

Numerical quality factor statistics for superconducting rf cavities with spatially inhomogeneous multilayer coatings modeled by Gaussian random fields

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Bulk niobium has long been the material of choice for superconducting radio-frequency applications. An alternative approach is the superconductor-insulator-superconductor (SIS) multilayer structure, which enables the use of brittle high- T_c materials, such as NbTiN. At present, SIS coatings are limited to flat samples, with the single-cell TESLA cavity representing a key milestone. Extending coating processes to nonflat geometries is expected to introduce macroscopic inhomogeneities in coating thickness. We model these variations using Gaussian random fields parameterized by a spatial correlation length and generated by solving a stochastic partial differential equation. The resulting field is incorporated into the boundary condition of the cavity eigenvalue problem, from which quantities of interest—such as the quality factor—are computed. This procedure is repeated for eight correlation lengths, with 2048 samples per correlation length, where the resulting quality factors are recorded. Our results show that the quality factors follow a normal distribution. The standard deviation increases with the correlation length and can be statistically distinguished. In contrast, the mean values remain largely unchanged, with only a few significant differences. In extreme cases, depending on the correlation length, the quality factor may differ from the uniform case by 2%–6%.

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I. INTRODUCTION

Bulk niobium (Nb) has long been the material of choice for superconducting radio-frequency (srf) cavities in accelerator physics applications and has therefore been extensively studied to approach its theoretical capabilities. A variety of surface preparation techniques have been developed to improve rf performance. Electropolishing produces smoother surfaces, reducing local enhancement due to roughness, but introduces hydrogen absorption that must be removed by subsequent heat treatment. High-pressure rinsing with ultrapure water is then applied to remove microparticles and reduce field emission [1].

More recent advances include nitrogen doping and diffusion [2], as well as low-temperature baking [3,4]. These methods have significantly improved the quality factor and the achievable accelerating fields. However, such approaches remain fundamentally limited by the critical magnetic field of Nb ($H_c \sim 200$ mT). Additionally, the

operating temperature of bulk Nb cavities is currently 2 K and is limited by the critical temperature T_c of Nb. Using higher T_c superconductors could enable operation at 4.2 K, substantially reducing cryogenic costs [5].

Because rf fields penetrate only a very thin surface layer in superconductors (typically less than 1 μm [5]), the superconducting properties relevant for cavity performance are governed almost entirely by this thin layer. This has motivated the development of thin-film cavities, in which a superconducting coating is deposited onto a bulk substrate. This approach has been successfully implemented in Nb/Cu cavities at CERN facilities such as LEP-II and LHC [6]. An extension of this concept is the use of superconductor-insulator-superconductor (SIS) multilayer structures [7]. Here, a bulk superconducting substrate is coated with a thin insulating layer $\mathcal{O}(10\text{ nm})$ followed by a superconducting thin film $\mathcal{O}(100\text{ nm})$ with higher T_c and H_c . The superconducting thin film partially screens the rf field, reducing the field strengths encountered by the substrate such that it maintains the superconducting state at higher fields than possible by bulk only. The insulator layer prevents vortex penetration and suppresses Josephson coupling effects [8]. One promising candidate is the NbTiN-AlN-Nb multilayer [9–11]. Other approaches include superconductor-superconductor bilayers, such as Nb₃Sn on Nb produced via e.g., co-sputtering [12,13], as

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well as Nb₃Sn thin films on Cu substrates, expanding on the successes of Nb/Cu cavities [14].

All of these approaches rely on thin-film deposition techniques, e.g., atomic layer deposition, co-sputtering, and dc magnetron sputtering, and are typically investigated on flat samples for quadrupole resonator measurements [14,15]. Under such conditions, film thickness is generally uniform, globally speaking. However, extending these techniques to complex geometries, e.g., TESLA-type cavities, is expected to produce large-scale thickness inhomogeneities, with spatial distributions that depend on the particular deposition method. The influences of such inhomogeneities on relevant quantities of interest, e.g., quality factor, remains largely unexplored and is difficult to assess experimentally. In this work, we perform a numerical study to quantify the impact of coating thickness variations on the quality factor of a single-cell TESLA cavity. To model these variations, the coating thickness distributions are emulated through Gaussian random fields (GRFs). In essence, GRFs are a continuous extension of the normal distribution. They consist of an infinite collection of random variables, indexed by spatial coordinates, such that any finite subset follows a joint multivariate normal distribution. Furthermore, a GRF is completely defined by its mean function and covariance kernel [16]. In this study, the chosen covariance kernel is parameterized by two parameters: the spatial correlation length and the standard deviation. Only the former is relevant, since the generated values are possibly unbound and are consequently rescaled to within a desired thickness range. The correlation length dictates the radius within which neighboring points display similar thickness values, thereby satisfying the expectation of any deposition method that thicknesses are locally similar. By using GRFs without any additional constraints, we remain agnostic of any particular deposition method. Since only the spatial correlation length is relevant, we sample GRFs over a range of correlation lengths and compare them statistically.

The generation of GRFs on complex geometries and the inclusion of them into the electromagnetic eigenvalue problem is not readily available in commercial software. We therefore made use of the open-source software FEniCSx [17–20], which is able to assemble the finite element matrices from the weak formulation of the problem. We outline a general simulation scheme in Fig. 3 that is applicable to all of the mentioned thin-film approaches, though we apply it specifically to the SIS structure NbTiN-AlN-Nb for illustrative purposes.

II. METHODS

A. Eigenproblem formulation

In order to obtain the quality factor of the cavity, we use the finite element (FE) method. In particular, we numerically solve the eigenvalue problem:

$$\nabla \times (\nabla \times \mathbf{E}) = \left(\frac{\omega}{c}\right)^2 \mathbf{E}, \quad \text{on } \Omega \quad (1)$$

$$\mathbf{n} \times \mathbf{E} = Z(\mathbf{r}, \omega)[\mathbf{n} \times (\mathbf{n} \times \mathbf{H})], \quad \text{on } \partial\Omega, \quad (2)$$

where $\mathbf{E}$ and $\mathbf{H}$ denote the electric and magnetic field strengths, respectively, $\omega \in \mathbb{C}$ is the angular frequency, c is the speed of light, $\mathbf{n}$ is the normal to the surface $\partial\Omega$ and $Z(\mathbf{r}, \omega)$ is the frequency-dependent and position-dependent surface impedance. The former, Eq. (1), is obtained from Maxwell's equations in frequency domain, without sources, and with homogeneous, linear and isotropic constitutive relations. The latter, Eq. (2), is the Leontovich surface impedance boundary condition (SIBC) [21]. The boundary conditions can be specified fully in terms of the electric field through Faraday's law $\mathbf{H} = (j/\omega\mu)\nabla \times \mathbf{E}$. To write down the weak formulation, we define the space

$$H_t(\text{curl}; \Omega) = \{\mathbf{u} \in H(\text{curl}; \Omega) | \gamma_t(\mathbf{u}) \in L^2(\partial\Omega)^3\}, \quad (3)$$

where the tangential trace operator γ_t is defined by

$$\gamma_t: H(\text{curl}; \Omega) \rightarrow H^{-1/2}(\text{div}_{\partial\Omega}; \partial\Omega): \mathbf{u} \mapsto \mathbf{n} \times \mathbf{u}|_{\partial\Omega}. \quad (4)$$

We further define the inner product

$$\langle \cdot, \cdot \rangle_{\Omega}: L^2(\Omega)^3 \times L^2(\Omega)^3: (\mathbf{u}, \mathbf{v}) \mapsto \int_{\Omega} d\mu \mathbf{v}^\dagger \mathbf{u}, \quad (5)$$

where $d\mu$ is the appropriate Lebesgue measure and $\mathbf{v}^\dagger$ denotes the Hermitian conjugate. Finally, we assume that, for all $\omega \in \mathbb{C}$, the function $Z^{-1}(\mathbf{r}, \omega) \in L^\infty(\partial\Omega)$, such that we may define the bounded operators

$$M_{Z^{-1}}^\omega: L^2(\partial\Omega)^3 \rightarrow L^2(\partial\Omega)^3: \mathbf{u} \mapsto Z^{-1}\mathbf{u} \quad (6)$$

whose action is effectively multiplication by Z^{-1} . The weak formulation then reads: find $\mathbf{u} \in H_t(\text{curl}; \Omega)$ and $\omega \in \mathbb{C}$ such that $\forall \mathbf{v} \in H_t(\text{curl}; \Omega)$ [22,23]:

$$\begin{aligned} \langle \nabla \times \mathbf{u}, \nabla \times \mathbf{v} \rangle_{\Omega} - jZ_0 \frac{\omega}{c} \langle M_{Z^{-1}}^\omega \gamma_t(\mathbf{u}), \gamma_t(\mathbf{v}) \rangle_{\partial\Omega} \\ - \left(\frac{\omega}{c}\right)^2 \langle \mathbf{u}, \mathbf{v} \rangle_{\Omega} = 0, \end{aligned} \quad (7)$$

where Z_0 is the impedance of free space. The domain Ω is discretized into tetrahedral elements, and the $H_t(\text{curl}; \Omega)$ space is approximated via Nédélec basis functions of the first kind $\{\mathbf{w}_i\}_{i=1,\dots,N}$, where N denotes the number of degrees of freedom (DOF) [24]. The test and trial functions are expanded in this basis, i.e.,

$$\mathbf{u}(\mathbf{r}, \omega) = \sum_{i=1}^N x_i(\omega) \mathbf{w}_i(\mathbf{r}), \quad (8)$$

ALGORITHM 1. Fixed point iteration for solving the non-linear eigenvalue problem for a single eigenmode

-
- 1 Choose an initial guess $\omega_0 \in \mathbb{C}$ and set $i = 0$;
 - 2 **repeat**
 - 3 Assemble the auxiliary matrix
 $A_i = \mathbf{K} - jZ_0 \frac{\omega_i}{c} \mathbf{B}(\omega_i)$;
 - 4 Solve the generalized eigenvalue problem
 $[A_i - (\frac{\omega_i}{c})^2 \mathbf{M}] \mathbf{x} = 0$;
 - 5 Extract the converged eigenpair $(\omega_{i+1}, \mathbf{x}_{i+1})$ closest to the previous eigenpair $(\omega_i, \mathbf{x}_i)$;
 - 6 Increment $i \leftarrow i + 1$;
 - 7 **until** $|\omega_i - \omega_{i-1}|/|\omega_i| < \epsilon$;
-

which reduces the continuous problem Eq. (7) to the discrete eigenvalue problem

$$\left[\mathbf{K} - jZ_0 \frac{\omega}{c} \mathbf{B}(\omega) - \left(\frac{\omega}{c} \right)^2 \mathbf{M} \right] \mathbf{x} = 0, \quad (9)$$

where the components of the stiffness matrix $\mathbf{K}$, boundary matrix $\mathbf{B}(\omega)$, and mass matrix $\mathbf{M}$ are given by

$$K_{ij} = \langle \nabla \times \mathbf{w}_i, \nabla \times \mathbf{w}_j \rangle_{\Omega}, \quad (10)$$

$$B_{ij}(\omega) = \langle M_{Z-1}^{\omega} \gamma_t(\mathbf{w}_i), \gamma_t(\mathbf{w}_j) \rangle_{\partial\Omega}, \quad (11)$$

$$M_{ij} = \langle \mathbf{w}_i, \mathbf{w}_j \rangle_{\Omega} \quad (12)$$

for $i, j = 1, \dots, N$. The resonance frequency of an eigenmode f and the quality factor Q , which quantifies the damping, can be obtained from the eigenvalue ω via

$$f = \frac{1}{2\pi} \text{Re}(\omega), \quad Q = -\frac{\text{Re}(\omega)}{2 \text{Im}(\omega)}. \quad (13)$$

We solve the eigenvalue problem Eq. (9) via a fixed point iteration as detailed in Algorithm 1. We choose an initial guess, then repeatedly reduce the nonlinear problem Eq. (9) to a generalized eigenvalue problem. The reduced problem is subsequently solved numerically. The solution of the problem yields the eigenpair for the next iteration. This process is repeated until relative convergence of the eigenvalue with tolerance ϵ .

B. Surface impedance model

Before the eigenvalue problem Eq. (9) can be solved, a surface impedance model, i.e., description of $Z(\mathbf{r}, \omega)$, must be specified. The material assigned to the boundary $\partial\Omega$ is a SIS multilayer, where the coating thickness is inhomogeneous over the boundary. However, due to the order of magnitude difference in length scales, ~ 1 –10 mm for the size of mesh cells on the boundary and ~ 10 –100 nm for

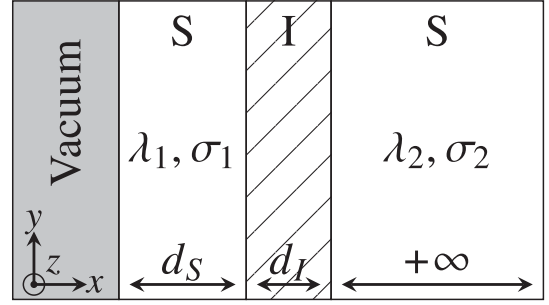


FIG. 1. SIS multilayer structure consisting of a superconducting coating layer of thickness d_S with London penetration depth λ_1 and normal conductance σ_1 , followed by an insulator layer of thickness d_I whose dielectric properties are considered negligible, all layered on top of a superconducting substrate with material parameters λ_2 and σ_2 , which is considered effectively infinitely thick. The layers are planar and parallel to the yz -plane and perpendicular to the x axis.

the layer thicknesses, we view each boundary cell as a semi-infinite space as illustrated in Fig. 1, with varying thickness parameter d_S . Hence, we need to determine $Z(\mathbf{r}, \omega)$ for such a semi-infinite space, at a specific boundary point $\mathbf{r} \in \partial\Omega$. The model by Kubo *et al.* [8,25,26] provides the magnetic fields under reasonable assumptions and simplifications for precisely this setup. The electric and magnetic fields are of the form $\mathbf{E} = E(x) \exp(-j\omega t) \mathbf{e}_y$ and $\mathbf{B} = B(x) \exp(-j\omega t) \mathbf{e}_z$. Solving Maxwell's equations and the London equations in each layer and imposing continuity in the fields at layer boundaries, one finds that the amplitudes of the magnetic fields in each layer are given by

$$\begin{aligned} B_I(x) &= \gamma B_0 \left(\cosh\left(\frac{d_S - x}{\lambda_1}\right) + r \sinh\left(\frac{d_S - x}{\lambda_1}\right) \right), \\ B_{II}(x) &= \gamma B_0, \\ B_{III}(x) &= \gamma B_0 \exp\left(-\frac{x - d_S - d_I}{\lambda_2}\right). \end{aligned} \quad (14)$$

The magnetic field value B_0 is the value at the vacuum and coating layer interface, and the constants γ and r are defined by

$$\gamma^{-1} = \cosh\left(\frac{d_S}{\lambda_1}\right) + r \sinh\left(\frac{d_S}{\lambda_1}\right), \quad r = \frac{\lambda_2 + d_I}{\lambda_1}. \quad (15)$$

The functions hold in a specific domain, namely $B_I(x)$ is valid for $0 \leq x \leq d_S$, $B_{II}(x)$ is valid for $d_S \leq x \leq d_S + d_I$ and $B_{III}(x)$ holds when $x \geq d_S + d_I$. We then obtain the surface impedance through the complex Poynting theorem in frequency domain [27]

$$\frac{1}{2} \int_{\Omega} dV (\mathbf{E} \cdot \mathbf{J}^*) + 2j\omega \int_{\Omega} dV (w_e - w_m) + \oint_{\partial\Omega} dS (\mathbf{n} \cdot \mathbf{S}) = 0, \quad (16)$$

where Ω is an arbitrary volume, $\partial\Omega$ its boundary, $\mathbf{n}$ the normal vector to that boundary and

$$w_e = \frac{1}{4} \mathbf{E} \cdot \mathbf{D}^*, \quad w_m = \frac{1}{4} \mathbf{B} \cdot \mathbf{H}^*, \quad \mathbf{S} = \frac{1}{2} \mathbf{E} \times \mathbf{H}^*. \quad (17)$$

Here, w_e and w_m are the energy densities represented by the electric and magnetic components, respectively, and $\mathbf{S}$ is the Poynting vector. We consider the $x \geq 0$ semi-infinite space as the volume Ω . Since the fields must decay to zero at infinity, this means only the $x = 0$ plane contributes to the surface integral. Using the Leontovich boundary condition Eq. (2), one also finds that

$$\mathbf{n} \cdot \mathbf{S}|_{x=0} = -\frac{1}{2} Z(\omega) |H_0|^2. \quad (18)$$

Using that the fields depends only on the x direction and rearranging for $Z(\omega)$, one finds that

$$Z(\omega) = \frac{1}{|H_0|^2} \left[\int_{\mathbb{R}^+} dx (\mathbf{E}(x) \cdot \mathbf{J}(x)^*) + 4j\omega \int_{\mathbb{R}^+} dx (w_e(x) - w_m(x)) \right]. \quad (19)$$

The electric field and current density are connected through the modification of Ohm's law by the two-fluid model [28]

$$\frac{\partial \mathbf{J}}{\partial t} = \sigma_n \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0 \lambda^2} \mathbf{E}, \quad (20)$$

where σ_n is the normal conductivity. In frequency domain, this yields,

$$\mathbf{J} = \sigma(\omega) \mathbf{E}, \quad \sigma(\omega) = \sigma_n + \frac{j}{\mu_0 \omega \lambda^2} := \sigma' + j\sigma''. \quad (21)$$

The quantities appearing in Eq. (19) can be expressed solely in terms of the magnetic fields by using the constitutive relation $\mathbf{B} = \mu_0 \mathbf{H}$ and Ampère's law with negligible displacement current:

$$\mathbf{E} \cdot \mathbf{J}^* = \frac{1}{\mu_0^2 \sigma} \left| \frac{dB}{dx} \right|^2, \quad (22)$$

$$w_e = \frac{\epsilon_0}{4\mu_0^2 |\sigma|^2} \left| \frac{dB}{dx} \right|^2, \quad (23)$$

$$w_m = \frac{1}{4\mu_0} |B|^2. \quad (24)$$

Substituting Eqs. (22)–(24) into Eq. (19), we obtain an expression for the surface impedance involving only the magnetic field amplitudes Eq. (14)

$$Z(\omega) = \frac{1}{|B_0|^2} \left[\left(\frac{1}{\sigma} + \frac{j\omega\epsilon_0}{|\sigma|^2} \right) \int_{\mathbb{R}^+} dx \left| \frac{dB}{dx} \right|^2 + j\omega\mu_0 \int_{\mathbb{R}^+} dx |B|^2 \right]. \quad (25)$$

For typical values $\sigma_n \sim 10^7 \text{ Sm}^{-1}$, $\omega \sim 10^{10} \text{ S}^{-1}$ and $\lambda \sim 10^{-7} \text{ m}$, one observes that $\sigma'/\sigma'' \sim 10^{-3} \ll 1$, so we expand $1/\sigma$ and $1/|\sigma|^2$ up to $\mathcal{O}(\sigma'^2/\sigma''^2)$, which yields

$$\frac{1}{\sigma} \approx \frac{\sigma'}{\sigma''^2} - \frac{j}{\sigma''}, \quad \frac{1}{|\sigma|^2} \approx \frac{1}{\sigma''^2} \quad (26)$$

and consequently

$$\frac{1}{\sigma} + \frac{j\omega\epsilon_0}{|\sigma|^2} \approx \sigma_n \mu_0^2 \omega^2 \lambda^4 - j\omega\mu_0 \lambda^2 \left(1 - \frac{\omega^2 \lambda^2}{c^2} \right). \quad (27)$$

Again, we note that $\omega^2 \lambda^2 / c^2 \sim 10^{-10} \ll 1$ and can therefore be neglected. This further simplifies Eq. (25) to

$$Z(\omega) \approx \frac{1}{|B_0|^2} \left[(\sigma_n \mu_0^2 \omega^2 \lambda^4 - j\omega\mu_0 \lambda^2) \int_{\mathbb{R}^+} dx \left| \frac{dB}{dx} \right|^2 + j\omega\mu_0 \int_{\mathbb{R}^+} dx |B|^2 \right]. \quad (28)$$

Naturally, Eq. (28) can be split into three components where the integration bounds are separated according to the domains of $B_I(x)$, $B_{II}(x)$, and $B_{III}(x)$, which we denote as

$$Z(\omega) = Z_I(\omega) + Z_{II}(\omega) + Z_{III}(\omega). \quad (29)$$

One can show that for the semi-infinite superconductor where $B(x) = B_0 \exp(-x/\lambda)$, Eq. (28) yields

$$Z_{\infty}(\omega) = \frac{1}{2} \sigma_n \mu_0^2 \omega^2 \lambda^3. \quad (30)$$

One then finds that

$$Z_I(\omega) = r^2 \left[Z_{\infty}^{(I)} \left\{ -(1-r^2) \frac{d_S}{\lambda_1} + \frac{1}{2} (1+r^2) \sinh \left(\frac{2d_S}{\lambda_1} \right) + r \left(\cosh \left(\frac{2d_S}{\lambda_1} \right) - 1 \right) \right\} + j\omega\mu_0 (1-r^2) d_S \right] \quad (31)$$

and

$$Z_{\text{II}}(\omega) = j\omega\gamma^2\mu_0 d_I, \quad Z_{\text{III}}(\omega) = \gamma^2 Z_{\infty}^{(\text{III})}(\omega). \quad (32)$$

We note that the resistive component of the surface impedance is identical to the surface resistance found in Ref. [8] through Joule dissipation. There, it is also mentioned that the resistance from dielectric losses is negligible compared to other contributions, concretely when $d_I = 100$ nm, the contribution from dielectric losses is less than 10^{-5} n Ω , while the other contributions range from 10 to 1000 n Ω . Furthermore, Eqs. (31) and (32) has not previously been mentioned for this model, but is required for the eigenvalue problem.

C. Gaussian random field coating

With the previous computation, we determined the surface impedance $Z(\omega)$ for a particular boundary element. To extend this to a position-dependent surface impedance $Z(\mathbf{r}, \omega)$, we make the coating thickness position-dependent, i.e., $d_S = d_S(\mathbf{r})$ for $\mathbf{r} \in \partial\Omega$. For this purpose, we use GRFs. It is known that for zero-mean GRFs with Matérn covariance kernel, a sample can be generated by solving a stochastic partial differential equation (SPDE) [29]. Here, we will consider the simplest nontrivial Matérn kernel,

$$c_h(\mathbf{r}, \mathbf{r}') = s^2 \exp\left(-\frac{\|\mathbf{r} - \mathbf{r}'\|}{l}\right), \quad (33)$$

where s is the standard deviation and l the spatial correlation length. A sample $h(\mathbf{r}) \sim \mathcal{GP}(0, c_h(\mathbf{r}, \mathbf{r}'))$ can then be generated by solving the SPDE

$$\left(\frac{1}{l^2} - \nabla^2\right)h(\mathbf{r}) = \sqrt{\frac{8\pi s^2}{l}}g(\mathbf{r}) := \alpha g(\mathbf{r}) \quad \text{on } \partial\Omega, \quad (34)$$

where $g(\mathbf{r}) \sim \mathcal{GP}(0, \delta(\mathbf{r} - \mathbf{r}'))$ is a Gaussian white noise field and ∇^2 is understood as the Laplace-Beltrami operator. Using the FE method, we can solve for $h(\mathbf{r})$ on complex geometries numerically. As before, we define the inner product

$$(\cdot, \cdot)_{\Omega}: L^2(\Omega) \times L^2(\Omega): (u, v) \mapsto \int_{\Omega} d\mu u \bar{v}. \quad (35)$$

Then, the weak formulation reads: for $g(\mathbf{r}) \sim \mathcal{GP}(0, \delta(\mathbf{r} - \mathbf{r}'))$, find $u \in H^1(\partial\Omega)$ such that $\forall v \in H^1(\partial\Omega)$

$$\frac{1}{l^2}(u, v)_{\partial\Omega} + (\nabla u, \nabla v)_{\partial\Omega} - \alpha(g, v)_{\partial\Omega} = 0. \quad (36)$$

Note that no boundary condition was required. Since $\Omega \subseteq \mathbb{R}^3$ is a Lipschitz domain, $\partial\Omega$ is closed and

$\partial(\partial\Omega) = \emptyset$; hence, integrals over $\partial(\partial\Omega)$ appearing due to Stokes' theorem vanish without imposing a boundary condition. The space $H^1(\partial\Omega)$ is approximated via Lagrange basis functions $\{\varphi_i(\mathbf{r})\}_{i=1, \dots, N_s}$, so the test and trial functions are expanded as

$$u(\mathbf{r}) = \sum_{i=1}^{N_s} u_i \varphi_i(\mathbf{r}), \quad (37)$$

where N_s denotes the scalar DOF. This leads to the discrete linear system

$$\left(\frac{1}{l^2} \mathbf{M}_S + \mathbf{K}_S\right) \mathbf{u} = \alpha \mathbf{g}, \quad (38)$$

where the components of the scalar mass matrix $\mathbf{M}_S$ and scalar stiffness matrix $\mathbf{K}_S$ are given by

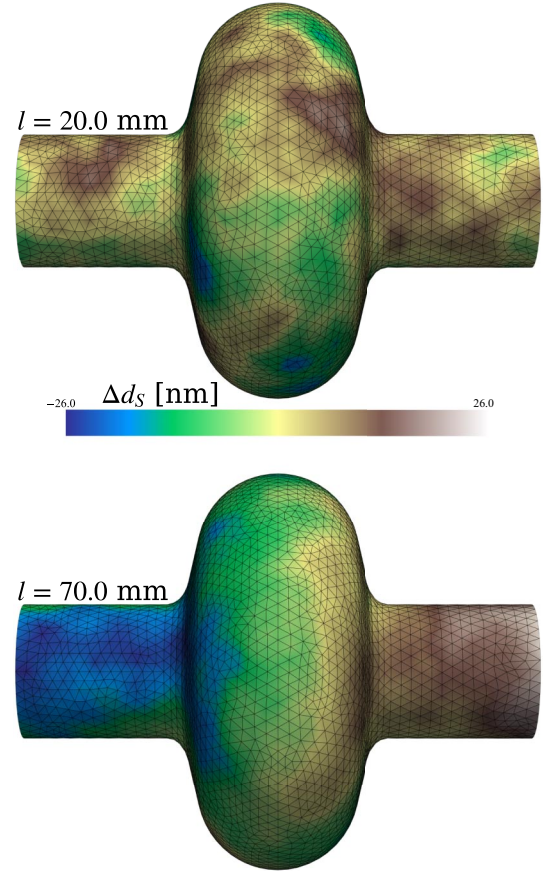


FIG. 2. Single cell TESLA cavity with Gaussian random field on the boundary for different spatial correlations lengths. The top plot displays the result for a correlation length of $l = 20$ mm and the bottom plot for a correlation length of $l = 70$ mm. The field is rescaled to the interval $[-26, 26]$ nm and represents the deviation from the global mean of the superconducting coating layer $d_S(\mathbf{r})$.

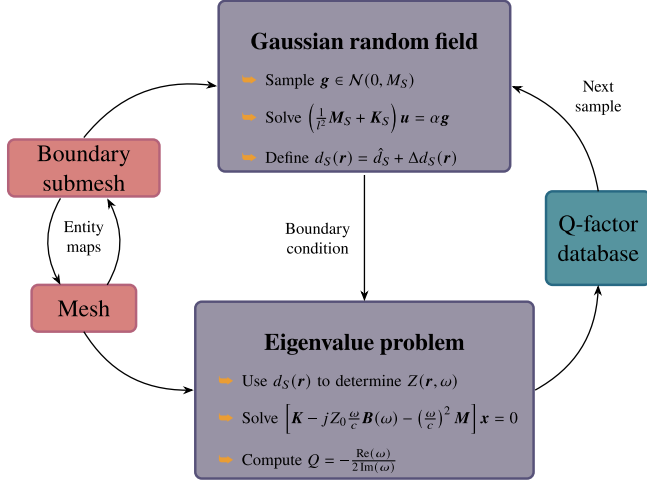


FIG. 3. Diagrammatic representation of the simulation scheme. We solve two FE problems, one on the boundary and one on the entire domain. The former yields the boundary condition for the latter. The latter yields the quality factor for a single sample of the coating distribution. Samples are collected in a database and used for statistical analysis.

$$M_{S,ij} = (\varphi_i, \varphi_j)_{\partial\Omega}, \quad (39)$$

$$K_{S,ij} = (\nabla\varphi_i, \nabla\varphi_j)_{\partial\Omega} \quad (40)$$

and the vectors $\mathbf{u}$ and $\mathbf{g}$ are the expansion coefficients in the FE basis. One can show that $\mathbf{g} \sim \mathcal{N}(0, M_S)$, i.e., $\mathbf{g}$ is sampled from a zero-mean multivariate normal distribution with the scalar mass matrix as covariance matrix. The position-dependent coating thickness is subsequently defined by

$$d_S(\mathbf{r}) = \hat{d}_S + \Delta d_S(\mathbf{r}), \quad (41)$$

where $\hat{d}_S$ is a global average coating thickness, and $\Delta d_S(\mathbf{r})$ is a rescaling of a sample $h(\mathbf{r}) \sim \mathcal{GP}(0, c_h(\mathbf{r}, \mathbf{r}'))$ to a desired interval. Note that the rescaling effectively makes the standard deviation s irrelevant. We illustrate $\Delta d_S(\mathbf{r})$ for different correlation lengths l in Fig. 2. We observe that, as the name suggests, l controls the length scale over which deviations occur.

D. Simulation scheme

The previously discussed topics are combined into a simulation scheme, which is illustrated as a flowchart in Fig. 3. The domain Ω is tessellated into a tetrahedral mesh, from which the boundary triangular mesh is extracted, forming the tessellation of $\partial\Omega$. For meshing the TESLA cavity, the software Gmsh was used [30]. On the boundary, we then solve Eq. (38), which yields a sample of the coating thickness distribution $d_S(\mathbf{r})$, parameterized by the spatial correlation length l . This in turn defines $Z(\mathbf{r}, \omega)$, which is present in the boundary condition Eq. (2) and

hence a component of the eigenvalue problem Eq. (9). Solving this problem yields an eigenvalue ω from which the quality factor Q can be extracted through Eq. (13). This process, which involves two FE simulations, returns a single sample of the quality factor. The assembly of the FE matrices in Eq. (9) and Eq. (38) was handled by the FE software FEniCSx [17–20]. Solving the linear system Eq. (38) was done using PETSc [31], and solving the eigenvalue problem of Eq. (9) was done using SLEPc [32].

III. RESULTS

A. Setup

The single-cell TESLA cavity is the first complex geometry many of the research groups working on coating srf cavities are targeting [33]. Therefore, this is the geometry we focus on. The exact parameters used to describe the geometry can be found in the top section of Table I, see Ref. [34] for how the parameters come together to form the geometry. We further choose to focus on the operating mode, which for the chosen parameters occurs at a resonance frequency of 1.288 GHz.

For the multilayer material on the boundary, we choose NbTiN-AlN-Nb, i.e., a bulk Nb substrate with an AlN insulator layer and a NbTiN coating. The relevant material parameters thereof can be found in the second and third sections of Table I. Note that a number of parameters, namely λ_{NbTiN} , λ_{Nb} , and σ_{Nb} , need to be rescaled to their appropriate values at 4.2 K. For the two-fluid model, the London penetration depth $\lambda(T)$ is scaled from $\lambda(0)$ by [35]

$$\lambda(T) = \lambda(0) \left[1 - \left(\frac{T}{T_c} \right)^4 \right]^{-1/2}. \quad (42)$$

For the normal conductance, we use the residual resistivity ratio. By definition,

$$\text{RRR} = \frac{\rho(300 \text{ K})}{\rho(4.2 \text{ K})} = \frac{\sigma(4.2 \text{ K})}{\sigma(300 \text{ K})}. \quad (43)$$

$$\text{Hence, } \sigma_{\text{Nb}}(4.2 \text{ K}) = \text{RRR}_{\text{Nb}} \sigma_{\text{Nb}}(300 \text{ K}).$$

Lastly, the thickness of the insulator and coating layer must be specified. We choose them to optimize for the maximum applicable magnetic field. This corresponds to the highest value of B_0 such that the SIS structures maintain the superconducting state. This value depends on the layer thicknesses and is given by [8]

$$B_{\text{max}} = \min \{ \tilde{\gamma}^{-1} B_{\text{sh, NbTiN}}, \gamma^{-1} B_{\text{max, Nb}} \}, \quad (44)$$

where γ is defined by Eq. (15) and $\tilde{\gamma}$ is given by

$$\tilde{\gamma} = \gamma \left[\sinh \left(\frac{d_S}{\lambda_1} \right) + r \cosh \left(\frac{d_S}{\lambda_1} \right) \right]. \quad (45)$$

TABLE I. Parameters used to perform the simulations.

Parameter	Value	Unit	Description	Source
a	35.0	mm	Iris radius	Ref. [34]
b	103.3	mm	Equator radius	Ref. [34]
h	57.7	mm	Half cell length	Ref. [34]
r_e	42.0	mm	Radius of equator	Ref. [34]
r_{iz}	12.0	mm	Horizontal ellipse radius of iris	Ref. [34]
r_{ez}	19.0	mm	Vertical ellipse radius of iris	Ref. [34]
l_p	75.0	mm	Length of pipes at either end of the TESLA cavity	...
σ_{NbTiN}	2.86×10^6	Sm^{-1}	Normal conductance of NbTiN at 4.2 K	Refs. [5,36]
λ_{NbTiN}	180.57	nm	London penetration depth of NbTiN at 0 K	Ref. [11]
$T_{c,\text{NbTiN}}$	15.4	K	Critical temperature of thin layer NbTiN	Refs. [10,11]
$B_{\text{sh,NbTiN}}$	439	mT	Superheating field of NbTiN	Ref. [37]
σ_{Nb}	6.58×10^6	Sm^{-1}	Normal conductance of Nb at 300 K	Ref. [38]
RRR_{Nb}	300	...	Residual resistivity ratio of Nb between 300 and 4.2 K	Ref. [39]
λ_{Nb}	39.0	nm	London penetration depth of Nb at 0 K	Ref. [40]
$T_{c,\text{Nb}}$	9.23	K	Critical temperature of bulk Nb	Refs. [5,38]
$B_{\text{max,Nb}}$	170	mT	Maximum applicable field to bulk Nb	Ref. [8]
$\hat{d}_S$	260	nm	Global average thickness of superconducting coating	Fig. 4
$\Delta d_{S,\text{max}}$	26	nm	Maximum deviation of d_S from the mean $\hat{d}_S$	...
d_I	10	nm	Thickness of insulator layer	Fig. 4

Here, $B_{\text{sh,NbTiN}}$ is the superheating field of NbTiN, and $B_{\text{max,Nb}}$ is the empirical field limit of the bulk Nb substrate. The values of both are listed in Table I. We illustrate a contour plot of B_{max} in Fig. 4. We find that the optimum is reached when $d_I = 10$ nm and $d_S = 260$ nm; hence, these parameters will be used. The optimum for the coating layer thickness is used for the global average $\hat{d}_S$. For the size of the deviations, we take 10% of this value, yielding

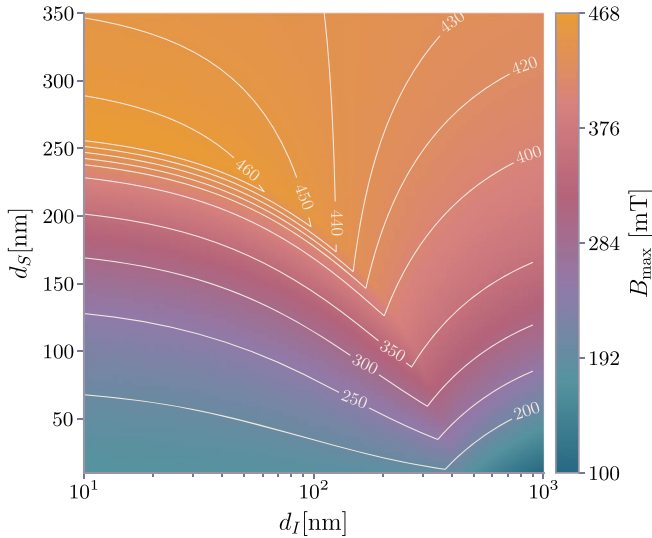


FIG. 4. Contour plot of the maximum applicable field B_{max} depending on thickness of the insulator (AlN) and superconducting coating (NbTiN) layers. The optimum is reached when $d_I = 10$ nm and $d_S = 260$ nm.

$\Delta d_{S,\text{max}} = 26$ nm. Hence, $d_S(\mathbf{r})$ lies in the interval [234, 286] nm.

The GRFs used to emulate the coating distribution $d_S(\mathbf{r})$ depends on the spatial correlation length l . We therefore repeat the simulation scheme Fig. 3 for a number of correlation lengths; in particular, we consider 4%, 8%, 12%, 16%, 20%, 24%, 28%, and 32% of the cavity length (265 mm). For each correlation length we sample 2048 times in the simulation scheme, in order to obtain a good approximation of the quality factor distributions. The maximum size of a triangle on the boundary of the TESLA cavity mesh was 8.7 mm. The mesh was refined purposefully such that this value is smaller than the smallest correlation length considered (10.6 mm).

B. Analysis

The distribution of quality factors does not necessarily have to be normal. Indeed, if one selects the thickness parameters to minimize the surface resistance, the highest quality factor would be reached. Deviating from this optimum would necessarily maintain or decrease the quality factor, yielding a very skewed distribution. By observing Q-Q plots of the data, we observed that, in this case, the quality factors for each l were normally distributed, i.e., $Q \sim \mathcal{N}(\mu_Q, \sigma_Q)$.

In Fig. 5, the spread of the quality factors for each l is illustrated using boxplots. We observe that the spread increases with increasing l , this trend is corroborated by the σ_Q observations in Fig. 6. In the most extreme cases, the quality factor deviates from that of a uniformly coated

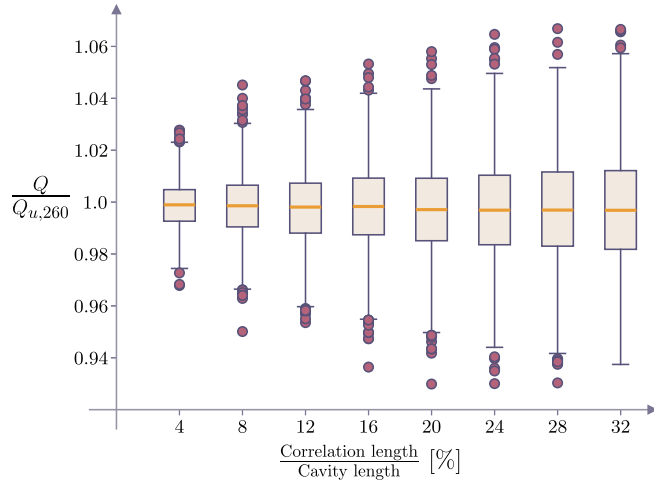


FIG. 5. Boxplot of the quality factors for each spatial correlation length. The quality factors are normalized to $Q_{u,260} = 1.869\,647 \times 10^8$, the quality factor of a uniform coating with coating thickness $d_s = 260$ nm. As the correlation length increases, the spread becomes larger. In extreme cases, deviations from uniform can reach approximately 6%.

cavity with coating thickness 260 nm by approximately 6%. Furthermore, the error bars of the σ_Q in Fig. 6 show barely any overlap, whereas the error bars of the μ_Q overlap substantially, particularly for the larger correlation lengths. We additionally observe that the μ_Q of inhomogeneously coated cavities are consistently smaller than the uniform value, including the error bars. Note that both Figs. 5 and 6 are normalized with respect to $Q_{u,260} = 1.869\,647 \times 10^8$, which corresponds to the quality factor of a uniform coating with a coating thickness of $d_s = 260$ nm. Applying the

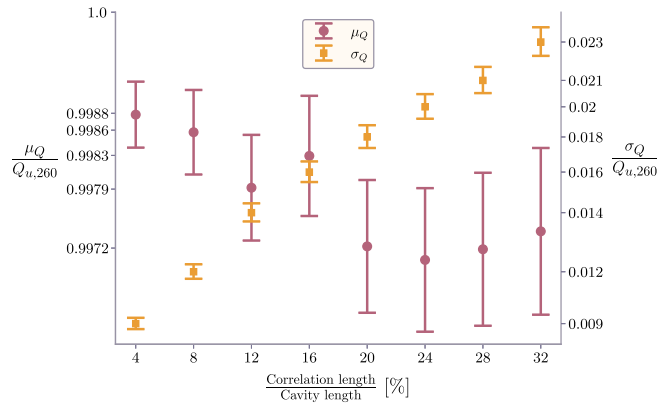


FIG. 6. Mean and standard deviation of the quality factor normal distribution for each spatial correlation length. The error bars refer to the 95% confidence interval of the quantity. The quantities are normalized to $Q_{u,260} = 1.869\,647 \times 10^8$, the quality factor of a uniform coating with coating thickness $d_s = 260$ nm. For all correlation lengths, the means lie below the uniform value with respect to the error bars. The error bars of the means have significant overlap, while the error bars of the standard deviations have barely any overlap.

classical formula $Q = G/R_s$ —using a geometric factor $G = 270\,\Omega$ [41] and surface resistance $R_s = 1476\,\Omega$ calculated via Eqs. (31) and (32)—yields $Q = 1.829 \times 10^8$, in good agreement with the simulation. In general, the classical formula is not applicable here since R_s is nonuniform. A similar limitation of the classical formula has been noted in other contexts, such as studies investigating the Q -slope problem [42,43].

To verify these observations, hypothesis tests were conducted. First, Levene’s test was applied to assess the equality of variances across all eight populations simultaneously [44]. The resulting p -value was extremely small, allowing rejection of the null hypothesis. Thus, the population variances are not all equal. To determine which variances differ, pairwise Levene’s tests were performed. The significance level $\alpha = 0.05$ was adjusted for multiple comparisons using the Holm-Bonferroni correction [45]. All resulting p -values were below their adjusted significance levels. The pair of correlation lengths with p -value closest to the adjusted significance level was (24%, 28%), with $p = 7.3 \times 10^{-3}$. In Fig. 6, the standard deviations corresponding to these correlation lengths exhibit error bars with slight overlap. From the pairwise comparisons, we conclude that all null hypotheses can be rejected. Hence, the variances of all populations are statistically distinct.

To compare the means, Welch’s ANOVA was used as we are dealing with unequal variances [46]. We find that the null hypothesis of equal means can be rejected with $p = 4.8 \times 10^{-5}$, indicating that not all means are equal. This analysis was followed by a *post hoc* Games-Howell test to examine pairwise equality of the means [47]. We find that the null hypothesis cannot be rejected in nearly all cases, and the few cases for which rejection is possible are only weakly significant. These pairs are: (4%, 20%), (4%, 24%), and (4%, 28%), with corresponding p -values 0.012, 0.008, and 0.033, respectively. Pairs that visually appear distinct in Fig. 6, such as (8%, 20%), (8%, 24%), and (8%, 28%), yield p -values close to the significance threshold, namely 0.085, 0.054, and 0.147, respectively.

IV. DISCUSSION

From our analysis, we find that the μ_Q cannot be statistically distinguished in most cases, and only weakly so in a few instances. In contrast, all variances differ significantly in the statistical sense. Hence, increasing l primarily results an increase in σ_Q , while no significant change in μ_Q is observed. Nevertheless, we find that the quality factor corresponding to a uniform coating lies significantly above the confidence intervals of all μ_Q w.r.t the 95% confidence interval. Depending on the correlation length, extreme observations show a 2% to 6% increase or decrease in the quality factor.

The simulation scheme presented in Fig. 3 can be modified and expanded upon in a number of ways; however, the general structure remains the same. First, we used GRFs to

emulate the distribution from a coating process. However, depending on the deposition method, certain random fields may not be representative. For example, using dc magnetron sputtering [6,14], the cavity is rotated around the sputtering device. Consequently, one expects an almost rotationally symmetric thickness distribution rather than an arbitrary GRF sample. More broadly, different deposition techniques, e.g., co-sputtering [12,13], dc magnetron sputtering [6,14], high-power impulse magnetron sputtering [48], atomic layer deposition [10], etc., likely yield distinct representative random fields. Hence, caution is required when applying GRFs without additional constraints. Note, however, that the use of GRFs in Fig. 3 can be substituted with any alternative random field generator, such as a surrogate model or a Karhunen-Loève expansion [49]. While this work applies random fields specifically to the thickness of the superconducting thin films, the approach generalizes to any scalar quantity defined on the boundary.

Second, the random fields enter the eigenvalue problem Eq. (7) via the surface impedance. While we derived a surface impedance formulation from the work and approximations of Ref. [8], any alternative surface impedance model $Z(r, \omega)$ can be integrated into the scheme. For example, one could replace the two-fluid model with more advanced descriptions of the optical conductivity of a superconductor, i.e., the complex-valued conductivity [50,51], cf. the models of Mattis-Bardeen [52], Zimmermann [53], Eliashberg [54,55] or Herman-Hlubina [56]. Furthermore, this work focuses on macroscale correlation lengths (on the order of millimeters or centimeters) arising from the challenges of depositing onto complex, nonflat geometries. We did not consider microscale correlation lengths (on the order of micrometers), i.e., surface roughness, see e.g., the atomic force microscopy images in Ref. [10]. Resolving these microscale geometric features directly through mesh refinement would cause a large increase in computational cost. Instead, surface roughness contributions could be incorporated directly into the surface impedance model, for instance via a correction factor [57], or higher-order SIBCs [21,58]. Modifying the surface impedance model also makes it possible to consider thin-layer approaches other than SIS structures, most notably Nb or Nb₃Sn thin films on Cu substrates [5].

Finally, while we selected the Q factor as the quantity of interest for statistical analysis, the overall framework can be applied to any quantity derivable from the eigenvalues and electromagnetic eigenmodes.

V. CONCLUSION

We investigated a single-cell TESLA cavity with an inhomogeneously coated NbTiN-AlN-Nb surface. The global average coating thickness was determined according to the configuration that optimizes the maximum applicable field. Coating inhomogeneities were modeled

using Gaussian random fields. To analyze their impact, we developed a simulation scheme consisting of two FE simulations: the first generates a Gaussian random field sample on the boundary, while the second incorporates this field into its boundary conditions to compute a quality factor sample. This procedure was carried out for eight spatial correlation lengths, with 2048 samples evaluated for each of them. The resulting quality factor distributions were subsequently compared across all correlation lengths. Our analysis revealed that the population means were mostly indifferent, with exceptions being only weakly significant; however, pronounced differences were observed in the standard deviations. We found that the standard deviations increase with correlation length. Furthermore, the quality factor of a homogeneous coating was found to lie above the confidence interval of all population means. Finally, extreme observations indicate that, depending on the correlation length, the quality factor may increase or decrease by approximately 2%–6%.

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DATA AVAILABILITY

The data and software that support the findings of this article are openly available on Zenodo [59].

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- [1] H. Padamsee, *RF Superconductivity: Science, Technology and Applications* (Wiley-VCH, Berlin, 2009).
 - [2] P. Dhakal, Nitrogen doping and infusion in SRF cavities: A review, *Phys. Open* **5**, 100034 (2020).
 - [3] A. Grassellino *et al.*, Accelerating fields up to 49 mv/m in TESLA-shape superconducting rf niobium cavities via 75 c vacuum bake, *arXiv:1806.09824*.
 - [4] M. Wenskat *et al.*, Vacancy-hydrogen interaction in niobium during low-temperature baking, *Sci. Rep.* **10**, 8300 (2020).
 - [5] A.-M. Valente-Feliciano, Superconducting RF materials other than bulk niobium: A review, *Supercond. Sci. Technol.* **29**, 113002 (2016).
 - [6] S. Calatroni, 20 years of experience with the Nb/Cu technology for superconducting cavities and perspectives for future developments, *Phys. C Supercond.* **441**, 95 (2006).
 - [7] A. Gurevich, Enhancement of rf breakdown field of superconductors by multilayer coating, *Appl. Phys. Lett.* **88**, 01511 (2006).
 - [8] T. Kubo, Multilayer coating for higher accelerating fields in superconducting radio-frequency cavities: A review of theoretical aspects, *Supercond. Sci. Technol.* **30**, 023001 (2016).

- [9] M. C. Burton *et al.*, Superconducting NbTiN thin films for superconducting radio frequency accelerator cavity applications, *J. Vac. Sci. Technol. A* **34**, 021518 (2016).
- [10] I. G. Díaz-Palacio *et al.*, Thermal annealing of superconducting niobium titanium nitride thin films deposited by plasma-enhanced atomic layer deposition, *J. Appl. Phys.* **134**, 035301 (2023).
- [11] M. Asaduzzaman *et al.*, Superconducting properties of thin film Nb_{1-x}Ti_xN studied via the NMR of implanted Li₈, *J. Phys. Condens. Matter* **37**, 395701 (2025).
- [12] N. Schäfer *et al.*, Kinetically induced low-temperature synthesis of Nb₃Sn thin films, *J. Appl. Phys.* **128**, 133902 (2020).
- [13] M. N. Sayeed, U. Pudasaini, G. V. Ereemeev, and H. E. Elsayed-Ali, Fabrication of superconducting Nb₃Sn film by co-sputtering, *Vacuum* **212**, 112019 (2023).
- [14] D. Fonnesu *et al.*, Recipe optimization and SRF test of c-compatible Nb₃Sn films by DC magnetron sputtering from a stoichiometric target, *Sci. Rep.* **16**, 3539 (2026).
- [15] D. Tikhonov *et al.*, Investigation of SIS multilayer films at HZB, in *Proceedings of the 20th Int. Conf. on RF Superconductivity* (JaCoW Publishing, East Lansing, MI, 2021), pp. 72–76.
- [16] R. J. Adler and J. Taylor, *Random Fields and Geometry* (Springer, New York, 2007).
- [17] I. A. Baratta *et al.*, DOLFINx: The next generation FEniCS problem solving environment, preprint (2023).
- [18] M. W. Scroggs, J. S. Dokken, C. N. Richardson, and G. N. Wells, Construction of arbitrary order finite element degree-of-freedom maps on polygonal and polyhedral cell meshes, *ACM Trans. Math. Soft.* **48**, 1 (2022).
- [19] M. W. Scroggs, I. A. Baratta, C. N. Richardson, and G. N. Wells, Basix: A runtime finite element basis evaluation library, *J. Open Source Softw.* **7**, 3982 (2022).
- [20] M. S. Alnaes *et al.*, Unified form language: A domain-specific language for weak formulations of partial differential equations, *ACM Trans. Math. Softw.* **40**, 1 (2014).
- [21] S. V. Yuferev and N. Ida, *Surface Impedance Boundary Conditions: A Comprehensive Approach* (CRC Press, Boca Raton, FL, 2018).
- [22] P. Monk, *Finite Element Methods for Maxwell's Equations* (Oxford University Press, Oxford, 2003).
- [23] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods* (Springer, New York, 2008).
- [24] J. C. Nedelec, Mixed finite elements in $\mathbb{R}^3$, *Numer. Math.* **35**, 315 (1980).
- [25] T. Kubo, Y. Iwashita, and T. Saeki, RF field-attenuation formulae for the multilayer coating model, in *Proceedings of the 4th International Particle Accelerator Conference* (JaCoW Publishing, Shanghai, 2013), pp. 2343–2345.
- [26] T. Kubo, Y. Iwashita, and T. Saeki, Radio-frequency electromagnetic field and vortex penetration in multilayered superconductors, *Appl. Phys. Lett.* **104**, 032603 (2014).
- [27] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1999).
- [28] H. London, The high-frequency resistance of superconducting tin, *Proc. R. Soc. Lond. A* **176**, 522 (1940).
- [29] K. J. Koh and F. Cirak, Stochastic PDE representation of random fields for large-scale Gaussian process regression and statistical finite element analysis, *Comput. Methods Appl. Mech. Eng.* **417**, 116358 (2023).
- [30] C. Geuzaine and J.-F. Remacle, Gmsh: A 3-D finite element mesh generator with built-in pre- and post-processing facilities, *Int. J. Number. Methods Eng.* **79**, 1309 (2009).
- [31] L. D. Dalcin, R. R. Paz, P. A. Kler, and A. Cosimo, Parallel distributed computing using Python, *Adv. Water Resour.* **34**, 1124 (2011).
- [32] V. Hernandez, J. E. Roman, and V. Vidal, SLEPc: A scalable and flexible toolkit for the solution of eigenvalue problems, *ACM Trans. Math. Softw.* **31**, 351 (2005).
- [33] A.-M. Valente-Feliciano *et al.*, Next-generation superconducting RF technology based on advanced thin film technologies and innovative materials for accelerator enhanced performance and energy reach, [arXiv:2204.02536](https://arxiv.org/abs/2204.02536).
- [34] R. Wanzenberg, *Monopole, Dipole and Quadrupole Passbands of the TESLA-Cell Cavity* (DESY, Hamburg, 2001).
- [35] C. J. Gorter and H. Casimir, On supraconductivity i, *Physica* **1**, 306 (1934).
- [36] R. Di Leo, A. Nigro, G. Nobile, and R. Vaglio, Niobium-titanium nitride thin films for superconducting rf accelerator cavities, *J. Low Temp. Phys.* **78**, 41 (1990).
- [37] T. Junginger *et al.*, Critical field of SRF materials, in *Proceedings of the 9th Int. Particle Accelerator Conf.* (JaCoW Publishing, Vancouver, BC, 2018), pp. 3921–3924.
- [38] J. Rumble *et al.*, *CRC Handbook of Chemistry and Physics* (CRC Press, Boca Raton, FL, 2017), Vol. 102.
- [39] M. Altarelli *et al.*, The European X-ray free-electron laser: Technical design report, Tech. Rep., European XFEL, Hamburg, Germany, 2007.
- [40] B. W. Maxfield and W. L. McLean, Superconducting penetration depth of niobium, *Phys. Rev.* **139**, A1515 (1965).
- [41] B. Aune *et al.*, Superconducting TESLA cavities, *Phys. Rev. ST Accel. Beams* **3**, 092001 (2000).
- [42] D. Longuevergne, Geometrical corrections for accurate fitting of the field dependent surface resistance for superconducting accelerating cavities, *Nuclear Instrum. Methods Phys. Res. Sec. A* **910**, 41 (2018).
- [43] J. R. Delayen, H. Park, S. U. De Silva, G. Ciovati, and Z. Li, Determination of the magnetic field dependence of the surface resistance of superconductors from cavity tests, *Phys. Rev. Accel. Beams* **21**, 122001 (2018).
- [44] M. B. Brown and A. B. Forsythe, Robust tests for the equality of variances, *J. Am. Stat. Assoc.* **69**, 364 (1974).
- [45] S. Holm, A simple sequentially rejective multiple test procedure, *Scand. J. Stat.* **2**, 65 (1979).
- [46] B. L. Welch, On the comparison of several mean values: An alternative approach, *Biometrika* **38**, 330 (1951).
- [47] P. A. Games and J. F. Howell, Pairwise multiple comparison procedures with unequal n's and/or variances: A Monte Carlo study, *J. Educ. Stat.* **1**, 113 (1976).
- [48] M. Arzeo, F. Avino, S. Pfeiffer, G. Rosaz, M. Taborelli, L. Vega-Cid, and W. Venturini-Delsolaro, Enhanced radio-frequency performance of niobium films on copper substrates deposited by high power impulse magnetron sputtering, *Supercond. Sci. Technol.* **35**, 054008 (2022).

- [49] R. G. Ghanem and P. D. Spanos, *Stochastic Finite Elements: A Spectral Approach* (Springer-Verlag, Berlin, 1991).
- [50] A. Gurevich and T. Kubo, Surface impedance and optimum surface resistance of a superconductor with an imperfect surface, *Phys. Rev. B* **96**, 184515 (2017).
- [51] H. Ueki, M. Zarea, and J. A. Sauls, The frequency shift and Q of disordered superconducting rf cavities, *Prog. Theor. Exp. Phys.* **2025**, 053I02 (2025).
- [52] D. C. Mattis and J. Bardeen, Theory of the anomalous skin effect in normal and superconducting metals, *Phys. Rev.* **111**, 412 (1958).
- [53] W. Zimmermann, E. H. Brandt, M. Bauer, E. Seider, and L. Genzel, Optical conductivity of BCS superconductors with arbitrary purity, *Physica C* **183**, 99 (1991).
- [54] G. M. Eliashberg, Interactions between electrons and lattice vibrations in a superconductor, *J. Exptl. Theoret. Phys. (U.S.S.R.)* **38**, 966 (1960).
- [55] F. Marsiglio, Eliashberg theory: A short review, *Ann. Phys. (New York)* **417**, 168102 (2020).
- [56] F. Herman and R. Hlubina, Microwave response of superconductors that obey local electrodynamics, *Phys. Rev. B* **104**, 094519 (2021).
- [57] Z. Wu and L. E. Davis, Surface roughness effect on surface impedance of superconductors, *J. Appl. Phys.* **76**, 3669 (1994).
- [58] T. B. A. Senior, Impedance boundary conditions for statistically rough surfaces, *Appl. Sci. Res., Sect. B* **8**, 437 (1960).
- [59] A. Gobeyn, W. Ackermann, and H. De Gerssem, Result data for manuscript: Numerical quality factor statistics for srf cavities with spatially inhomogeneous multi-layer coatings modeled by Gaussian random fields (2026).