

Surface synthesis of multipolar TE-mode cavities

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The design of transverse electric (TE) mode cavities offers new opportunities for beam manipulation and beam-quality improvement. We present a field-driven methodology for synthesizing multipolar TE-mode cavities by superimposing cylindrical standing-wave solutions and enforcing perfect electric conductor boundary conditions through sequential quadratic programming. This procedure produces hybrid cavity geometries that are validated against eigenmode simulations in CST, with examples including $TE_{\{0,2\}1}$ and $TE_{\{2,4\}1}$ hybrid modes. The results demonstrate that prescribed multipolar content can be realized with high modal purity and initial superposition of the multipolar components is preserved. While the present study focuses on methodology and validation, the approach provides the foundation for future applications such as integrated emittance correction and chromaticity compensation.

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I. INTRODUCTION

The design of advanced rf cavities plays a central role in addressing long-standing challenges in beam dynamics, such as space-charge-driven emittance growth and chromatic aberrations. Recent work by Wroe *et al.* [1] demonstrated how multipolar TM-mode cavities can be systematically designed using an azimuthal modulation method. An equivalent methodology for TE modes, however, has not yet been established. Along with Wroe's work, this now completes the design of azimuthally modulated cavities for both subclasses of rf resonator fields in standing-wave structures: transverse electric (TE) and transverse magnetic (TM).

Various applications of TE modes have been found and are discussed briefly. A prominent application of TE modes in linacs also called H-mode structures—resonators dominated by magnetic fields—such as the interdigital H-mode (IH), cross-bar H-mode (CH), which are types of drift-tube linacs (DTL). These cavities operate in TE-like modes (with magnetic fields along the beam axis) to achieve high shunt impedance at low beam velocities [2,3]. The IH-DTL is based on the TE_{11} mode of a pillbox cavity, while the multistem CH-DTL is based on the TE_{21} mode. In an IH-DTL or CH-DTL, drift tubes mounted on ridges or stems

form accelerating gaps; although the axial E -field is confined to the gaps, the overall rf field pattern corresponds to a transverse-electric cavity mode. These TE-mode DTLs have proven advantageous immediately after an RFQ for heavy-ion or proton linacs [2]. Simulation studies show that TE_{m1} structures with lower azimuthal index m generally provide higher shunt impedances (more efficient acceleration) than conventional (Alvarez) DTLs in the low-energy regime. These H-mode cavities have also seen applications in accelerating and controlling a deuteron beam when combined with more traditional beamline elements such as a permanent magnet quadrupole [4]. Another concept for future heavy-ion accelerators, Helmholtz linear accelerator, also uses IH-DTL modes for phase focusing [5].

TE modes also find use in specialized linac cavities designed to address rf breakdown limits. In conventional accelerating modes, electric and magnetic surface fields are inherently coupled—regions of high E often coincide with high B on cavity walls—which complicates studies of breakdown physics. A TE mode, however, can be configured to have no surface electric field, only magnetic field on the walls. SLAC researchers exploited this by designing a dual-mode X-band cavity that can operate in two resonances: an accelerating TM_{020} mode and a TE_{011} mode [6]. The TE_{011} mode in this cavity has strong surface magnetic fields but negligible surface E -field, thus any breakdown events in that mode would be driven purely by high magnetic flux. By constructing a cavity where the TM and TE modes are degenerate in frequency, they could “stack” a high magnetic field on top of the accelerating field and study their separate contributions.

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The radio-frequency quadrupole (RFQ) linac exemplifies how TE-like fields can be exploited to deliver both focusing and acceleration. Introduced by Kapchinskii and Teplyaev in 1970 [7], the RFQ was developed to efficiently bunch and accelerate low-energy ion beams [8]. The unmodulated four-vane structure supports a TE₂₁₀-like quadrupolar field distribution (two periods in azimuth with no axial variation), which provides purely transverse focusing. However, when the vane tips are longitudinally modulated, the violation of axial homogeneity generates a longitudinal electric field component, enabling acceleration in addition to focusing. In this case, the field pattern can no longer be regarded as a pure TE mode; rather, it becomes a hybrid with all six field components. Separately, in the superconducting regime, H-modes have been proposed to replace a static solenoid for emittance compensation [9].

In this work, we introduce a field-driven methodology for TE cavity synthesis. By superimposing analytical standing-wave solutions in cylindrical coordinates and enforcing boundary conditions via numerical optimization, we obtain closed cavity geometries that support prescribed hybrid TE fields. The method is validated against CST eigenmode simulations, with examples of hybrid TE_{0,2}₁ and TE_{2,4}₁ cavities, where the notation TE_{m₁,m₂}₁ is adopted to represent the superposition of solutions with m_1 and m_2 . This study is intended as a foundational proof-of-concept; a companion paper will address beam-dynamics applications in the context of space charge compensation.

II. RELATIONSHIP BETWEEN EM FIELD COMPONENTS IN CYLINDRICAL COORDINATES

In this section, a brief derivation of the electromagnetic fields within a cylindrical cavity is presented. Detailed derivations can be found in various texts [10–13].

Beginning with Maxwell's equations, we consider the curl relationships between the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$:

$$\nabla \times \mathbf{E} = -i\omega\mu_0\mathbf{H}, \quad (1)$$

$$\nabla \times \mathbf{H} = i\omega\epsilon_0\mathbf{E}, \quad (2)$$

where the fields are assumed to have a harmonic time dependence $e^{i\omega t}$, with an angular frequency ω .

Utilizing the Helmholtz equation—omitting the field's time dependence—and expressing it in cylindrical coordinates via the Laplacian operator, we solve it using separation of variables to obtain the general spatially varying form for H_z ,

$$H_z(r, \phi, z) = A_m J_m(\gamma r) \sin(m\phi + \phi_0) \sin[\beta(z - L/2)]. \quad (3)$$

$J_m(\gamma r)$ represents the standard Bessel function of the first kind of m th order. And we note that the Bessel function

of the second kind is excluded from consideration due to its singularity at the cavity axis, making it physically non-viable. Finally, ϕ_0 represents a simple phase shift and A is a constant that can typically be determined from the input power of the cavity. The boundary conditions applied on the end caps are from $-L/2$ to $+L/2$.

The transverse propagation constant γ , linked to the indices m and n , naturally arises upon applying boundary conditions. The relationship

$$k^2 = \gamma^2 + \beta^2, \quad (4)$$

which connects the longitudinal (β) and transverse (γ) constants with the overall frequency ($k = (\omega/c)$) is an important relation that is later used to scale our cavities. Equations for $\mathbf{E}$ and $\mathbf{H}$ can be derived using Eqs. (1)–(3), where we can write the complete set of spatially varying fields as

$$E_r = \frac{-i\omega\mu_0 m}{\gamma^2 r} A_m J_m(\gamma r) \cos(m\phi + \phi_0) \sin[\beta(z - L/2)], \quad (5)$$

$$E_\phi = \frac{i\omega\mu_0}{\gamma} A_m J'_m(\gamma r) \sin(m\phi + \phi_0) \sin[\beta(z - L/2)], \quad (6)$$

$$H_r = -\frac{i\beta}{\gamma} A_m J'_m(\gamma r) \sin(m\phi + \phi_0) \cos[\beta(z - L/2)], \quad (7)$$

$$H_\phi = -\frac{i\beta m}{\gamma^2 r} A_m J_m(\gamma r) \cos(m\phi + \phi_0) \cos[\beta(z - L/2)], \quad (8)$$

$$H_z = A_m J_m(\gamma r) \sin(m\phi + \phi_0) \sin[\beta(z - L/2)]. \quad (9)$$

These equations are used to model the fields in our rf cavities.

A. Perfect electric conducting (PEC) boundary conditions

Applying the boundary condition of vanishing tangential E-fields at a perfect electric conductor (PEC) boundary yields the standard eigenfrequency relation:

$$\gamma^2 + \beta^2 = k^2 \Rightarrow \left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{L}\right)^2 = \left(\frac{\omega}{c}\right)^2, \quad (10)$$

where a and L are the cavity radius and length, respectively. Here, x'_{mn} denotes the n th zero of the J'_m . The three indices (m, n, p) label azimuthal, radial, and longitudinal mode numbers, typically denoted TE _{mnp} , where $p \geq 1$ and throughout this work, p is taken to be 1.

III. FIELD-DRIVEN SURFACE SYNTHESIS

Direct superposition of TE eigenmodes in a circular cavity is not self-consistent. Each eigenmode is tied to a distinct transverse propagation constant γ through the

dispersion relation, Eq. (10), so that fixing the cavity radius results in different eigenfrequencies, while fixing the frequency requires different radii. As a consequence, no single circular geometry can simultaneously support multiple TE eigenmodes at a common frequency.

To enable hybridization, we therefore move away from bounded eigenmode solutions and instead employ the homogeneous separable solutions of the Helmholtz equation in cylindrical coordinates as an analytic basis. These fields, expressed in dimensionless form, can be interpreted as analytic continuations of conventional TE modes beyond a resonator boundary. The cavity surface is then reconstructed a posteriori by enforcing boundary conditions, at which point the superposition becomes reinstated as a genuine eigenmode of the synthesized geometry.

Constructing consistent superpositions, we introduce the dimensionless radial coordinate $\chi = \gamma r$, so that the Bessel functions are expressed directly in terms of χ . This change of variables maps the fields into a dimensionless space, where they are interpreted as analytic continuations of the separable solutions of the conventional cavity eigenmodes.

For numerical stability we further scale the wave number as $k = 1$, so that the field magnitude remains of order unity. This avoids poorly conditioned terms in the optimization and ensures stable convergence without the need for logarithmic transforms on the constraints. After a valid surface has been determined in this dimensionless basis, the geometry is converted back into physical units through the nodal-scaling procedure described in Sec. V.

In a circular waveguide the condition $E_\phi = 0$ is sufficient, but for noncircular geometries both radial (E_r) and azimuthal (E_ϕ) components contribute to the tangential field. Hence, the correct boundary must satisfy

$$\mathbf{E} \cdot \mathbf{S} = 0 \quad \forall \phi, \quad (11)$$

with $\mathbf{S}$ being the tangent vector to the surface, necessitating a constraint-based optimization approach.

To solve this, we model TE fields on a polar mesh (r, ϕ) at the midplane transverse slice ($z = 0$) using analytical expressions for $\mathbf{E}$ and $\mathbf{H}$. Each selected TE_{mn1} mode is assigned a relative phase ϕ_0 and weight, with rotational symmetry ($2\pi/m$ -fold) used to align modes prior to superposition. A Fourier series is chosen to parametrize the unknown cavity cross section:

$$r(\phi) = a_0 + \sum_{v=1}^{v_{\max}} [a_v \cos(v\phi) + b_v \sin(v\phi)], \quad (12)$$

so that the design variables are $\{a_v, b_v, x_0, y_0\}$, with $\{a_0, 0, 0, 0\}$ corresponding to a circle centered at the origin, our trial boundary. Fourier representation naturally enforces periodicity, smoothness, and symmetry, avoiding explicit closure constraints such as $r(0) = r(2\pi)$.

PEC conditions are imposed at the trial boundary by enforcing a vanishing tangential electric field. To improve robustness near global field 0 s, where gradients deteriorate, we apply a smooth weight

$$w(\phi) = 1 + \frac{\alpha}{|H_z(\phi)| + \delta}, \quad (13)$$

with $\alpha = \delta = 0.1$ in the present study, and constrain $w(\phi) \times (\mathbf{E} \cdot \mathbf{S}) = 0$ for all sampled ϕ . Where H_z is nonzero then the optimizer is naturally quick to find solutions that satisfy $\mathbf{E} \cdot \mathbf{S}$. When the boundary crosses through global 0 s ($H_z \approx 0$), a larger weight is produced $w(\phi) \approx 2$, guiding the optimizer's attention toward these uncertain regions. δ is chosen simply to avoid a division by zero in these regions, whilst α was chosen such that these regions have double the weight; increasing the value of α draws more attention to these regions. This helps to improve the satisfaction of the boundary condition as the precision is often limited by these regions. Overall w serves only to condition the optimizer.

The procedure is performed in MATLAB using *fmincon* with the SQP algorithm [14], which handles nonlinear constraints better than an interior-point framework. The objective is set to zero (a feasibility problem), with nonlinear equality constraints enforcing $w(\phi) \times (\mathbf{E} \cdot \mathbf{S}) = 0$ at the discretized boundary points. The optimizer iteratively adjusts the Fourier coefficients, deforming the initial circular guess into a geometry that satisfies the given constraints. Finally, the synthesized boundary must be rescaled from dimensionless units to physical dimensions before validation. An RK4 approach was also attempted; see the Appendix.

IV. PURE MODES

To illustrate the analytic basis used in this work, Figs. 1(a)–1(c) show the electric field magnitude of representative TE modes with azimuthal indices $m = 0, 2, 4$. The white contours mark where $E_\phi = 0$, corresponding to the nodal lines that determine possible cavity boundaries. For example, the innermost nodal circle of the TE_{0n} mode defines the radius of the lowest-order solution TE_{01} , while successive outer rings correspond to higher radial indices ($\text{TE}_{02}, \text{TE}_{03}, \dots$). By plotting over a sufficiently large domain, multiple radial solutions can be visualized simultaneously.

Additionally, within this set of TE_{mn1} modes, we exclusively consider even azimuthal harmonic numbers (m), although any superposition of harmonics can be used. This selection is motivated by symmetry properties relevant for future applications involving symmetrical electron or ion bunches. Modes with even harmonic numbers exhibit symmetrical field distributions and thus provide distinct focusing characteristics. Such applications are beyond the scope of this paper and not discussed here.

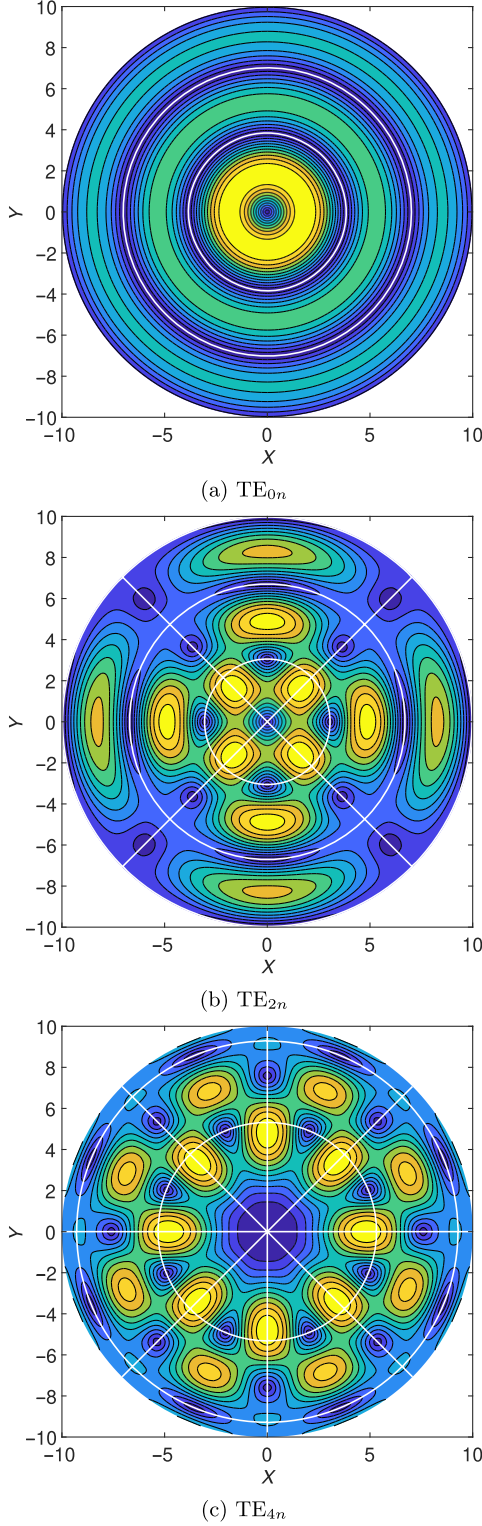


FIG. 1. Analytical continuation of $|E|$ of TE modes: (a) TE_{0n} , (b) TE_{2n} , and (c) TE_{4n} , in dimensionless transverse coordinates, where $X = \gamma x$ and $Y = \gamma y$. White lines indicate zero contours of E_ϕ .

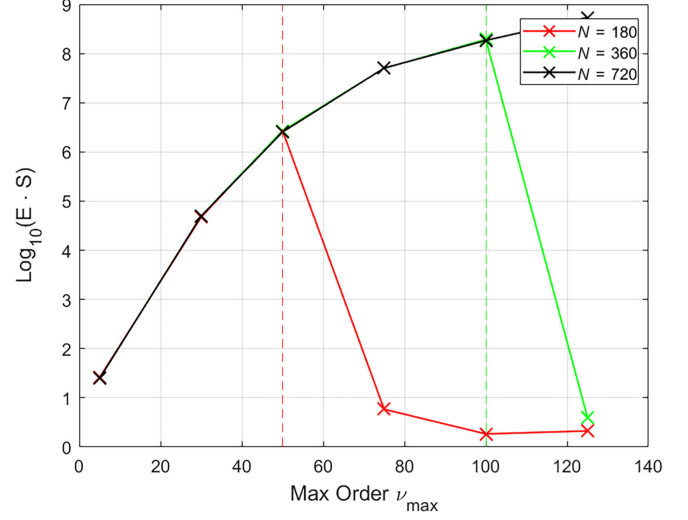


FIG. 2. Convergence study of the maximum boundary constraint violation for varying Fourier series orders and different azimuthal discretization. The logarithm has been multiplied by -1 for better clarity.

V. HYBRID $TE_{\{0,2\}1}$ MODE

Unless otherwise specified, we discretize the azimuth with $N = 360$ points and retain $\nu_{\max} = 100$ Fourier terms, which was found sufficient for convergence (see Fig. 2). The sharp fall-off observed for $N = 180$ and $N = 360$ demonstrates that when too many Fourier coefficients are introduced, the azimuthal discretization can no longer resolve the higher-order terms. This leads to an ill-conditioned Jacobian and the optimizer generates unphysical cavity shapes. By construction, the boundary is represented by a truncated Fourier series and is therefore band-limited in the azimuthal coordinate. For a maximum order $\nu_{\max}$, the sampling theorem requires at least $2\nu_{\max} + 1$ angular points to avoid aliasing. We use $N = 360$, which exceeds this requirement for $\nu_{\max} = 100$, ensuring that the numerical sampling fully resolves the chosen Fourier subspace [15].

As a first example we consider the hybrid formed by combining the TE_{01} and TE_{21} solutions with equal weights and zero relative phase. Figure 3 shows the resulting cavity cross section (green line) overlaid on the electric field distribution and nodal contours of $E_\phi = 0$. The geometry naturally adopts a smooth elliptical shape following the zero of E_ϕ .

The PEC boundary condition is verified in Fig. 4, which plots the weighted residual $\mathbf{E} \cdot \mathbf{S}$ along the cavity surface. The residuals lie at the level of numerical round-off ($\approx 10^{-9}$) demonstrating that the tangential electric field is suppressed. Small numerical instabilities appear where the boundary intersects global field nodes. At these points both

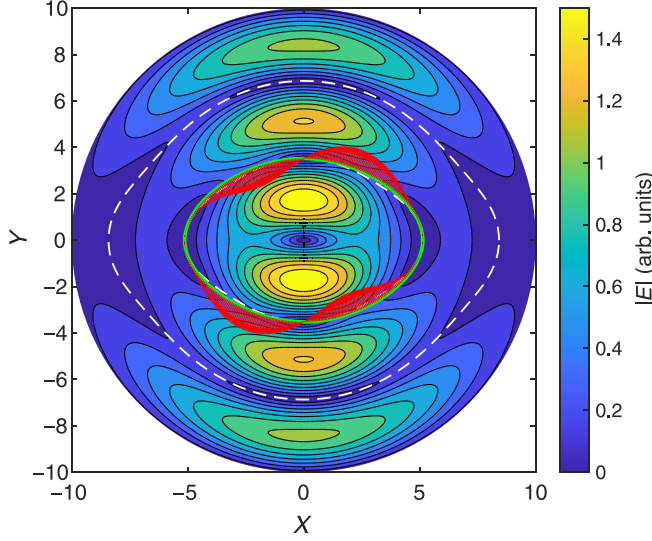


FIG. 3. Contour plot of the electric field magnitude (arb. units) for the hybrid $TE_{\{0,2\}1}$ mode, shown with vector field quivers in red. Dashed white lines indicate zero contours of the E_ϕ component, and the solid green line outlines the cavity cross section.

E_r and E_ϕ vanish, producing indeterminate forms in the differential condition, Eq. (A4), causing the optimizer to lose gradient information.

A. CST validation: Hybrid $TE_{\{0,2\}1}$ mode

Numerical eigenmode solutions were obtained using the eigenmode solver in CST. This solver computes the resonant frequencies and field distributions of a cavity geometry directly from Maxwell's equations, providing access to and

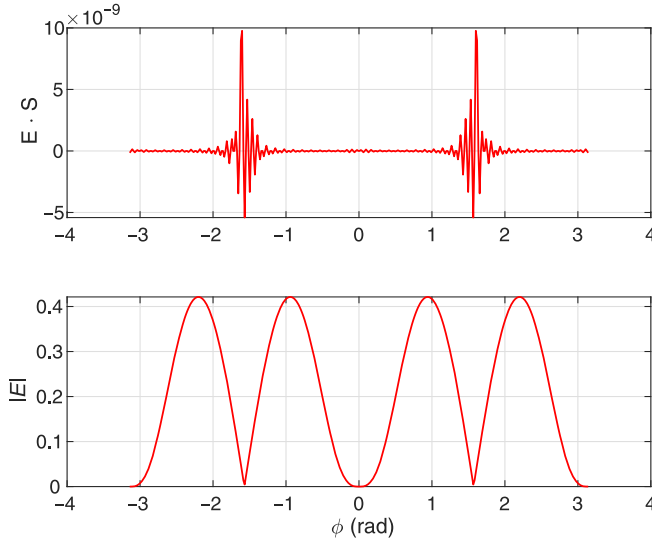


FIG. 4. Electric boundary condition ($E \cdot S$ -top) and the magnitude of the electric field ($|E|$ -bottom) as a function of the azimuthal angle for the hybrid $TE_{\{0,2\}1}$ mode.

validation of hybrid TE modes. This section directly follows from our cavity designs from MATLAB and is imported into CST for validation. Since the cavity presented in MATLAB is in dimensionless units, it must be rescaled.

The cross-sectional contour generated in MATLAB is imported into CST using the numerical coordinates produced by MATLAB, while preserving the unit system defined in CST. For an arbitrary test cavity length L_t , we impose *magnetic wall* boundary conditions at the cavity end caps ($H_t = 0$ on z as seen in CST) and run an eigenmode simulation in CST. This reveals ω_t , which corresponds to the $p = 0$ solution, otherwise absent under PEC boundary conditions and is therefore independent of L_t . We then define the corresponding transverse wave number as

$$\gamma_t = \frac{\omega_t}{c}. \quad (14)$$

Using Eq. (10) with the $\gamma = \gamma_t$ the length of the cavity can be determined at the desired angular frequency ω for any mode with $p > 0$. Mathematically, we use

$$L = \frac{p\pi c}{\sqrt{\omega^2 - \omega_t^2}}, \quad (15)$$

where it becomes obvious that ω_t is the cut-off angular frequency. If the mode cannot be realized because $\omega < \omega_t$ then we can apply an arbitrary global scale factor (SF) to the cross-sectional contour where we define ω_{ref} at $SF = 1$ (original scale). This allows us to rewrite Eq. (15) as

$$L = \frac{p\pi c}{\sqrt{\omega^2 - \left(\frac{\omega_{\text{ref}}}{SF}\right)^2}}. \quad (16)$$

Setting the cavity length to the calculated L from Eq. (16) and reverting back to the physical boundary conditions, we find the mode realized at the desired frequency. For the hybrid $TE_{\{0,2\}1}$, $f_{\text{ref}} = 4.77$ GHz, however for other studies, we wanted a cavity operating at 1.3 GHz, so we applied $SF = 4$ to reduce the cut-off

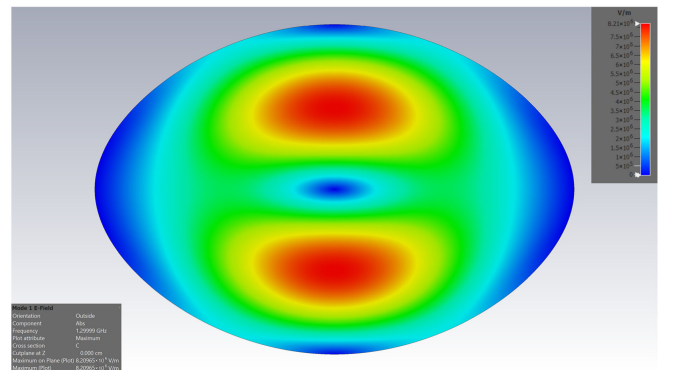


FIG. 5. CST simulation of the hybrid $TE_{\{0,2\}1}$ mode.

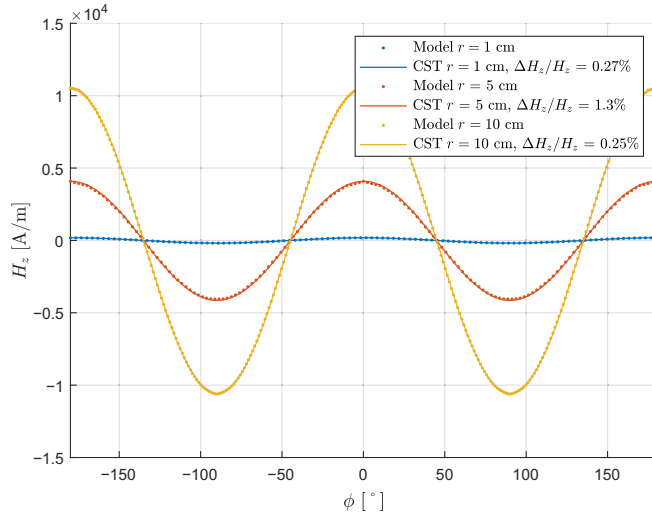


FIG. 6. Comparison of H_z between CST and the MATLAB model on circular contours in the midplane ($z = 0$) for the hybrid $\text{TE}_{\{0,2\}1}$ mode. Markers show CST and solid lines show the MATLAB model at $r = 1, 5$, and 10 cm, with the model amplitude scaled to match CST. The rms relative deviations $\Delta H_z/H_z$ are 0.27%, 1.3%, and 0.25%, respectively.

frequency to $f_i = 1.19$ GHz. This gives $\gamma = 0.25 \text{ cm}^{-1}$ and the cavity length to be 29.00 cm for it to be realized at 1.30 GHz, as seen in Fig. 5.

To validate that the intended modal superposition is preserved after importing the MATLAB-synthesized cross section into CST, we compare the longitudinal magnetic field H_z extracted from CST with the MATLAB prediction on circular contours in the cavity midplane ($z = 0$) at several radii. The MATLAB fields are evaluated using the same TE_{01} and TE_{21} components used to generate the boundary, with equal weights and the physical transverse wave number $\gamma = 0.25 \text{ cm}^{-1}$, so that the comparison is performed in physical units. Figure 6 shows the resulting azimuthal traces.

Figure 6 shows excellent agreement between CST and the MATLAB model across all radii, with rms relative deviations at the subpercent level. These radii have been chosen to compare behavior near-axis, cavity midpoint, and close to the boundary. The residual discrepancies could be attributed to the meshing and postprocessing interpolation in CST, rather than a systematic change in field structure. In particular, the azimuthal dependence matches the expected hybrid $\text{TE}_{\{0,2\}1}$ pattern, providing strong evidence that the prescribed superposition (including the chosen relative weights) is preserved and that no significant higher-order modal content is introduced.

VI. HYBRID $\text{TE}_{\{2,4\}1}$ MODE

In this section, we examine the combination of TE_{21} and TE_{41} in equal weights with $\phi_0 = \pi/4$.

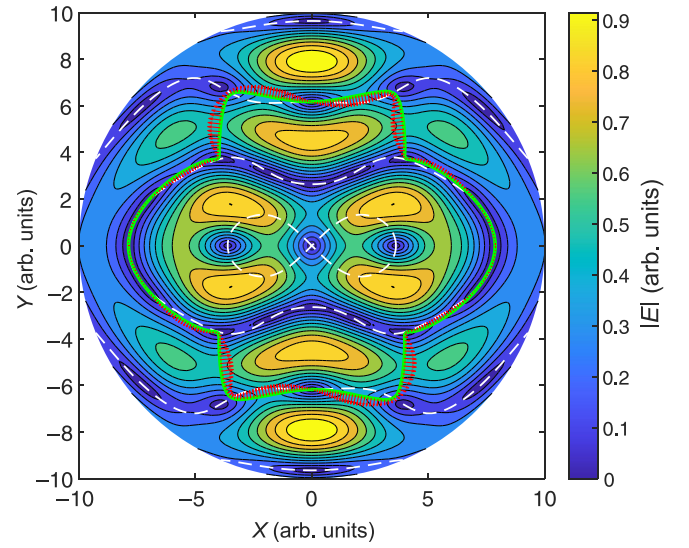


FIG. 7. Contour plot of the magnitude of the electric field in arbitrary units for the hybrid mode $\text{TE}_{\{2,4\}1}$ aided with quivers (in red). Dashed white lines indicate zero contours of the E_ϕ component. Solid green line outlines the cross section of the cavity.

The resultant hybrid mode cavity cross section is illustrated in Fig. 7, outlined in green, alongside zero-line contour of E_ϕ . Although the boundary condition has not achieved full numerical precision, see Fig. 8, a deeper analysis provides further insights. Observing Fig. 7, which highlights the zero equipotential contours of E_ϕ (white dashed lines), one can intuitively identify three distinct potential cavity solutions. The first solution, a central “bow-tie” shaped region intersecting the origin, is immediately discarded as it would physically obstruct particle passage. The second solution, a closed-loop contour

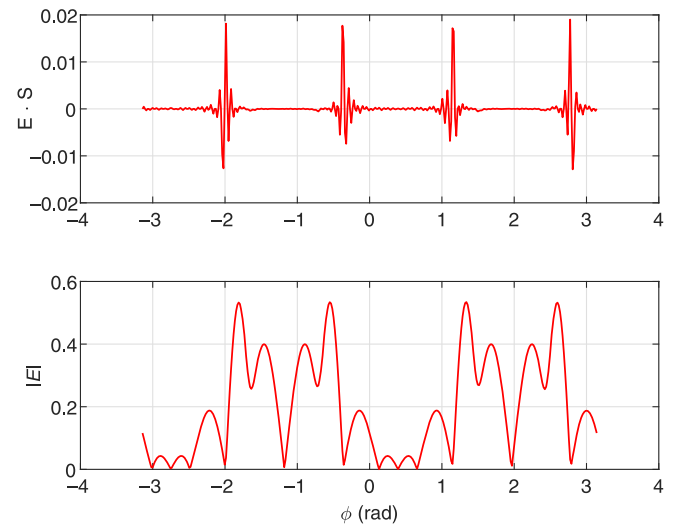
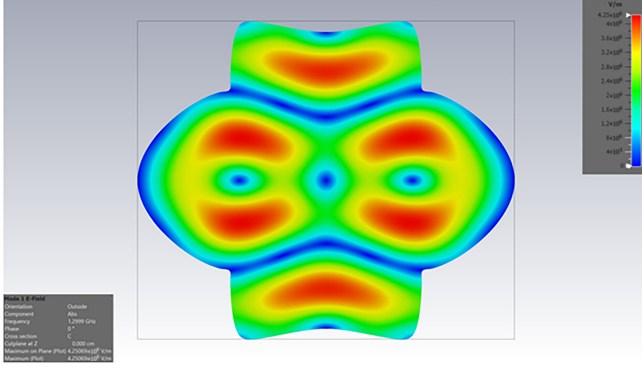


FIG. 8. Electric boundary condition (top) and the magnitude of the electric field (bottom) as a function of the azimuthal angle.

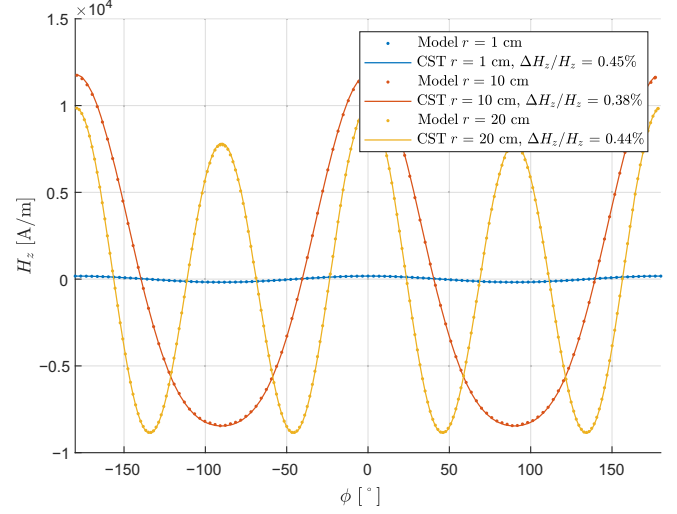
FIG. 9. CST simulation of the hybrid cavity $\text{TE}_{\{2,4\}1}$ mode.

situated slightly further outward, represents the primary region that the optimizer tracks for the cavity boundary. However, the optimizer unexpectedly jumps to the outermost zero contour of E_ϕ , momentarily deviating from the primary loop, before subsequently returning. To prevent the unwanted solutions, we enforce bounds such that the domain is restricted, i.e., not allowing the optimizer to see the “bow-tie.”

This abrupt transition between solutions directly corresponds to the prominent spikes observed in Fig. 8, significantly hindering the optimizer’s progress. The primary cause of these jumps is twofold: first, the loss of crucial directional information when approaching global 0 s of the field, and second, a continuous and physically viable solution along the desired closed-loop contour may not exist throughout the entire region. As a result, the optimizer is compelled to “jump” onto a neighboring contour, typically a higher-order solution before eventually returning to the original loop. Wroe *et al.* have also documented and experienced these issues [1]. For more complex combinations of modes, it is possible to observe solutions that cannot form a closed loop and instead continue to “jump” to the next higher-order solution indefinitely. These are referred to as spiral or discontinuous solutions, and unlike TM modes, these are incredibly difficult to visualize and are not readily apparent in TE modes.

Despite numerical difficulties and the jumps between solutions (Fig. 8), the resulting cavity cross section in Fig. 7 appears physically reasonable given the complexity of the hybrid field distribution. The geometry notably retains a twofold symmetry, as a result of the fact that the TE_{21} mode can be considered a subset of the TE_{41} modes when the nodal lines are appropriately constrained. Additionally, the cavity boundary does not exhibit any abrupt or sharp features, remaining smooth and continuous as expected from our Fourier series representation.

Unexpectedly, the resulting surface remains sufficiently accurate to assert its validity confidently. Extracting the boundary coordinates subsequently enables verification through CST. Such validation provides conclusive evidence

FIG. 10. Comparison of H_z between CST and the MATLAB model on circular contours in the midplane ($z = 0$) for the hybrid $\text{TE}_{\{2,4\}1}$ mode. Markers show CST and solid lines the MATLAB model at $r = 1, 10$, and 20 cm, with the model amplitude scaled to match CST. The rms relative deviations $\Delta H_z/H_z$ are 0.45%, 0.38%, and 0.44%, respectively.

of both the physical existence and operational viability of the cavity.

A. CST validation: Hybrid $\text{TE}_{\{2,4\}1}$ mode

Performing the same analysis to check that both CST and MATLAB are consistent with each other, we find $f_{\text{ref}} = 4.77$ GHz, so we apply $SF = 4$, and find $\gamma = 0.25 \text{ cm}^{-1}$ for an operating frequency of 1.3 GHz. As expected, Fig. 9 reveals the field distribution as determined by CST, which is consistent with the fields modeled in MATLAB with the observed frequency 1.2999 GHz.

Figure 10 shows the comparison of H_z between CST and our MATLAB model with subpercent level of deviation from MATLAB.

VII. FINAL REMARKS

We have introduced a field-driven methodology for synthesizing multipolar TE-mode cavities, based on the superposition of cylindrical standing-wave solutions and numerical enforcement of boundary conditions. Using sequential quadratic programming, we demonstrated that closed cavity surfaces supporting prescribed hybrid TE fields can be obtained and validated against eigenmode solutions in CST. Examples of $\text{TE}_{\{0,2\}1}$ and $\text{TE}_{\{2,4\}1}$ hybrids confirm both the feasibility of the approach and the preservation of modal purity.

Compared with TM modes, enforcing TE boundary conditions requires simultaneous control of both radial and azimuthal electric components, making the synthesis more challenging. The method presented remains general

and can be extended to higher-order hybrids and to more practical geometries, including beam pipes [16].

This work should be regarded as a proof-of-concept for TE cavity surface synthesis. While the present study focused on low-order hybrids, the approach provides a systematic framework that can be applied to more complex structures. Future work will integrate these cavities into beam-dynamics simulations to assess their performance for emittance compensation, chromaticity correction, and other advanced accelerator applications.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this article are openly available [17].

APPENDIX: ALTERNATIVE RUNGE-KUTTA (RK4) APPROACH

For completeness, we also explored an alternative method using a fourth-order Runge-Kutta (RK4) solver to address the ordinary differential equations (ODEs) derived from the boundary conditions. We define our boundary surface S implicitly as

$$S: r - r(\phi) = 0, \quad (\text{A1})$$

with its gradient:

$$\nabla S = \left(\frac{\partial S}{\partial r}, \frac{1}{r} \frac{\partial S}{\partial \phi}, \frac{\partial S}{\partial z} \right) = \left(1, -\frac{1}{r} \frac{dr}{d\phi}, 0 \right). \quad (\text{A2})$$

Requiring the tangential electric field components at the boundary to vanish, we have:

$$\mathbf{E}_{\parallel} = \mathbf{E} - \frac{(\mathbf{E} \cdot \nabla S)}{|\nabla S|^2} \nabla S. \quad (\text{A3})$$

Expanding explicitly, the tangential field at the boundary condition reduces to solving the following ODE:

$$\frac{dr}{d\phi} = -\frac{rE_{\phi}^S}{E_r^S}. \quad (\text{A4})$$

From here, we see that when our bound intersects a point which $E_r \approx 0$, it becomes numerically unstable, hence why we have chosen a surface deformation procedure as opposed to a direct solve.

Similarly, applying magnetic field boundary conditions yields:

$$\frac{dr}{d\phi} = \frac{rH_r^S}{H_{\phi}^S}. \quad (\text{A5})$$

An intuitive and reassuring consequence of these equations is the orthogonality of the fields at the boundary, by equating both ODEs:

$$r(E_r^S H_r^S + H_{\phi}^S E_{\phi}^S) = r(\mathbf{E} \cdot \mathbf{H})^S = 0, \quad (\text{A6})$$

serving as a consistency check. Despite its theoretical elegance and superior local accuracy compared to global boundary deformation methods, the RK4 method encountered practical challenges. It proved highly sensitive to initial conditions and local field minima, frequently spiraling toward local minima or global field 0 s and producing unphysical solutions. Although conceptually robust, this sensitivity to numerical precision and local field structures significantly limited the practical applicability of the RK4 approach. Nonetheless, the mathematical framework presented in this section can also be used as a constraint or as a soft objective function to guide the solution.

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