

Nonlinear dynamics optimization based on minimizing resonance driving terms along a storage ring

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The optimization of nonlinear dynamics for a storage ring lattice aims to achieve large dynamic aperture (DA) and momentum aperture (MA), which are critical for beam injection and lifetime. One of the main factors affecting MA is off-momentum DA. In our previous work, it was demonstrated that reducing on- and off-momentum resonance driving terms (RDTs) along the ring is very effective in enlarging on- and off-momentum DAs. In this paper, the previous effectiveness analysis is formulated into a practical nonlinear optimization method, in which the on- and off-momentum ring-averaged RDTs are minimized simultaneously while constraints on tune shifts and the working point are imposed. Off-momentum ring-averaged RDTs are evaluated over a range of momentum deviations to optimize MA more effectively. The dependence of the ring-averaged RDT and DA on tune is also analyzed, showing that reducing the ring-averaged RDT can help guide the working point away from structural resonance lines and control tune shifts with momentum to avoid dangerous resonance lines. Applied to the SOLEIL storage ring, this efficient ring-averaged RDT optimization enables a rapid iterative workflow of optimization, analysis, and refinement, providing physical insight into the lattice and leading to improved nonlinear solutions. One optimized solution achieves a DA comparable to that of the reference solution obtained from tracking-based optimization while improving the beam lifetime, as validated experimentally on the SOLEIL storage ring.

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I. INTRODUCTION

Dynamic aperture (DA) and local momentum aperture (LMA) are crucial aspects of the nonlinear dynamics optimization for a storage ring lattice, as they significantly influence beam injection and lifetime [1,2]. Some analytical and numerical methods have been developed for nonlinear dynamics optimization [3–11]. Controlling nonlinear driving terms, including resonance driving terms (RDTs), amplitude-dependent tune shift (ADTS) driving terms, and chromatic terms, is a widely used traditional analytical method [3,12–18]. In this method, reducing RDTs helps to suppress corresponding resonances, controlling ADTS terms helps to avoid dangerous resonance lines, and reducing chromatic terms helps to mitigate the discrepancy between on- and off-momentum dynamics.

In our previous work, the analysis based on the Baker-Campbell-Hausdorff (BCH) formula revealed that by reducing the variation of lower-order RDTs along the longitudinal position of a storage ring, the cross terms contributing to higher-order nonlinear terms are reduced, which in turn suppresses higher-order resonances and controls ADTSs [19–21]. The longitudinal variation of RDTs is quantified using the ring-averaged RDT. As a consequence, it was demonstrated that reducing the ring-averaged RDT is much more effective in enlarging DA than reducing the commonly used one-turn RDTs. In Ref. [22], we introduced off-momentum nonlinear driving terms, which are calculated using off-momentum linear optics and describe off-momentum nonlinear dynamics directly. The analysis above was then extended to the off-momentum case. For LMA, it is determined by the rf acceptance, DA, and off-momentum closed orbit [1,23]. Therefore, reducing the off-momentum ring-averaged RDT helps enlarge LMA by improving off-momentum DAs.

While our previous work revealed the effectiveness of the ring-averaged RDT as an indicator for DA optimization, those studies focused on analyzing its relationship with nonlinear dynamics rather than on practical nonlinear

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optimization. In those earlier studies, the on- and off-momentum nonlinear dynamics were analyzed separately, with a fixed working point and no constraints imposed. However, in a practical nonlinear optimization, simply reducing the on- and off-momentum ring-averaged RDTs may not yield satisfactory results, because the working point, ADTS terms, and tune shifts with momentum also have important influences on nonlinear dynamics. This paper presents a practical nonlinear optimization method based on minimizing the ring-averaged RDT, in which the on- and off-momentum nonlinear dynamics are optimized simultaneously while accounting for the working point and tune-shift terms and then applies it to the SOLEIL storage ring. Leveraging the high efficiency of minimizing the ring-averaged RDT in DA optimization, the method analyzes the optimization results to gain deeper insights into the nonlinear dynamics of the lattice, iteratively refines the optimization, and finally yields improved nonlinear solutions for the SOLEIL storage ring.

The paper is organized as follows. In Sec. II, we briefly review the calculation of the ring-averaged RDT and its relation to higher-order nonlinear terms and then analyze the influence of tune variation and structural resonances on ring-averaged RDT to bridge the gap toward practical optimization. In Sec. III, the SOLEIL storage ring lattice is introduced and the nonlinear dynamics of the reference solution is analyzed. Then Sec. IV details the practical optimization method and its application to the nonlinear optimization of SOLEIL, which involves an iterative process of optimization, analysis, and refinement that yields deeper insight into the lattice's nonlinear dynamics. In Sec. V, the optimized solutions are validated through machine experiments. Finally, the conclusion is provided.

II. REDUCING RING-AVERAGED RDT

A. Ring-averaged RDT and higher-order terms

The one-turn map of a storage ring lattice can be concatenated from individual element maps, and the Lie algebra is widely used to analyze these transfer maps. Let h_n and $\hat{V}_n^k$ be the n th-order Lie generators in the normal form for the ring Hamiltonian and k th nonlinear kick in the ring, respectively. According to the BCH formula, we have [3,24]

$$h_3 = \sum_{k=1}^N \hat{V}_3^k, \quad (1)$$

$$h_4 = \sum_{k=1}^N \hat{V}_4^k + \frac{1}{2} \sum_{j>k=1}^N [\hat{V}_3^k, \hat{V}_3^j], \quad (2)$$

where N is the number of nonlinear kicks. The fourth-order nonlinear terms are contributed by two parts. One is the sum of fourth-order nonlinear terms of each element,

and the other part is the cross terms of third-order nonlinear terms. The second part can also be written as $\frac{1}{2} \sum_{j=2}^N [\sum_{k=1}^{j-1} \hat{V}_3^k, \hat{V}_3^j]$. This form clearly shows that the accumulated third-order nonlinear terms $\sum_{k=1}^{j-1} \hat{V}_3^k$ contribute to the fourth-order terms. Therefore, reducing the accumulated third-order nonlinear terms helps to reduce the cross terms and thereby suppress the fourth-order terms. In our previous work [19], this analysis was extended to higher orders. It was demonstrated that reducing lower-order accumulated RDTs effectively reduces higher-order nonlinear terms and suppresses higher-order resonances. Consequently, reducing accumulated RDTs is much more effective in improving DA than reducing the commonly used one-turn RDTs. The effectiveness of reducing accumulated RDTs was also demonstrated in other studies [25]. In addition, by recording the accumulated third-order terms, the nested loops in Eq. (2) are reduced to a single loop, lowering the computational cost from $O(N^2)$ to $O(N)$.

The ring-averaged RDT was used to quantify the accumulated RDTs along the ring [19], where RDTs at different longitudinal positions are calculated as in Ref. [26]. In this paper, the n th-order ring-averaged RDT is calculated as

$$f_{n,\text{rms}} = \sqrt{\sum_{j+k+l+m=n} f_{jklm,\text{rms}}^2}. \quad (3)$$

For each RDT, we have

$$f_{jklm,\text{rms}} = \sqrt{\sum_{i=1}^N |f_{jklm}(s_i)|^2 / N}, \quad (4)$$

where s_i is the position of the i th nonlinear magnet. Moreover, in Ref. [22], we introduced off-momentum ring-averaged RDTs, which were demonstrated to be very effective in enlarging off-momentum DAs. Instead of analyzing the deviation from on-momentum dynamics using chromatic terms, the basic idea is to directly focus on the off-momentum nonlinear dynamics with given momentum deviation values and analyze the nonlinear driving terms for these cases. We denote the n th-order off-momentum ring-averaged RDT as $f_{n,\text{rms}}(\delta)$. In this paper, the longitudinal variations of on- and off-momentum RDTs are calculated using our own code [27]; equivalent calculations are also available through the author-contributed `computeRDTfluctuation` function in MATLAB AT [28,29].

B. Ring-averaged RDT and tune variation

The working point has an important effect on both RDTs and DA, which was not studied in our previous work. Here, we analyze its influence on ring-averaged RDT and DA.

We denote the mode of an RDT by $\mathbf{m} = (j - k, l - m)$, and let $h_{w,\mathbf{m}}$ represent the contribution of the w th nonlinear kick to the RDT of mode $\mathbf{m}$. For the third-order RDTs, we have [19]

$$|f_{jklm}(s_i)| = \left| \sum_{w=1}^i h_{w,\mathbf{m}} - \frac{\sum_{w=1}^N h_{w,\mathbf{m}}}{1 - e^{im\mu}} \right|, \quad (5)$$

where $\mu = 2\pi(\nu_x, \nu_y)$. Equation (5) indicates that when the tune is near the corresponding resonance line,

$$|f_{jklm}(s_i)| \approx \left| \frac{\sum_{w=1}^N h_{w,\mathbf{m}}}{2 \sin \frac{m\mu}{2}} \right| \approx \left| \frac{\sum_{w=1}^N h_{w,\mathbf{m}}}{2\pi \|\mathbf{m}\|_2 \Delta\nu_m} \right|, \quad (6)$$

where $\Delta\nu_m$ is the distance to the resonance line, and $\|\mathbf{m}\|_2 = \sqrt{(j-k)^2 + (l-m)^2}$. As the tune approaches the resonance line, $\Delta\nu_m$ decreases, causing the ring-averaged RDT to increase sharply. The numerator $\sum_{w=1}^N h_{w,\mathbf{m}}$ corresponds to the conventional one-turn RDT, characterizing the intrinsic perturbations induced by nonlinear kicks. In contrast, $|f_{jklm}|$, which accounts for the tune distance from the resonance, represents the effective resonance driving effect. Furthermore, if the ring consists of N_p superperiods, then we have

$$\sum_{w=1}^N h_{w,\mathbf{m}} = \frac{1 - e^{im\mu}}{1 - e^{im\mu/N_p}} \sum_{w=1}^{N/N_p} h_{w,\mathbf{m}}. \quad (7)$$

In this case, the one-turn RDT can be cancelled by periodic phase advances, and the approximation in Eq. (6) no longer holds. Therefore, the significant increase in the ring-averaged RDT occurs only near the structural resonance lines. The case of fourth-order RDTs is similar but more complex. Explicit formulas are given in Refs. [19,26]. A fourth-order term with mode $\mathbf{m}$ can arise from the cross term of two third-order RDTs with modes $\mathbf{m}_1$ and $\mathbf{m}_2$, satisfying $\mathbf{m} = \mathbf{m}_1 + \mathbf{m}_2$. The fourth-order ring-averaged RDT increases near these structural resonance lines.

Although this pronounced increase occurs only near structural resonance lines, it has implications for the optimization of the working point and nonlinear dynamics. In practice, field errors and local linear-optics adjustments can reduce the symmetry of a storage ring to a single superperiod, such as the current SOLEIL lattice, to be optimized later. Figure 1 presents the result of a working point scan for SOLEIL, where the sextupole strengths remained unchanged during the scan. As the tunes approach the third- and fourth-order resonances, the ring-averaged RDTs increase, rendering the corresponding resonance lines clearly visible in the plots. The variations of ring-averaged RDTs in the plots primarily arise from the changing tune distance from the resonance lines, while

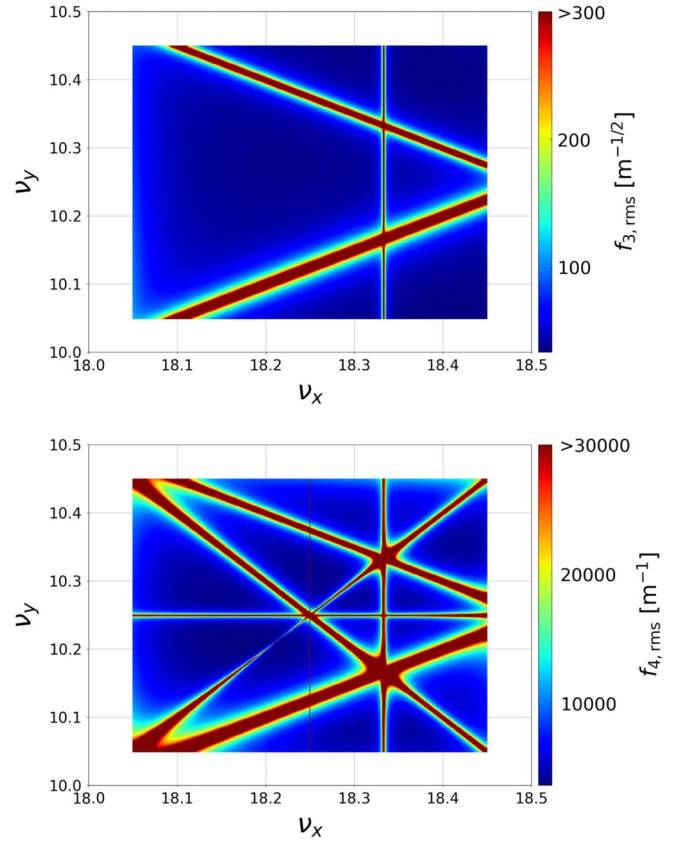


FIG. 1. Ring-averaged RDTs of the SOLEIL storage ring lattice at different working points. The color bars represent the values of the third-order ring-averaged RDT $f_{3,\text{rms}}$ (top) and fourth-order ring-averaged RDT $f_{4,\text{rms}}$ (bottom).

changes in the linear optics also contribute to variations in RDTs, which is why the width of the $2\nu_x - 2\nu_y$ resonance line is not uniform. The width of each resonance line in the plots roughly reflects the strength of the resonance effect.

Figure 2 shows the DAs tracked at two working points, both located at a distance $\Delta\nu_{(2,-2)} = 0.01$ from the $2\nu_x - 2\nu_y$ resonance line. One of them, (18.220, 10.234), lies near the narrowest region of the resonance line, while the other is (18.116, 10.130). As shown in the frequency map, the resonance has a weaker influence at (18.220, 10.234), resulting in a correspondingly larger DA. Reducing the ring-averaged RDT can help avoid resonance lines and identify regions with weaker resonance effects. However, this method yields only a rough optimization of the working point. A more precise adjustment of the working point, or active control of ADTS to ensure that resonance lines lie outside the DA, is still required. The stopband of a nonlinear resonance depends on the amplitude of particle motion, and particles with larger amplitudes are more susceptible to the resonance. Therefore, if the separation from the resonance line is not sufficiently large, shifting the working point away may cause particles with even larger amplitudes to encounter the resonance.

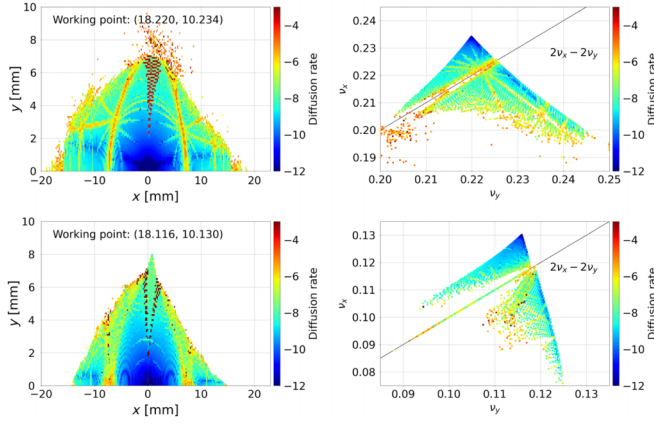


FIG. 2. DAs and frequency maps of the SOLEIL storage ring for two different working points (18.220, 10.234) and (18.116, 10.130), both at the same tune distance of 0.01 from the $2\nu_x - 2\nu_y$ resonance line. The $f_{2002,\text{rms}}$ values at the two working points are 1515.75 and 8218.22 m^{-1} , respectively.

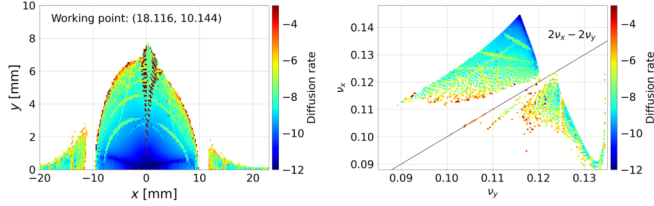


FIG. 3. DA and frequency map of the SOLEIL storage ring at the working point (18.116, 10.144), which lies at a tune distance of 0.02 from the $2\nu_x - 2\nu_y$ resonance line. The ring-averaged RDT $f_{2002,\text{rms}}$ at this working point is 3679.19 m^{-1} .

Consequently, the impact of the resonance on DA may be amplified rather than suppressed. We slightly shift the working point at (18.116, 10.130) to (18.116, 10.144), increasing $\Delta\nu_{(2,-2)}$ from 0.01 to 0.02. As shown in Fig. 3, particles are lost near the resonance $2\nu_x - 2\nu_y$, and the DA is smaller than that obtained for $\Delta\nu_{(2,-2)} = 0.01$. Moreover, the actual effect of a resonance also depends on its intrinsic properties. For instance, when the working point of the SOLEIL lattice is near the $2\nu_x + \nu_y$ or $3\nu_x$ resonance, the DA is nearly zero.

The relationship between off-momentum ring-averaged RDTs and off-momentum tunes is similar to the on-momentum case. Again using the SOLEIL lattice as an example, Fig. 4 shows the off-momentum tunes and corresponding ring-averaged RDTs. The off-momentum ring-averaged RDTs increase sharply as off-momentum tunes approach the third- and fourth-order resonance lines. Reducing the off-momentum ring-averaged RDTs primarily serves to avoid resonance crossings whenever possible. When a crossing is unavoidable, reducing the off-momentum ring-averaged RDTs mitigates the influence of the crossing resonance.

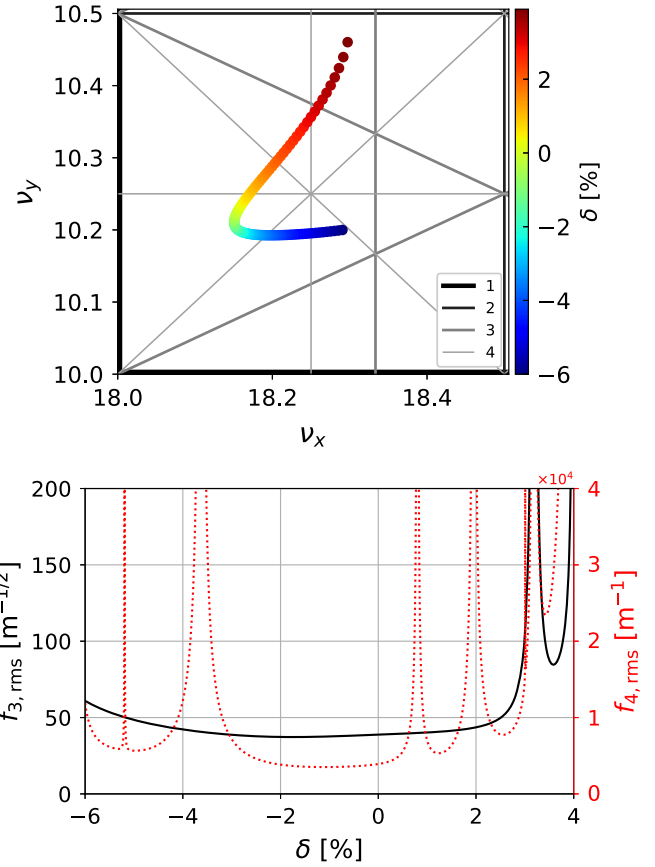


FIG. 4. Off-momentum tunes (top) and off-momentum ring-averaged RDTs of the reference nonlinear solution of SOLEIL.

In our previous study, the off-momentum RDTs and off-momentum DAs were analyzed for given momentum deviations δ . However, LMA optimization requires enlarging the overall off-momentum DAs for different values of δ , which entails reducing the overall off-momentum ring-averaged RDTs. A practical means is to select a series of sampling points at suitable intervals in δ and compute the off-momentum ring-averaged RDTs at each point. A possible concern is that when an off-momentum tune lies exactly on a resonance line, the nonlinear normal-form transformation cannot be performed, and the ring-averaged RDT is therefore not well defined. In the optimization, however, the optimization algorithm can naturally find a nearby solution with nearly identical nonlinear dynamics but a nonzero denominator. As a result, exactly on-resonance solutions are effectively excluded by the optimization itself and do not require special treatment. Excessively dense sampling increases the computation time and compromises efficiency, whereas an overly coarse spacing may fail to capture the resonance-crossing trend. The analysis of working point, ADTS, and off-momentum tunes in this subsection will be taken into account in the nonlinear optimization of SOLEIL presented later.

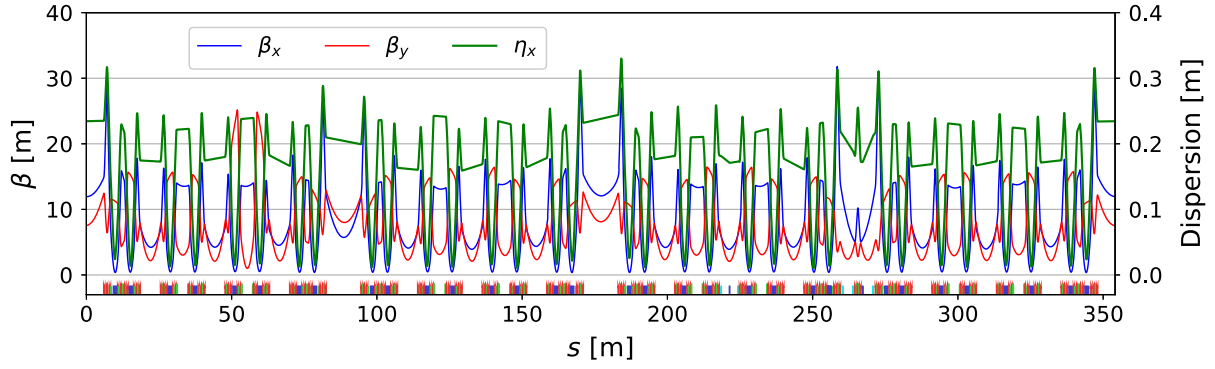


FIG. 5. Linear optics and magnet layout of the SOLEIL storage ring lattice.

III. NONLINEAR ANALYSIS OF THE SOLEIL STORAGE RING LATTICE

SOLEIL is a third-generation storage ring light source. In the initial design, the storage ring consisted of four superperiods with a beam energy of 2.75 GeV [30]. To provide more suitable linear optics for the insertion devices and superbends, the optical functions were locally adjusted, breaking the symmetry of the lattice and making it more complex. A recent modification involved the addition of a permanent magnet superbend in the storage ring [31], and the current linear optics of SOLEIL is shown in Fig. 5. Table I lists some parameters of the storage ring.

The DA needs to be sufficiently large for off-axis injection. At SOLEIL, the natural emittance of the booster is 140 nm rad, and the beam is injected into the storage ring with a horizontal amplitude of -7 mm. In addition, enlarging the DA also reduces beam losses caused by elastic scattering with residual gas, increasing the beam lifetime. The LMA, which is determined by both the transverse nonlinear dynamics and longitudinal oscillation, limits the Touschek lifetime and the inelastic scattering lifetime. Due to the influence of high-order momentum compaction factors, the rf bucket is distorted, resulting in an rf acceptance of approximately 4% in the positive direction and -6% in the negative direction. Within this range of about -6% to 4% , the LMA is mainly constrained by off-momentum transverse nonlinear dynamics.

The nonlinear solution currently used for the operation of the SOLEIL storage ring lattice was optimized by a

tracking-based multiobjective genetic algorithm (MOGA) [31]. This solution is hereafter referred to as the reference solution. Figure 6 shows the DA, frequency map, and LMA of the reference solution, where the physical aperture is not considered in the particle tracking. The DA is sufficiently large and has low diffusion rates in the central region. However, the LMA can be further improved. For many longitudinal locations, the LMA is much smaller than the rf acceptance and limited by the transverse nonlinear dynamics.

To further analyze the transverse nonlinear dynamics of the reference solution, four-dimensional tracking was performed to calculate off-momentum horizontal DAs, as shown in Fig. 7. Resonance effects are clearly visible in the diffusion map. Among these, the most critical one occurs when δ ranges from 3% to 4%, where the DA drops drastically. This resonance corresponds to $\nu_x + 2\nu_y$, as shown in Fig. 4. As a result, the LMA boundary varies between 3% and 4% in the positive direction. When $\delta = 4\%$, the linear optics no longer has a stable periodic solution. For cases with $\delta < 0$, the off-momentum horizontal DAs are relatively large. However, the calculation of LMA is performed with 6D tracking in the presence of longitudinal oscillations. Consequently, the small DA for 3%–4% also restricts the LMA in the negative direction. This analysis suggests that optimizing the transverse nonlinear dynamics can potentially enlarge LMA and improve beam lifetime.

IV. NONLINEAR OPTIMIZATION OF SOLEIL

We now proceed to optimize the nonlinear dynamics of the SOLEIL storage ring by minimizing the on- and off-momentum ring-averaged RDTs. The optimization is not formulated as a single-step minimization of the RDT objectives; instead, the high computational efficiency of RDT evaluation is exploited to support an iterative workflow with multiple optimization runs. The solutions obtained from each run are analyzed to gain further insight into the nonlinear dynamics of the lattice. These insights then guide the refinement of the optimization setup in

TABLE I. Some ring parameters of SOLEIL.

Beam energy (GeV)	2.75
Circumference (m)	354.097
Natural emittances (nm rad)	3.9
Working point (H, V)	(18.1565, 10.2282)
Corrected chromaticities (H, V)	(1.3, 2.5)
rf voltage (MV)	2.6
Momentum compaction (first order)	4.182×10^{-4}
Momentum compaction (second order)	4.675×10^{-3}

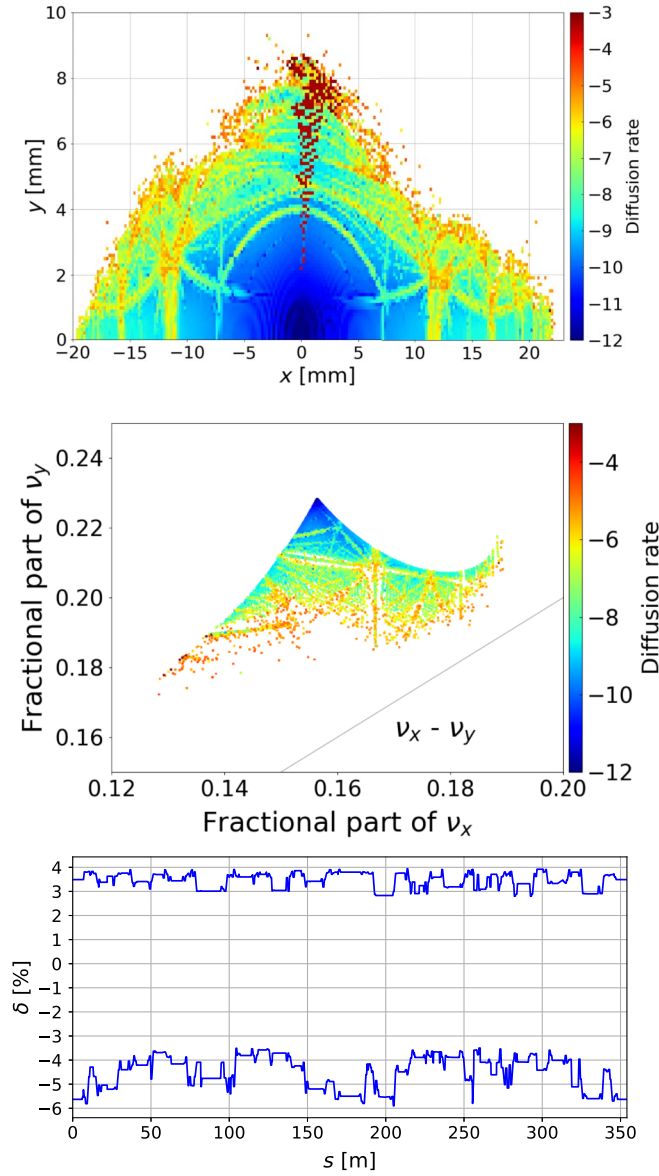


FIG. 6. DA (top and middle plots) and LMA (bottom plot) of the reference nonlinear solution of SOLEIL. The physical aperture is not considered in the tracking.

subsequent runs, including adjustments to variable ranges, inclusion of additional constraints, and modifications to the objective functions. The overall workflow is illustrated in Fig. 8.

A. Optimization settings

The NSGA-II algorithm [32] implemented in pymoo [33] was employed, and two objective functions based on ring-averaged RDTs were utilized to optimize the on- and off-momentum nonlinear dynamics simultaneously. It has been demonstrated that the fourth-order ring-averaged RDT can be well controlled by minimizing the third-order

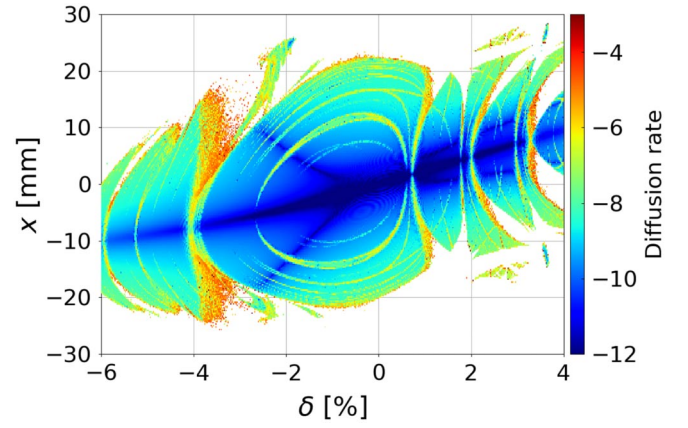


FIG. 7. Off-momentum horizontal DAs of the reference solution.

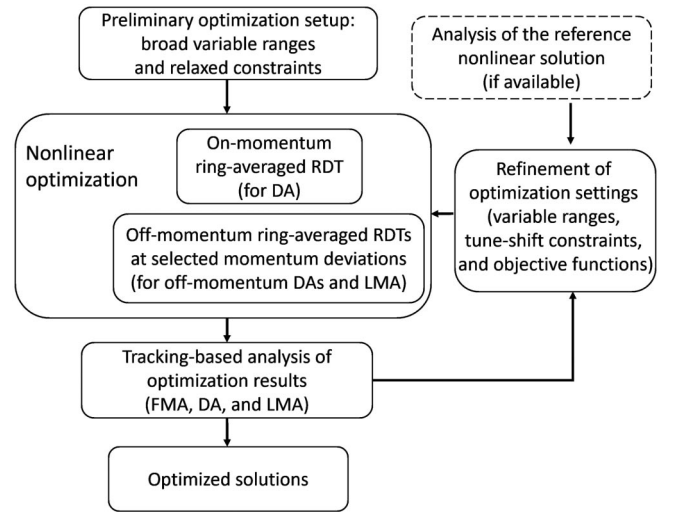


FIG. 8. Workflow of the nonlinear optimization based on minimizing ring-averaged RDTs.

ring-averaged RDT [19]. However, to directly suppress fourth-order resonances and optimize the working point to avoid them, here the third- and fourth-order ring-averaged RDTs are controlled together via the combined term:

$$f_{\text{rms}}(\delta) = f_{3,\text{rms}}(\delta) + w \cdot f_{4,\text{rms}}(\delta), \quad (8)$$

where the coefficient w is estimated according to the action J_x at the edge of the desired DA. For the on-momentum ring-averaged RDT, w is $1/200$. For the off-momentum ring-averaged RDT, w depends on the off-momentum β_x . The two objective functions were

$$f_1 = f_{\text{rms}}(0), \quad (9)$$

$$f_2 = \text{rms}\left(\frac{f_{\text{rms}}(\delta)}{\sqrt{\beta_x(\delta)}}\right), \quad \text{for } \delta = -6\%, -5\%, -4\%, \\ -3\%, -2\%, -1\%, 1\%, 2\%, 3\%, 4\%. \quad (10)$$

The off-momentum β function is included in f_2 because DA is related to $\sqrt{\beta_x}$, and we aim for more uniform off-momentum DAs. Given that adjusting the working point in the optimization induces only minor changes in on-momentum β functions, f_1 does not take on-momentum β_x into account. As discussed in Sec. II, the ring-averaged RDT increases sharply when the tune approaches resonance lines. Therefore, reducing f_2 can also help control the tune shifts with momentum.

There are 11 families of sextupoles, named S1–S11, respectively. Among them, S9 and S10 are used for correcting the chromaticities to the desired values, and the strengths of the other nine families are independent variables. The working point can also be tuned by adjusting the strengths of the quadrupole families Q7 and Q9. Therefore, the number of optimization variables is 11. To avoid being limited by the experience of the reference solution and also explore more feasible solutions, the optimization started with large variable ranges and without constraints. The ranges of sextupole strengths were from 0 to the upper limit of the magnets, with polarities matching those of the reference solution. The upper limit is 73 m^{-3} , which corresponds to the following definition of the normalized sextupole strength:

$$k_2 = \frac{q}{P_0} \frac{\partial^2 B_y}{\partial x^2}, \quad (11)$$

where q is the electron charge, P_0 is the reference momentum, and B_y is the vertical magnetic field. The variation ranges of horizontal and vertical betatron tunes were (18.1, 18.4) and (10.1, 10.4), respectively.

In addition, it is worth mentioning that these two objectives are for optimizing the on- and off-momentum DAs, while LMA is also influenced by off-momentum closed orbit. For the SOLEIL lattice, the impact of the off-momentum closed orbit is less significant, so we consider it when selecting solutions from the optimized solution set but do not use it as an optimization objective.

B. Preliminary optimization and analysis

In the preliminary optimization, the algorithm ran 100 generations with a population size of 5000. The distribution of optimized solutions across generations in the tune space is shown in Fig. 9. The results show that reducing the ring-averaged RDT helps the working points avoid regions with dense resonance lines. As the number of generations increases, the optimized working points gradually converge near that of the reference solution. Moreover, the vertical tune ν_y values of the optimized solutions in the last

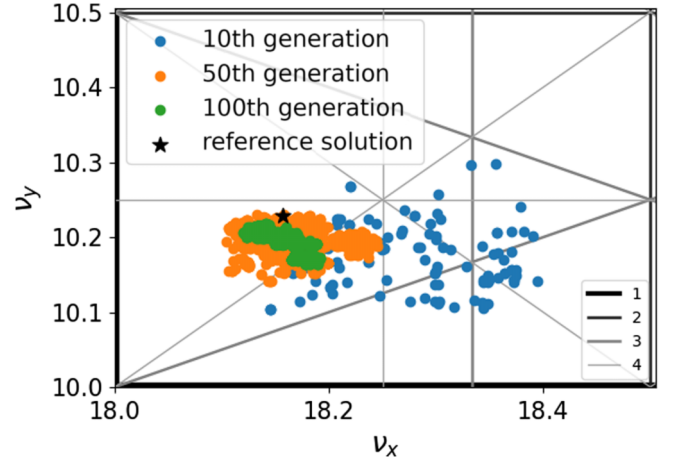


FIG. 9. Evolution of optimized solutions in the tune space across 10th, 50th, and 100th generations. The working point of the reference solution is marked with a star.

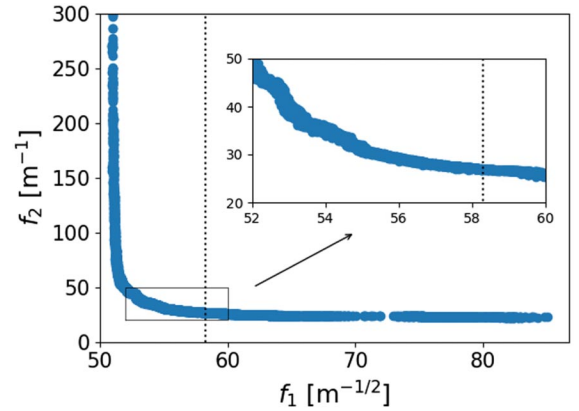


FIG. 10. Part of the Pareto front of on- and off-momentum ring-averaged RDTs.

generation were all smaller than that of the reference solution. According to the analysis of the reference solution, the off-momentum tune gradually approaches the half-integer resonance as δ increases, which makes off-momentum ring-averaged RDT significantly large. To achieve a smaller f_2 , the optimization algorithm tended to obtain a smaller ν_y and smaller tune shifts with momentum, thereby keeping the off-momentum tunes away from the dangerous half-integer resonance. This is the reason why ν_y values of the optimized solutions were smaller, which will be demonstrated in the following analysis.

We then reformed the optimization with smaller variable ranges for a more refined search and obtained similar results. Figure 10 shows part of the Pareto front. For the reference solution, there is no stable linear optics when $\delta = 4\%$ and thus f_2 cannot be calculated. Therefore, a dashed vertical line is used to show the reference solution in the objective space. It can be seen that solutions with

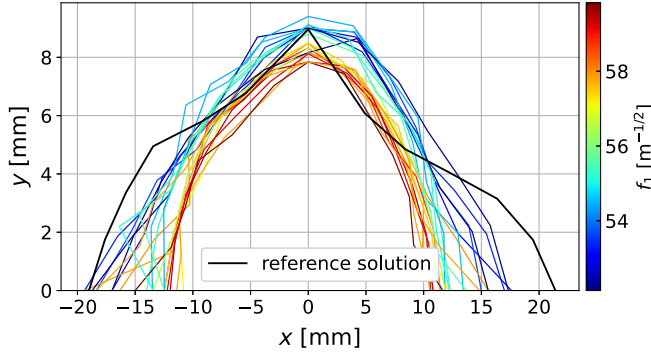


FIG. 11. Comparison of DAs of preliminarily optimized solutions.

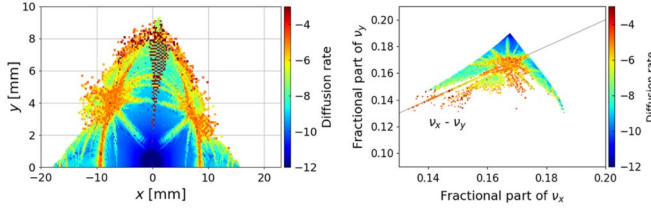


FIG. 12. Frequency map of one preliminarily optimized nonlinear solution in Fig. 11.

stable off-momentum linear optics and a smaller f_1 were found. Next, we will analyze the optimization results based on particle tracking.

The DAs of some solutions, with f_1 being close to that of the reference solution, were calculated, as shown in Fig. 11. Overall, reducing on-momentum ring-averaged RDT f_1 leads to an increase in DA. However, compared to the reference solution with f_1 around $58 \text{ m}^{-1/2}$, these solutions have smaller DAs. The FMA results in Fig. 12 show that the new working point is closer to the difference

resonance $\nu_x - \nu_y$ and a dangerous resonance node, which have a significant impact on DA. These results were obtained without constraining ADTSs. Although reducing ring-averaged RDT can roughly adjust the working point and reduce ADTSs, further precise control of both the working point and ADTSs is still required to improve the DA.

For the analysis of optimization results of off-momentum nonlinear dynamics, four sets of solutions were selected. Figure 13 shows their off-momentum tunes and off-momentum horizontal DAs. The variation trends of the off-momentum tunes of these solutions are roughly the same as that of the reference solution. As δ increases, the off-momentum tunes gradually approach the third-order resonance $\nu_x + 2\nu_y$ and the half-integer resonance $2\nu_y$. As δ decreases, the off-momentum tunes gradually approach or cross another third-order resonance $\nu_x - 2\nu_y$. When the off-momentum tunes approach and cross these dangerous resonance lines, the off-momentum horizontal DAs shrink accordingly. From solution set A to solution set C, reducing off-momentum ring-averaged RDT helps adjust the off-momentum tunes to avoid dangerous resonance lines, resulting in more uniform off-momentum DAs. However, solution set D seems to be an exception. These solutions have f_2 values as small as those in set B, but their off-momentum DAs are significantly worse. The main reason is that the solutions in set D cross the third-order resonance $\nu_x - 2\nu_y$ when $\delta \approx -4.5\%$, while at $\delta = -4\%$ or $\delta = -5\%$, the corresponding off-momentum RDTs are comparable to those of set B, as shown in Fig. 14. This example shows that a resonance crossing between two sampled momentum deviations can be missed if the δ spacing is too coarse. If a smaller δ interval is used in the calculation of f_2 , then solution set D can be identified and excluded in the optimization. In addition

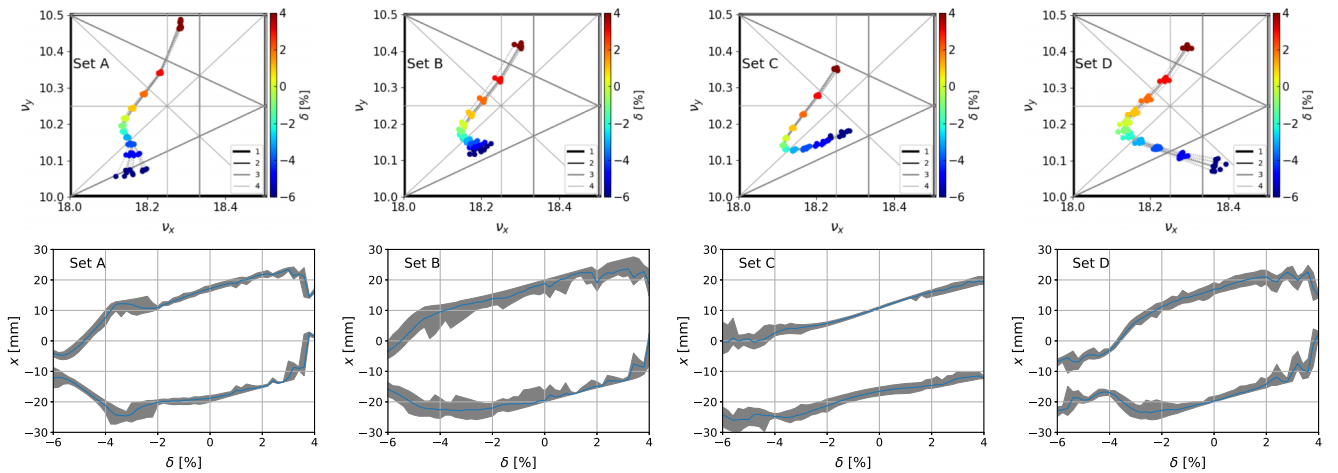


FIG. 13. Comparison of off-momentum tunes (top) and off-momentum horizontal DAs (bottom) for preliminarily optimized nonlinear solutions. The solutions of set A have f_2 over 200 m^{-1} , while sets B, C, and D have f_2 around 35, 23, and 35 m^{-1} , respectively.

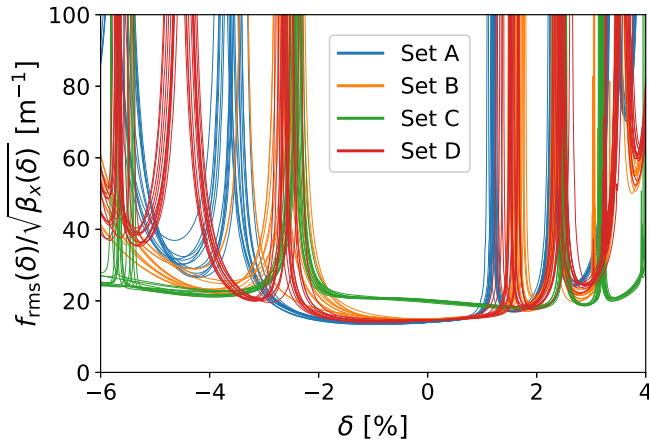


FIG. 14. Off-momentum ring-averaged RDTs of the solutions in Fig. 13.

to adjusting the objective function, we can also avoid such crossing by setting constraints.

C. Further optimization and analysis

The following adjustments were made for further optimization. First, the ranges of variables were reduced. In addition, since most of the solutions obtained in the previous optimization have relatively uniform off-momentum horizontal DAs within the range of δ from -3% to 2% , we modified the objective function to focus the optimization on large momentum deviations, using f'_2 instead of f_2 as one of the optimization objectives:

$$f'_2 = \text{rms} \left(\frac{f_{\text{rms}}(\delta)}{\sqrt{\beta_x(\delta)}} \right), \quad \text{for } \delta = -6\%, -5\%, -4\%, 3\%, 4\%. \quad (12)$$

Unlike f_2 , which includes ten momentum-deviation points from -6% to 4% , f'_2 uses only five sampling points at large momentum deviations. This choice is based on the analysis above, which shows that cases with small momentum deviations are less important for LMA optimization. This can further reduce the computation. Finally, based on the analysis above, the difference resonance $\nu_x - \nu_y$ has a relatively large impact on DA, and the third-order resonances have a relatively large impact on the off-momentum DAs. It is necessary to introduce constraints on tune shifts with amplitude and momentum to prevent them from crossing these dangerous resonance lines.

The results of the previous optimization showed that most good solutions have working points above the resonance line $\nu_x - \nu_y$. Therefore, the constraints for working point and ADTSs were set as follows:

$$\nu_x + \frac{x_m^2}{2\beta_x} \frac{d\nu_x}{dJ_x} - \nu_y - \frac{x_m^2}{2\beta_x} \frac{d\nu_y}{dJ_x} < 0, \quad (13)$$

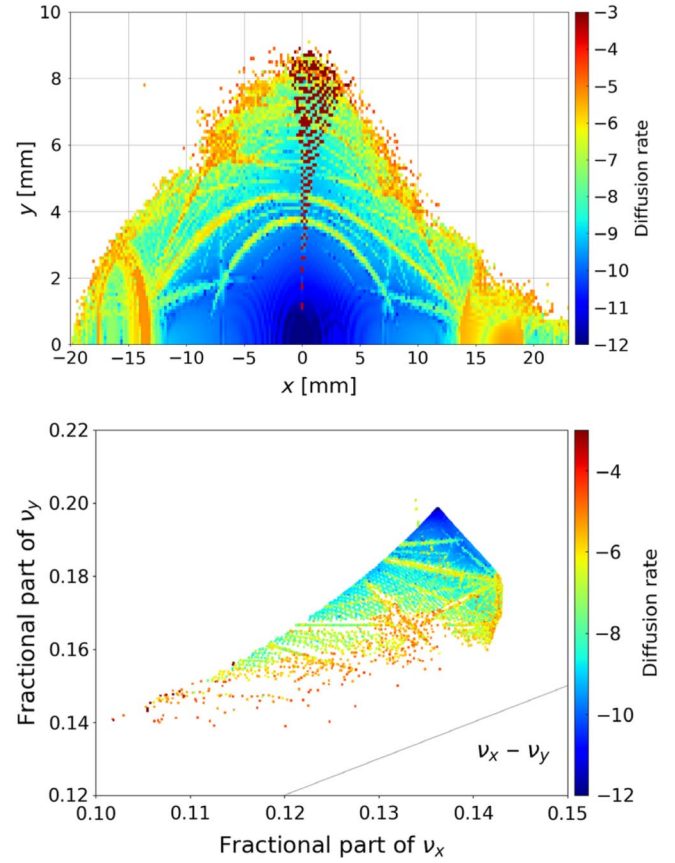


FIG. 15. Frequency map of the DA of solution 1.

$$\nu_x + \frac{y_m^2}{2\beta_y} \frac{d\nu_x}{dJ_y} - \nu_y - \frac{y_m^2}{2\beta_y} \frac{d\nu_y}{dJ_y} < 0, \quad (14)$$

where x_m and y_m are parameters for constraining ADTSs. These constraints can ensure that the tune shifts induced by first-order ADTSs do not cross $\nu_x - \nu_y$ within the ranges $|x| < x_m$ or $|y| < y_m$. It should also be noted that higher-order ADTSs further complicate the tune shifts. Therefore, x_m and y_m should not be set too small in order to retain sufficient margin in the optimization. For lattices with very strong nonlinear effects, it is preferable to directly control the tracked ADTSs, which include higher-order effects.

For the off-momentum nonlinear dynamics, the off-momentum tunes were constrained to avoid crossing $\nu_x - 2\nu_y$:

$$\nu_x(\delta) - 2\nu_y(\delta) < 0, \quad \text{for } \delta \in [-6\%, 4\%]. \quad (15)$$

Although other third-order resonances and the half-integer resonance also affect off-momentum DAs, the previous optimization showed that reducing off-momentum ring-averaged RDTs effectively avoided crossing or approaching these resonance lines. Therefore, they were not considered in the constraints.

We first tried constraint settings with $x_m = 20$ mm and $y_m = 6$ mm. Compared with the preliminary optimization results, after the constraints were applied, the DAs of the optimized solutions were obviously improved. The DA of an optimized solution, named solution 1, is shown in Fig. 15. Compared with the reference solution, this solution has a comparable DA, but there are no obvious resonance lines near the injection position in the DA. As mentioned above, the DA of the reference solution optimized using particle-tracking-based MOGA is large enough. Therefore, achieving a comparable or even slightly better DA also validates our method. Moreover, the off-momentum DAs of solution 1 were improved, especially in the range of $\delta > 0$, as shown in Fig. 16. Consequently, its LMA is better than that of the reference solution at most longitudinal positions, as shown in Fig. 17.

The optimization and analysis above were all based on the ideal lattice. However, in practice, the effects of errors and physical apertures also need to be considered. In order to protect in-vacuum undulators, when the vertical scraper is closed, the vertical physical aperture of the SOLEIL storage ring ranges only from -3 to 3 mm. The horizontal physical aperture is limited to -21 mm with the horizontal scraper closed. To simulate a more realistic situation, TRACY was used to track LMA with multipolar components and physical apertures included. The result is shown

in Fig. 18. Compared with the ideal case, LMAs of both solution 1 and the reference solution decrease, but solution 1 still has a larger LMA.

According to FMA and the statistics of lost particles, the decrease in LMA is mainly due to the linear coupling resonance. The tune shifts with momentum cross the linear coupling resonance at approximately $\delta = -3.5\%$ to -4% , and Figs. 7 and 16 also show its impact. On the resonance, the relatively large horizontal amplitude caused by dispersion is coupled to the vertical direction, and the particles are lost due to the limited vertical physical aperture.

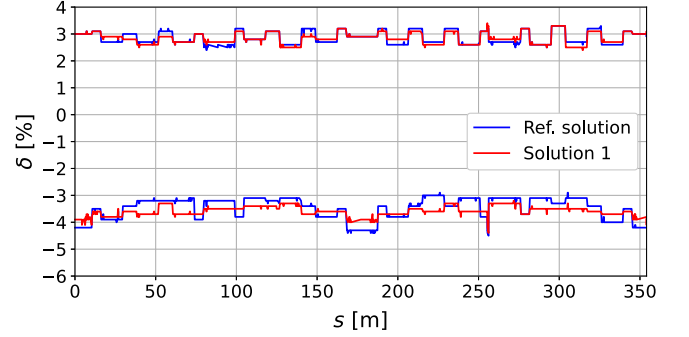


FIG. 18. LMA of solution 1 with multipolar components and physical apertures included.

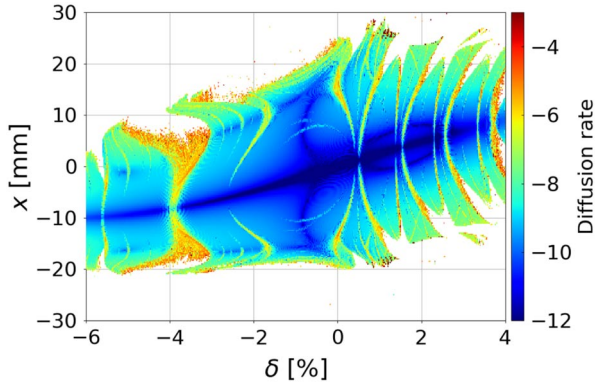


FIG. 16. Off-momentum horizontal DAs of solution 1.

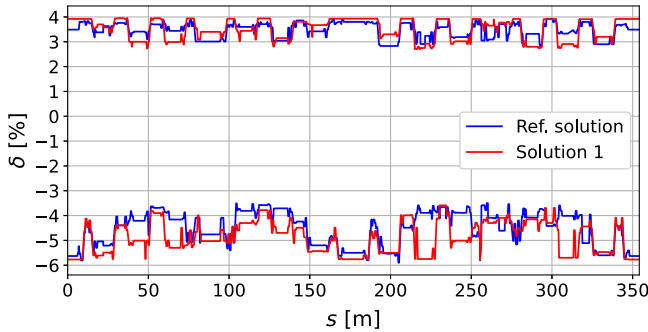


FIG. 17. LMA of solution 1 in the ideal case.

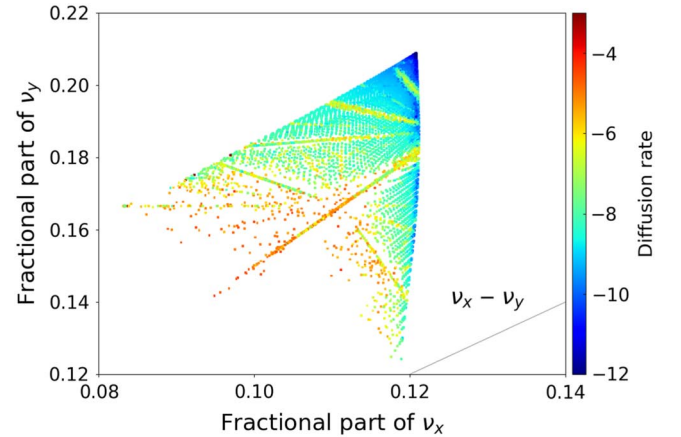
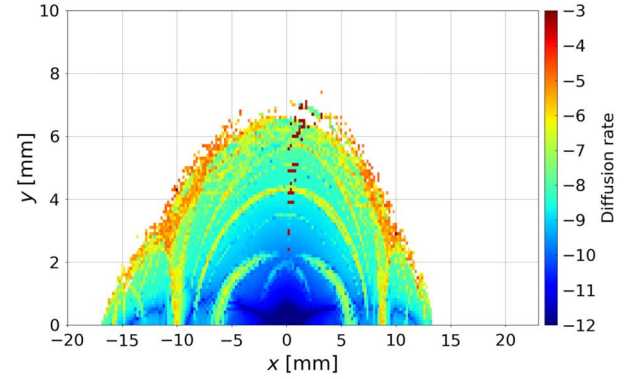


FIG. 19. Frequency map of the DA of solution 2.

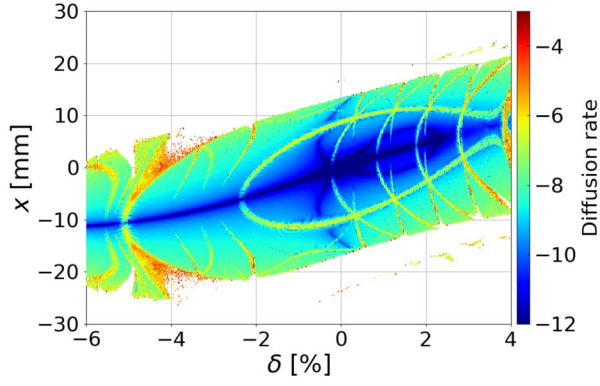


FIG. 20. Frequency map of off-momentum horizontal DAs of solution 2.

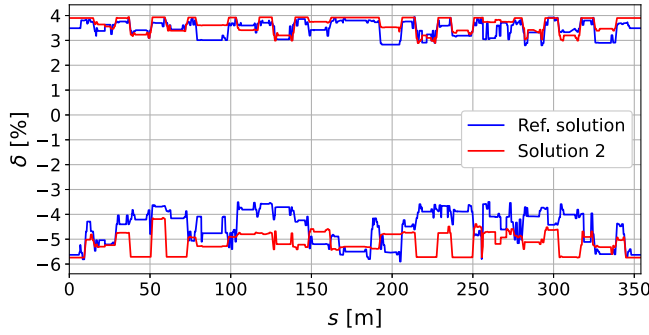


FIG. 21. LMA of solution 2 with ideal situation.

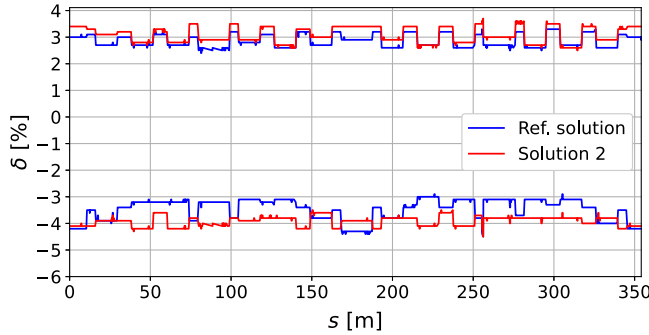


FIG. 22. LMA of solution 2 with multipolar components and physical apertures included.

The pursuit of a DA comparable to the reference solution was primarily for validating the method based on reducing the ring-averaged RDT. In practice, however, such a large DA already exceeds the injection requirement of SOLEIL. Therefore, from a practical standpoint, constraints on DA can be relaxed, while achieving a larger LMA becomes more critical. A constraint setting of $x_m = 15$ mm and $y_m = 5$ mm was adopted, and off-momentum tunes were constrained not to cross the coupling resonance $\nu_x - \nu_y$ before $\delta = -4\%$.

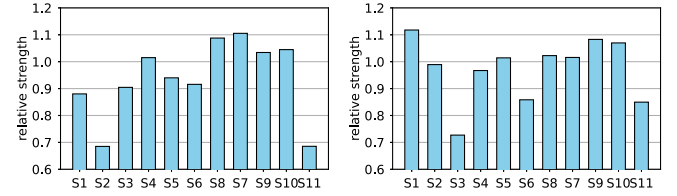


FIG. 23. Sextupole strengths of solution 1 (left) and solution 2 (right) relative to the reference solution.

With these new constraints, another optimized solution, named solution 2, was obtained. Although its DA is smaller than that of the reference solution, as shown in Fig. 19, it still meets the off-axis injection requirement. The main improvement of this solution lies in the off-momentum DAs, as shown in Fig. 20. This solution achieves relatively uniform off-momentum horizontal DAs, and the tune shifts with momentum cross the coupling resonance at about $\delta = -4\%$. As a consequence, solution 2 shows a significant improvement in LMA compared with the reference solution. The LMAs for the ideal lattice and practical lattice are shown in Figs. 21 and 22, respectively.

V. EXPERIMENTS WITH OPTIMIZED SOLUTIONS

In order to validate the nonlinear optimization for SOLEIL, an experiment was designed to measure and compare the nonlinear dynamic performance of the reference solution, solution 1 and solution 2. Figure 23 shows the sextupole strengths of solutions 1 and 2 relative to the reference solution. The three solutions exhibit marked differences, with maximum deviations of about 30%.

Before the experiment, we calculated the Touschek lifetimes using the LMAs obtained with multipolar components and physical apertures included. For a single-bunch current of 1.2 mA and a transverse coupling of 1%, the Touschek lifetime of the reference solution was 14.31 h. The lifetime increased to 15.43 h for solution 1, corresponding to an improvement of approximately 8%, and to 19.80 h for solution 2, corresponding to an improvement of approximately 38%.

During the experiment, the gaps of insertion devices were kept open to the maximum to reduce the influence of the magnetic fields of insertion devices on the lattice nonlinear dynamics, and both horizontal and vertical scrapers were kept closed. The experiment was carried

TABLE II. Measured injection efficiency.

	Injection efficiency
Ref. solution	70%
Solution 1	75%
Solution 2	70%

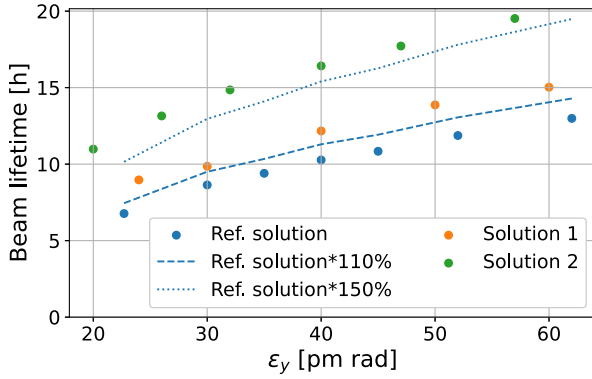


FIG. 24. Measured total beam lifetime for three SOLEIL lattice nonlinear solutions.

out in two stages. First, the working point and chromaticity were measured in low-intensity mode to verify the correct parameter settings, and the injection efficiency was measured. Subsequently, the total beam lifetime was measured under the 1/4 filling mode with a total current of 125 mA (single-bunch current of 1.2 mA). Since the Touschek lifetime, which is the primary limitation of the beam lifetime, is influenced by the vertical beam size, the beam lifetime was measured at different vertical emittance values for a better comparison. The vertical beam size was controlled via white noise excitation.

The measurement results of the injection efficiency and beam lifetime are shown in Table II and Fig. 24, respectively. Although the DAs of these three solutions differ, they all meet the injection requirements, and the injection efficiency shows no significant differences, remaining around 70% due to other practical factors. The beam lifetime of solution 1 increased by more than 10%, while that of solution 2 increased by more than 50%.

The beam lifetime improvement observed experimentally is not fully consistent with the tracking simulation results. A possible explanation is the presence of additional error sources in the actual machine beyond multipolar components, which reduce the beam lifetimes of these nonlinear solutions differently. Another possibility is that the sextupole strength adjustments may have redistributed the linear coupling around the ring, thereby also contributing to the observed lifetime change. Overall, however, the trend of lifetime variation remains consistent.

VI. CONCLUSION

This study presents a systematic optimization based on the ring-averaged RDT for the nonlinear dynamics of the SOLEIL lattice. Two nonlinear solutions were obtained: one offers a larger LMA and a DA comparable to the reference solution obtained from tracking-based MOGA; the other provides a much larger LMA while maintaining a DA sufficient for injection. The optimized nonlinear solutions were experimentally validated, demonstrating

beam lifetime increases of over 10% and 50%, respectively, while maintaining the current injection efficiency.

The optimization reduces the on-momentum ring-averaged RDT to enlarge DA and reduces the off-momentum ring-averaged RDT to enlarge the off-momentum DAs, thereby enlarging LMA. Concurrently, it controls the tune shifts with amplitude and momentum to avoid dangerous resonance lines. Due to the high computational efficiency of minimizing ring-averaged RDT, each round of this optimization requires only a few hours. This enables rapid, multi-iteration cycles following a “ring-averaged RDT optimization–nonlinear dynamics analysis–refined setup” loop, which provides deeper physical insights and enables continuous improvement of the optimization strategy, resulting in a highly effective and efficient process. This rapid iterative optimization capability is crucial. It not only helps designers accumulate valuable experience but is also inherently necessary, as it is generally impossible to know all constraints during the early stages of lattice design, which itself requires continuous iteration.

While other studies also improved off-momentum dynamics by enlarging off-momentum DA [6,34,35], our method follows a similar rationale but employs a different and more efficient mechanism: the off-momentum DA is enlarged by reducing the off-momentum ring-averaged RDT. The high computational efficiency enables the use of a finer δ spacing, as well as the iterative optimization loop described above.

In addition, this work analyzed the influence of tunes on both the ring-averaged RDT and DA. The results demonstrated that reducing the ring-averaged RDT aids in adjusting tunes away from structural resonance lines, and that minimizing the off-momentum ring-averaged RDT helps control the tune shifts with momentum.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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