Space charge effects on coupled-bunch instability in high-intensity proton rings

Kazuhito Ohmi,[1,2,3] Liangsheng Huang,[1] Hanyang Liu,[1] and Li Rao[1]

[1]*Spallation Neutron Source Science Center, Dongguan 523803, China*
[2]*IHEP, Institute of High Energy Physics, Yuquan Road, Beijing 100049, China*
[3]*KEK, High Energy Accelerator Organization, Tsukuba, Ibaraki 305-0801, Japan*

(Received 26 November 2024; accepted 30 May 2025; published 7 July 2025)

High-intensity proton accelerators are utilized for neutron, neutrino, and hadron sources. Increasing the beam intensity is important for science that utilizes these particles. However, as beam intensity increases in these proton accelerators, beam instability has emerged as a limiting factor. Coupled-bunch instability, resulting from impedance with frequency components near or below the revolution frequency, stemming from the electrical resistance of beam chambers, ceramic chambers in rapid cycling synchrotrons, and kicker magnets, poses a significant challenge in these proton accelerators. The oscillation frequency of beams observed during operation with chromaticity is considerably higher than the impedance frequency. Space charge forces exhibit significant strength, with a tune shift exceeding 0.1 as an absolute value. In single-bunch behavior, space charge forces dominate, influencing the oscillation modes of instability. It is inevitable that the effect will extend to coupled-bunch modes. This paper provides a theoretical discussion on the coupled-bunch instability of beams subjected to intense space charge forces.

DOI: 10.1103/c876-vj6w

I. INTRODUCTION

In high-intensity proton accelerators such as the China Spallation Neutron Source (CSNS) [1] and the Japan Proton Accelerator Research Complex (J-PARC) [2], coupled-bunch instabilities due to impedance with frequencies ranging from $0.1 \sim 10$ times the revolution frequency and with $Q = 10 \sim 100$ have been observed [3,4], where Q is the quality factor of the beam coupling impedance. The tune shift due to space charge forces is larger than 0.1 as an absolute value. We discuss the coupled-bunch instabilities of beams experiencing strong space charge forces.

The collective motion of particles freely undergoing synchro-betatron oscillations can be represented in the transverse amplitude distribution on the longitudinal phase space as $v(z,\delta) = y(z,\delta) + ip_y(z,\delta)$, where y is used for the transverse coordinate, which can be considered as either horizontal or vertical coordinate. $z = c\beta(t_0 - t)$ and $\delta = \Delta p/p_0$ are the time advance of the particle arrival multiplied by the velocity ($c\beta$) and momentum deviation from the reference particle, which are the canonical variables in the longitudinal motion. From now on, we will use a more convenient representation $v(J,\phi) = [y(J,\phi) + ip_y(J,\phi)]$, where y and p_y are normalized by Twiss parameters. J and ϕ are the amplitude and phase of the synchrotron motion, $z = \sqrt{2J\beta_z}\cos\phi$ and $\delta = \sqrt{2J/\beta_z}\sin\phi$, where β_z is the aspect ratio of the particle trajectory in the longitudinal phase space. Considering $v(J,\phi)$ as the base vector, we can derive the revolution transformation that includes impedance and space charge forces. By solving the eigenvalue problem, we can evaluate the tune and growth of instabilities.

For particles moving freely, the distribution after one revolution can be expressed as $e^{-i\mu_y}v(J,\phi-\mu_s)$ based on betatron oscillation with the tune of $\nu_y = \mu_y/(2\pi)$ and synchrotron oscillation with the tune of $\nu_s = \mu_s/(2\pi)$. Since v is periodic with respect to the synchrotron phase ϕ, it is Fourier expanded:

$$y(J,\phi) = \sum_l y_l(J)e^{il\phi}, \qquad p_y(J) = \sum_l p_l(J)e^{il\phi}, \quad (1)$$

where the summation for l is $-\infty$ to ∞, but in practice, it is truncated at a suitable integer $\pm l_{\max}$. Using v expression, $v(J,\phi) = \sum v_l(J)e^{il\phi}$, $y_l = (v_l + v^*_{-l})/2$, $p_l = (v_l - v^*_{-l})/(2i)$. We consider an oscillation with only one l component. After one revolution, the distribution changes $v(J,\phi)_0$ to $v(J,\phi)_1$ as

$$v(J,\phi)_1 = e^{-i\mu_y}v(J,\phi-\mu_s)_0 = e^{-i(\mu_y+l\mu_s)}v(J,\phi)_0. \quad (2)$$

The presence of $\phi - \mu_s$ in the relation indicates that the distribution at ϕ after one revolution is shifted from the initial

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

distribution at $\phi - \mu_s$ as described by the so-called Vlasov equation. Equation (2) means that the eigenvector $v_l(J)e^{il\phi}$ has an eigenvalue $e^{-i(\mu_y + l\mu_s)}$ with respect to one revolution.

Due to space charge forces, particles become correlated with each other, deviating from the free oscillation. Here, linear space charge forces in the transverse direction are assumed to follow a KV distribution. A uniform water bag or Gaussian distribution is assumed in the longitudinal phase space. Space charge forces can be effectively represented as wakefields and impedance [5]. Transverse tune shift due to the space charge force depends on z within a bunch. Integrating over the synchrotron period, the tune shift becomes a function of the amplitude J. Coupling can occur in the J direction as well as in the l mode, leading to the emergence of collective motion.

Chromaticity is an important parameter affecting instability in proton accelerators. Variations in betatron phase within the bunch arise from chromaticity, which can be characterized by the head-tail phase (χ). While this parameter is typically small in electron storage rings and large proton colliders ($\chi \ll 1$), it exceeds 1 in high-intensity proton synchrotrons such as CSNS and J-PARC. The bunch exhibits sinusoidal modulation in the z direction $e^{i\chi z/\sigma_z}$, where σ_z is the bunch length. The impedance considered here is a low-frequency component that causes correlations between bunches. The oscillation component of a bunch, when coupled with the impedance, is only the overall dipole moment of the bunch. If the head-tail phase increases due to chromaticity, the overall dipole moment decreases. There are various oscillation modes within the bunch. The dipole moment of the entire bunch is determined by the combination of the head-tail phase and the oscillation modes within the bunch.

The oscillation modes within the bunch are altered by the space charge force. The nature of the instability changes significantly depending on whether space charge effects are considered. Understanding the oscillation modes caused by space charge is essential for studying the coupled-bunch instability.

Many studies have been conducted on the impact of space charge effects on single-bunch instability [6–11]. The oscillation modes due to space charge have been studied in several works. Initially, Blaskiewicz [6] discussed these modes using an airbag model within a square potential. This work was later expanded by Burov [7], and further analyzed by Macridin *et al.* [8], and Kornilov and Boine-Frankenheim [9] using particle tracking simulations that account for space charge.

The intrabunch oscillation modes are significantly altered by space charge. In the context of coupled-bunch instability, the most important factor is the overall dipole moment of the bunch. Among the intrabunch oscillation modes that consider chromaticity and space charge forces, the mode with the largest overall dipole moment becomes the dominant unstable mode. The influence of space charge forces in coupled-bunch instability is significantly different from that in single-bunch cases.

A study on coupled-bunch instability was conducted with a focus on the space charge damping effect [12]. In Ref. [13], coupled-bunch instability was investigated for two limiting cases: zero space charge and the strong space charge. The paper presented results similar to those discussed in this work.

In this work, we adopt a linear model with minimal approximations and employ numerical analysis to investigate how coupled-bunch instability is influenced by space charge, particularly focusing on the overall dipole moment. The strength of the space charge force, synchrotron tune, and chromaticity primarily target the parameter space of accelerators in spallation neutron sources such as SNS, CSNS, and J-PARC.

This paper is structured as follows: the analysis formalism is discussed in Sec. II. Chromaticity and coupled-bunch instability, considering the internal structure, are addressed in Secs. II A–II C. Oscillation modes due to space charge forces are examined in Sec. III. The comparison with previous studies is discussed in Sec. III B. Coupled-bunch instability under the influence of space charge forces is discussed in Sec. IV. The conclusion is presented in Sec. V.

II. ANALYSIS FORMALISM

A. Chromaticity

Chromaticity is a quantity that characterizes the dependence of the betatron tune on the momentum deviation δ in the motion of a single particle. The tune is expressed as a polynomial expansion with respect to δ, and the coefficients of this expansion are referred to as chromaticity:

$$\nu_y(\delta) = \sum_{n=0} \nu_n \delta^n. \tag{3}$$

The linear chromaticity corresponds to the coefficient of the first order, $\xi = \nu_1$. When chromaticity is present in an initial dipole amplitude distribution with an azimuthal mode l, it undergoes the following transformation over one revolution:

$$\begin{aligned} y(J,\phi)_1 &= e^{-2\pi i \nu_y(\delta)} y(J, \phi - \mu_s)_0 \\ &= e^{-2\pi i(\nu_y(\delta) + l\nu_s)} y(J,\phi)_0. \end{aligned} \tag{4}$$

The dipole amplitude after one revolution contains Fourier components other than l, indicating it is not an eigenvector with respect to l.

We seek the eigenmodes for l in the free betatron-synchrotron oscillations with chromaticity. The change in betatron phase due to chromaticity is divided into two parts that do not depend on the angle variable ϕ and those that do:

$$d\phi_\beta = \nu_0 d\theta + \sum_{n=1} \nu_n \delta^n d\theta = [\bar{\nu}_0(J) + \tilde{\nu}_\xi(J,\phi)]d\theta. \quad (5)$$

Here, $\theta = 2\pi s/L$ is the angle indicating the positon s in the ring with circumference L. $\tilde{\nu}_\xi(J,\phi)$ can be expressed as a Fourier expansion for $\sin n\phi$ and $\cos n\phi$.

Considering the variation in oscillation modes due to chromaticity $\zeta(J,\phi)$, we expand as follows:

$$y(J,\phi) = \zeta(J,\phi)\sum_l y_l(J)e^{il\phi}. \quad (6)$$

The transformation for l mode is:

$$y(J,\phi)_{d\theta} = e^{-i(\bar{\nu}_0(J)+l\nu_s)d\theta} \times \exp\left(-\frac{\nu_s}{\zeta}\frac{d\zeta}{d\phi}d\theta - i\tilde{\nu}_\xi(J,\phi)d\theta\right)y(J,\phi)_0, \quad (7)$$

where $d\phi/d\theta = \nu_s$. Setting the expression inside the exponential to zero, the distribution represented by Eq. (6) becomes an eigenvector with respect to the revolution transformation ($d\theta \to 2\pi$) for the chromaticity with an eigenvalue of $e^{-2\pi i(\bar{\nu}_0(J)+l\nu_s)}$. The value of ζ is determined as follows:

$$\zeta(J,\phi) = \exp\left(-\frac{i}{\nu_s}\int \tilde{\nu}_\xi(J,\phi)d\phi\right). \quad (8)$$

In the case of linear chromaticity, with $\tilde{\nu}_\xi = \xi\delta = \xi\sqrt{2J/\beta_z}\sin\phi$, the aforementioned results are obtained, $\zeta(J,\phi) = e^{2\pi i\xi z/\eta L} = e^{i\chi z/\sigma_z}$, using $\beta_z = \eta L/(2\pi\nu_s)$ and $z = \sqrt{2J\beta_z}\cos\phi$, where η is the slippage factor, and $\chi = 2\pi\xi\sigma_z/(\eta L)$ is the head-tail phase.

Here, we separated the phase component with respect to chromaticity and sought eigenstates, but there may be alternative methods using Eq. (1) without focusing on eigenstates for chromaticity. Particularly, there might be advantages in doing so when looking at the final results. However, for a clearer physical interpretation, we believe it is preferable to follow the conventional method.

B. Analysis formalism for coupled-bunch instability

The dipole amplitude in the phase space (J,ϕ) for bunch b is denoted by $v_b(J,\phi)$. Assuming that bunches are stored at equal intervals with equal bunch population in the ring, the correlation between bunches is characterized by the mode number m. The dipole amplitudes must be periodic with respect to the circumference when observed in a snapshot. For mode m, the dipole amplitude for each bunch is expressed as follows [14]:

$$v_{m,b}(J,\phi;s+bL/H) = e^{2\pi imb/H}v_{m,0}(J,\phi;s), \quad (9)$$

where s represents the longitudinal position. The bth bunch is located bL/H ahead of the 0th bunch. The number of modes is the same as the number of bunches H, i.e., ranging from $m = 0$ to $H-1$. We observe a frequency signal $(m + \nu_y + pH)\omega_0$ for mode m oscillation using a beam position monitor, where the integer p ranges from $p = -\infty$ to ∞, and $\omega_0 = 2\pi c\beta/L$ is the revolution frequency.

The coupled-bunch modes are considered to be uncoupled from each other. Once the modes are determined, the interbunch correlation is described by Eq. (9). Therefore, if we know the internal motion of the $b = 0$ bunch, we can understand the internal oscillations of all bunches. From now on, we will treat only the oscillation of the 0th bunch as $y_m = y_{m,0}$. Although bunch internal oscillation modes with respect to l may potentially couple as seen in the transverse mode coupling instability, in general coupled-bunch instability (when the single-bunch tune shift is small), they do not couple. Bunch internal oscillation with J can be correlated by impedance; therefore, a distribution as a function of J appears as an internal oscillation mode. The modes are characterized by m, l and the function of J, each having its own eigen oscillation.

The equation of motion for the 0th bunch is as follows:

$$\delta p_{y,0}(z;s) = -C_p \times \sum_{b=-\infty}^{\infty}\int W_D(z-z'-bL/H)y_b(z';s)\rho(z')dz', \quad (10)$$

where $C_p = N_b r_p/\gamma = N_b e^2/(4\pi\epsilon_0 E)$ and N_b are the bunch population. $W_D(z-z')$ is the dipolar wake function, expressed in unit of m^{-2}, which induces a dipole kick at z according to the dipole moment at z' in bunches. Here, by summing over b infinitely, we incorporate the effects for all bunches and all revolutions. The slow-frequency quadrupolar wake in a multibunch system causes a tune shift in all bunches, but since it does not affect the intrabunch oscillations, we only consider the dipolar wake/impedance (W_D/Z_D) as a source of coupled-bunch instability.

The dipole amplitude of the 0th bunch oscillating in mode m is expanded in the azimuthal modes, taking into account the chromaticity as follows:

$$v_m(J,\phi) = e^{2\pi i\xi z/\eta L}\sum_{l=-\infty}^{\infty} v_{ml}(J)e^{il\phi}. \quad (11)$$

The bth bunch in mode m is related to the 0th bunch at a different longitudinal position $s_b = s + bL/H$ at the same time, according to Eq. (9). To relate the y of the bth bunch that arrives at the same s as the 0th bunch, the phase change during travel $\Delta s = bL/H$ is taken into account, and it is related as follows:

$$v_{b,m}(z;s) = e^{2\pi imb/H}v_m(z;s-bL/H) = e^{2\pi i(mb/H+\nu b/H)}v_m(z;s), \quad (12)$$

where ν is expressed as the frequency, but it represents the eigen frequency when expanded into eigenmodes, so it is an implicit relation.

The equation of motion Eq. (10) can be written for each m mode as follows:

$$\delta p_{ml}(J) = -C_p\beta_y \int \sum_{l'} W_{mll'}(J,J')\psi(J')y_{ml'}(J')dJ', \tag{13}$$

where

$$W_{mll'}(J,J') = i^{l-l'-1}H\omega_0 \times \sum_{p=-\infty}^{\infty} Z_D(\omega)J_l[(\kappa-\chi)r]J_{l'}[(\kappa-\chi)r']. \tag{14}$$

Here ψ is the distribution function in the longitudinal phase space, and the distribution in z is expressed by $\rho(z) = \int d\delta\psi(z,\delta)$. ψ is a function that depends only on J and not on ϕ. $J_l(x)$ is a Bessel function of order l.

The integration in Eq. (13) is carried out over the range of 0 to ∞ for J', where $r' = \sqrt{2J'/\epsilon_z}$. Here $\kappa = \omega\sigma_z/\beta c$ represents the phase angle of the frequency ω in terms of the bunch length, and the sum over p is performed using $\omega = (m+\nu+pH)\omega_0$.

Discretizing with respect to J in n_J steps and transforming into matrix form yield

$$\delta p_{ml}(J) = -2\sum_{l'J'} M_{Wm,lJl'J'}y_{ml'}(J'). \tag{15}$$

The matrix element is written as follows,

$$M_{Wm,lJl'J'} = \frac{C_p\beta_y}{2}W_{mll'}(J,J')\psi(J')\Delta J. \tag{16}$$

If the impedance is uniformly distributed along s within the ring, the eigenvalue problem for the revolution can be replaced by an eigenvalue problem with respect to the tune $\nu = \mu/(2\pi)$ [5],

$$\mu = \mu_y + l\mu_s + M_{Wm}. \tag{17}$$

C. Numerical solutions using the parameters of the CSNS

The CSNS is a high-intensity proton accelerator-based facility in Dongguan, China, including a rapid cycling synchrotron (RCS). The circumference of the RCS is $L = 227.92$ m, and protons are accelerated from a kinetic energy of $E_k = 80$ MeV to 1.6 GeV at a repetition rate of 25 Hz, being utilized as a neutron source. An instability has been observed around the energy of $E_k = 120$ MeV during the acceleration process with a proton population of $N_b = 0.5\times10^{13}$. In this analysis, we will examine the instability using the parameters corresponding to that observation. The absolute value of the space charge tune shift is $\Delta\nu_{\rm sc} = 0.27$ at $N_b = 0.78\times10^{13}$/bunch. The slippage factor is $\eta = -0.74$ at $E_k = 120$ MeV, and the synchrotron tune is $\nu_s = 0.01$. The longitudinal distribution is assumed to be water bag, uniform in the phase space, with a half length of $\sqrt{2}\sigma_z = 35$ m, and an energy spread $\pm2\%$. The head-tail phase per chromaticity is $\chi/\xi = 1.3$.

A resonator model is employed with a peak impedance $Z_{\rm peak} = 3$ kΩ/m, resonant frequency $\omega_R/(2\pi) = 123$ kHz, and quality factor $Q = 40$ to represent the impedance. The ceramic chamber is identified as the primary source of impedance [3]. In the sum of Eq. (14), ν is replaced with ν_y. This implies that we are assuming that the effective Q is small. The results here deal with cases where l is relatively small or the tune is close to ν_y, so the impact seems to be minimal. It would be better to perform the iteration in the future.

In the absence of impedance, the bunch oscillation modes are characterized by azimuthal tunes of $\nu_y + l\nu_s$, and the radial modes are degenerate. When the chromaticity is zero, the oscillation mode with $l = 0$ azimuthally and uniform radially, which has the largest dipole moment, is chosen as the intrabunch oscillation mode for the coupled-bunch instability. When impedance/wake forces are introduced, the intrabunch mode is affected. For finite chromaticity, the growth rate is determined based on the oscillation mode in the bunch where the radial degeneracy is resolved. The growth rate and radial distribution for l are obtained by solving the eigenvalue problem of Eqs. (16) and (17).

Figure 1 displays the tunes $\nu = {\rm Re}(\mu)/2\pi$ and growth rate ${\rm Im}(\mu)$ as functions of the bunch population for $\xi = 0$. The tunes remain nearly unchanged at $\nu = \nu_y + l\nu_s$, but only the mode with $\nu_y = 0.8$, i.e., $l = 0$, exhibits unstable growth. The coupled-bunch mode is at $m = 1$, with a frequency $(5-\nu_y)\omega_0 = 0.2\omega_0$, representing the slowest frequency for the betatron tune. The growth rate is proportional to the bunch population, with the growth rate ${\rm Im}(\mu) = 0.0106$/turn at $N_b = 10^{13}$. The amplitude distribution of the unstable mode $y(z,\delta)$ is nearly flat, with a slight structure of about 1%, although not shown here. This indicates predominantly dipole oscillations that have shifted almost entirely.

Similar calculations were conducted for chromaticities of $\xi = -2, -4, -6$, and -9.7. The tunes remained unchanged as depicted in Fig. 1(a). The point at $\xi = 0$ is omitted because it is exceptionally large (0.0106 for $l = 0$), as shown in Fig. 1. Several unstable modes were observed, and the growth rate was proportional to the bunch population for all unstable modes. Figure 2 illustrates the results for $-2, -4, -6$, and -9.7. In plot (a), the growth rates of five unstable modes at $N_b = 10^{13}$ are plotted as functions of chromaticity. The $l = 0$ mode is found among

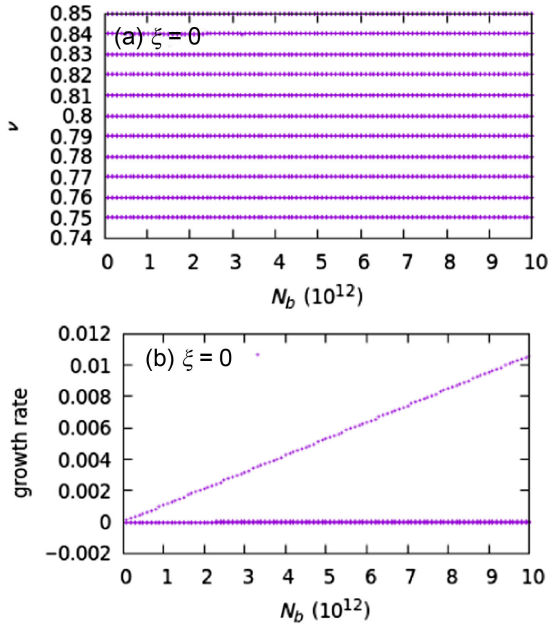


FIG. 1. Eigenvalues for $\xi = 0$. Tune $\nu = \mathrm{Re}(\mu)/2\pi$ and growth rate $\mathrm{Im}(\mu)$ are plotted in (a) and (b), respectively.

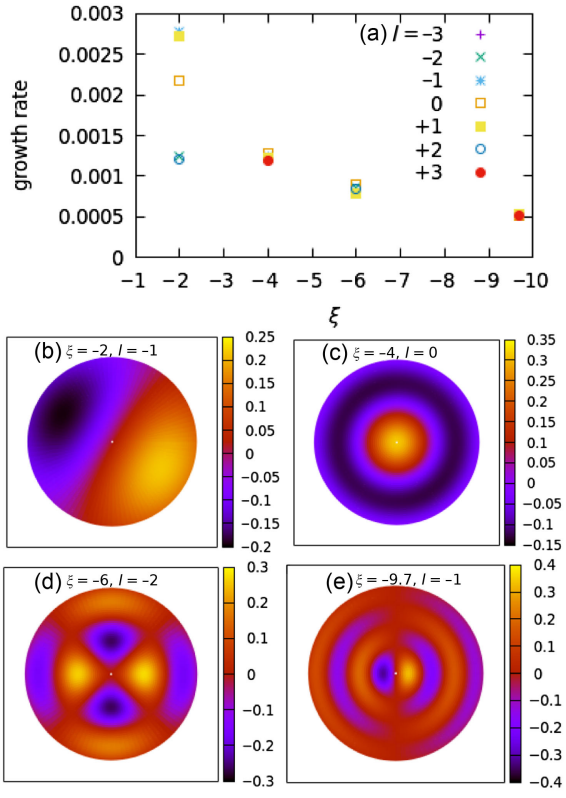


FIG. 2. Eigenvalues for $\xi = -2, -4, -6$, and -9.7. The growth rate of 5 unstable modes at $N_b = 10^{13}$ are plotted as functions of chromaticity shown in plot (a). The amplitude distributions $y_l(J)e^{il\phi}$ for the unstable modes with $|l| = 0 \sim 2$ are drawn in (b)–(e).

the five modes for all chromaticities. The largest azimuthal modes, $l = \pm 4$, are observed at $\xi = -4$ and -9.7, with lower l modes prevailing overall. The instability weakens as chromaticity increases.

Plots 2(b)–2(e) display samples of the amplitude distribution $y_l(J)e^{il\phi}$ for the unstable modes with $|l| = 0$ to 2 for each chromaticity. The tune remains almost unchanged, but the degeneracy with respect to J has been resolved.

When applying a phase change $e^{-i\mu s/L}$ to these distributions as they evolve in time, modes with $l < 0$ rotate clockwise, while modes with $l > 0$ rotate counterclockwise. It is important to note that the actual measured distribution is obtained by multiplying by the head-tail phase $e^{i\chi z/\sigma_z}$.

III. SPACE CHARGE FORCE

The oscillation modes of a bunch due to space charge forces are significantly different from free oscillations. We discuss the modes that oscillate under the influence of space charge forces. The space charge force is represented by the following equation:

$$\Delta p_y(z,\delta) = k_{\mathrm{sc}}\rho(z)[y(z,\delta) - \bar{y}(z)]\Delta s. \tag{18}$$

The distribution in the longitudinal phase space $y(z,\delta)$ experiences the space charge force depending on the deviation from the centroid position $\bar{y}(z)$ at z. The force is also proportional to the local charge density of $\rho(z)$. The coefficients are as follows:

$$k_{\mathrm{sc}} = \frac{2Nr_e}{\beta^2\gamma^3}\frac{1}{\sigma_y(\sigma_x + \sigma_y)}. \tag{19}$$

The effective wake and impedance due to the space charge force are given by

$$W_{\mathrm{sc}}(z - z') = \frac{k_{\mathrm{sc}}L}{C_p}\delta(z - z'), \qquad Z_{\mathrm{sc}}(\omega) = i\frac{k_{\mathrm{sc}}L}{\beta c C_p}, \tag{20}$$

where the dipolar and quadrupolar components of the wake and impedance are represented by

$$W_{D/Q} = \pm W_{\mathrm{sc}}, \qquad Z_{D/Q} = \pm Z_{\mathrm{sc}}. \tag{21}$$

For the space charge force, the quadrupolar term is essential and has no correlation with other bunches. Therefore, the sum over revolutions (p) in the previous section is performed by integration (numerically, over a frequency step smaller than the revolution frequency).

The equations for $W_{lJl'J'}, \bar{W}_{ll'}$ are given by Eqs. (14) and (16) in Ref. [5]. All bunches oscillating in the coupled-bunch mode m are related by Eq. (12). Since the space charge force acts on all bunches, it is sufficient to analyze a single bunch, thus m is omitted in the description.

TABLE I. $D(\kappa)$ for several longitudinal distribution cases. $h(x)$ is the step function. The beam size is defined by $\sigma_z = \sqrt{\varepsilon_z \beta_z}$, and particle distributes $0 \le \hat{J} \le 1$ and $|z| \le \sqrt{2}\sigma_z$ except for Gaussian, where $\hat{J} = J/\varepsilon_z$.

Type	$\psi(\hat{J})$	$\rho(z)$	$D(\kappa)$
0	$\frac{1}{2\pi} e^{-\hat{J}}$	$\frac{1}{\sqrt{2\pi}\sigma_z} e^{-z^2/2\sigma_z^2}$	$\frac{1}{2\pi} e^{-\kappa^2/2}$
1	$(1/\pi)(1-\hat{J})$	$\frac{2}{3\pi\sigma_z^4}(2\sigma_z^2 - z^2)^{3/2}$	$(2/\pi)\frac{J_2(\sqrt{2}\kappa)}{\kappa^2}$
2	$\frac{3}{4\pi}\sqrt{1-\hat{J}}$	$\frac{3}{8\sqrt{2}\sigma_z^3}(2\sigma_z^2 - z^2)$	$\frac{3}{2\pi}\frac{\sin\sqrt{2}\kappa - \sqrt{2}\kappa\cos\sqrt{2}\kappa}{(\sqrt{2}\kappa)^3}$
3	$\frac{1}{2\pi}h(1-\hat{J})$	$\frac{1}{\pi\sigma_z^2}\sqrt{2\sigma_z^2 - z^2}$	$(1/\pi)\frac{J_1(\sqrt{2}\kappa)}{\sqrt{2}\kappa}$
4	$\frac{1}{4\pi}\frac{1}{\sqrt{1-\hat{J}}}$	$\frac{1}{2\sqrt{2}\sigma_z}$	$\frac{1}{2\pi}\frac{\sin\sqrt{2}\kappa}{\sqrt{2}\kappa}$

Here, we provide expressions using impedance:

$$M_{W,lJl'J'} = \frac{C_p \beta_y}{2}\left[\sum_{l'} \bar{W}_{ll'}(J)\delta_{JJ'} + \sum_{l'} W_{ll'}(J,J')\psi(J')\Delta J\right], \tag{22}$$

$$\bar{W}_{ll'}(J) = i^{l-l'-1}\int d\omega Z_Q(\omega) J_{l-l'}(\kappa r) D(\kappa), \tag{23}$$

$$W_{ll'}(J,J') = i^{l-l'-1}\int d\omega Z_D(\omega) J_l[(\kappa-\chi)r] J_{l'}[(\kappa-\chi)r']. \tag{24}$$

$D(\kappa)$ is obtained by integrating over r' in $\bar{W}_{ll'}(J)$,

$$D(\kappa) = \int_0^\infty r' dr' J_0(\kappa r')\psi(r'). \tag{25}$$

Integrations have been conducted for several longitudinal distribution cases, and the results are summarized in Table I.

The $\bar{W}_{ll'}(J)$ related to the quadrupolar term influences the tune shift. By considering the diagonal components, it is possible to evaluate the tune shift dependent on J,

$$\Delta\nu_{y,l}(J) = \frac{C_p\beta_y}{4\pi}\bar{W}_{ll}(J) = -\Delta\nu_{\rm sc}\int_0^\infty d\sqrt{2}\kappa \frac{J_0(\kappa r)J_1(\sqrt{2}\kappa)}{\sqrt{2}\kappa}, \tag{26}$$

where J_0 and J_1 are the Bessel functions of order 0 and 1. The tune shift is distributed between $-\Delta\nu_{\rm sc}$ at $J = 0$ and $-2\Delta\nu_{\rm sc}/\pi$ at $J = \varepsilon_z$.

A. Oscillation modes of a bunch subject to space charge force

We calculate the oscillation modes of a bunch subject only to space charge forces using the parameters of CSNS-RCS. The key parameters are the tune shift due to space charge forces and the synchrotron tune, so the results are not limited to CSNS-RCS.

Figure 3 illustrates the oscillations of a bunch subject to space charge forces for the water bag distribution in longitudinal phase space. The space charge tune shift and that normalized by ν_s are $\Delta\nu_{\rm sc} = 0.35$ and $\Delta\nu_{\rm sc}/\nu_s = 35$ at $N_b = 1\times10^{13}$. For each $l \ge 1$, the tune of one mode is separated from those of other modes that are continuously distributed. The $l = 0$ mode has a tune independent of the bunch population. The tune of the $l = 0$ mode is equal to the original tune, ν_y. The phase space distribution is flat with uniform displacement, $y(J,\phi) = \text{constant}$, although not shown here. This is a straightforward result indicating that a bunch oscillating entirely in a dipole motion experiences no space charge tune shift.

The isolated modes with $l > 0$ experience a space charge tune shift, but the slope gradually decreases with an increase in the bunch population, and they never fall below the original betatron tune $\nu_y = 0.8$.

The other continuously distributed modes arise from the dependence on J in the tune shift due to space charge, as given by Eq. (26). The distribution resembles a circular distribution seen in Fig. 7(a) of Ref. [15]. The larger the radius of the circle, the smaller the tune shift (going upward within the range of J in Fig. 3). Since the impedance of

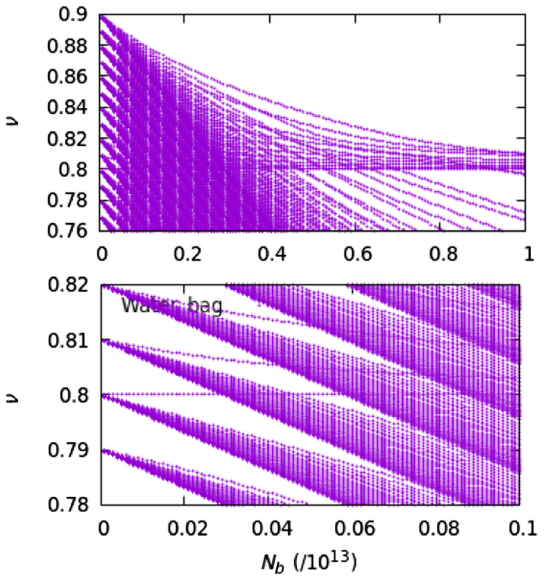


FIG. 3. Tune of oscillation modes experiencing space charge force, where $l_{\rm max} = 10$ and $n_J = 40$. The space charge tune shift and that normalized by ν_s are $\Delta\nu_{\rm sc} = 0.35$ and $\Delta\nu_{\rm sc}/\nu_s = 35$ at $N_b = 1\times10^{13}$. The bottom plot enlarges the region of tune and beam intensity to make the mode behavior more visible.

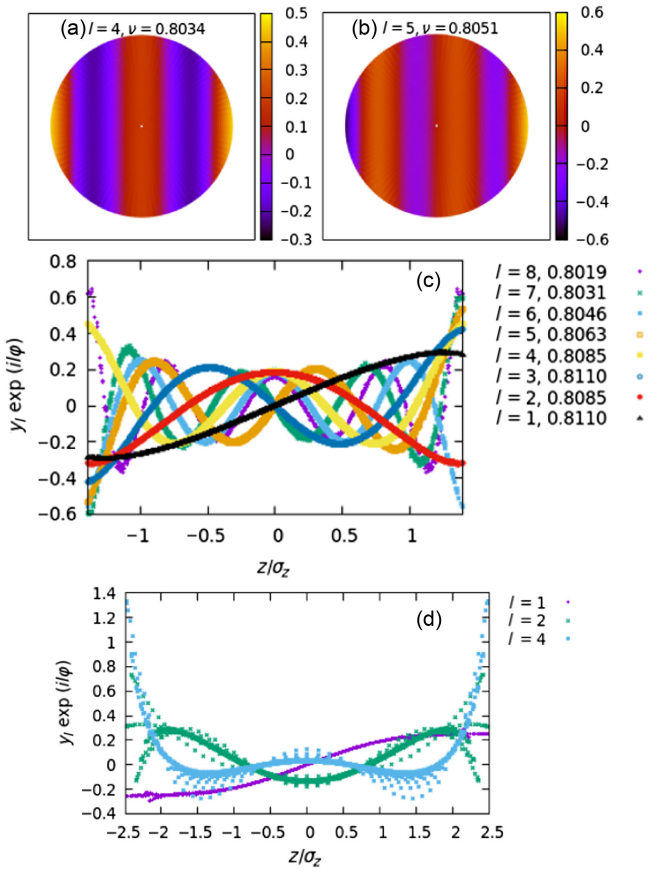


FIG. 4. Oscillation modes experiencing space charge force. Amplitude distributions in the longitudinal phase space are plotted for $l = 4$ and 5 in (a) and (b). The distribution projected onto the z axis for each l using the water bag model is drawn in (c). That for Gaussian model is drawn in (d).

space charge forces is purely imaginary, instability does not occur unless the tune reaches a half integer.

Figure 4 displays the amplitude distributions $y(z,\delta)$ for each l. The amplitude distribution takes the form of a sinusoidal function with respect to z and is independent of δ, as shown in plots (a) and (b), where the horizontal and vertical axes represent z and δ, respectively. The distributions projected onto the z axis for all l are drawn in (c). The oscillation pattern is cosinelike for even l and sinelike for odd l. The oscillation period decreases as l increases, scaling as $y_l(z) \sim i^l e^{ilz/\sigma_z}$. It deviates from sinusoidal shape near the edge of z. Even when multiplied by $e^{-i\mu n}$, the amplitude oscillates but the shape remains unchanged, always becoming a function of z only. In other words, these distributions do not collectively rotate in the longitudinal phase space with respect to the synchrotron oscillation. These states can no longer be referred to as synchro-beta modes. In these modes, where $y(z,\delta) - \bar{y}(z) = 0$, it can be considered almost certain to approach the 0 mode tune ν_y.

This oscillation pattern varies depending on the density distribution of the bunch. In the case of a Gaussian distribution in plot (d), the deviation from the sine and cosine functions is larger compared to the water bag distribution, making it somewhat nonuniform in δ.

Chromaticity introduces a frequency shift in the bunch oscillation, Eq. (14), using the parametrization of Eq. (6), but since the impedance of space charge forces remains constant with respect to frequency, the tune spectrum in Fig. 3 and the amplitude distribution $y_l e^{il\phi}$ in Fig. 4 are independent of chromaticity.

However, in the presence of chromaticity, the actual distribution is obtained by multiplying by $e^{i\chi z/\sigma_z}$. Depending on the value of chromaticity, the oscillations of modes with similar frequencies cancel each other out, resulting in the appearance of a uniform dipole component. This has a significant impact on coupled-bunch instability.

B. Comparison of the space charge oscillation mode with previous studies

The oscillation modes due to space charge were initially discussed using an airbag model within a square potential [6]. Subsequently, they were analytically studied in a generalized potential where the space charge force is dominant compared to synchrotron oscillations, i.e., $\Delta\nu_{sc} \gg \nu_s$ [7]. In both cases, the eigenmodes were solved under the condition that the dipole amplitude $y(J,\phi)$ is solely a function of z, i.e., $y(J,\phi) = y(z)$. A similar approach was also applied to coupled-bunch instability in [13].

In this paper, the eigenvalue problem for the dipole amplitude $y(J,\phi)$ was numerically solved for any density distribution within a parabolic potential. It was demonstrated that as the space charge force becomes dominant, $y(J,\phi)$ tends to take the form $y(z)$, and the results agree with previous works.

References [8,9] employ a macroparticle tracking code that incorporates space charge forces. Notably, Ref. [8] utilizes a method for extracting modes based on the behavior of the computed particle distribution. Without imposing any conditions on $y(J,\phi)$, it demonstrates how $y(J,\phi)$ transitions into $y(z)$ as the space charge force shifts from weak to strong. The distinction from this paper lies in whether the study reduces to a semianalytical eigenvalue problem based on simulations.

Reference [10] solves the eigenvalue problem using a two-particle model with finite size, assuming that the space charge force is constant, regardless of the z positions of the two particles. Reference [11] employs a model with multiple macroparticles distributed in the J and ϕ directions. The space charge force is represented as $\delta(z_i - z_j)$ for the ith and jth particles, but because it is integrated over the region containing the particles, the singularity of the delta function is eliminated. Although there is a difference in handling the wake force—whether in the time domain or the frequency domain—the matrix treated in this model is similar to that discussed in this paper. The distinction lies in whether the distribution is represented as a collection of macroparticles or handled as a discretized distribution of space.

The aforementioned methods treat the wake force in the time domain. In contrast, the method in this paper treats it in the frequency domain as an impedance. The space charge force is expressed by a frequency-independent impedance, and the narrowband impedance can be directly addressed. Therefore, its application to coupled-bunch instabilities is straightforward, and it is currently the only method available for addressing coupled-bunch instabilities.

IV. COUPLED-BUNCH INSTABILITY IN BEAMS SUBJECT TO SPACE CHARGE FORCE

The eigenvalue problem for coupled-bunch oscillation described in Eq. (17) is solved by combining the effects of the space charge force and the previously mentioned resonator impedance, with parameters $Z_{\mathrm{peak}} = 3\ \mathrm{k\Omega/m}$, $\omega_R/(2\pi) = 123$ kHz, and $Q = 40$. The matrix used for solving the eigenvalue problem is the sum of Eqs. (16) and (22). It is sufficient to consider only $m = 1$, since the coupled-bunch mode is already determined.

Figure 5 displays the results for chromaticity values $\xi = 0, -2, -4$, and -6. Plot (a) shows the tunes as functions of the bunch population for these chromaticity values, both with and without the impedance. The tune spectra, at all chromaticities, remain the same as those without the impedance. The tune spectrum characterizes the intrabunch oscillation mode. The resonator impedance does not seem to alter the intrabunch oscillation modes in the presence of the space charge force. Therefore, we must consider the coupled-bunch instability based on the space charge oscillation modes. It appears that the unstable modes seen in Fig. 2 are not realized here. The radial distribution in Fig. 2 can be represented as a linear combination of the continuously distributed modes in Fig. 3. However, the mode will likely be smeared out because the frequency spreads by $\sim\Delta\nu_{\mathrm{sc}}$, depending on J. Thus, it is natural to focus on the isolated modes in the space charge oscillation modes.

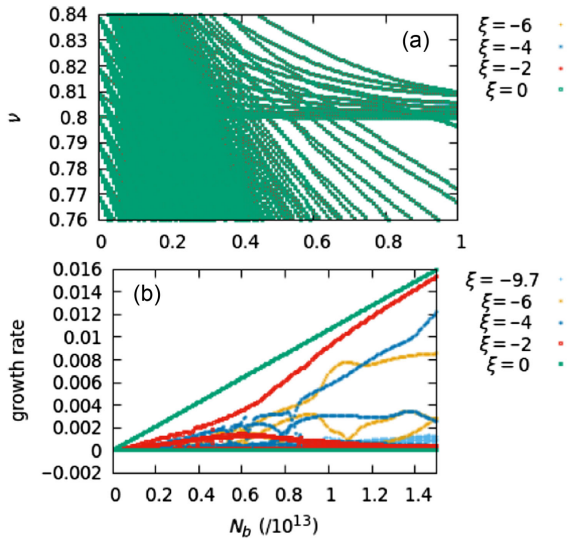


FIG. 5. Tune (a) and growth rate (b) for chromaticity values $\xi = 0, -2, -4$, and -6, where $l_{\max} = 10$ and $n_J = 40$.

Plot (b) depicts the growth rate as a function of the bunch population. For $\xi = 0$, the growth rate is the same as that without space charge in Fig. 1(b). The $y(J, \phi)$ for the unstable mode without space charge was nearly flat. The flat dipole distribution was also an isolated mode for $\xi = 0$. It is logical that the unstable mode and its growth rate for $\xi = 0$ are independent of the presence or absence of space charge.

The growth rates of finite chromaticities are quite different from those without space charge seen in Fig. 1(b). The growth at $\xi = -2$ is initially lower than that at $\xi = 0$, matches at $N_b = 1 \times 10^{13}$, and then increases proportionally with N_b, as it does at $\xi = 0$. The growth at $\xi = -4$ and -6 begins to increase from $N_b \approx 0.8 \times 10^{13}$, but does not reach the value at $\xi = 0$ for $\leq 1.5 \times 10^{13}$. For $\xi = -9.7$, there are no prominent growth modes, and the maximum growth rate is around 0.0005 at $N_b = 1.5 \times 10^{13}$.

The growth rate is determined by the overall dipole moment of the intrabunch oscillation mode, including chromatic modulation for the slow-frequency impedance. Figure 6 displays the intrabunch oscillation modes with the highest growth rate. In plots (a) and (b), the amplitude distributions of the modes at $N_b = 0.75 \times 10^{13}$ and $N_b = 1.5 \times 10^{13}$ for each chromaticity are shown. The distributions appear as curves since they do not depend on δ. The curves are somewhat asymmetric, unlike those in Fig. 4(c). It can be observed that the amplitude distribution formed by the space charge force is somewhat disrupted by the impedance. The $l = 2$, 4, and 6 modes in Fig. 4(c) are observed for $\xi = -2$, $\xi = -4$, and -6, respectively, in plots (a) and (b).

The actual amplitude distribution is obtained by multiplying the chromatic phase, $e^{i\chi z/\sigma_z}$. Plots (c) and (d) show the actual distributions for $N_b = 0.75 \times 10^{13}$ and $N_b = 1.5 \times 10^{13}$, respectively. In plots (c) and (d), the distribution at each chromaticity is shifted in the positive direction, though oscillating components are observed. In plot (d), the distribution at $\xi = -2$ closely matches that at $\xi = 0$, resulting in a flat dipole oscillation. At $\xi = -4$ and -6, the distribution shifts further in the positive direction compare to plot (c), and the oscillating components decrease but remain observable. These plots (c) and (d), which clearly indicate the presence of overall dipole oscillation components, explain the growth observed in Fig. 5(b).

In Fig. 7, the growth of instability with respect to chromaticity is summarized for cases with no space charge force, $N_b = 0.75 \times 10^{13}$, 1.0×10^{13}, and 1.5×10^{13}. It is observed that for $\xi = -2$, -4, and -6, the suppression of instability is weaker compared to the case without space charge force, as discussed above. For higher negative chromaticity values ($\xi = -9.7$), chromaticity effectively suppresses the instability. Space charge modes with higher

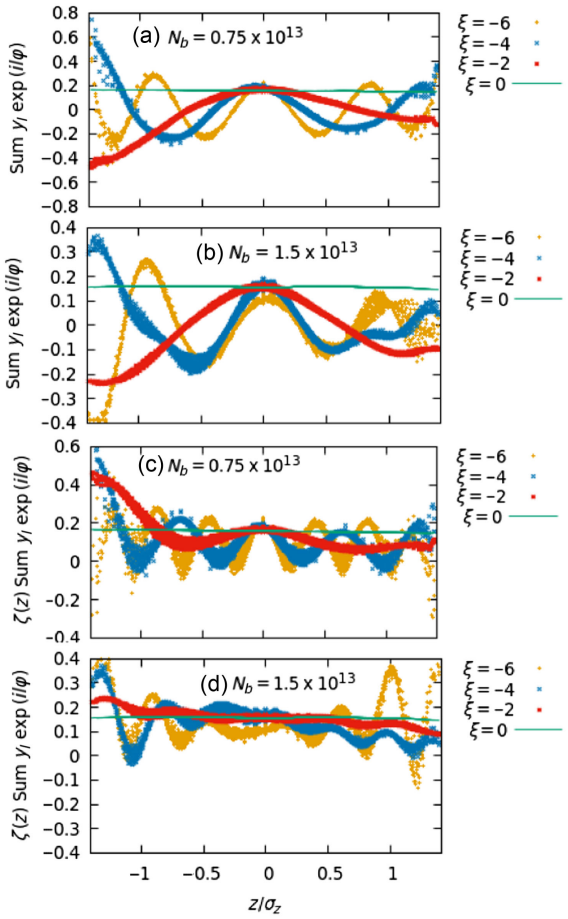


FIG. 6. Unstable modes for chromaticity values $\xi = 0, -2, -4$, and -6. Plots (a) and (b) depict the amplitudes $y_l e^{il\phi}$ of the modes with the fastest growth at $N_b = 0.75 \times 10^{13}$ and 1.5×10^{13}, respectively. Plots (c) and (d) depict the actual distribution $e^{i\chi z/\sigma_z} y_l e^{il\phi}$ at $N_b = 0.75 \times 10^{13}$ and 1.5×10^{13}.

l values are necessary to compensate for the chromatic phase. However, $l_{\max} = 10$ in this analysis may be insufficient, or the distribution of the modes deviates from a sinusoidal shape.

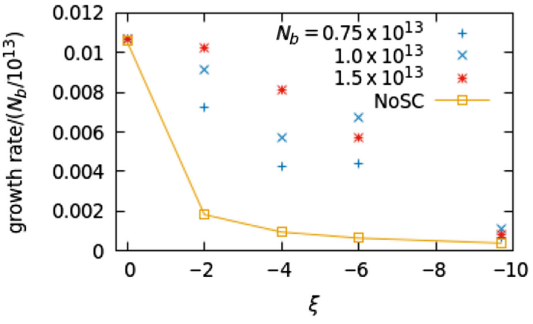


FIG. 7. Growth rate of the instability with respect to chromaticity for cases with no space charge force, $N_b = 0.75 \times 10^{13}$, 1×10^{13}, and 1.5×10^{13}.

V. CONCLUSION

We have been conducting research on the effects of space charge on coupled-bunch instability in high-intensity proton synchrotrons. Here, we consider the impedance as a narrow-band impedance whose frequency is lower than the revolution frequency. The coupled-bunch mode (m) corresponds to an integer p that satisfies $m + \nu + l\nu_s + pH \approx -\omega_R/\omega_0$. The intrabunch oscillation modes in the m mode are analyzed using the same method as for single-bunch oscillation mode analysis. The impedance used in the analysis includes multibunch and multiturn effects. The oscillation modes are described by the amplitude $y(z, \delta)$ in the longitudinal phase space. Conventionally, chromaticity is treated by separating the chromatic phase from the amplitude distribution expanded by the azimuthal (l) and radial modes (J).

In the absence of space charge forces, instability occurs at $l = 0$ in the dipole mode at chromaticity $\xi = 0$. The unstable mode is a dipole distribution that is entirely shifted. By decreasing the chromaticity from $\xi = 0$ to $-2, -4, -6$, and -9.7, complex azimuthal and radial modes emerge, leading to a reduction in the growth of coupled-bunch instability. The impedance amplifies instability but has little effect on the tune.

Next, we examined the bunch oscillation modes in the presence of only the space charge force. Two types of oscillation modes arise due to space charge forces. The first type consists of modes with continuous tune distributions for each J in each azimuthal mode l. The second type consists of modes that occur uniquely for each $l \geq 0$. The amplitude distributions are of cosine and sine types depending on the parity of l with respect to z, independent of δ. In these modes, the tune shift due to the space charge force disappears asymptotically, and characteristics of collective synchrotron oscillations are also lost. The tunes of all these modes converge toward the original tune. This phenomenon appears to result from the interplay between the rotation of the distribution by the synchrotron motion and the momentum change induced by the space charge force.

In the conventional treatment, where the chromatic phase is separated, the space charge modes do not change with chromaticity because the effective impedance of the space charge force is frequency independent.

Then, we investigated the bunch oscillations considering both the space charge force and the impedance. The tune spectrum was independent of the presence or absence of the impedance and the value of chromaticity. This indicates that the intrabunch oscillation is determined solely by the space charge force. The instability is induced by the isolated modes that depend solely on z, rather than the continuously distributed modes with respect to J, which couple with the impedance. By examining the intrabunch oscillation of the isolated modes, we observed that it is slightly modified by the impedance but is largely determined by the space charge force alone.

As expected, the low-frequency impedance couples only with the overall dipole component of the bunch. When considering the chromatic phase, the sinusoidal z components are canceled out, and an overall dipole component emerges. Consequently, the suppression of coupled-bunch instability due to chromaticity becomes less effective.

Finally, to relate this study to experiments, we discuss the response to forced oscillations using a transverse rf exciter. For oscillations at $\omega = \omega_0(m + \nu_y + pH) = \chi c\beta/\sigma_z$, the $l = 0$ mode is excited. This excitation is independent of the strength of the space charge force. In the regime where $\Delta\nu_{\rm sc} \gg \nu_s$, the eigenmode oscillation is $\sim e^{i(l+\chi)z/\sigma_z}$, so oscillations at $\omega = \omega_0(m + \nu_y + pH) = (l + \chi)c\beta/\sigma_z$ are excited. As the space charge force weakens, the response is expected to diminish. In particular, $l \approx -\chi$ corresponds to the region addressed in this paper. Changes in bunch oscillations with respect to chromaticity and beam intensity are expected to be observed in the evolution of the overall dipole moment.

ACKNOWLEDGMENTS

This work was supported by the Chinese Academy of Sciences President's International Fellowship Initiative (Grant No. 2024PVA0057) and innovation Study of IHEP. One of the authors (K. O.) thanks Mingyang Huang, Sheng Wang, and Yuan Zhang for their hospitality in IHEP and CSNS. He also thanks E. Metral at CERN for fruitful discussions.

[1] S. Wang, Y. An, S. Fang *et al.*, An overview of design for CSNS/RCS and beam transport, Sci. China: Phys., Mech. Astron. **54**, 239 (2011).

[2] High-Intensity Proton Accelerator Project Team, Accelerator technical design report for high-intensity proton accelerator facility project, JAERI Report No. JAERI-Tech 2003-044 and KEK Report No. 2002-13.

[3] L. Huang, S. Wang, S. Xu, X. Liu, M. Huang, R. Liu, B. Tan, H. Liu, L. Rao, and Y. Han, Source of instability in the rapid cycling synchrotron of the China spallation neutron source, Eur. Phys. J. Plus **140**, 71 (2025).

[4] Y. Shobuda, Y. H. Chin, N. Hayashi, Y. Irie, T. Takayanagi, T. Togashi, T. Toyama, K. Yamamoto, and M. Yamamoto, Reducing the beam impedance of the kicker at the 3-GeV rapid cycling synchrotron of the Japan Proton Accelerator Research Complex, Phys. Rev. Accel. Beams **21**, 061003 (2018).

[5] K. Ohmi and Y. Zhang, Effects of space charge force on the beam-beam mode coupling instability, Phys. Rev. Accel. Beams **27**, 101001 (2024).

[6] M. Blaskiewicz, Fast head-tail instability with space charge, Phys. Rev. ST Accel. Beams **1**, 044201 (1998).

[7] A. Burov, Head-tail mode for strong space charge, Phys. Rev. ST Accel. Beams **12**, 044202 (2009).

[8] A. Macridin, A. Burov, E. Stern, J. Amundson, and P. Spentzouris, Simulation of transverse modes with their intrinsic Landau damping for bunched beams in the presence of space charge, Phys. Rev. ST Accel. Beams **18**, 074401 (2015).

[9] V. Kornilov and O. Boine-Frankenheim, Head-tail instability and Landau damping in bunches with space charge, Phys. Rev. ST Accel. Beams **13**, 114201 (2010).

[10] Y. H. Chin, A. W. Chao, and M. M. Blaskiewicz, Two particle model for studying the effects of space-charge force on strong head-tail instabilities, Phys. Rev. Accel. Beams **19**, 014201 (2016).

[11] X. Buffat, E. Metral, E. Gottlob, C. Hogh, A. Oeftiger, and T. Pieloni, Description of beam instabilities in synchrotrons with wakefields and space charge forces using the criculant matrix model, Phys. Rev. Accel. Beams **24**, 060101 (2021).

[12] Y. Shobuda, Y. H. Chin, P. K. Saha, H. Hotchi, H. Harada, Y. Irie, F. Tamura, N. Tani, T. Toyama, Y. Watanabe, and M. Yamamoto, Theoretical elucidation of space charge effects on the coupled-bunch instability at the 3 GeV rapid cycling synchrotron at the Japan Proton Accelerator Research Complex, Prog. Theor. Exp. Phys. **2017**, 013G01 (2017).

[13] A. Burov, Coupled-beam and coupled-bunch instabilities, Phys. Rev. Accel. Beams **21**, 114401 (2018).

[14] A. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators* (Wiley-Interscience Publication, New York, 1993).

[15] K. Ohmi, Y. Zhang, and C. Lin, Beam-beam mode coupling in collision with a crossing angle, Phys. Rev. Accel. Beams **26**, 111001 (2023).