq. (16) can be written as

$$\mu A_z \left(\frac{H_z^{n+1} - H_z^n}{\Delta t} \right) = -S_z^{n+1/2} + c_z E_{tz}^{n+1/2} \tilde{l}_z, \quad (22)$$

where the superscript n denotes the time level, and we have defined

$$S_z^n = E_y l_y |_{i+1}^n - E_y l_y |_i^n - E_x l_x |_{j+1}^n + E_x l_x |_j^n. \quad (23)$$

This quantity accounts for the path-integral contributions along grid-conformal cell edges, which make use of only standard Yee-cell electric field values. In the limit of a perfect conductor, $E_{tz} = 0$, and Eq. (22) reverts to the treatment of Dey and Mittra. A second-order accurate implicit treatment can be obtained from the substitutions of

$$\begin{aligned} S_z^{n+1/2} &\rightarrow (S_z^{n+1} + S_z^n)/2 \\ E_{tz}^{n+1/2} &\rightarrow (E_{tz}^{n+1} + E_{tz}^n)/2 \end{aligned} \quad (24)$$

into the right-hand side of Eq. (22).

Equations (10) and (11) may be used to eliminate E_{tz}^{n+1} from Eq. (24). Combining these results, we obtain, after some rearranging,

$$\begin{aligned} &\left[1 + c_z \alpha_z \tilde{l}_z \left(\eta_2 - \sum_i p_{i1} \right) \right] H_z^{n+1} + \alpha_z S_z^{n+1} \\ &= \left(1 + c_z \alpha_z \tilde{l}_z \sum_i p_{i2} \right) H_z^n - \alpha_z S_z^n + \alpha_z c_z \tilde{l}_z \left(E_{tz}^n - \sum_i p_{i3} A_{iz}^n \right), \end{aligned} \quad (25)$$

where

$$\alpha_z = \Delta t/2 \mu A_z. \quad (26)$$

We note also that in the derivation in Eq. (25), we have made use of the fact that $\hat{n} \cdot \vec{H} \simeq 0$. This approximation is appropriate for $(\omega\epsilon/\sigma)^{1/2} \ll 1$ (see, for example, Ref. [12]).

A completely analogous treatment is performed for the x and y components of Faraday's law, with the results obtained from simple cyclic permutations of the formulas given above for H_z . As in the original Dey and Mittra algorithm, there is no modification of Ampere's law due to the presence of partially filled cells. The resulting coupled field equations are solved implicitly using iterative methods. In this formulation, only E and H components at standard positions in the Yee cell are treated implicitly. The solution of the field equations advances the grid field values from

time step n to $n+1$, by using only old values (time level n) of $\vec{E}_t$ and $\vec{A}_i$. That is, the number of linear equations involved in the field solution does not need to be extended to include the tangential electric fields along partial-cell edges. After the field solution yields the updated magnetic fields, $\vec{A}_i$ can be updated by Eq. (11), and $\vec{E}_t$ by Eq. (10). The use of the partial-cell treatment restores second-order spatial-accuracy and the time-centered temporal advance of Eq. (25) ensures second-order temporal accuracy.

We now consider briefly how the generalized partial-cell treatment reverts to the original grid-conformal boundary treatment of the original SIBC algorithm. In Fig. 6, we have modified the 2D Cartesian cell-face plot of Fig. 5 so that the conductor edge is now parallel to the y -axis and $\hat{n}^{(z)} = -\hat{x}$. Note that, in the partial-cell treatment, the conductor edge need not be aligned with the cell edge, i.e., $0 \leq l_x|_j \leq \Delta x$. For $l_x|_j < \Delta x$, the update of H_z is still given by Eq. (25),

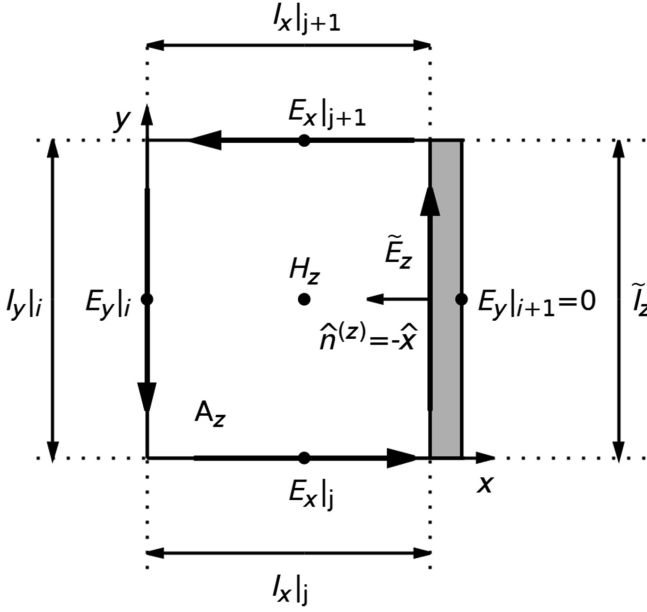


FIG. 6. Plot of 2D cell-face for cell intersected by a lossy half-space.

with the simple modification that $E_y|_{i+1}$ is set to zero, because it is completely embedded within a conductor. The surface tangential field, $\tilde{E}_z$, is, of course, nonzero. We consider now the limit that $l_x|_j \rightarrow \Delta x$. In this case, for which $\tilde{E}_z \rightarrow E_y|_{i+1}$ and $A_z \rightarrow \Delta x \Delta y$, the partial-cell treatment reverts to the grid-conformal Oh and Schutt-Aine model. For $l_x|_j < \Delta x/2$, the magnetic field value, H_z , at its location on the Yee cell is also fully embedded in the conductor. In contrast with the embedded electric field component, H_z is *not* set to zero, and its value can be thought of as representing the constant field value in the vacuum nonconducting area, $A_z = l_x|_j \Delta y$.

In the limit that $l_x|_j \rightarrow 0$, the area A_z becomes vanishing small. As discussed by Dey and Mittra [11], this can result in strong constraints on the Courant-Friedrichs-Lewy (CFL) stability condition for an explicit solver, with the minimum vacuum area on a cell face constraining the maximum allowed time step rather than the full cell size. The implicit treatment, however, is found to be insensitive to CFL instabilities due to small vacuum areas in partially filled cells. The iterative solution of the implicit field equations may, however, require more iterations when small cell face areas are present, i.e., numerical stability is retained but run time increases. Large iteration numbers can be avoided, if desired, by setting a minimum surface area, $A_{\min}$. If the cell-face area is less than $A_{\min}$, it is replaced by $A_{\min}$. There are no similar modifications to the l coefficients. In our numerical studies, a nominal value of $A_{\min} = 10^{-3} A_{\text{face}}$ was found to curtail large iteration numbers without any noticeable degradation of the results, where A_{face} is the full cell-face area.

V. SAMPLE SIMULATION RESULTS

A. Reflection from lossy dielectric half-space

As an initial test problem, we consider reflection of a normally incident TEM wave on a lossy dielectric half-space. This is the same test used by Oh and Schutt-Aine which led to Fig. 4 of Ref. [21]. We consider a half-space ($\hat{n} = -\hat{x}$) characterized by $\sigma = 2$ S/m and a vacuum permittivity and permeability. The y-polarized electric field is injected into the simulation at the open vacuum boundary with a temporal pulse shape given by

$$E_y(t) \propto e^{-(t-t_o)^2/t_w^2}, \quad (27)$$

where $t_o = 1.3$ ns and $t_w = 37.5$ ps. The reflection coefficient is calculated by obtaining $E_y(t)$ and $H_z(t)$ at the vacuum-conductor interface. The frequency-domain impedance is calculated from Fourier transforms of the fields, $Z(f) = E_y(f)/H_z(f)$, and the reflection coefficient is given by

$$\Gamma = \frac{Z(f) - Z_o}{Z(f) + Z_o}. \quad (28)$$

The 1D simulation cell size is $\Delta x = 0.01875$ cm, and the time step is $\Delta t = 0.625$ ps. The fact that $c\Delta t/\Delta x \simeq 1$ minimizes the effect of numerical dispersion of the TEM wave propagation in the vacuum region. Due to the implicit treatment, larger time steps are not, however, CFL unstable. It is also possible to model the conducting region with a volume conductivity model in which a term of $-\sigma \vec{E}$ is added to the right-hand side of Ampere's law, as in Eq. (1). The use of such a model requires the adequate resolution of the skin depth, which in this case is well fulfilled out to frequencies of many GHz. In this regime, it is possible to compare the surface-impedance model to the volume conductivity model. This test is performed in order to validate the model. However, it is important to recall that the major utility of SIBC model lies in the fact that it does *not* require resolution of the skin depth. This comes, of course, at the expense of loss of knowledge of the exact field structure within the conducting medium.

In Fig. 7, the magnitude of the reflection coefficient, Γ , is plotted as a function of frequency out to a value of 10 GHz ($\sigma/\epsilon_o = 225$ GHz). The analytic result, given by Eq. (14), is shown as a black curve. The simulation results for the volume conductivity model and the SIBC model are shown as red and blue squares, respectively. The figure shows that there is excellent agreement between the two simulation models with the exact solution. This is true over nearly the entire frequency range. Only at frequencies above about 6 GHz is there some evidence that the volume conductivity model yields a slightly higher value of $|\Gamma|$ than SIBC. At higher frequencies, the skin depth, δ , decreases and discretization errors on the order of $\Delta x/\delta$ become more

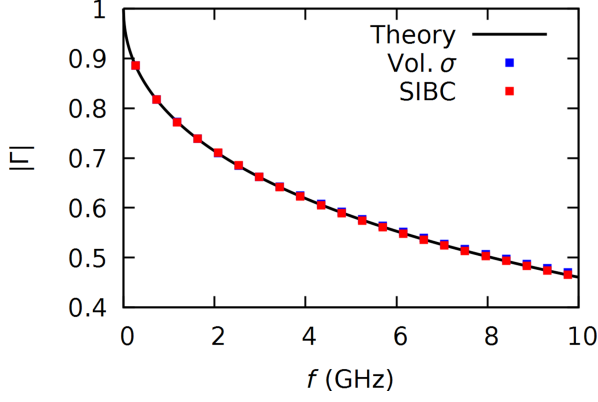


FIG. 7. Magnitude of the reflection coefficient of a TEM wave normally incident on a half-space characterized by $\sigma = 2$ S/m, ϵ_o , and μ_o . Results are shown for the exact analytic result given by Eq. (14) (black) and simulations using a volume (bulk) conductivity model (blue squares) and the SIBC model (red squares).

prominent for the bulk treatment. The observed discrepancy decreases when the cell size is dropped further. At larger cell sizes, when the volume conductivity model breaks down, the surface-impedance model continues to give good agreement with the analytic result for this test problem.

B. Attenuation in coaxial transmission line

We now consider TEM wave propagation in a lossy coaxial transmission line with an inner radius a and outer radius b , in which the inner and outer conductors are characterized by a conductivity of σ . The area between the conductors is taken to be vacuum. A schematic illustration of the problem is shown in Fig. 8. We consider a forward propagating TEM wave:

$$\vec{E}, \vec{H} \propto e^{i(kz - \omega t)}. \quad (29)$$

In the high-conductivity limit ($\omega\epsilon/\sigma \ll 1$), the wave number can be expressed as (see, for example, Ref. [28])

$$k = \sqrt{\omega C(\omega L + iR)} \equiv k_r + ik_i \quad (30)$$

where

$$\begin{aligned} L &= \frac{\mu_o}{2\pi} \ln(b/a), \\ C &= \frac{2\pi\epsilon_o}{\ln(b/a)}, \\ R &= \frac{1}{2\pi\delta\sigma} \left(\frac{1}{a} + \frac{1}{b} \right), \end{aligned} \quad (31)$$

are the inductance, capacitance, and resistance per unit length, respectively, and the skin depth is given by

$$\delta = \sqrt{\frac{2}{\omega\mu_o\sigma}}. \quad (32)$$

The expression for the resistance per unit length also assumes that both a , and the thickness of the outer conductor, are $\gg \delta$.

To compare the algorithm to the high-conductivity theory outlined above, we perform a 2D- rz simulation, in which a sinusoidal TEM wave of frequency, f , is injected into the left-hand boundary of a coaxial line, with $a = 10$ cm, $b = 20$ cm, and $\sigma = 50$ S/m. We consider frequencies in the range of 25–200 MHz, for which $10^{-6} < \omega\epsilon_o/\sigma \ll 1$, and δ ranges from around 1.5–0.5 cm.

In the simulations, we calculate the time-dependent Poynting flux of the fields over the cross-sectional area at fixed longitudinal positions: $W(z, t)$. Choosing “input” and “output” longitudinal locations, z_i and z_o , as in Fig. 8, we can calculate the attenuation coefficient as

$$k_i = -\frac{1}{2(z_o - z_i)} \ln\left(\frac{W_o}{W_i}\right), \quad (33)$$

where

$$W_i = \langle W(z_i, t) \rangle, \quad (34)$$

$$W_o = \langle W(z_o, t) \rangle, \quad (35)$$

and the brackets $\langle \rangle$ denote time-averaging over many periods (at steady state). These numerical results can then be compared to Eq. (30).

We first perform a 2D cylindrical simulation using the volume conductivity model. Square cells are used in the simulation: $\Delta r = \Delta z$, with the time step set to 0.9 times the CFL limit, i.e.,

$$c\Delta t = 0.9 \times (\Delta r^{-2} + \Delta z^{-2})^{-1/2}. \quad (36)$$

The simulation domain is extended to a maximum radius, $r_{\max}$, such that $r_{\max} - b$ is many skin depths thick, as in Fig. 8. In the top plot of Fig. 9, we have plotted

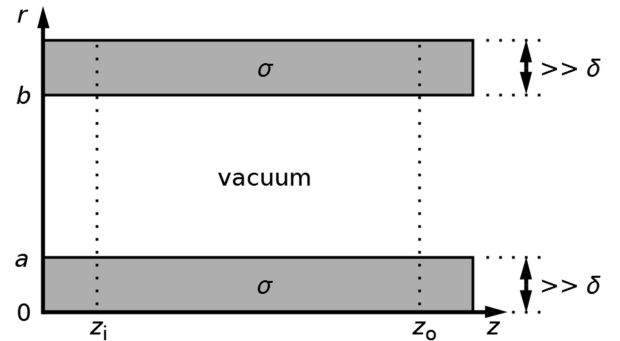


FIG. 8. Schematic illustration of geometry of a lossy coaxial transmission line.

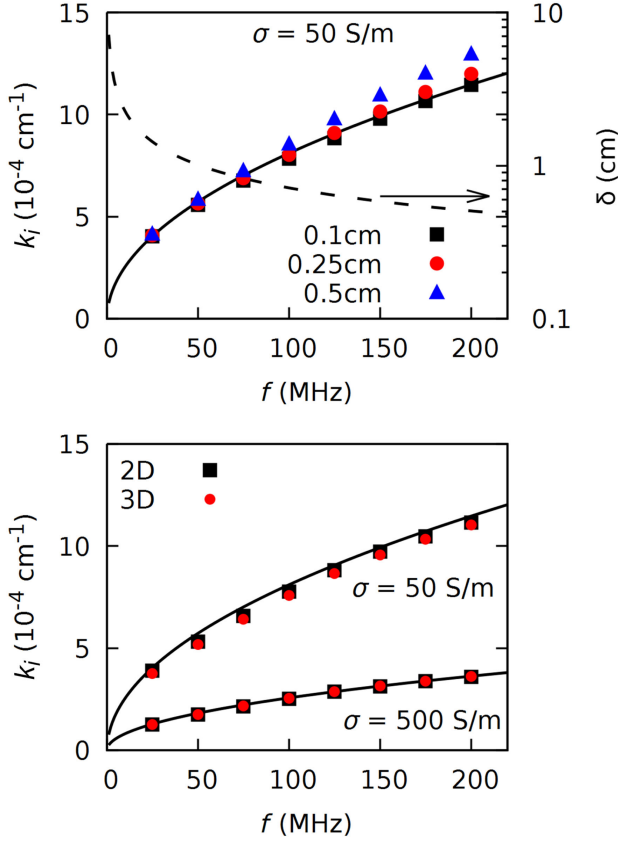


FIG. 9. Theory and simulation results for propagation of the TEM mode in a lossy transmission line with $a = 10$ cm, $b = 20$ cm. Top: attenuation in 2D-cylindrical simulations using the volume conductivity model with $\sigma = 50$ S/m, and varying cell sizes. The approximate theory result is shown as a black line. The skin depth as a function of frequency is plotted as a dashed line. Bottom: results using SIBC model with partially filled cells in both 2D-cylindrical and 3D- Cartesian coordinates (cell size of 2 cm in both cases).

simulation results for k_i as a function of frequency for cell sizes of 0.1 cm (black squares), 0.2 cm (red circle), and 0.5 cm (blue triangles). The approximate theoretical result, obtained from Eq. (30), is shown as a black line. For reference, we have also plotted the skin depth as a function of frequency as a dashed line. For a cell size of 0.1 cm, there is good agreement between the theory and volume conductivity model. In this case, the skin depth is reasonably well resolved in the simulation throughout the entire frequency range. At larger cell sizes, the simulation results begin to diverge from the theory at higher frequencies for which the skin depth becomes increasingly poorly resolved.

In the bottom plot of Fig. 9, 2D-cylindrical results, using the SIBC method in conjunction with partially filled cells, are shown as black squares. These results, which show generally good agreement with the theory, were obtained with a cell size of 2 cm, which exceeds the skin depth across the entire chosen frequency range. The results at this coarse grid resolution (the vacuum gap is modeled by only

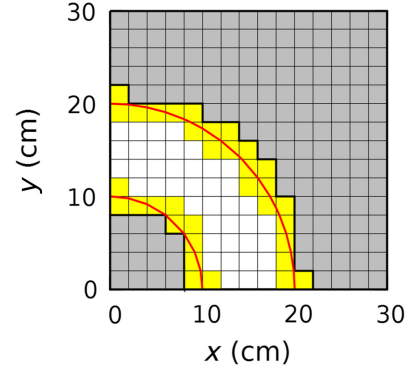


FIG. 10. Spatially discretized cross section of coaxial transmission line in 3D-Cartesian coordinates. Grid cells ($\Delta x = \Delta y = 2$ cm) are shown as thin black lines. Red lines show the boundary between conducting and nonconducting regions in partially filled cells (yellow). Thick black lines show the boundary between partially filled cells and fully conducting cells (gray).

five cells in the radial direction) are essentially identical to simulations with greater spatial resolution, demonstrating that, in contrast to the bulk-conductivity model, the SIBC method does not require resolution of the skin depth.

A more careful examination of the results in the bottom plot of Fig. 9 shows that, at 50 S/m, the simulation results are slightly lower than the theory across most of the frequency range. This discrepancy largely vanishes at a conductivity value of 500 S/m. This is believed to be due to the fact that the approximate theory result given above makes use of the approximation that $a \gg \delta$, which is better satisfied at higher conductivities.

In 2D-cylindrical coordinates, the coaxial transmission line problem is not a particularly stringent test of the partial-cell modeling, as the conductor surfaces are parallel to the grid lines, as in Fig. 6. This corresponds to an effectively infinite radius-of-curvature. In 3D-Cartesian coordinates, the situation is more complex. For a cubic grid with 2-cm cells, the cross section of the transmission line, in spatially discretized form, is shown in Fig. 10. Red lines show the boundary between conducting and nonconducting regions in the partially filled cells (yellow), while thick black lines show the boundary between partially filled cells and cells fully embedded in the conducting region (gray). Cells shown in white are fully embedded in vacuum.

The 3D-Cartesian results, performed with 2-cm cubic cells and a time step 0.9 times the CFL limit, are shown as red circles in the bottom plot of Fig. 9. Very good agreement is found between the 2D and 3D results at both 50 and 500 S/m, even for the extremely coarse transverse spatial resolution seen in Fig. 10.

C. Generalized coaxial transmission line

As a final example, we consider again the case of a coaxial transmission line but now allow the inner and outer

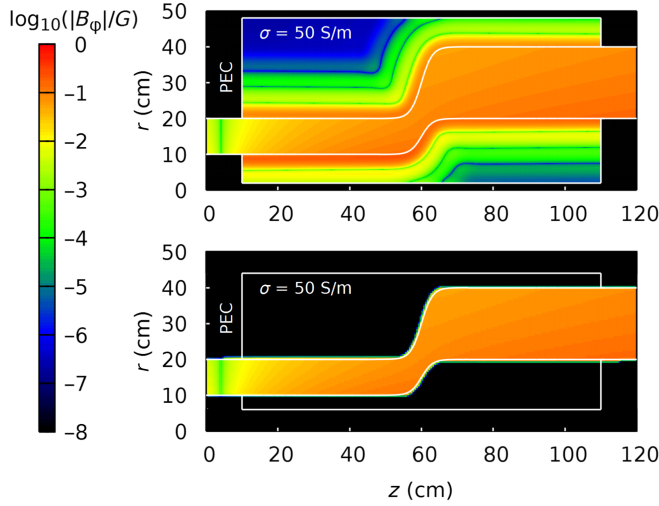


FIG. 11. Results from 2D-cylindrical generalized transmission line simulations with $\sigma = 50$ S/m, $f = 25$ MHz. The ratio of $b(z)/a(z) = 2$ is constant throughout. Contours of the azimuthal component of B are shown for (top) the volume conductivity model ($\Delta r = \Delta z = 0.1$ cm) and (bottom) the SIBC model with partially filled cells ($\Delta r = \Delta z = 0.5$ cm).

conductor radii to be a functions of z . The conductor boundaries are shown as white lines in Fig. 11. We note the ratio of $b(z)/a(z)$ is fixed at a constant value of 2. The impedance of the transmission line is then constant and reflection is negligible. There is a rather drastic and nonadiabatic change in conductor radius in the longitudinal region between $z = 60$ cm and 70 cm. This shape is chosen to introduce relatively small radii of curvature into the simulation geometry in both 2D-cylindrical and 3D-Cartesian coordinates. The conductivity is again chosen to be 50 S/m, and the line is driven by a sinusoidal TEM wave of 25 MHz ($\delta \approx 1.5$ cm), with a linear temporal ramp of many wave periods.

The top plot of Fig. 11 shows the contours of $|B_\phi|$ (on a log scale) at a fixed time value for a 2D-cylindrical simulation using the bulk conductivity model for cells with $\Delta r = \Delta z = 0.1$ cm. The field value is seen to fall off rapidly normal to the conductor surface, although a small residual field remains at the boundary with the PEC region at large and small radii. The bottom plot shows the corresponding simulation results for the SIBC model with partially filled cells ($\Delta r = \Delta z = 0.5$ cm). Comparison with the top plot shows that the field contours in the vacuum region are essentially the same for the two methods.

In Fig. 12, we have plotted the cycle-averaged ratio, W_o/W_i , as a function of frequency, for $z_i = 5$ cm and $z_o = 115$ cm. At 50 S/m, there is relatively good agreement between the 2D-bulk conductivity model (blue triangles) and the 2D-SIBC model (black squares). The bulk model gives slightly different results, in part, due to the effect of non-negligible reflections of the field at the interface between the finite and infinite conductivity

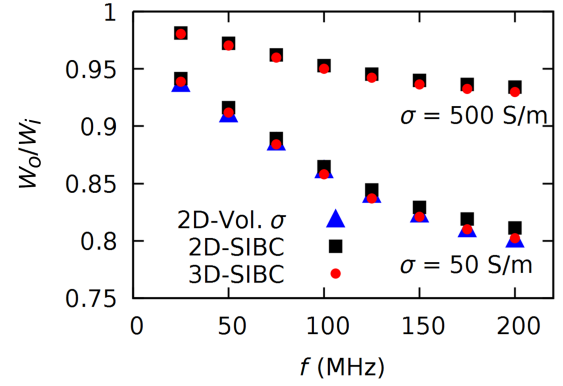


FIG. 12. Time-averaged attenuation as a function of frequency for a generalized coaxial transmission line (geometry shown in Fig. 11) from 2D-cylindrical and 3D-Cartesian simulations.

regions (see top plot of Fig. 11) which are caused by the artificial truncation of the medium in the simulation domain. The 2D-SIBC results shown in Fig. 12 are performed with a very coarse spatial resolution using 2-cm cubic cells, but the results converge to better than a percent across the frequency range to simulations with smaller cell sizes. We have also plotted in Fig. 12 results from 3D-Cartesian SIBC simulations (red circles). There is a good agreement between the 2D- and 3D-SIBC, although the 3D values are consistently a bit lower at 50 S/m. The agreement improves at 500 S/m. This is believed to be due to the fact that the radius of curvature of the conductor surface in the expanding region becomes comparable to the skin depth, which introduces errors into the SIBC method. At the higher conductivity value, the skin depth decreases relative to the local radius of curvature of the conductor surfaces and the agreement improves.

VI. CONCLUSIONS

We have extended the SIBC model of Oh and Schutt-Aine to allow for its use with arbitrarily shaped conducting objects, in which curved objects are treated by the locally conformal partial-cell algorithm of Dey and Mittra. A second-order accurate implicit implementation of the model has been incorporated into CHICAGO. A series of test problems was used to demonstrate the capability, as well as its limitations. The use of the SIBC assumes that both the conducting object size and local radius of curvature are large compared to the skin depth, $\propto (\sigma\omega)^{-1/2}$. Moreover, the fitting of surface impedance to a sum of rational functions also imposes a lower bound on $\omega\epsilon/\sigma$. When both these approximations are satisfied, the SIBC has a great advantage over bulk modeling of conducting regions. In the bulk model, spatial resolution of skin depths is required which can lead to prohibitively small cell sizes at high conductivities.

The implicit field solver outlined above largely removes the CFL instabilities described by Dey and Mittra for the case of partial cells with small vacuum volumes, but the

implicit solution may require more iterations in such cases. In our test problems, a floor value applied to the vacuum cell-face area was found to mitigate large iteration numbers. A default value of 0.001 times the full cell-face area was found to curtail large iteration numbers (a reduction from hundreds to tens of iterations in some cases) without any noticeable degradation in accuracy, but a proper choice of floor value may be problem dependent.

The combination of the locally conformal partial-cell approach with the SIBC should allow for higher fidelity simulations of the MITLs in high-current pulsed-power accelerators, as well as for any other application in which resistive losses in arbitrarily shaped objects are significant. The incorporation of these models into larger-scale simulations is ongoing and will be the focus of future work.

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DATA AVAILABILITY

The data are not openly available. The data are available from the authors upon reasonable request.

APPENDIX: TABLE OF FITTING CONSTANTS FOR NORMALIZED IMPEDANCE

In this paper, we have generated a new set of fitting constants, C_i , and ω_i , to the normalized impedance. The values are obtained from nonlinear curve fitting of Eq. (9). The sum over i in Eq. (9) runs from 1 to L , where L is denoted the “order.” Fits for $L = 6, 7$, and 8 , were calculated previously by Oh and Schutt-Aine and are tabulated in Ref. [21]. For the new fitting given here, for which $L = 9$, the coefficients C_i and ω_i are given in Table I

TABLE I. Fitting constants obtained from nonlinear fitting of Eq. (9) for $L = 9$ to the normalized impedance given by Eq. (7).

i	C_i	ω_i
1	$4.1106072907211746 \times 10^{-10}$	$5.517388379207408 \times 10^{-7}$
2	$1.922943828770872 \times 10^{-8}$	$8.465278932327191 \times 10^{-6}$
3	$5.157673763195173 \times 10^{-7}$	0.000083084089156992
4	$9.720105093900001 \times 10^{-6}$	0.0006309072760888785
5	0.00014135673816517415	0.0039813814792659
6	0.0016664884844947796	0.02165983963468775
7	0.01592489600691925	0.10180526962911413
8	0.10957721636514209	0.382321311952006
9	0.37261933638441014	0.8877803322577745

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