Erratum: Betatron frequency and the Poincaré rotation number [Phys. Rev. Accel. Beams 23, 054001 (2020)]

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The partial action, $J'(\mathcal{K})$, in Eq. (13) is not defined correctly in the paper. The correct definition of $J'(\mathcal{K})$ is a curvilinear sector area (divided by 2π), as shown in Fig. 1. A convenient choice for this sector is to start with a ray from the fixed point passing through the initial point (q_n, p_n) [1]. This forms the first boundary. The two other boundaries of this sector are: (1) a segment of the invariant curve, $\mathcal{K}(p, q) = \text{const}$, from the initial point (q_n, p_n) to point (q_{n+1}, p_{n+1}) after one map iteration and (2) a curve, obtained by a single map iteration of the initial straight-line boundary. Notice that for a McMillan-type map, Eq. (22), an initial ray with $p_n = 0$ is always transformed into a ray with $q_{n+1} = 0$ [see Fig. 1(b)]. For maps with a constant rotation number (Appendix A), $J'(\mathcal{K})$ is a true sector because any initial ray is transformed into another ray.

Similarly, Eq. (16) also needs to be corrected to account for a "curved" sector boundary:

$$\frac{1}{2\pi}\int_{q_n}^{q_{n+1}}\left(\frac{\partial\mathcal{K}}{\partial p}\right)^{-1}\mathrm{d}q = \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\mathcal{K}}\left(\frac{1}{2}q_n p_n - I_1 + \int_{q_n}^{q_{n+1}} p\mathrm{d}q\right) = \frac{\mathrm{d}J'}{\mathrm{d}\mathcal{K}}, \tag{1}$$

where the first term, $\frac{1}{2}\, q\, p$, is the area of the triangle, defined by the initial interval, and I_1 is the area under the transformed curve, passing through the point (q_{n+1}, p_{n+1}),

$$I_1 = \int_0^{q_{n+1}} p_{n+1}(q)\mathrm{d}q = \int_0^{\mathcal{K}} p_{n+1}(\mathcal{K}')\frac{\mathrm{d}q(\mathcal{K}')}{\mathrm{d}\mathcal{K}'}\mathrm{d}\mathcal{K}'. \tag{2}$$

The rightmost term is helpful in understanding how to apply the Leibniz integral rule in Eq. (16). This Erratum does not change the conclusions of the paper, but clarifies the definitions and the numerical procedure in Appendix B as it is now better defined how to calculate $\Delta J'$ for two initial conditions in the numerical derivative, Eq. (B3). The numerical derivative Eq. (B3) is defined as the ratio of $dJ'(\mathcal{K})$ (Fig. 2, top-left) to $dJ(\mathcal{K})$ (Fig. 2, bottom-left). Depending on the map and the choice of initial conditions, there could be a number of numerical algorithms to calculate the corresponding phase-space areas. Note that using a sector definition of J', as in Appendix B, may yield sufficient accuracy as the difference is in the second order of $\Delta J'$.

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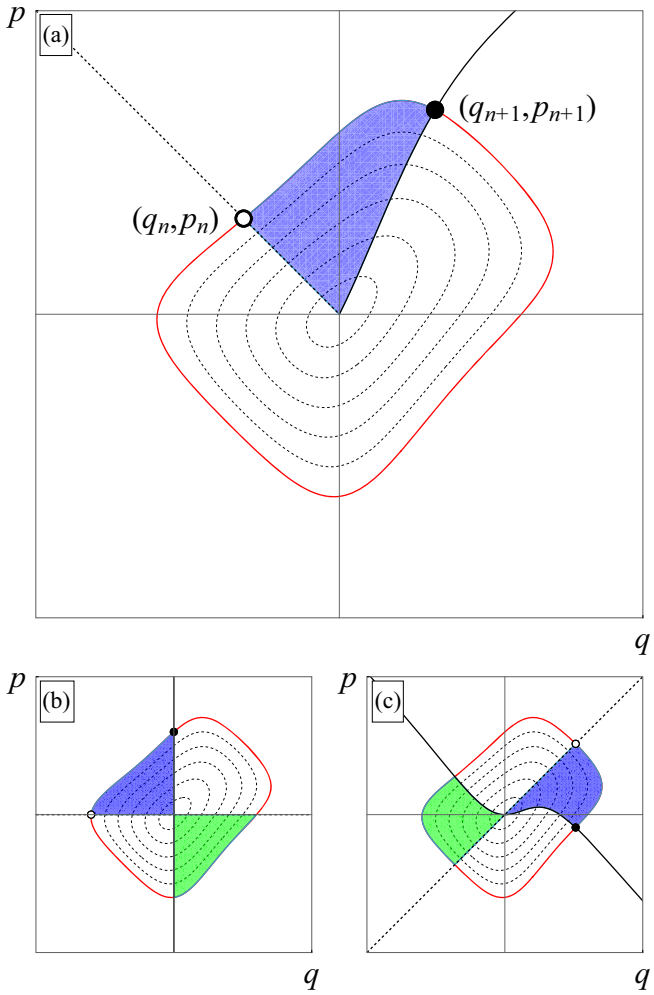


FIG. 1. Representative examples illustrating the construction of a partial action J' for the symmetric McMillan map, defined by the invariant $\mathcal{K}(p,q) = p^2q^2 - (p^2q + pq^2)/2 + p^2 - pq + q^2$. For a selected invariant set, $\mathcal{K} = 9/4$ (solid red curve), an initial point (q_n, p_n) (hollow) is first selected. The dashed line denotes a ray, originating at the origin (or the fixed point) and passing through (q_n, p_n), while the solid black curve shows the image of this ray after one iteration of the map, with the image of the point $(q_n, p_n) \to (q_{n+1}, p_{n+1})$ indicated in black. Panels (a)–(c) correspond to the choices of $p_n = -q_n$, $p_n = 0$, and $p_n = q_n$, respectively. The blue- (or green-) shaded region has an invariant area, independent of the choice of (q_n, p_n), and is equal to $2\pi J'$.

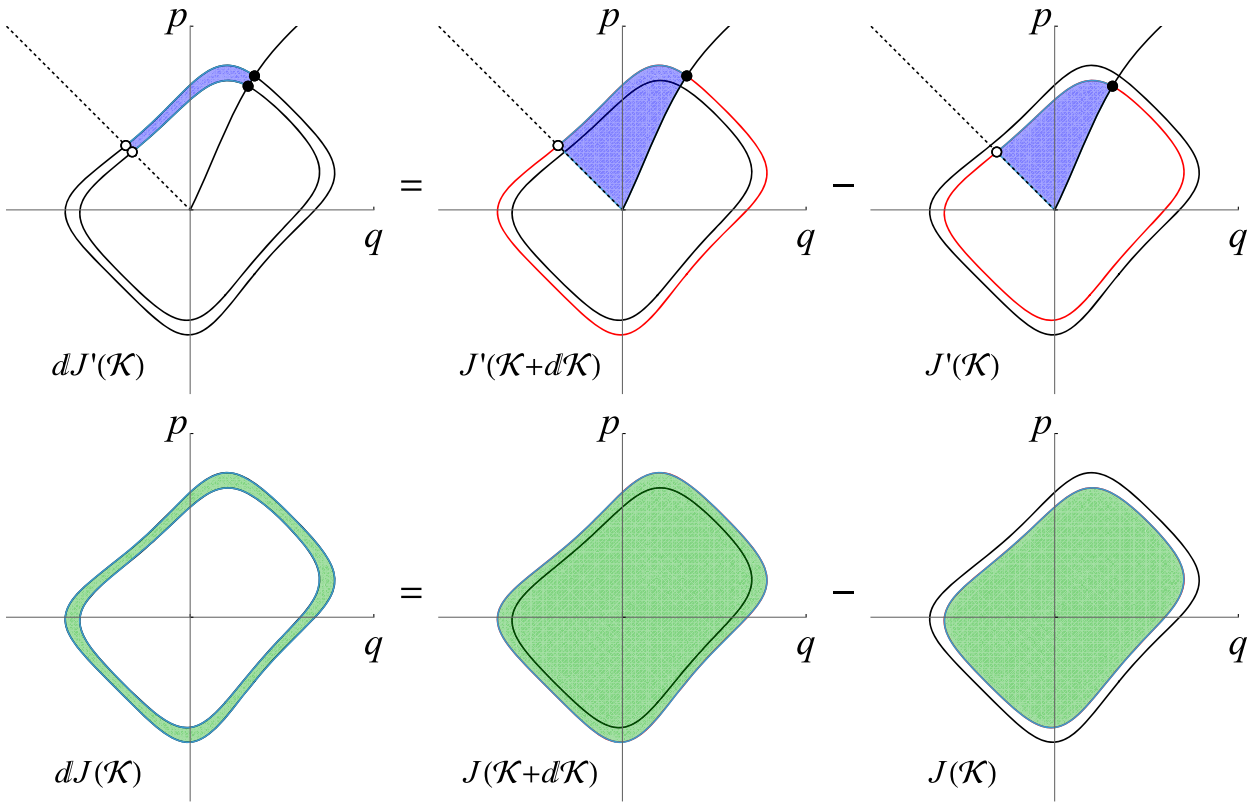


FIG. 2. Graphic representations of partial action (top) and full action (bottom) for evaluating the numerical derivative, Eq. (B3). Note that action is the area divided by 2π.

[1] The value of J' is independent of its initial boundary, which can be any continuous piecewise differentiable curve originating from the fixed point.