

Resistive wall wake for nonultrarelativistic beam in ring model

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The nonultrarelativistic wake function theory is not only applicable to proton or heavy ion accelerators, but with the emergence of new accelerator physics, it has been found to be equally applicable to electron accelerators such as the steady-state microbunching (SSMB) storage rings. However, existing results are still far from practical applications. In this paper, building on previous work, we further generalize the previous point source model into a more universal ring model. When considering the nonultrarelativistic limit, we discover a completely new lowest-order transverse wakefield and estimate its intensity to be comparable to the dipole field under certain circumstances. Moreover, by separating the space charge component, we propose a numerical algorithm that can compute the resistive wall field more efficiently. Finally, we calculate the parasitic loss and kick angle using SSMB parameters as simple applications.

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I. INTRODUCTION

The electromagnetic interaction between the beam and the imperfect conductor pipe is referred to as resistive wall wakefield. It is a constant interaction that goes along with the beam. Since it is a fundamental wakefield in accelerator collective instability physics, textbooks frequently start with it. Nonetheless, $\gamma \rightarrow \infty$ is assumed in the straightforward, classical, and useful equation of the resistive wall wake function. Under the conditions of some low-energy accelerators, such as proton and ion beam accelerators, where the beam energy is low compared to the particle's rest-mass energy, this assumption might not hold true.

Even in medium energy electron accelerators, if the bunch length is short enough, the classical equation may not apply [1]. This is due to the Coulomb field of the image charge caused by the pipe of radius b , and the range it can influence on the axis can be estimated as b/γ . We shall shortly discover that, because of the microbunching structure of the beam, an electron storage ring exhibiting steady-state microbunching (SSMB) [2,3], with energy of hundreds of MeV still requires the formula for $\gamma \rightarrow \infty$.

Thus, the nonultrarelativistic resistive wall wake function calculation becomes necessary. A few attempts have

tried to compute the wake and impedance, starting from the literature [1,4–6]. Zimmermann and Oide's study [6] served as the foundation for our research. They provide a thorough derivation for the case of monopoles and dipoles that are not ultrarelativistic.

This paper is organized as follows. First, we claim the ring model and definition of our solution in Secs. II and III, respectively. Then, we have followed the derivation of the point source of the dipole and monopole based on published literature in Sec. IV A. And we have discovered a new lowest transverse wake function for monopole which is comparable with the dipole's transverse wake on certain circumstance in Sec. IV C. Next, we proceed to create the ring model in order to use the result to numerically analyze the dynamical problem in Secs. IV E and V. This allows us to approximate the strip as a ring and approach any form of distribution of the beam by slicing it into many strips, as we said in Sec. II. The analysis becomes more precise as we slice more strips. Benchmarks are set for the outcomes in Sec. VI. We split the impedance into the field from the resistive wall and the source space charge field especially in Sec. VI C 2. When calculating the numerical inverse Fourier transform from the so-called impedance to wake function, this will be quite beneficial. As straightforward applications, we computed the parasitic loss and kick angle utilizing the parameters of SSMB in Sec. VII.

This work lays the foundation and clarifies the ideas for subsequent beam dynamics analysis.

II. RING MODEL

We mentioned that to facilitate numerical calculations of the wakefield for arbitrary transverse charge distributions,

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FIG. 1. Ring model.

we employ the ring current model. And the basic idea of this model is the principle of superposition. Here, we will briefly explain how to analyze arbitrary transverse charge distributions using this model.

As shown in Fig. 1, we divide the beam transversely into concentric strips, and roughly represent each strip with a ring. Suppose the radius of the ring can be represented by the average of the inner and outer radius of the strip. Based on this, each ring is further expanded into a Fourier series in the azimuthal direction to achieve multipole expansion. The finer the division of the strips, the higher the accuracy of this process.

III. USAGE OF LORENTZ FORCE

In the ultrarelativistic situation, we observe that the wake function is independent of radius r . According to Chao's textbook [7], the wake function takes the shape of Eq. (2.50)

$$\int_{-L/2}^{L/2} ds \mathbf{F}_{\perp} = -eI_m W_m(z) m r^{m-1} (\hat{\mathbf{r}} \cos m\theta - \hat{\boldsymbol{\theta}} \sin m\theta), \quad (1)$$

$$\int_{-L/2}^{L/2} ds \mathbf{F}_{\parallel} = -eI_m W'_m(z) r^m \cos m\theta, \quad (2)$$

$$\int_{-L/2}^{L/2} ds eB_s = eI_m W'_m(z) r^m \sin m\theta. \quad (3)$$

We find that the right-hand side of Eq. (1) is zero for $m = 0$. However, when it comes to the nonultrarelativistic case, it is noted that the wake function is radius-dependent. Meanwhile, the transverse force $\int_{-L/2}^{L/2} ds \mathbf{F}_{\perp}$ for $m = 0$ is not zero according to the Panofsky-Wenzel theorem

$$\nabla_{\perp} \int_{-L/2}^{L/2} ds \mathbf{F}_{\parallel} = \frac{\partial}{\partial z} \int_{-L/2}^{L/2} ds \mathbf{F}_{\perp}. \quad (4)$$

What needs to be pointed out here is that Eq. (4) is only based on the rigid beam assumption. This assumption will be the only fundamental assumption for calculating our wakefield. It should be noted that the assumption of ultrarelativistic and the rigid beam assumption are inherently self-consistent. Nevertheless, the beam bunches are

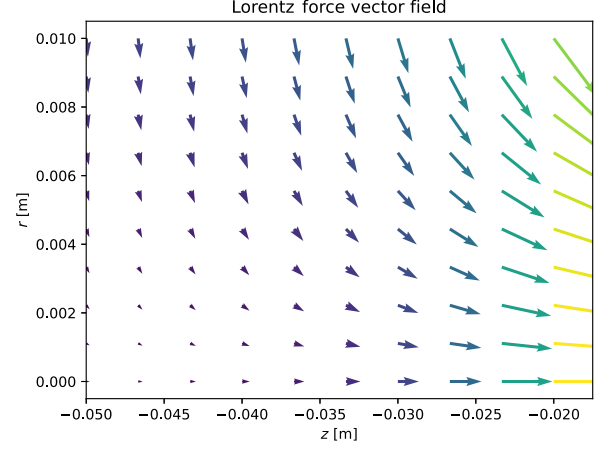


FIG. 2. Lorentz force vector of which the longitudinal component is the second term of Eq. (32) and the radial component is Eq. (28).

unable to provide invariantly rigid shape if nonultrarelativistic is accepted. The analysis must be integrated with dynamics for a discussion to be self-consistent. This means that the wakefield for a given distribution of beam is calculated first. Then, estimate the feedback on the distribution in terms of dynamics and proceed as the first step. This goes beyond the scope of this paper and will be the subject of future work.

We illustrate the monopole Lorentz force vector field, Fig. 2, in order to provide a better understanding of the Lorentz force that satisfies the Panofsky-Wenzel theorem. In the circumstance when a longitudinal Lorentz force is dependent on radial position, the transverse Lorentz force will also exist. Meanwhile, the radial Lorentz force will be directed in a negative direction as the longitudinal Lorentz force diminishes when the radial position increases.

In analogy to the nonultrarelativistic case given by Zimmermann and Oide [6] and to fit in the theoretical framework of the Panofsky-Wenzel theorem, we define the so-called *general Green's function wakefields* per unit length—also called *general wake function*—moment by moment as

$$-E_s^{(0)} = q \frac{\partial W_0(z, r, \gamma)}{\partial z}, \quad (5)$$

$$-(E_r + (\mathbf{v} \times \mathbf{B})_r)^{(0)} = q \frac{\partial W_0(z, r, \gamma)}{\partial r}, \quad (6)$$

$$-E_s^{(1)} = qarW_{1,\parallel}(z, r, \gamma) \cos \varphi, \quad (7)$$

$$\begin{aligned} & -(\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\perp}^{(1)} \\ & = qa[\hat{\mathbf{r}}W_{1,r}(z, r, \gamma) \cos \varphi - \hat{\boldsymbol{\theta}} \sin \varphi W_{1,\varphi}(z, r, \gamma)], \end{aligned} \quad (8)$$

where the Panofsky-Wenzel theorem, Eq. (4), indicates that $W_{1,r}$, $W_{1,\varphi}$, and $W_{1,\parallel}$ have the relation

$$W_{1,\parallel}(z, r, \gamma) + r \frac{\partial}{\partial r} W_{1,\parallel}(z, r, \gamma) = \frac{\partial}{\partial z} W_{1,r}(z, r, \gamma), \quad (9)$$

$$W_{1,\parallel}(z, r, \gamma) = \frac{\partial}{\partial z} W_{1,\varphi}(z, r, \gamma). \quad (10)$$

Here, we also find that for the transverse Lorentz force, there exist two different wake functions in the radial direction $W_{1,r}(z, r, \gamma)$ and in the azimuthal direction $W_{1,\varphi}(z, r, \gamma)$.

And when we reach the ultrarelativistic limit, the wake functions W_0 , $W_{1,r}$, $W_{1,\varphi}$, and $W_{1,\parallel}$ are independent of r and γ and equal to the functions $W_0(z)$, $W_1(z)$, $W_1(z)$, and $W'_1(z)$. By giving the definition above, we can unify the monopole and dipole wakefield in the ultra or nonultra relativistic limit.

However, in our opinion, the wake function in the nonultrarelativistic case becomes complicated because it is not factorized. Direct use of the Lorentz force ought to be preferable.

IV. MONOPOLE

A. Lowest order longitudinal and transverse moment

The longitudinal wake force scales as $a^{2m} W'_m/L \sim (a/b)^{2m} (1/b^2)$, while the transverse wake force scales as $a^{2m-1} W_m/L \sim (a/b)^{2m-1} (1/b^2)$, where a is the transverse size of the beam, b is the pipe radius, according to Chao's result [7]. The lowest mode dominates because typically $b \gg a$. Intuitively, the monopole mode should be the lowest mode in both longitudinal and transverse direction. However, due to the Panofsky-Wenzel theorem, the transverse wake force of a monopole is actually zero. The scaling law we quote is valid for $m \geq 1$ when it refers to the transverse case.

Nonetheless, the above result is from the definition and result of the ultrarelativistic approximation. The monopole moment continues to be the lowest mode in transverse wake force, as shown by the definition in the nonultrarelativistic case in Eq. (6). Therefore, when we calculate the transverse wake force, i.e., transverse Lorentz force, we need to determine whether a monopole or a dipole generates a stronger transverse wake force.

B. Monopole's longitudinal solution by Zimmermann and Oide

It should be noted that we already have a complete theory that relies on the idea that the particle's velocity is c , as we reviewed in Appendix A. There will be a different solution if the particle's speed is slower than that of light in free space. Below we will briefly introduce the processing methods of Zimmermann and Oide [6], as our processing

method will be similar to theirs. They focused on the point source which is Eq. (11)

$$\rho = \frac{Q}{2\pi r} \delta(s - ct) \delta(r). \quad (11)$$

The Maxwell equation to be solved is

$$-\Delta A + \frac{\mu_r \epsilon_r}{c^2} \frac{\partial^2 A}{\partial t^2} + \mu_0 \mu_r \sigma \frac{\partial A}{\partial t} = \mu_0 \mu_r (\mathbf{j} - \sigma \mathbf{E}), \quad (12a)$$

$$-\Delta \phi + \frac{\mu_r \epsilon_r}{c^2} \frac{\partial^2 \phi}{\partial t^2} + \mu_0 \mu_r \sigma \frac{\partial \phi}{\partial t} = \frac{\rho}{\epsilon_r \epsilon_0}. \quad (12b)$$

We establish a cylindrical coordinate system r, φ, s with the inner radius of the pipe denoted as b . The solution domain is divided into two parts.

According to symmetry, the vector potential $\mathbf{A}$ and the potential ϕ are independent of φ , and the components of the vector potential have the relationships $A_\varphi = 0$, $A_r = 0$. We take the Fourier transform, which is

$$\begin{aligned} \mathcal{F}[g(z, r)] &= \int_{-\infty}^{\infty} g(z, r) e^{-\frac{i\omega z}{v}} dz \\ &= \int_{-\infty}^{\infty} g(z, r) e^{-ikz} dz = \tilde{g}(\omega, r), \end{aligned} \quad (13)$$

for the equation. For $r < b$, we can get

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{A}_s}{\partial r} \right) - k_r^2 \tilde{A}_s = -\mu_0 \tilde{j}_s, \quad (14a)$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{\phi}}{\partial r} \right) - k_r^2 \tilde{\phi} = -\frac{\tilde{\rho}_0}{\epsilon_0}, \quad (14b)$$

where

$$k_r^2 = \left(\frac{\omega}{c\beta\gamma} \right)^2. \quad (15)$$

For $r > b$, we have

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{A}_s}{\partial r} \right) - \lambda^2 \tilde{A}_s = 0, \quad (16a)$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{\phi}}{\partial r} \right) - \lambda^2 \tilde{\phi} = 0, \quad (16b)$$

where

$$\begin{aligned} \lambda^2 &= \lambda_0^2 + k_r^2 = -(i\mu_0 \sigma \omega - k_r^2), \\ \lambda_0 &= \frac{1 - i \operatorname{sgn}(\omega)}{\delta_{\text{skin}}}. \end{aligned} \quad (17)$$

The general solution of Eqs. (14) and (16) would be *modified Bessel function*. Using the boundary condition to determine the unknown coefficients, we have the formula for the electric field in the frequency domain as Eq. (18)

$$\tilde{E}_s(\omega, r) = -Qv \frac{iZ_0 c k_r^2}{2\pi\omega} \left[K_0(k_r r) + I_0(k_r r) \frac{\omega^2 \lambda K_1(bk_r) K_0(b\lambda) + k_r c^2 (\lambda^2 - k^2) K_0(bk_r) K_1(\lambda b)}{\omega^2 \lambda I_1(bk_r) K_0(b\lambda) - k_r c^2 (\lambda^2 - k^2) I_0(bk_r) K_1(\lambda b)} \right]. \quad (18)$$

It is observed that when it comes to ultrarelativistic case, the impedance will become only frequency-dependent; when it comes to nonultrarelativistic, the impedance derived from the electromagnetic field will become both frequency-dependent and radius-dependent.

Zimmermann and Oide assumed that the relative permittivity $\epsilon_r = 1$ and relative magnetic permeability $\mu_r = 1$ when they arrived at this solution. Since permittivity is inherently frequency-dependent, this would render the governing equations significantly more intractable. Therefore, in their first calculations, they ignore this more accurate effect. Another thing that needs to be observed is that even though we can view ϵ_r and μ_r as constant, if they are not equal to unity, some replacements need to be made to modify the equation.

In the meantime, dividing the solution domain into the vacuum region inside the pipe $r < b$ and the region of the metal wall $r > b$ implies that physically this model is applied to the situation that the thickness of the pipe is much greater than the skin depth. If we focus on some effect whose frequency is low, we should check whether the metal pipe in the real world is thick enough.

Other questions, such as ac conductivity, anomalous skin effect, and magnetoresistance, are discussed by Zimmermann and Oide [6].

Finally, we have the longitudinal Lorentz force

$$\tilde{F}_s(\omega, r) = q\tilde{E}_s(\omega, r). \quad (19)$$

C. Monopole transverse Lorentz force

Zimmermann and Oide's analysis [6] for the monopole was concluded in the previous section. However, since we have $\tilde{E}_s(\omega, r)$ as the longitudinal Lorentz force, we also find the transverse Lorentz force in the frequency domain to be

$$\begin{aligned} \tilde{E}_r(\omega, r) &= -\frac{c^2 k k_r \mu_0 Q v}{\omega} \frac{1}{2\pi} \left[\frac{N_1}{D_1} I_1(k_r r) - K_1(k_r r) \right] e^{ikz}, \\ \tilde{B}_\phi(\omega, r) &= k_r \frac{\mu_0 Q v}{2\pi} \left[-\frac{N_1}{D_1} I_1(k_r r) + K_1(k_r r) \right] e^{ikz}, \end{aligned} \quad (20)$$

$$\begin{aligned} \tilde{F}_r(\omega, r) &= q(\tilde{E}_r - v\tilde{B}_\phi) \\ &= qk_r \frac{c^2 - v^2 \mu_0 Q v}{v} \frac{1}{2\pi} \left[-\frac{N_1}{D_1} I_1(k_r r) + K_1(k_r r) \right] e^{ikz}, \end{aligned} \quad (21)$$

$$\begin{aligned} N_1 &= k_0^2 \lambda K_1(bk_r) K_0(b\lambda) \\ &\quad + k_r (\lambda^2 - k^2) K_0(bk_r) K_1(\lambda b), \end{aligned}$$

$$\begin{aligned} D_1 &= k_0^2 \lambda I_1(bk_r) K_0(b\lambda) \\ &\quad - k_r (\lambda^2 - k^2) I_0(bk_r) K_1(\lambda b). \end{aligned}$$

Via assuming $|\lambda b| \gg 1$, $k_r \delta_{\text{skin}} \ll 1$ and $\lambda \approx \lambda_0$ as Zimmermann and Oide [6] did, we can simplify Eq. (21) to the form,

$$\begin{aligned} \tilde{F}_r(\omega, r) &= \beta^2 Q q \left\{ \frac{k_r c^2 \mu_0}{\gamma^2 \beta^2 2\pi} \right. \\ &\quad \times \left[K_1(k_r r) - \frac{N_2}{D_2} I_1(k_r r) \right] \Big\}, \\ N_2 &= \delta_{\text{skin}} \omega^2 K_1(k_r b) + k_r c^2 [1 - i \text{sgn}(\omega)] K_0(k_r b), \\ D_2 &= \delta_{\text{skin}} \omega^2 I_1(k_r b) - k_r c^2 [1 - i \text{sgn}(\omega)] I_0(k_r b), \end{aligned} \quad (22)$$

whose unit is [N m]. Expanding F_r to second order of $1/\beta\gamma$ when γ approaches infinity, the result is

$$\begin{aligned} \tilde{F}_r(\omega, r) &\approx Q q \left(\frac{c^2 \mu_0}{2\pi r} \right. \\ &\quad \left. + \frac{c^2 r \sqrt{\frac{\mu_0}{\sigma}} |\omega|^{\frac{3}{2}}}{2\sqrt{2} b c^2 \pi [1 - i \text{sgn}(\omega)] - 2b^2 \pi \sqrt{\frac{1}{\mu_0 \sigma}} |\omega|^{\frac{3}{2}}} \right) \frac{1}{\gamma^2}. \end{aligned} \quad (23)$$

If we have the condition,

$$|2\sqrt{2} b c^2 \pi [1 - i \text{sgn}(\omega)]| \gg 2b^2 \pi \sqrt{\frac{1}{\mu_0 \sigma}} |\omega|^{\frac{3}{2}}, \quad (24)$$

i.e.,

$$|\omega| \ll \left(\frac{c^2}{b} \sqrt{\mu_0 \sigma} \right)^{\frac{2}{3}}, \quad (25)$$

we can get the low-frequency approximation, which is

$$\tilde{F}_r(\omega, r) \approx Qq \left(\frac{c^2 \mu_0}{2\pi r} + \frac{c^2 r \sqrt{\frac{\mu_0}{\sigma}} |\omega|^{\frac{3}{2}}}{2\sqrt{2} b c^2 \pi [1 - i \operatorname{sgn}(\omega)]} \right) \frac{1}{\gamma^2}. \quad (26)$$

Take aluminum pipe of radius 1 cm, for example, when $\omega \ll 1.5403 \times 10^{13}$ Hz, the low-frequency approximation is applied. It should be noticed that when we are talking about low frequency, this approximation is equivalent to the far distance approximation. The two are connected by the Fourier transform, and the corresponding region of z is

$$|z| \gg \frac{v}{\omega} \sim 1.95 \times 10^{-5} \text{ m}. \quad (27)$$

Because the Fourier pair of the first term of Eq. (26) is $\delta(z)$ function, which is not in the relevant region of z , we only perform the inverse Fourier transform on the second term. When $z \ll -1.95 \times 10^{-5}$ m, the result is

$$F_r(z, r, \gamma) = \frac{1}{2\pi v} \int_{-\infty}^{\infty} \tilde{F}_r(\omega, r, \gamma) e^{\frac{i\omega z}{v}} d\omega = -Qq \frac{3rv^{\frac{3}{2}} \sqrt{\frac{\mu_0}{\sigma}}}{16b\pi^{\frac{3}{2}} \gamma^2 |z|^{\frac{5}{2}}}, \quad (28)$$

while when $z \gg 1.95 \times 10^{-5}$ m, we have

$$F_r(z, r, \gamma) = 0. \quad (29)$$

This finding implies a causality nature similar to that derived from ultrarelativistic assumption. Since the result for the space charge component, which is the first term of Eq. (26), is a pancake field, this outcome is predictable. We anticipate that when expanded to the fourth order in $1/\beta\gamma$, the Lorentz force will not be zero when z is greater than zero.

Compared to the dipole transverse Lorentz force under ultrarelativistic case Eq. (A5), if we choose the dipole moment to be Qa , the monopole moment to be Q and the test charge to be q , we have the ratio R of the monopole transverse force Eq. (28) to the dipole transverse force equation (30) as Eq. (31)

$$F_r^{\text{dipole}} = -\frac{c}{\pi b^3} \sqrt{\frac{Z_0}{\pi \sigma}} \frac{1}{|z|^{\frac{1}{2}}} a Q q, \quad (30)$$

$$R = \frac{3rb^2}{16a|z|^2} \frac{1}{\gamma^2 \beta^{-\frac{3}{2}}}. \quad (31)$$

We think the interested region of r has the same order of magnitude as the bunch shape a . Demanding $R \approx 1$, if the beam pipe radius $b \sim 10^{-2}$ m and $z \sim 10^{-4}$ m, $\gamma^2 \beta^{-\frac{3}{2}}$ should have the order of magnitude as 10^4 . This means $\gamma \sim 10^2$.

Therefore, in some low energy cases, the strongest order moment of the transverse Lorentz force might not be

$m = 1$. Attention should be paid when focusing on the transverse case.

The above result also reminds us that we have the series of F_s [6] to the orders of $1/\gamma^2$ when $z < 0$,

$$F_s(r, z, \gamma) = Qq \frac{v^{\frac{3}{2}}}{4\pi^{\frac{3}{2}} b} \sqrt{\frac{\mu_0}{\sigma}} \left(\frac{1}{|z|^{\frac{3}{2}}} + \frac{15(2b^2 - r^2)}{16\gamma^2 |z|^{\frac{7}{2}}} \right). \quad (32)$$

We can verify that they satisfy the Panofsky-Wenzel theorem

$$\nabla_{\perp} \int_{\frac{z}{2}}^{-\frac{z}{2}} ds F_{\parallel} = \frac{\partial}{\partial z} \int_{\frac{z}{2}}^{-\frac{z}{2}} ds F_{\perp}, \quad (33)$$

where the left-hand side and right-hand side equal

$$\frac{\partial}{\partial r} F_s = -Qq \frac{15rv^{\frac{3}{2}} \sqrt{\frac{\mu_0}{\sigma}}}{32b\pi^{\frac{3}{2}} \gamma^2 |z|^{\frac{7}{2}}} = \frac{\partial}{\partial z} F_r. \quad (34)$$

We can follow the definition of the general Green's function wakefields, Eqs. (5) and (6), then we will find W_0 to be

$$W_0(z, r, \gamma) = -\frac{v^{\frac{3}{2}}}{2\pi^{\frac{3}{2}} b} \sqrt{\frac{\mu_0}{\sigma}} \left(\frac{1}{|z|^{\frac{3}{2}}} + \frac{3(2b^2 - r^2)}{16\gamma^2 |z|^{\frac{7}{2}}} \right). \quad (35)$$

D. Inverse Fourier transform

For more general situation, using the numerical inverse Fourier transform, we are able to obtain the electromagnetic field in the time domain once we have the expression in the frequency domain. The coefficient in the denominator of Eq. (36), called v and representing the particle's speed, should be noted before performing the integration. This is because we need to take the Fourier transform pair corresponding to Eq. (13)

$$\begin{aligned} \mathcal{F}^{-1}[\tilde{g}(\omega, r)] &= \frac{1}{2\pi v} \int_{-\infty}^{\infty} \tilde{g}(\omega, r) e^{\frac{i\omega z}{v}} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}(\omega, r) e^{ikz} dk = g(z, r). \end{aligned} \quad (36)$$

This coefficient has already appeared at Eq. (28) when we calculate the analytical solution.

It is necessary to use some skills to make sure the numerical procedure does not encounter any issues, which are mainly overflow warnings. Appendix C illustrates how we apply these skills. In summary, what we have done is to keep the first term in the asymptotic expansion and derive simpler expressions, Eqs. (C3) and (C4), which will be effective in various frequency ranges. This allows us to avoid data overflow in the actual computational process. Figure 3 shows the effective ranges of Eqs. (18), (C3), and (C4) by the translucent shadow regions in different colors. To give a better understanding, the primary focus of our

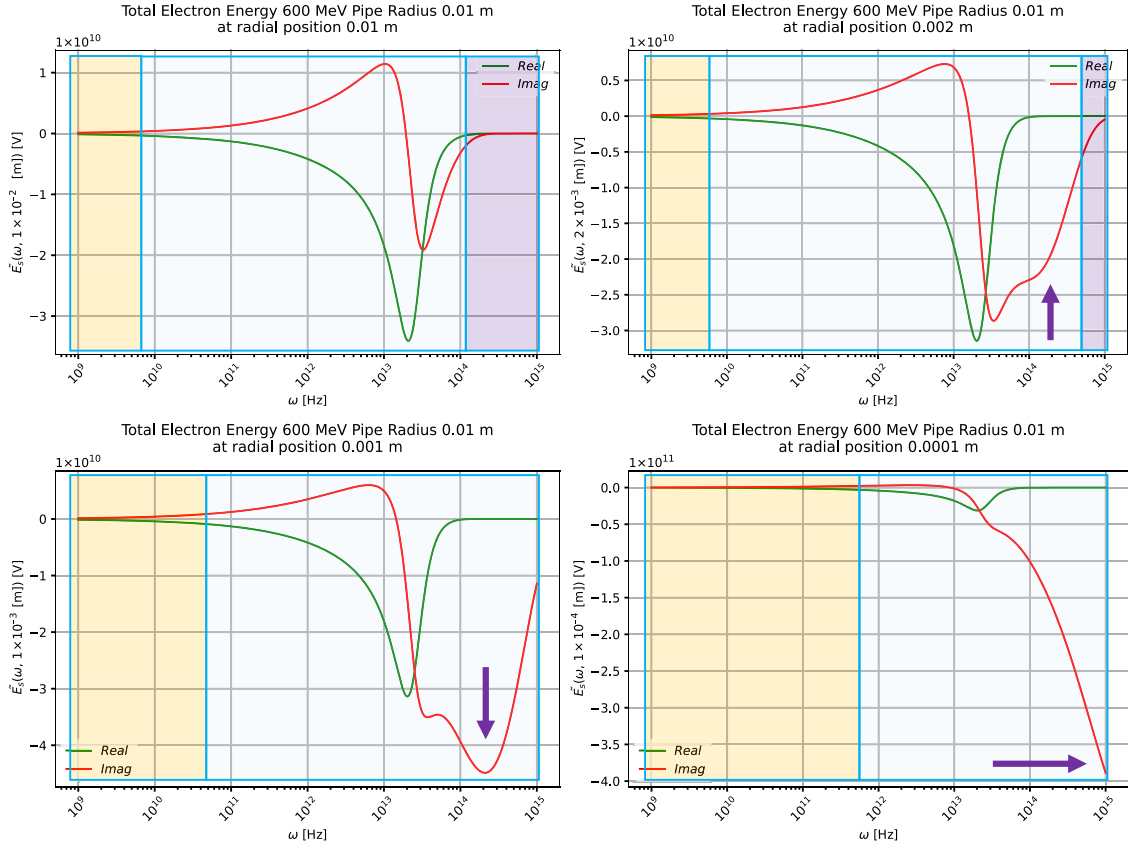


FIG. 3. Plots of the longitudinal electric field of the monopole at various radial positions such as 0.01, 0.002, 0.001, and 0.0001 m. Energy is 600 MeV and pipe radius is 0.01 m. Three colored translucent shadow regions in the image indicate the corresponding ranges of effectiveness for various asymptotic formulas. The effective ranges of Eqs. (18), (C3), and (C4) are represented by the yellow, blue, and purple regions, respectively.

numerical skills is to deal with the problem of numerical overflow that results from calculating the second term in Eq. (18).

Implementing the inverse Fourier transform by directly computing the numerical integral is a laborious procedure, though, and most importantly takes a lot of time. As we dig deeper into the understanding of the nonultra-relativistic resistive wall wake, we will soon discover that most of the time is consumed by the first term of Eq. (18). In Fig. 3, this term manifests as a high-frequency resonance phenomenon, with the resonance peak marked by the purple arrow. We will illustrate that the physical meaning of this modified Bessel K function term is the space charge and present a faster algorithm to calculate the field in Sec. VIC.

As for the resonance phenomenon occurring in the frequency range of 10^{12} to 10^{14} Hz, it is the result of the second term in Eq. (18). Its shape is similar to that under the ultrarelativistic assumption, but there are some differences depending on the energy and transverse position, among which the differences caused by variations in energy are greater than those caused by variations in transverse position. The physical explanation of this kind of resonance has been given by Bane and Sands [8]. It can

be understood as the matching between the phase velocity of electromagnetic waves in a nonperfect conductor pipe and the particle velocity. Using the perturbation method shown in Jackson [9], Sections 8.5–8.6, they give an estimate of the position of the resonance peak, which is $\omega \approx (\sigma b^2 \omega_\lambda^4 / 2\epsilon_0 c^2)^{1/3}$, with ω_λ the cutoff frequency of the mode. We find the resonance peak to be 2.07522×10^{13} Hz when the electron energy is taken to be infinite, while the estimated value is 3.83056×10^{13} Hz, which is 1.846 times larger than the former. If we return to the 600 MeV case, it will be 2.02145×10^{13} Hz, and the ratio of the estimated value to it is 1.895. Since this is a rough estimation, we consider the calculated deviation to be acceptable. It is noticed that the resonance peak will shift to lower frequency as the energy decreases. Considering that the lower the energy, the slower the speed, which means there are no high-frequency components at all when at rest, we believe this trend is physically consistent. Furthermore, if we want to improve the estimation accuracy of the position and width of the resonance peak and take into account the influence brought by the energy, we believe that a solution to the dispersion relation with less approximation is necessary.

It should be emphasized that the numerical skills we proposed in Appendix C are unrelated to the resonance phenomenon caused by velocity matching, even though the resonance peaks in the blue shadow region of Fig. 3 were obtained using those skills. The occurrence of the numerical overflow is only limited by the computational precision of the computer. If we had more bits to store a number, there would be no need for us to introduce the numerical skills.

E. Monopole ring model

Previously, we have analyzed the calculation results of the point model in detail. Here, we give the ring model of a monopole,

$$\begin{aligned}\rho_0 &= \frac{I_0}{2\pi a} \delta(s - vt) \delta(r - a) \\ &= -\frac{Q_0}{a} \delta(s - vt) \delta(r - a),\end{aligned}\quad (37)$$

$$\mathbf{j}_0 = v\rho_0 \hat{s}. \quad (38)$$

Solving Eqs. (14) and (16) in different areas like we did in the point case. We express the general solution of the vector potential and potential in three regions.

$$\begin{aligned}r &< a, \\ A_r &= 0, \\ A_\varphi &= 0, \\ A_s &= p_s^c I_0(k_r r) e^{ikz}, \\ \phi &= p_0^c I_0(k_r r) e^{ikz},\end{aligned}\quad (39)$$

where Lorentz gauge requires that

$$p_0^c = \frac{c^2 k}{\omega} p_s^c. \quad (40)$$

Then,

$$\begin{aligned}a &< r < b, \\ A_r &= 0, \\ A_\varphi &= 0, \\ A_s &= [p_s I_0(k_r r) + q_s K_0(k_r r)] e^{ikz}, \\ \phi &= [p_0 I_0(k_r r) + q_0 K_0(k_r r)] e^{ikz},\end{aligned}\quad (41)$$

where Lorentz gauge requires that

$$p_0 = \frac{c^2 k}{\omega} p_s, \quad q_0 = \frac{c^2 k}{\omega} q_s \quad (42)$$

and

$$\begin{aligned}r &> b, \\ A_r &= 0, \\ A_\varphi &= 0, \\ A_s &= q_s^w K_1(\lambda r) e^{ikz}, \\ \phi &= q_0^w K_1(\lambda r) e^{ikz},\end{aligned}\quad (43)$$

where Lorentz gauge requires that

$$q_0^w = \frac{1}{\frac{\omega}{kc^2} + i\frac{\mu_0 \sigma}{k}} q_s^w. \quad (44)$$

So we have four unknown parameters to determine, which are

$$p_s^c, \quad p_s, \quad q_s, \quad q_s^w. \quad (45)$$

We can decide the four parameters via boundary conditions. We will introduce this procedure in Appendix D. Finally, the four parameters could be solved as

$$p_s^c = p_s - \frac{\mu_0 Q_0 \omega K_0(ak_r)}{k}, \quad (46)$$

$$q_s = -\frac{\mu_0 Q_0 \omega I_0(ak_r)}{k}, \quad (47)$$

$$q_s^w = \frac{-k_r p_s I_1(bk_r) + k_r q_s K_1(bk_r)}{\lambda K_1(b\lambda)}, \quad (48)$$

$$\begin{aligned}p_s &= q_s \frac{N_{p_s}}{D_{p_s}}, \\ N_{p_s} &= \omega^2 \lambda K_0(\lambda b) K_1(bk_r) \\ &\quad + k_r c^2 (\lambda^2 - k^2) K_1(b\lambda) K_0(bk_r), \\ D_{p_s} &= \omega^2 \lambda K_0(\lambda b) I_1(bk_r) \\ &\quad - k_r c^2 (\lambda^2 - k^2) K_1(\lambda b) I_0(bk_r).\end{aligned}\quad (49)$$

We observe that the ring model turns into the point model when a gets closer to zero. The structure of Eq. (49) is identical to that of the point model.

V. DIPOLE

We can solve the electromagnetic field surrounding a *dipole ring*. We provide the source term of dipole ring, Eq. (50), explicitly,

$$\begin{aligned}
\rho_1 &= \frac{I_1}{\pi a^2} \delta(s - vt) \delta(r - a) \cos \varphi \\
&= -\frac{Q_1}{a} \delta(s - vt) \delta(r - a) \cos \varphi, \\
j_1 &= c\rho_1 \hat{s},
\end{aligned} \tag{50}$$

where I_1 is the dipole moment, a is the radius of the ring, and v is the velocity of the particle.

The Maxwell equation under Lorentz gauge is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{A}_s}{\partial r} \right) - \left(\frac{1}{r^2} + k_r^2 \right) \tilde{A}_s = -\mu_0 \tilde{j}_s, \tag{51}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{\phi}}{\partial r} \right) - \left(\frac{1}{r^2} + k_r^2 \right) \tilde{\phi} = -\frac{\rho_0}{\epsilon_0}, \tag{52}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial \tilde{A}_r}{\partial r} \right] - \frac{2i}{r^2} \tilde{A}_\varphi - \left[\frac{2}{r^2} + k_r^2 \right] \tilde{A}_r = -\mu_0 \tilde{j}_r, \tag{53}$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial \tilde{A}_\varphi}{\partial r} \right] - \left[\frac{2}{r^2} + k_r^2 \right] \tilde{A}_\varphi + \frac{2i}{r^2} \tilde{A}_r = -\mu_0 \tilde{j}_\varphi. \tag{54}$$

According to Zimmermann and Oide, Eqs. (53) and (54) can be decoupled as

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{A}_\pm}{\partial r} \right) - \left(\frac{(1 \pm 1)^2}{r^2} + k_r^2 \right) \tilde{A}_\pm = -\mu_0 \tilde{j}_\pm, \tag{55}$$

by introducing $A_+ = A_r + iA_\varphi$, $A_- = A_r - iA_\varphi$, and $j_\pm = j_r \pm ij_\varphi$. Because $j_\varphi = 0$ and $j_r = 0$, Eq. (55) becomes homogeneous.

Solving the equation, we give the general solution of vector potential A_r, A_φ, A_s and potential ϕ . In different areas, we have

$$\begin{aligned}
r &< a, \\
A_r &= \frac{1}{2} [p_+^c I_2(k_r r) + p_-^c I_0(k_r r)] \cos \varphi e^{ikz}, \\
A_\varphi &= \frac{1}{2} [p_+^c I_2(k_r r) - p_-^c I_0(k_r r)] \sin \varphi e^{ikz}, \\
A_s &= p_s^c I_1(k_r r) \cos \varphi e^{ikz}, \\
\phi &= p_0^c I_1(k_r r) \cos \varphi e^{ikz},
\end{aligned} \tag{56}$$

where Lorentz gauge requires that

$$p_+^c = -p_-^c, \quad p_0^c = \frac{c^2 k}{\omega} p_s^c. \tag{57}$$

Then

$$a < r < b,$$

$$\begin{aligned}
A_r &= \frac{1}{2} [p_+ I_2(k_r r) + q_+ K_2(k_r r) + p_- I_0(k_r r) \\
&\quad + q_- K_0(k_r r)] \cos \varphi e^{ikz}, \\
A_\varphi &= \frac{1}{2} [p_+ I_2(k_r r) + q_+ K_2(k_r r) - p_- I_0(k_r r) \\
&\quad - q_- K_0(k_r r)] \sin \varphi e^{ikz}, \\
A_s &= [p_s I_1(k_r r) + q_s K_1(k_r r)] \cos \varphi e^{ikz}, \\
\phi &= [p_0 I_1(k_r r) + q_0 K_1(k_r r)] \cos \varphi e^{ikz},
\end{aligned} \tag{58}$$

where Lorentz gauge requires that

$$p_- = -p_+, \quad p_0 = \frac{c^2 k}{\omega} p_s, \quad q_- = -q_+, \quad q_0 = \frac{c^2 k}{\omega} q_s. \tag{59}$$

And

$$\begin{aligned}
r &> b, \\
A_r &= \frac{1}{2} [q_+^w K_2(\lambda r) + q_-^w K_0(\lambda r)] \cos \varphi e^{ikz}, \\
A_\varphi &= \frac{1}{2} \{q_+^w [K_2(\lambda r) - q_-^w K_0(\lambda r)]\} \sin \varphi e^{ikz}, \\
A_s &= q_s^w K_1(\lambda r) \cos \varphi e^{ikz}, \\
\phi &= q_0^w K_1(\lambda r) \cos \varphi e^{ikz},
\end{aligned} \tag{60}$$

where Lorentz gauge requires that

$$q_-^w = -q_+^w, \quad q_0^w = \frac{1}{\frac{\omega}{kc^2} + i \frac{\mu_0 \sigma}{k}} q_s^w. \tag{61}$$

So we have eight unknown parameters to determine, which are

$$p_+^c, \quad p_s^c, \quad p_+, \quad q_+, \quad p_s, \quad q_s, \quad q_+^w, \quad q_s^w. \tag{62}$$

A. Result of eight coefficients

Using the boundary condition, we will have eight independent linear equations. This procedure will be illustrated in Appendix D. By linear algebra manipulation, we solve eight coefficients in the end, which are

$$p_+^c = \frac{q_+ K_2(ak_r)}{I_2(ak_r)} + p_+, \tag{63}$$

$$p_s^c = -\frac{N_{p_s^c}}{akk_r I_0(ak_r) - k I_1(ak_r)}, \tag{64}$$

$$\begin{aligned}
N_{p_s} &= -akk_r p_s I_0(ak_r) + kp_s I_1(ak_r) \\
&+ kq_s [ak_r K_0(ak_r) + K_1(ak_r)] + \mu_0 Q_1 \omega, \\
q_+ &= 0, \\
q_s &= -\frac{\mu_0 Q_1 \omega I_1(ak_r)}{k}, \\
p_s &= \frac{N_{p_s}}{D_{p_s}}, \\
p_+ &= \frac{N_{p_+}}{D_{p_+}},
\end{aligned} \tag{66}$$

$$\begin{aligned}
N_{p_s} &= \omega(\lambda^2 k_r \mu_0 Q_1 I_1(ak_r)(b\lambda K_0(b\lambda)K_1(bk_r)(c^2 k^2 - \omega^2)(c^2 \mu_0 \sigma - i\omega) \\
&+ K_1(b\lambda)(c^4 k^2 \mu_0 \sigma K_1(bk_r) + bk_r \omega K_0(bk_r)(c^2(\mu_0 \sigma \omega + ik^2) - i\omega^2))) \\
&+ k^2 p_+(k_r^2 - \lambda^2)K_1(b\lambda)I_1(b\lambda)(-\omega^2 + c^2(k^2 - i\mu_0 \sigma \omega))), \\
D_{p_s} &= k\lambda^2 k_r (ibk_r \omega K_1(b\lambda)I_0(bk_r)(\omega^2 - c^2(k^2 - i\mu_0 \sigma \omega)) \\
&+ I_1(bk_r)(c^4 k^2 \mu_0 \sigma K_1(b\lambda) + b\lambda K_0(b\lambda)(c^2 k^2 - \omega^2)(c^2 \mu_0 \sigma - i\omega))), \\
p_+ &= \frac{N_{p_+}}{D_{p_+}}, \\
N_{p_+} &= 2ic^4 \lambda k_r \mu_0^2 Q_1 \sigma \omega I_1(ak_r)K_1^2(b\lambda),
\end{aligned} \tag{68}$$

$$\begin{aligned}
D_{p_+} &= (ib^2 \lambda k_r^2 \omega K_1^2(b\lambda)I_0(bk_r)(I_0(bk_r) + I_2(bk_r))(\omega^2 - c^2(k^2 - i\mu_0 \sigma \omega)) \\
&+ 2I_1^2(bk_r)(b^2 \lambda k_r^2 K_0^2(b\lambda)(c^2 k^2 - \omega^2)(c^2 \mu_0 \sigma - i\omega) + c^4 k^2 \lambda \mu_0 \sigma K_1^2(b\lambda) \\
&+ bk_r^2 K_0(b\lambda)K_1(b\lambda)(2c^4 k^2 \mu_0 \sigma - c^2 \omega(\mu_0 \sigma \omega + ik^2) + i\omega^3)) \\
&+ bk_r K_1(b\lambda)I_1(bk_r)(I_0(bk_r)(c^4 k^2 \lambda \mu_0 \sigma K_1(b\lambda) + ibk_r^2 \omega K_2(b\lambda)(\omega^2 - c^2(k^2 - i\mu_0 \sigma \omega)) \\
&+ bK_0(b\lambda)(c^4 k^2 \lambda^2 \mu_0 \sigma - c^2 \omega(\lambda^2 + k_r^2)(\mu_0 \sigma \omega + ik^2) + i\omega^3(\lambda^2 + k_r^2))) \\
&\lambda I_2(bk_r)(c^4 k^2 \mu_0 \sigma K_1(b\lambda) + b\lambda K_0(b\lambda)(c^2 k^2 - \omega^2)(c^2 \mu_0 \sigma - i\omega))).
\end{aligned}$$

$$q_+^w = -\frac{k_r p_+ I_1(bk_r)}{\lambda K_1(b\lambda)}, \tag{69}$$

$$q_s^w = -\frac{N_{q_s^w}}{k\lambda k_r [b\lambda K_0(b\lambda) + K_1(b\lambda)]}, \tag{70}$$

$$\begin{aligned}
N_{q_s^w} &= \lambda k_r \mu_0 Q_1 \omega I_1(ak_r)[bk_r K_0(bk_r) + K_1(bk_r)] \\
&+ ik^2 k_r q_+^w K_1(b\lambda) + b\lambda k k_r^2 p_s I_0(bk_r) \\
&+ \lambda k I_1(bk_r)(-k_r p_s + ik p_+),
\end{aligned}$$

After we get the eight parameters, we can express the potential and vector potential, then we can express the electromagnetic field via

$$E_s = \frac{1}{2\pi v} \int_{-\infty}^{\infty} (-ik\tilde{\phi} + i\omega\tilde{A}_s) \cos \varphi e^{ikz} d\omega, \tag{71}$$

$$B_s = \frac{1}{2\pi v} \int_{-\infty}^{\infty} \left(\frac{1}{r} \tilde{A}_r + \frac{\partial \tilde{A}_\varphi}{\partial r} + \frac{1}{r} \tilde{A}_\varphi \right) \sin \varphi e^{ikz} d\omega, \tag{72}$$

$$E_\varphi = \frac{1}{2\pi v} \int_{-\infty}^{\infty} \left(\frac{1}{r} \tilde{\phi} + i\omega\tilde{A}_\varphi \right) \sin \varphi e^{ikz} d\omega, \tag{73}$$

$$B_\varphi = \frac{1}{2\pi v} \int_{-\infty}^{\infty} \left(ik\tilde{A}_r - \frac{\partial}{\partial r} \tilde{A}_s \right) \cos \varphi e^{ikz} d\omega, \tag{74}$$

$$E_r = \frac{1}{2\pi v} \int_{-\infty}^{\infty} \left(-\frac{\partial}{\partial r} \tilde{\phi} + i\omega\tilde{A}_r \right) \cos \varphi e^{ikz} d\omega, \tag{75}$$

$$B_r = \frac{1}{2\pi v} \int_{-\infty}^{\infty} \left(-\frac{1}{r} \tilde{A}_s - ik\tilde{A}_\varphi \right) \sin \varphi e^{ikz} d\omega. \tag{76}$$

We combine the field to form the Lorentz force, which is

$$F_s = qE_s, \tag{77}$$

$$F_r = q(E_r - vB_\varphi), \tag{78}$$

$$F_\varphi = q(E_\varphi + vB_r). \tag{79}$$

When we calculate the Lorentz force numerically, we should remember that the same skill in Sec. IV D is used. Though, the process is complicated.

Again, it is noticed that when a approaches zero, the ring model result becomes the point source model given by Zimmermann and Oide [6].

VI. VERIFICATION

The Lorentz force calculations in frequency and time domain are followed by the step of verification. In addition to extrapolation to Zimmermann and Oide's result [6], we will verify our result in three more extrapolations. The three limits listed below are for ultrarelativistic extrapolation, far distance extrapolation, and space charge extrapolation, respectively,

$$\begin{aligned} \gamma &\rightarrow \infty, \\ z &\gg \chi^{1/3}b, \\ \sigma &\rightarrow 0. \end{aligned} \quad (80)$$

To find agreement with the previous findings, we return to the prior definition of the wake function, Eqs. (5)–(8), when analyzing the first two extrapolations in Secs. VI A and VI B. Furthermore, it should be noted that the third extrapolation gave us the idea to develop a numerical computation algorithm that is quicker than before.

A. Ultrarelativistic extrapolation

The wake function we are solving is a function of γ . As we said in Sec. III, when γ approaches infinity, the wake functions W_0 , $W_{1,r}$, $W_{1,\phi}$, and $W_{1,\parallel}$ are independent of r and γ and equal to the functions $W_0(z)$, $W_1(z)$, $W_1(z)$, and $W_1'(z)$. For instance, the complete form of the longitudinal wake function for monopole and transverse wake function for dipole given by Bane and Sands are

$$W_0'(z) = \frac{16}{4\pi\epsilon_0 b^2} \left(\frac{1}{3} e^{z/s_0} \cos \frac{\sqrt{3}z}{s_0} - \frac{\sqrt{2}}{\pi} \int_0^\infty \frac{dx x^2 e^{x^2 z/s_0}}{x^6 + 8} \right), \quad z < 0, \quad (81)$$

$$W_0'(z) = 0, \quad z > 0.$$

$$W_1'(z) = \frac{32}{4\pi\epsilon_0 b^4} \left(\frac{1}{3} e^{z/s_0} \cos \frac{\sqrt{3}z}{s_0} - \frac{\sqrt{2}}{\pi} \int_0^\infty \frac{dx x^2 e^{x^2 z/s_0}}{x^6 + 8} \right), \quad z < 0, \quad (82)$$

$$W_1'(z) = 0, \quad z > 0.$$

$$W_1(z) = - \int_z^0 W_1'(s) ds$$

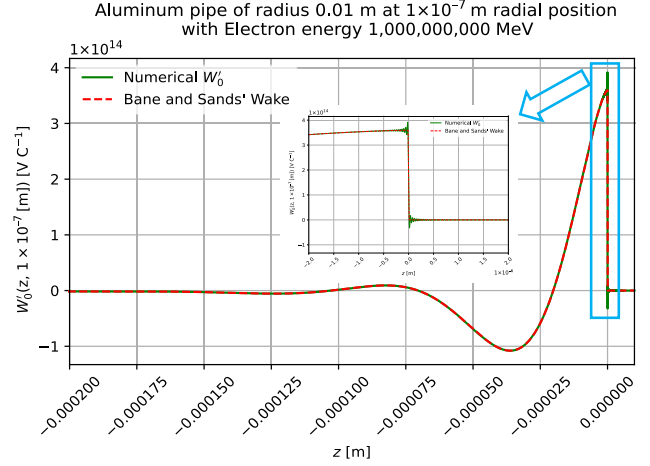


FIG. 4. Wake function for a monopole whose energy is large enough. The green solid curve is the result given by extrapolation of numerical solution, while the red dashed curve is the outcome of the complete form of wake function Eq. (81) given by Bane and Sands. The length L is 1 m.

We can see from Figs. 4 and 5 that the results obtained from ultrarelativistic extrapolation match well with the analytical solution, Eqs. (81) and (82).

Despite the analytical expression, there is an easy way to check the correctness qualitatively, which is to check if there is causality. We can learn that when the velocity of the electron equals the speed of light, the electromagnetic field cannot propagate faster than the electron. Therefore, in Figs. 4 and 5, the green curve should exhibit behavior that equals zero when $z > 0$. We do observe this phenomenon.

It is worth noting that the green curve shows a certain degree of fluctuation around the step point $z = 0$ in Fig. 4. It can be described by Gibbs phenomenon, because we cutoff the whole axis of integration with a certain frequency.

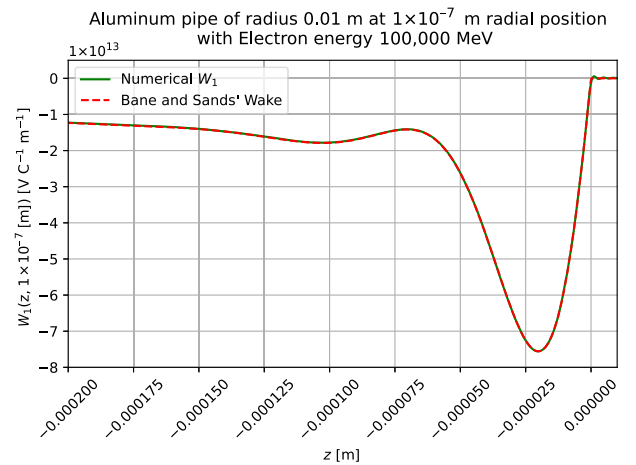


FIG. 5. Wake function for dipole whose energy is large enough. The green solid curve is the result given by extrapolation of numerical solution, while the red dashed curve is the outcome of the complete form of wake function Eq. (82) given by Bane and Sands. The length L is 1 m.

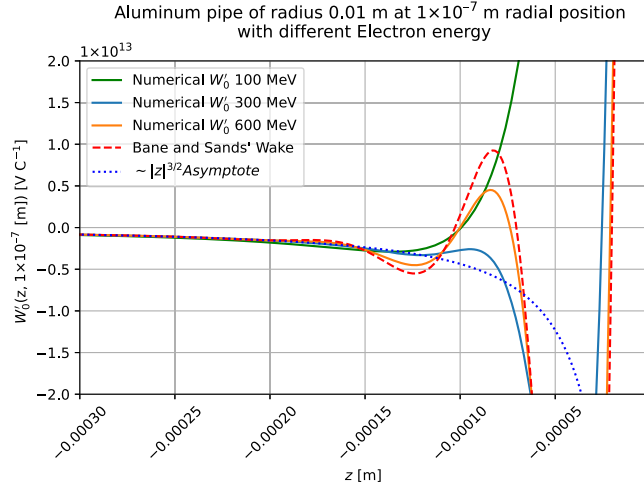


FIG. 6. Monopole wake calculated by numerical (green solid, orange solid, and blue solid) and analytical (blue dotted) methods. They agree well when z gets larger. The red dashed curve is the monopole wake when energy approaches infinity. The length L is 1 m.

In other words, the properties of the step function must be characterized by infinitely high frequencies.

B. Far distance extrapolation

We know that in the far distance we have the asymptotic function for ultrarelativistic case,

$$W'_0(z) = -\frac{c}{4\pi b} \sqrt{\frac{Z_0}{\pi\sigma}} \frac{L}{|z|^{\frac{3}{2}}}, \quad z \ll -\chi^{\frac{1}{3}}b, \quad (83)$$

$$W_1(z) = -\frac{c}{\pi b^3} \sqrt{\frac{Z_0}{\pi\sigma}} \frac{L}{|z|^{\frac{1}{2}}}, \quad z \ll -\chi^{\frac{1}{3}}b. \quad (84)$$

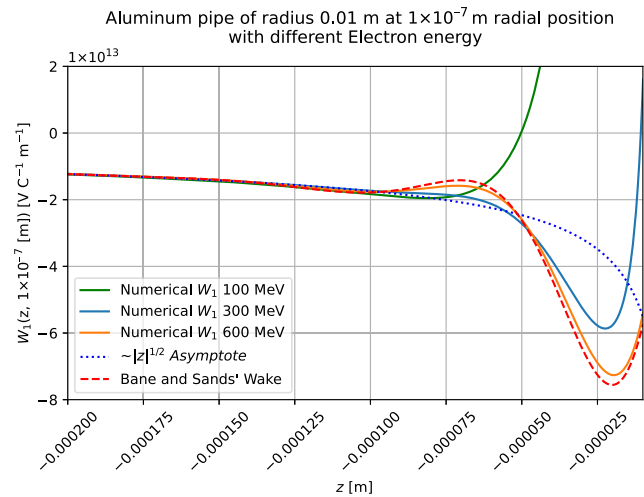


FIG. 7. Dipole wake calculated by numerical (green solid, orange solid, and blue solid) and analytical (blue dotted) methods. They also agree well when z gets larger. The red dashed curve is the dipole wake when energy approaches infinity. The length L is 1 m.

We plot the asymptotic function in Figs. 6 and 7 as the blue dotted curve, and the numerical wakefield at different energies as the green solid, orange solid, and blue solid curves. We give a numerical example to compare the blue dotted and green solid curves. Consider the wake function at the point $z = -0.001$ m when energy of the beam is 600 MeV, the characteristic parameter $\chi^{\frac{1}{3}}b$ for particles travel through 1 cm radius aluminum pipe equals 1.95×10^{-5} m. This satisfies the condition $z \ll -\chi^{\frac{1}{3}}b$. The length L is 1 m. In that case, $W'_0(z)$ given by Eq. (83) equals -1.377×10^{11} V/C, while the method of numerical integration gives -1.365×10^{11} V/C. In the meantime, $W_1(z)$ given by Eq. (84) equals -5.508×10^{12} V/(C m), and the method of numerical integration gives -5.431×10^{12} V/(C m).

One may be curious about the fact that we used the asymptotic formula Eqs. (83) and (84) to describe the results of ultrarelativistic limit, while providing examples of finite energy like 100, 300, and 600 MeV, and yet all of them exhibit good agreement in the far distance. The following explains why the wake function values in the far distance obtained using the two distinct methods are equal. In fact, we are discussing the low frequency domain impedance while we are considering the wakefield in the far distance. This is because when z is large, the high-frequency component of the impedance is averaged when doing inverse Fourier transform. The more distant the z , we are interested in, the lower the dominant frequency. On the other hand, it should be mentioned that frequency ω and γ are frequently combined together. Specifically, the modified Bessel function contains the variable $k_r = \omega/c\beta\gamma$. We can conclude that even though γ is not sufficiently large, in a far enough region where lower frequencies dominate, since k_r is still small, the wakefield under ultra- and nonultra-relativistic conditions remains the same, as sketched in Fig. 8.

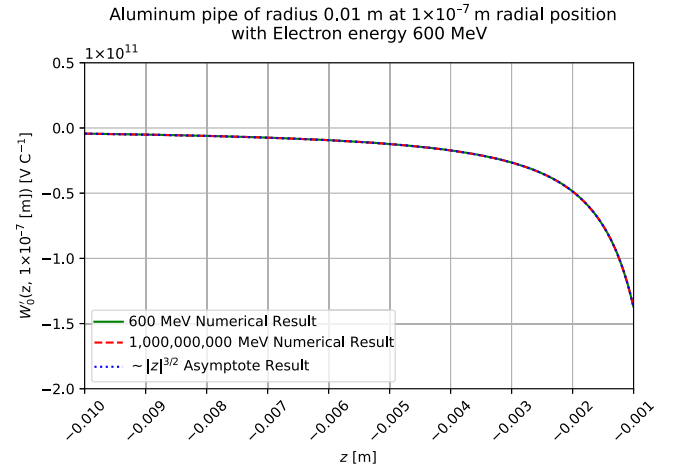


FIG. 8. The green solid curve, red dashed curve, and blue dotted curve represent the calculated results for 600 MeV energy, large enough energy, and the asymptotic formula Eq. (83), respectively. They exhibit good agreement with the monopole wake in the far distance. The length L is 1 m.

C. Space charge extrapolation

1. Monopole—verification

To illustrate the physics, we start with the simplest model, the point source monopole model.

If we let $\sigma \rightarrow 0$, we can establish the metal pipe as vacuum at $r > b$, and if the region is still divided into two districts, within the so-called pipe, there is

$$\begin{aligned}
 r &< b, \\
 E_r &= -\frac{\partial \phi}{\partial r} = -\frac{c^2 k k_r}{\omega} [p I_1(k_r r) - q K_1(k_r r)] e^{ikz}, \\
 E_s &= -\frac{\partial \phi}{\partial s} - \frac{\partial A_s}{\partial t} \\
 &= -i\omega \frac{k_r^2}{k_0^2} [p I_0(k_r r) + q K_0(k_r r)] e^{ikz}, \\
 B_\varphi &= -\frac{\partial A_s}{\partial r} = [-p k_r I_1(k_r r) + q k_r K_1(k_r r)] e^{ikz}. \quad (85)
 \end{aligned}$$

Outside the pipe, we have

$$\begin{aligned}
 r &> b, \\
 E_r &= q_{rw} K_1(\lambda r) e^{ikz}, \\
 E_s &= q_{sw} K_0(\lambda r) e^{ikz}, \\
 B_\varphi &= q_{\varphi w} K_1(\lambda r) e^{ikz}. \quad (86)
 \end{aligned}$$

By the continuity condition, the expression of the frequency domain of the vacuum electromagnetic field (space charge) can be written as

$$\begin{aligned}
 E_r &= \frac{c^2 k k_r \mu_0 Q v}{\omega} \frac{1}{2\pi} K_1(k_r r) e^{ikz}, \\
 E_s &= -\frac{i\omega}{\beta^2 \gamma^2} \frac{\mu_0 Q v}{2\pi} K_0(k_r r) e^{ikz}, \\
 B_\varphi &= k_r \frac{\mu_0 Q v}{2\pi} K_1(k_r r) e^{ikz}. \quad (87)
 \end{aligned}$$

In the meantime, we have the electromagnetic field of a point charge moving along the axis with speed v

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q \left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \left(\frac{v^2}{c^2}\right) \sin^2 \theta\right)^{\frac{3}{2}}} \frac{\hat{\mathbf{R}}}{R^2}, \quad (88)$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{Q v \left(1 - \frac{v^2}{c^2}\right) \sin \theta}{\left(1 - \left(\frac{v^2}{c^2}\right) \sin^2 \theta\right)^{\frac{3}{2}}} \frac{\hat{\boldsymbol{\phi}}}{R^2}, \quad (89)$$

where θ is the angle between the observation direction and the forward direction. Those expressions in the two different domains are, in fact, the Fourier transform counterparts. We in fact solve the Fourier transform by a physical

method. In Appendix E, we give a not very rigorous verification for this Fourier pair numerically.

Space charge extrapolation, Eq. (87), serves as a source of inspiration for numerical computation optimization. Upon thorough investigation, we find that the modified Bessel K function term in the electric field, Eqs. (18) and (20), is in fact Eq. (87). Such a clean separation indicates that the electric field of nonultrarelativistic resistive wall is composed of the space charge field and the residual resistive wall field. The latter originates from the fluctuating image space charge on the metal wall. We will soon investigate that the singularities in the time domain brought by the space charge effect are the source of the time consuming issue we encountered when numerically calculating the integration. Since we are able to employ the Lorentz transform in the time domain to solve the space charge of a moving particle in vacuum space, we can split out the singularities in the frequency domain. After that, the resistive wall field is simple to compute numerically.

2. Dipole—algorithm

We take the dipole ring model, for example, to illustrate the general case of the algorithm. And we will present the monopole's result in Appendix F. First, we need to calculate the space charge of a dipole ring in the frequency domain to separate the residual resistive wall field. Next, we directly calculate the space charge in the time domain by Lorentz transform. Then we combine the results together and compare the outcomes with straightforward numerical calculation.

Consider the vacuum limitation, this time the ring dipole divides the area into two sections.

$$\rho_1 = -\frac{Q_1}{a} \delta(s - vt) \delta(r - a) \cos \varphi. \quad (90)$$

We have

$$\begin{aligned}
 r &< a, \\
 A_r &= 0, \\
 A_\varphi &= 0, \\
 A_s &= p_{s-\text{free}}^c I_1(k_r r) \cos \varphi e^{ikz}, \\
 \phi &= \frac{c^2 k}{\omega} p_{s-\text{free}}^c I_1(k_r r) \cos \varphi e^{ikz}, \quad (91)
 \end{aligned}$$

and

$$\begin{aligned}
 r &> a, \\
 A_r &= 0, \\
 A_\varphi &= 0, \\
 A_s &= q_{s-\text{free}} K_1(k_r r) \cos \varphi e^{ikz}, \\
 \phi &= \frac{c^2 k}{\omega} q_{s-\text{free}} K_1(k_r r) \cos \varphi e^{ikz}, \quad (92)
 \end{aligned}$$

where

$$p_{s\text{-free}}^c = -\frac{kq_s[ak_rK_0(ak_r) + K_1(ak_r)] + \mu_0Q_1\omega}{akk_rI_0(ak_r) - kI_1(ak_r)},$$

$$q_{s\text{-free}} = -\frac{Q_1\mu_0\omega I_1(ak_r)}{k}.$$
(93)

So, the residual resistive wall should have the form

$$r < a,$$

$$A_r = \frac{1}{2} \{p_+^c [I_2(k_r r) - I_0(k_r r)]\} \cos \varphi e^{ikz},$$

$$A_\varphi = \frac{1}{2} \{[p_+^c I_2(k_r r) + I_0(k_r r)]\} \sin \varphi e^{ikz},$$

$$A_s = (p_s^c - p_{s\text{-free}}^c) I_1(k_r r) \cos \varphi e^{ikz},$$

$$\phi = \frac{c^2 k}{\omega} (p_s^c - p_{s\text{-free}}^c) I_1(k_r r) \cos \varphi e^{ikz}.$$
(94)

Then

$$a < r < b,$$

$$A_r = \frac{1}{2} [p_+ I_2(k_r r) + q_+ K_2(k_r r) - p_+ I_0(k_r r) - q_+ K_0(k_r r)] \cos \varphi e^{ikz},$$

$$A_\varphi = \frac{1}{2} [p_+ I_2(k_r r) + q_+ K_2(k_r r) + p_+ I_0(k_r r) + q_+ K_0(k_r r)] \sin \varphi e^{ikz},$$

$$A_s = [p_s I_1(k_r r) + (q_s - q_{s\text{-free}}) K_1(k_r r)] \cos \varphi e^{ikz},$$

$$\phi = \frac{c^2 k}{\omega} [p_s I_1(k_r r) + (q_s - q_{s\text{-free}}) K_1(k_r r)] \cos \varphi e^{ikz}.$$
(95)

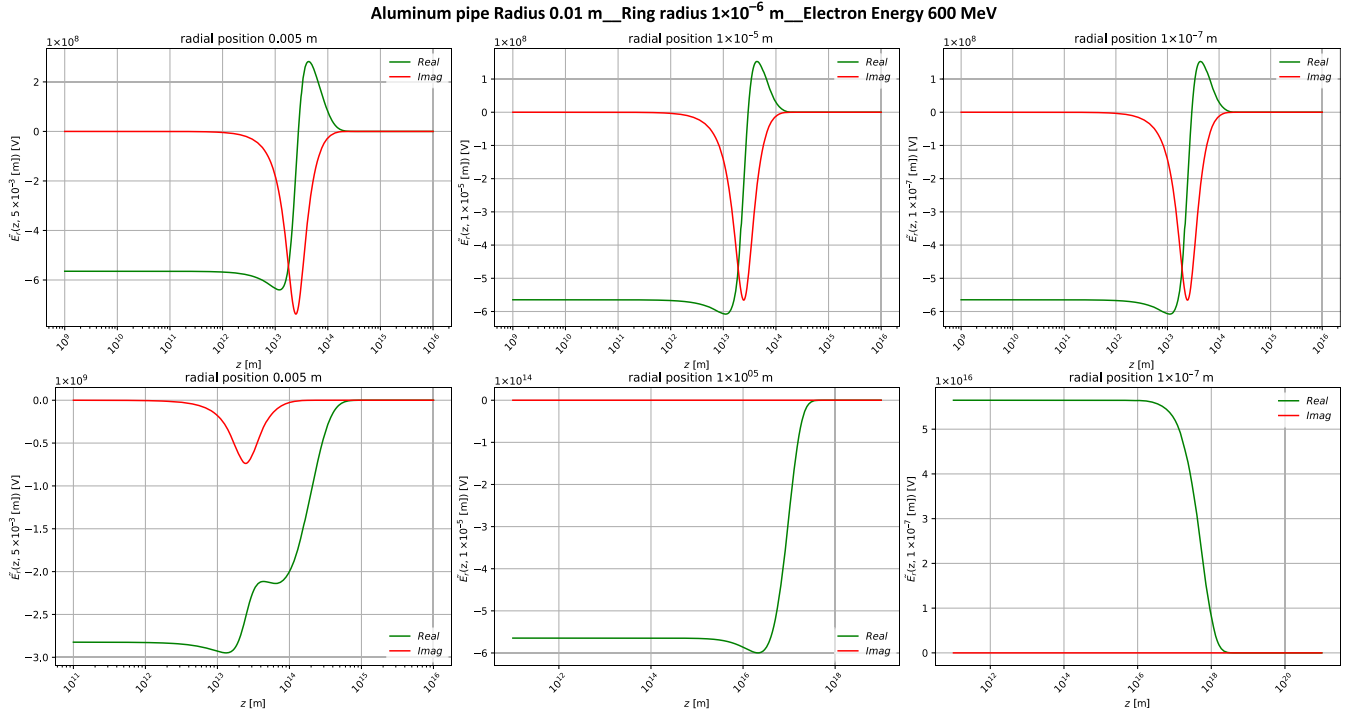


FIG. 9. Frequency domain of E_r . The upper three subplots are the residual resistive wall effect, while the lower three subplots are the total nonultrarelativistic effect.

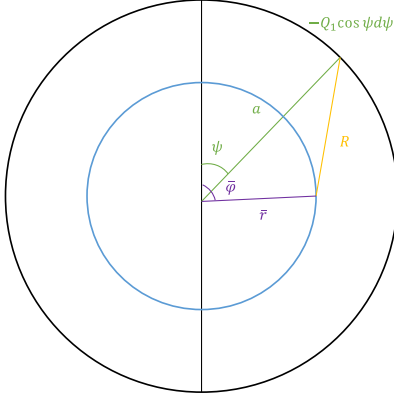


FIG. 10. Relevant parameters of the dipole in the rest frame. The black ring is the dipole ring. We will derive the electric field at $(\bar{r}, \bar{\varphi}, \bar{z})$, which can be projected onto the blue ring along z axis.

For instance, we draw plots to see the frequency domain of E_r induced by the residual resistive wall effect or total nonultrarelativistic effect, as Fig. 9 showed.

Before we move on, we will give a brief explanation of the singularities brought by space charge. When we place some charge at $r = a$, as the radius r approaches a , in the space-time domain, the electromagnetic field will tend to infinity and exhibit a significant variation. The pronounced variation implies the presence of fine structures. And those structures mean that as the radius approaches a , the spectrum will include progressively higher frequency components.

From the lower three subplots of Fig. 9, we can learn the singularities mentioned above. When the radius changes from 0.005 to 10^{-7} m, the cutoff frequency increases from

approximately 10^{15} to 10^{19} Hz. However, we notice that the residual resistive wall in the frequency domain is constrained in a relatively low frequency domain no matter how small the radius we are interested from the upper three subplots of Fig. 9. This means the numerical integration will have a relatively low cutoff frequency, which will significantly save the time that we consume.

After we separate out the residual resistive wall part, we need to analytically solve the ring dipole space charge in the free space.

First, we consider the ring in the rest frame. In this frame, there will only exist electric field. Then, we use Lorentz transform to get the space charge in the laboratory frame. The settings of relevant parameters are shown in Fig. 10.

We set the source term as

$$q = -Q_1 \cos \psi d\psi. \quad (96)$$

The expressions of the electromagnetic field in the laboratory frame are as follows, where $E(x)$ and $K(x)$ are the complete elliptic integrals of the second kind and the first kind, respectively,

$$B_r = -\frac{1}{c^2} v E_\varphi = \frac{1}{c^2} v \gamma \frac{Q_1}{2\pi\epsilon_0} \times \left(\frac{(a^2 + r^2 + (\gamma z)^2) K(x)}{ar^2 \sqrt{(a+r)^2 + (\gamma z)^2}} - \frac{((a+r)^2 + (\gamma z)^2) E(x)}{ar^2 \sqrt{(a+r)^2 + (\gamma z)^2}} \right) \sin \varphi, \quad (97)$$

$$B_\varphi = \frac{1}{c^2} v E_r = \frac{1}{c^2} v \gamma \frac{Q_1}{2\pi\epsilon_0} \times \left(\frac{(a^4 - a^2(r^2 - 2(\gamma z)^2) + (\gamma z)^2(r^2 + (\gamma z)^2)) E(x)}{ar^2((a-r)^2 + (\gamma z)^2) \sqrt{(a+r)^2 + (\gamma z)^2}} - \frac{(a^2 + (\gamma z)^2)((a-r)^2 + (\gamma z)^2) K(x)}{ar^2((a-r)^2 + (\gamma z)^2) \sqrt{(a+r)^2 + (\gamma z)^2}} \right) \cos \varphi, \quad (98)$$

$$B_s = 0, \quad (99)$$

$$E_s = \frac{Q_1 \gamma z}{2\pi\epsilon_0} \left\{ \frac{[(a-r)^2 + (\gamma z)^2] K(x)}{ar[(a-r)^2 + (\gamma z)^2] \sqrt{(a+r)^2 + (\gamma z)^2}} - \frac{[a^2 + r^2 + (\gamma z)^2] E(x)}{ar[(a-r)^2 + (\gamma z)^2] \sqrt{(a+r)^2 + (\gamma z)^2}} \right\} \cos \varphi, \quad (100)$$

where

$$x = \frac{4ar}{(a+r)^2 + (\gamma z)^2}. \quad (101)$$

In Appendix G, we provide the derivation process of the aforementioned expressions, as well as the original integral forms before simplification for reference. Sometimes, the original integral forms are more useful and brief than the closed forms, in practice. If we are interested in the space

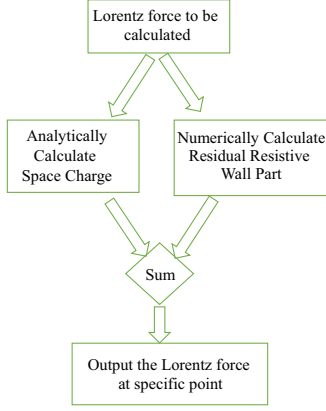


FIG. 11. Flowchart of the faster algorithm to calculate Lorentz force of nonultrarelativistic resistive wall.

charge between the bunches. Take SSMB parameters, for example, since $r \sim a \approx 10 \mu\text{m}$, $\gamma \approx 1200$, and smallest spacing between bunches $z \approx 1 \mu\text{m}$, we have $\gamma z \gg r \sim a$. We truncate the series from Eqs. (97) to (100) to the lowest order. Finally, we have the asymptotic form

$$E_s \approx -\frac{Q_1}{2\pi\epsilon_0} \left(\frac{3\pi a r \operatorname{sgn}(\gamma z)}{2(\gamma z)^4} \right) \cos \varphi, \quad (102)$$

$$B_r = -\frac{1}{c^2} v E_\varphi \approx \frac{1}{c^2} v \gamma \frac{Q_1}{2\pi\epsilon_0} \left(\frac{\pi a}{2|\gamma z|^3} \right) \sin \varphi, \quad (103)$$

$$B_\varphi = \frac{1}{c^2} v E_r \approx \frac{1}{c^2} v \gamma \frac{Q_1}{2\pi\epsilon_0} \left(\frac{\pi a}{2|\gamma z|^3} \right) \cos \varphi. \quad (104)$$

As a result, we are able to develop an algorithm that calculates the Lorentz force in the time domain faster. The algorithm's steps are as follows. First, we determine the ring model's space charge and provide the space charge's Lorentz force in the time domain. In the meantime, we compute the residual resistive wall field by numerical inverse Fourier transform in the frequency domain. Finally, to obtain the overall impact of the nonultrarelativistic resistive wall, we sum those results above. We plot a flowchart to explain this process, as Fig. 11.

As an example, Fig. 12 shows that the mixed method, which is the faster numerical algorithm agrees well with the original numerical calculation. The latter took approximately several days of computing time, while the former only cost a few minutes on a high performance computing cluster with 1024 cores in total.

Those familiar with the nonultrarelativistic resistive wall field may be aware of a limit called reduced resistivity where $\sigma \rightarrow +\infty$, representing the choice of a perfect conductor for the pipe material. Attention should be paid that this is not equivalent to the space charge extrapolation, since the former includes the latter. We will

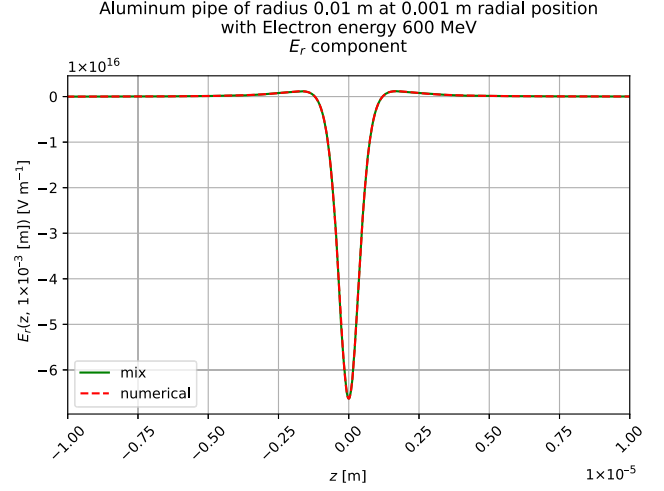


FIG. 12. E_r field in the time domain by original straightforward numerical calculation (red dashed) and mix method (green solid), which is faster.

illustrate in Appendix H why we do not use the reduced resistivity limit to build our calculation algorithm.

VII. APPLICATION

A. SSMB case parasitic loss

Here we give an application of the Lorentz force in the longitudinal direction with SSMB parameters, Table I, which is the parasitic loss.

If we assume the beam's longitudinal distribution to be a Gaussian distribution, it should be

$$\rho(z) = \frac{q}{\sqrt{2\pi}\sigma_z} e^{-\frac{z^2}{2\sigma_z^2}},$$

$$\tilde{\rho}(\omega) = q e^{-\frac{\omega^2 \sigma_z^2}{2\rho^2 c^2}}. \quad (105)$$

We modify the parasitic loss formula in the nonultrarelativistic case, the energy loss per bunch is

$$\begin{aligned} \frac{\Delta\mathcal{E}}{L} &= \int_{-\infty}^{\infty} \rho(z') \int_{-\infty}^{\infty} \rho(z) E_s^{\text{per}}(z' - z) dz dz' \\ &= \frac{1}{2\pi v} \int_{-\infty}^{\infty} |\tilde{\rho}(\omega)|^2 \tilde{E}_s^{\text{per}}(\omega) d\omega, \end{aligned} \quad (106)$$

TABLE I. SSMB parameters.

Parameter	Value	Purpose
Length	10 nm	Longitudinal coherent
Transverse size	10 – 100 μm	
Electron number	10^4	
Spacing	1 μm	High average power $\gamma \approx 1175.2$
Typical energy	600 MeV	

where we use the below equation as the transition of the derivation

$$\frac{\Delta\mathcal{E}}{L} = \int_{-\infty}^{\infty} \rho(z') \int_{-\infty}^{\infty} \rho(z) \times \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{i\omega(z'-z)}{v}} \tilde{E}_s^{\text{per}}(\omega) \frac{d\omega}{v} dz dz'.$$

$\tilde{E}_s^{\text{per}}, E_s^{\text{per}}$ represents the electric field per unit source charge and has the connection with the impedance and wake. In the ultrarelativistic case, if we substitute Eq. (A2) into Eq. (106), we get the classical formula [7] in the limit $\gamma \rightarrow \infty$,

$$\frac{\Delta\mathcal{E}}{L} = -\frac{2q^2}{4\pi\epsilon_0 b^2}, \quad \sigma_z \ll \chi^{\frac{1}{3}} b, \quad (107)$$

$$\frac{\Delta\mathcal{E}}{L} = -\frac{q^2 c}{4\pi^2 b \sigma_z^{\frac{3}{2}}} \Gamma\left(\frac{3}{4}\right) \sqrt{\frac{Z_0}{2\sigma}}, \quad \sigma_z \gg \chi^{\frac{1}{3}} b. \quad (108)$$

In the ultrarelativistic limit, inside an aluminum pipe of radius 1 cm, for SSMB bunch parameters in Table I, Eq. (107) applies and gives the parasitic loss per particle as

$$\frac{\Delta\mathcal{E}}{L} \sim -0.28799290 \text{ eV/m}$$

while the numerical result of Eq. (106) divided by the particle number of bunch gives,

$$\frac{\Delta\mathcal{E}}{L} \sim -0.28799397 \text{ eV/m}$$

Since the modified parasitic loss is the function of γ , we can plot the dependence of the energy, as shown in Fig. 13.

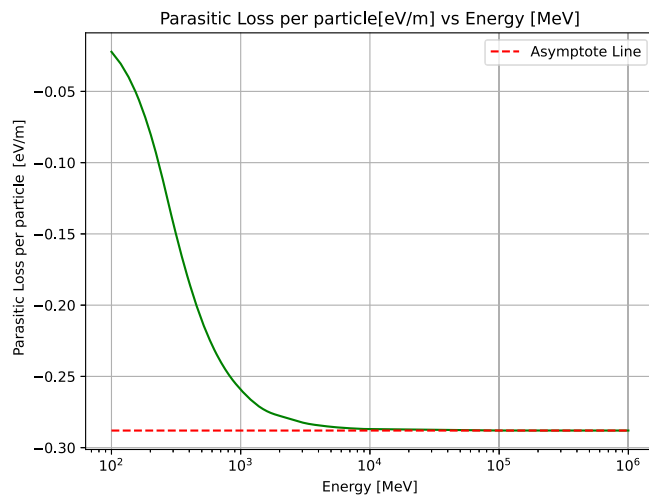


FIG. 13. Parasitic loss per particle at different energy. The red straight dashed line is the asymptotic value when energy approaches infinity.

We find that as the particle energy increases, the absolute value of parasitic loss gradually increases. Finally, it will match the results of ultrarelativistic.

B. SSMB case kick angle

As for the application for the transverse result, in analogy to the parasitic loss of longitudinal Lorentz force, we can calculate the transverse kick angle by Eqs. (109) and (110).

$$\begin{aligned} \Delta\theta_r(z, r, \varphi) &= \frac{-e}{\gamma m_e v^2} \int_{-\infty}^{\infty} \rho(z') [E_r^{\text{per}}(z - z', r, \varphi) \\ &\quad - v B_\varphi^{\text{per}}(z - z', r, \varphi)] dz' \\ &= \frac{-e}{\gamma m_e v^2} \frac{1}{2\pi v} \int_{-\infty}^{\infty} [\tilde{E}_r^{\text{per}}(\omega, r) \\ &\quad - v \tilde{B}_\varphi^{\text{per}}(\omega, r)] \cos m\varphi \tilde{\rho}(\omega) e^{\frac{i\omega}{v}z} d\omega, \end{aligned} \quad (109)$$

$$\begin{aligned} \Delta\theta_\varphi(z, r, \varphi) &= \frac{-e}{\gamma m_e v^2} \int_{-\infty}^{\infty} \rho(z') [E_\varphi^{\text{per}}(z - z', r, \varphi) \\ &\quad + v B_r^{\text{per}}(z - z', r, \varphi)] dz' \\ &= \frac{-e}{\gamma m_e v^2} \frac{1}{2\pi v} \int_{-\infty}^{\infty} [\tilde{E}_\varphi^{\text{per}}(\omega, r) \\ &\quad + v \tilde{B}_r^{\text{per}}(\omega, r)] \sin m\varphi \tilde{\rho}(\omega) e^{\frac{i\omega}{v}z} d\omega. \end{aligned} \quad (110)$$

Again, $E_r^{\text{per}}, B_\varphi^{\text{per}}, B_r^{\text{per}}$, and E_φ^{per} with or without tilde above are the electric field or the magnetic field per unit source charge and have the connection with the impedance and wake. The physical interpretation of the above formula can be better understood by looking at Fig. 14, which means the radial and azimuthal forces at any point in three-dimensional space of a beam distribution can cause a kick angle to the test charge.

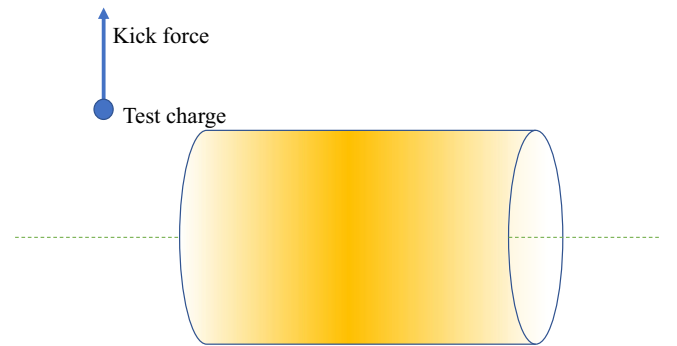


FIG. 14. Kick force provided by a beam distribution that can be decoupled into a ring in transverse direction and Gaussian distribution in longitudinal direction.

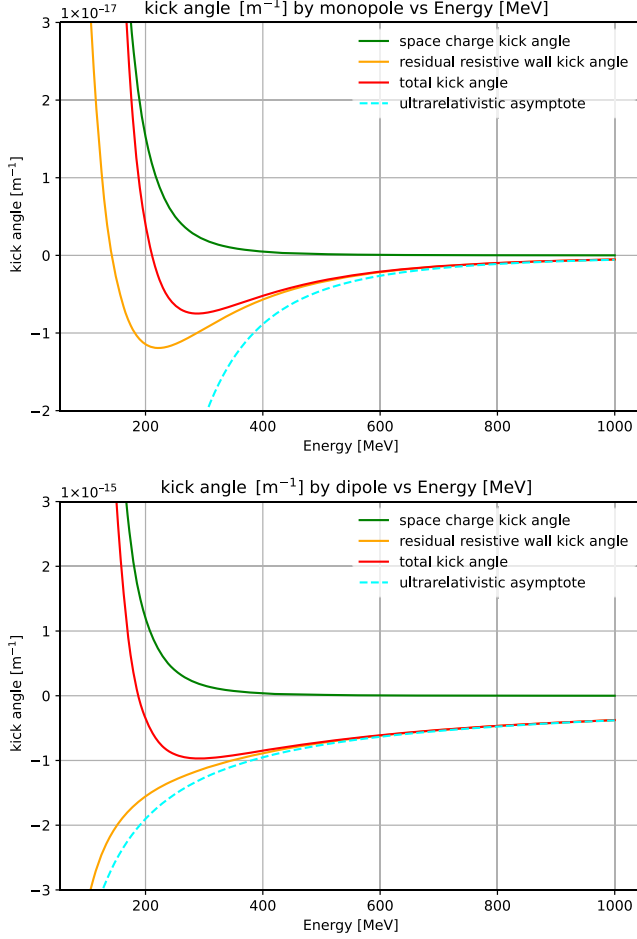


FIG. 15. Kick angle caused by the electron beam bunches with different energy at $z = -2 \times 10^{-5}$ m, $r = 1 \times 10^{-7}$ m, and $\varphi = 0$. The ring radius of the monopole and dipole is 1×10^{-6} m. The monopole ring is made of electrons and the maximum positive charge of the dipole is placed at $\varphi = 0$. The kick angle from space charge and residual resistive wall are the green solid and orange solid curves, respectively. The total kick angle is the red solid curve, while the cyan dashed curve is the ultrarelativistic asymptote for reference.

In practical applications, when we use the SSMB parameters in Table I, we have observed that the difference between the radial and azimuthal forces from the dipole is so small that it is difficult to distinguish them under a specific numerical calculation precision. Therefore, in the following discussion, we will focus on studying the kick angle brought by the radial direction component.

In Sec. IV C, our analysis revealed that the transverse Lorentz force induced by the monopole with $m = 0$ and the dipole with $m = 1$ can be of the same order of magnitude. Therefore, we must take into account both of these circumstances when examining the radial component, while we just need to take the dipole's influence into consideration for the azimuthal component.

First, we will calculate the kick angle caused by electron beam bunches with different energy at specific position in

three-dimensional space. Based on its mechanism of interaction, we split the kick angle effect into two parts, which are the space charge kick angle and residual resistive wall kick angle. As shown in Fig. 15, we present the magnitude of total kick angle (red solid) at point $(r, \varphi, z) = (1 \times 10^{-7} \text{ m}, 0, -2 \times 10^{-5} \text{ m})$, along with the contributions from space charge (green solid) and residual resistive wall (orange solid). As for the dashed curve in cyan, it is the ultrarelativistic asymptote for reference, which means that the total kick angle (red solid) will incline toward zero as energy gets closer to infinity, following the dashed cyan curve's trend. We obtain the ultrarelativistic asymptote as follows. We use Eq. (21) to obtain the result for monopole at different energy, while we derive the solution for dipole using Eq. (82). In other words, we replace $(E_r^{\text{per}} - vB_\varphi^{\text{per}})$ or $(\tilde{E}_r^{\text{per}} - v\tilde{B}_\varphi^{\text{per}})$ in Eq. (109) by the corresponding ultrarelativistic wake force.

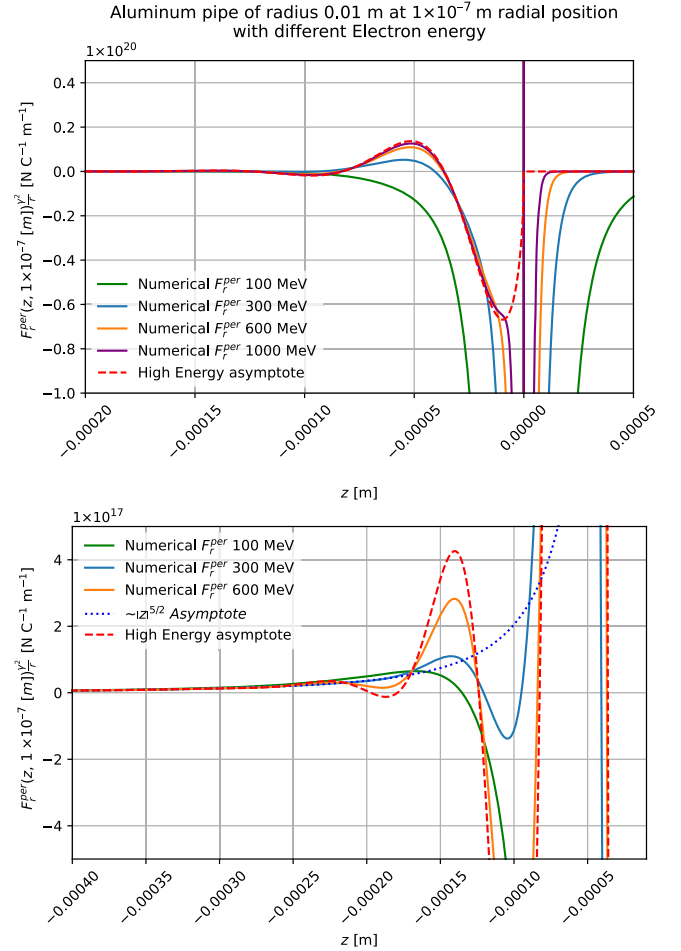


FIG. 16. F_r^{per} of the monopole ring with $Q_0 = 1$ C and $a = 1 \times 10^{-6}$ m at $r = 1 \times 10^{-7}$ m. It should be noted that the setting of Q_0 means the ring carries a negative charge. To compare the outcomes of different energy, we multiply the result, F_r^{per} , with a factor of γ^2/r . A closer longitudinal distance is displayed for the image of F_r^{per} in the upper half of the figure, while a further longitudinal distance is exhibited in the lower half.

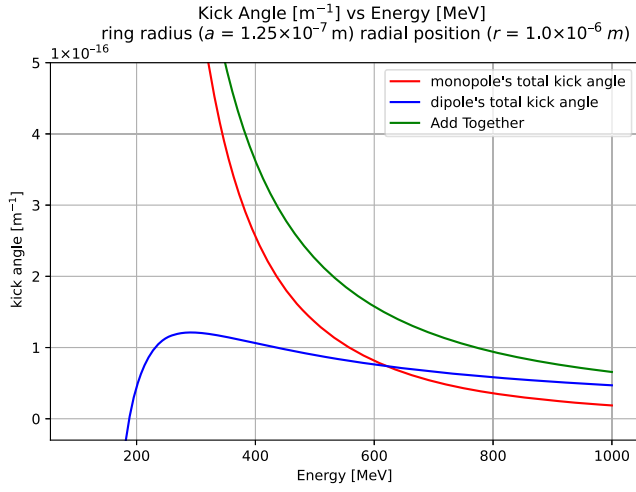


FIG. 17. Kick angle at $z = -2 \times 10^{-5}$ m, $r = 1 \times 10^{-6}$ m, and $\varphi = \pi$. The ring radius of the monopole and dipole is 1.25×10^{-7} m. The monopole ring is made of electrons and the maximum positive charge of the dipole is placed at $\varphi = 0$. The monopole results are shown by the red solid line, the dipole results by the blue solid line, and their combined overall effect by the green solid line.

We can find that the test particle at the studied location will be defocused transversely at lower energy due to the resistive wall effect and focused at higher energy. This effect is relatively weak and gradually accumulates as the particles move forward.

Next, we will discuss the differences in kick angle at different longitudinal positions. Since σ_z of the SSMB case is significantly smaller than the scale of the variation of Lorentz force at the points in which we are interested. The kick angle is actually a sampling of the Lorentz force over a very small region in the time domain. So, the figures like Figs. 7 and 16 of the Lorentz force for monopole and dipole, respectively, are enough for informational purposes. We are able to observe that the magnitude and direction of the kick angle vary at different longitudinal positions.

Finally, since we previously provided an estimation formula applicable at long distances, Eq. (31), indicating that the magnitudes of the transverse Lorentz forces of the monopole and dipole are approximately at the same level with the order of hundreds MeV energy, we want to examine whether such an estimation also holds some validity at short distances. Figure 17 shows that at the same point, when we compare the kick angle of the monopole and dipole, the validity still holds to a certain extent. We can even achieve a comparable magnitude of effects for the monopole and dipole around 600 MeV (SSMB's typical energy) by combining parameters within a reasonable range. Figure 18 further illustrates the ratio R of the transverse Lorentz force of the monopole to that of the dipole when the energy of the electron beam is 100 and 600 MeV within the interval of the longitudinal coordinate

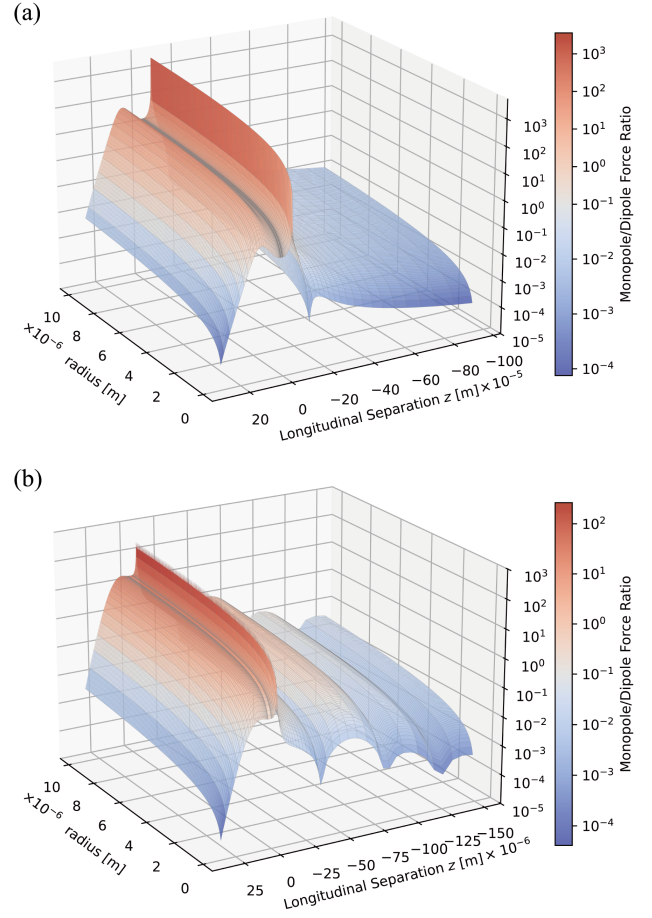


FIG. 18. Ratio of the transverse Lorentz force of the monopole to that of the dipole as a function of radial and longitudinal position in the aluminum pipe with $Q_0 = 1$ C, $Q_1 = -1$ C, $\psi = 0$ b = 0.01 m, and $a = 1 \times 10^{-6}$ m at 100 MeV or 600 MeV plotted on a logarithmic scale. Regions where the ratio is greater than 0.1 are colored in warm tones, while those where it is less than 0.1 are in cool tones. (a) Electron energy 100 MeV and (b) Electron energy 600 MeV.

where Eq. (31) is not applicable. It can be observed that the fundamental law revealed by Eq. (31) still holds even in this parameter space: as the beam energy increases, the interval of the longitudinal coordinate required to maintain the ratio above a specific value decreases; there exists a relatively linear relationship between the transverse coordinate and the ratio, with the ratio increasing as the transverse coordinate grows larger, which is not obvious on a logarithmic scale. However, we do not need to plot additional figures to verify this linear relationship, as shown by the second fundamental law. Since we have previously decomposed the wake function into two components, the space charge and the residual resistive wall wake, we will provide an interpretation from the perspective of this decomposition. For the space charge component, Eqs. (104) and (F18) demonstrate that the transverse Lorentz force of the dipole is independent of r , whereas the

force of the monopole exhibits a linear correlation with r . As for the residual resistive wall wake component, the asymptotic analysis presented earlier indicates that the first term in their $1/\gamma$ expansion possesses the same properties as those of the space charge. Therefore, the nonlinearity of the ratio with respect to r is a higher-order effect of the $1/\gamma$ expansion, which is difficult to observe within our energy range.

VIII. SUMMARY

In this paper, we have redefined the so-called general wake function for nonultrarelativistic case using the Panofsky-Wenzel theorem. For simplicity, we carried out subsequent analysis using the Lorentz force instead. We discovered the existence of a completely new transverse wake function, or called transverse Lorentz force, for the monopole case with $m = 0$. We conducted a detailed analysis of this field, obtaining its expansion to the second orders of $1/\gamma$ at far distance

$$F_r(z, r, \gamma) = -Qq \frac{3rv^{\frac{3}{2}} \sqrt{\frac{\mu_0}{\sigma}}}{16b\pi^2 \gamma^2 |z|^{\frac{3}{2}}}. \quad (111)$$

Furthermore, we compared its strength with the magnitude of the transverse Lorentz force for the dipole case with $m = 1$, deriving an estimation formula for the magnitude to decide which one is bigger,

$$R = \frac{F_r^{\text{monopole}}}{F_r^{\text{dipole}}} = \frac{3rb^2}{16a|z|^2} \frac{1}{\gamma^2 \beta^{-\frac{3}{2}}}, \quad (112)$$

when

$$z \ll -\frac{v}{\left(\frac{c^2}{b} \sqrt{\mu_0 \sigma}\right)^{\frac{2}{3}}}.$$

In comparison to previous work, we have taken further steps to compute the electromagnetic field of the ring model, which is beneficial for calculating the field under

arbitrarily charge distributions through slicing into strips. Additionally, we have shared our technique of overcoming data overflow in numerical computations by employing asymptotic functions.

We validated our results through three extrapolation methods: ultrarelativistic, far distance, and space charge extrapolation. The agreement between the results was satisfying. During the third extrapolation process, we were inspired to propose a numerical algorithm that significantly enhances computation speed. The core of this algorithm involves splitting the results into two parts: the space charge component, which is calculated analytically, and the residual resistive wall component, which is computed numerically.

As for applications, based on a static perspective, we estimated the parasitic loss and kick angle experienced by a single bunch under the SSMB parameters in terms of dynamics. We found significant discrepancies between the calculated results and those of the ultrarelativistic limit at the MeV energy scale.

ACKNOWLEDGMENTS

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APPENDIX A: PREVIOUS RESULT IN ULTRARELATIVISTIC LIMIT

The multipole used in the previous result [7] is

$$\rho_m = \frac{I_m}{\pi a^{m+1} (1 + \delta_{m0})} \delta(s - ct) \delta(r - a) \cos m\varphi, \quad (A1)$$

where a is the radius of the infinitesimally thin ring.

$$\frac{Z_m^{\parallel}(\omega)}{L} = \frac{\frac{4}{4\pi\epsilon_0 b^{2m}}}{(1 + \delta_{m0})bc \sqrt{\frac{2\pi\sigma}{4\pi\epsilon_0 |\omega|}} (1 + \text{sgn}(\omega)i) - \frac{ib^2}{m+1} \omega + \frac{imc^2}{\omega}}, \quad (A2)$$

$$\frac{Z_m^{\parallel}(\omega)}{L} = \frac{1}{4\pi\epsilon_0} \sqrt{\frac{8\epsilon_0}{\sigma}} \frac{1}{(1 + \delta_{m0})b^{2m+1}c} |\omega|^{\frac{1}{2}} (1 - \text{sgn}(\omega)i). \quad (A3)$$

The complete form of the resistive wall impedance is Eq. (B1). Furthermore, we will introduce its origins in Appendix B. The commonly encountered expression for impedance in the literature, as given by Eq. (A2), is the asymptotic form of Eq. (B1) when $|\omega| \gg (2\epsilon_0 c^2)/(b^2 \sigma)$. And the asymptotic form after taking relatively low-frequency approach,

$$\frac{2\varepsilon_0 m^2 c^2}{b^2 \sigma} \ll |\omega| \ll \left(\frac{c\varepsilon_0}{\sigma b}\right)^{-\frac{1}{3}} \frac{c}{b},$$

is Eq. (A3) [7]. The classical wake function for multipole has an analytical expression [7]

$$W_m(z) = -\frac{c}{\pi b^{2m+1}(1+\delta_{m0})} \sqrt{\frac{Z_0}{\pi\sigma}} \frac{L}{|z|^{\frac{1}{2}}}, \quad (\text{A4})$$

$$W'_m(z) = -\frac{c}{2\pi b^{2m+1}(1+\delta_{m0})} \sqrt{\frac{Z_0}{\pi\sigma}} \frac{L}{|z|^{\frac{3}{2}}}. \quad (\text{A5})$$

where b is the radius of the pipe, σ is the conductivity of the surrounding metal. This result is obtained by ultrarelativistic limit $\gamma \rightarrow +\infty$, meaning that the speed of the electron is the speed of light c in free space. Meanwhile, the range of the longitudinal coordinate for which the formula is applicable is that

$$\chi^{\frac{1}{3}} b \ll |z| \ll \frac{b}{\chi}, \quad z < 0, \quad (\text{A6})$$

where $\chi = \frac{1}{\mu_0 \sigma b c}$. For aluminum pipe with a radius ranging from several centimeters, the valid region will be $-1.54 \times 10^{13} \text{ m} \ll z \ll -1.95 \times 10^{-5} \text{ m}$. For the majority of accelerator facilities, this limit is acceptable. However, it is ineffective for certain light sources whose bunch length is tens of micrometers or even nanometers like that in SSMB.

In order to get the formula for the wake function calculation valid in the range comparable with $\chi^{\frac{1}{3}} b$, in

Bane and Sands' research [8], there is a formula for m th longitudinal electric field in the time-space domain

$$E_z^{(m)}(r, z) = \frac{2I_m(-16\nu)}{4\pi\varepsilon_0(1+\delta_{m0})} \frac{r^m}{b^{2(m+1)}} \left(\frac{1}{3} e^{\nu^{\frac{2}{3}} z/s_0} \cos \frac{\sqrt{3}\nu^{\frac{2}{3}} z}{s_0} - \frac{\sqrt{2}\nu}{\pi} \int_0^\infty \frac{dx x^2 e^{x^2 z/s_0}}{x^6 + 8\nu^2} \right), \quad (\text{A7})$$

where $\nu = 1$ when $m = 0$ and $\nu = (m+1)/2$ when $m > 0$. And we define

$$s_0 = \left(\frac{2\varepsilon_0 b^2 c}{\sigma} \right)^{\frac{1}{3}}. \quad (\text{A8})$$

Equation (A7) is valid for $-(b/\chi) \ll z < 0$. And we can determine the transverse Lorentz force by the Panofsky-Wenzel theorem.

If we are interested in the range $-\chi^{\frac{1}{3}} b \ll z < 0$, we also have the wake function, take $m = 0$ for instance [7],

$$\frac{W_0(z)}{L} = \frac{1}{4\pi\varepsilon_0} \left(\frac{4}{b^2} z + \frac{64}{15} \frac{1}{b^3} \sqrt{\frac{\sigma Z_0}{\pi}} |z|^{\frac{3}{2}} \right). \quad (\text{A9})$$

APPENDIX B: IMPEDANCE OF ULTRARELATIVISTIC CASE

The complete formula of the impedance of resistive wall under ultrarelativistic assumption is that

$$\frac{Z_m^{\parallel}(\omega)}{L} = -\frac{1}{4\pi\varepsilon_0 c} \frac{4ik\lambda^2(m+1)b^{-2m}H_m^{(1)}(b\lambda)}{\{k^2[4m(m+1)-b^2\lambda^2] + 2\lambda^2(m+1)m\}H_m^{(1)}(b\lambda) - b\lambda(m+1)[2k^2 + \lambda^2(1+\delta_{m0})]H_{m-1}^{(1)}(b\lambda)}, \quad (\text{B1})$$

where $H_m^{(1)}, H_{m-1}^{(1)}$ are the m th and $(m-1)$ th order Hankel functions of the first kind, respectively. And we define

$$k = \omega/c, \quad (\text{B2})$$

$$\lambda = \sqrt{\frac{\sigma|k|}{2\varepsilon_0 c}} [i + \text{sgn}(k)]. \quad (\text{B3})$$

It should be noticed that the λ defined here differs from the λ in the main text by an imaginary unit, i . In other words, there is a negative sign separating the squares of the former from the nonultrarelativistic form in Eq. (17).

We get Eq. (B1) by ignoring the approximation $|\omega| \gg (2\varepsilon_0 c^2)/(b^2 \sigma)$. And this means that Eqs. (2.5) and (2.36) in Chao's book [7] should be solved rigorously. To illustrate the difference between Eqs. (A2) and (B1), we take $Z_1^{\perp}/L$ as an example. From Fig. 19, we can find that at relatively high frequency those two impedances are

identical to each other, while at relatively low frequency there are some differences. In fact, we can derive Eq. (A2) from Eq. (B1) using the asymptotic expression

$$H_m^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} \exp \left[i \left(z - \frac{m\pi}{2} - \frac{\pi}{4} \right) \right], \quad z \rightarrow \infty. \quad (\text{B4})$$

Furthermore, the limit value of $Z_1^{\perp}/L$ from Eq. (A2) when ω approaches to zero is

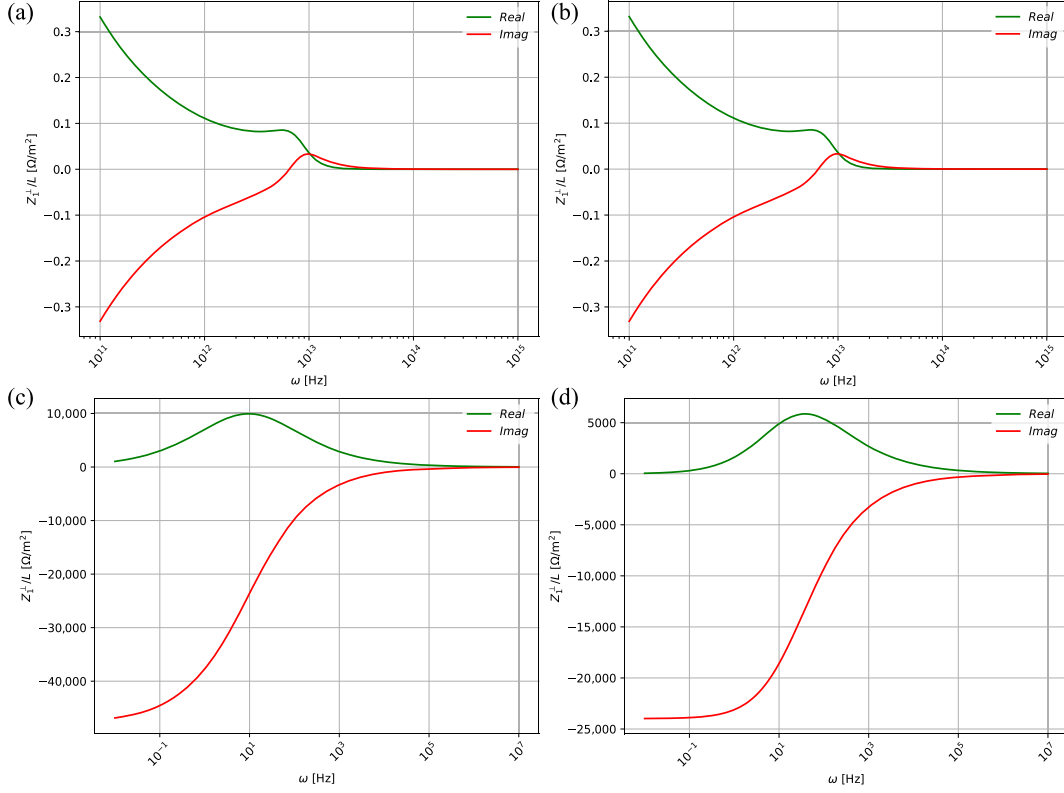


FIG. 19. (a) Behavior of $Z_1^\perp/L$ at relatively high frequency according to Eq. (A2) after the Panofsky-Wenzel theorem. (b) Same of $Z_1^\perp/L$ at relatively high frequency by Eq. (B1). (c) $Z_1^\perp/L$ at relatively low frequency from Eq. (A2). (d) $Z_1^\perp/L$ at relatively low frequency by Eq. (B1). An aluminum pipe with radius $b = 0.05$ m is assumed.

$$\lim_{\omega \rightarrow 0} \frac{Z_1^\perp}{L} = -\frac{1}{4\pi\epsilon_0} \frac{4}{b^2 c} i, \quad (\text{B5})$$

while the outcome of $Z_1^\perp/L$ from Eq. (B1) is

$$\lim_{\omega \rightarrow 0} \frac{Z_1^\perp}{L} = -\frac{1}{4\pi\epsilon_0} \frac{2}{b^2 c} i. \quad (\text{B6})$$

Since we did not use the approximation $|\omega| \gg (2\epsilon_0 c^2)/(b^2 \sigma)$ when solving the ring model, the monopole ring results from Eqs. (46) to (49), and the dipole ring results from Eqs. (63) to (68), have the same behavior as the result from Eq. (B1) at both high and low frequencies, when taking the limit $\gamma \rightarrow \infty$.

Next, we will give the rigorous solution. Take Eq. (2.5) as an example, the equation that $\tilde{E}_s$ satisfies is a Bessel function.

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{E}_s}{\partial r} \right) + \lambda^2 \tilde{E}_s = 0. \quad (\text{B7})$$

We can use Hankel function to form the general solution of Eq. (B7), which is

$$\tilde{E}_s = CH_0^{(1)}(\lambda r) + DH_0^{(2)}(\lambda r) \quad (\text{B8})$$

where the Hankel function of the first kind represents the radiating wave, while the Hankel function of the second kind represents the focused wave. There should only be a radiating wave because the metal pipe has no boundaries to reflect the wave. So, we drop the $H_0^{(2)}(\lambda r)$ term. Next, we will follow the same process as described in the textbook [7] to determine the coefficients based on the continuity conditions of the electromagnetic field.

As for Eq. (2.36), combined with Eq. (2.34), we solve the Bessel function of $\tilde{E}_s, \tilde{B}_s$

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{E}_s}{\partial r} \right) + \left(\lambda^2 - \frac{m^2}{r^2} \right) \tilde{E}_s &= 0, \\ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{B}_s}{\partial r} \right) + \left(\lambda^2 - \frac{m^2}{r^2} \right) \tilde{B}_s &= 0. \end{aligned} \quad (\text{B9})$$

And we get

$$\tilde{E}_s = -\tilde{B}_s = CH_m^{(1)}(\lambda r). \quad (\text{B10})$$

Next, we can proceed with solving for the unknown coefficients.

The process described above helps us obtain the complete expression for the impedance, denoted as Eq. (B1).

APPENDIX C: NUMERICAL SKILL

We integrate using Eq. (18), which is a somewhat complex expression. The computer program will quickly alert us to a numerical overflow warning if we attempt to perform the numerical integration. Further investigation will reveal that it is an overflow of the 0/0 kind, occurring in the second term of Eq. (18). To help illustrate this, we will use a numerical example. For an electron with an energy of 600 MeV, if we set the radius of the cylindrical pipe to $b = 0.01$ m and the material as aluminum, when the frequency approaches 10^{15} Hz, we will have

$$k_r \sim 2841 \text{ m}^{-1}, \quad \lambda \sim 1.5 \times 10^8 (1 - i) \text{ m}^{-1}.$$

Consequently, compared to modified Bessel functions with an argument of bk_r , the modified Bessel functions with $b\lambda$ in the expression will display the anomalous feature faster, i.e., when frequency gets closer to infinity, the terms $K_0(b\lambda)$ and $K_1(b\lambda)$ in the fractional expression of Eq. (18) will become very small. This results in the

fraction taking the form of 0/0, which raises computational problems. Nevertheless, since $K_\nu(z)$ satisfies, as $|z| \rightarrow +\infty$,

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} \exp^{-z} \left[1 + \sum_{n=1}^{\infty} \frac{(\nu, n)}{(2z)^n} \right], \quad |\arg z| < \frac{3\pi}{2}, \quad (\text{C1})$$

where

$$(\nu, n) = \frac{\Gamma\left(\frac{1}{2} + \nu + n\right)}{n! \Gamma\left(\frac{1}{2} + \nu - n\right)}, \quad (\text{C2})$$

we are able to derive an asymptotic expression, Eq. (C3), for the electric field at higher frequency by using the first term of Eq. (C1) as an approximation, which is independent of ν . With Eq. (C3), we can avoid the overflow caused by the argument $b\lambda$.

$$\tilde{E}_s(\omega, r) \approx -Qv \frac{iZ_0 c k_r^2}{2\pi\omega} \left[K_0(k_r r) + I_0(k_r r) \frac{\omega^2 \lambda K_1(bk_r) + k_r c^2 (\lambda^2 - k^2) K_0(bk_r)}{\omega^2 \lambda I_1(bk_r) - k_r c^2 (\lambda^2 - k^2) I_0(bk_r)} \right], \quad (\text{C3})$$

$$\tilde{E}_s(\omega, r) \approx -Qv \frac{iZ_0 c k_r^2}{2\pi\omega} \left[K_0(k_r r) + e^{k_r(r-b)} \frac{\omega^2 \lambda K_1(bk_r) + k_r c^2 (\lambda^2 - k^2) K_0(bk_r)}{\omega^2 \lambda - k_r c^2 (\lambda^2 - k^2)} \right]. \quad (\text{C4})$$

As the frequency continues to increase, $I_0(k_r r)$ in the numerator, along with $I_1(bk_r)$ and $I_0(bk_r)$ in the denominator, of the asymptotic expression, Eq. (C3), starts to become significantly large, causing the fraction to become a form of ∞/∞ . This again leads to computational overflow issues as it exceeds the calculation precision. We notice once more that $I_\nu(z)$ satisfies Eqs. (C5) and (C6), when $|z| \rightarrow +\infty$. Then we have the high frequency

approximation, Eq. (C4), by taking the first term as approximation similarly. And because $r < b$, we can find that $\tilde{E}_s(\omega, r)$ approaches to zero $\tilde{E}_s(\omega, r) \rightarrow 0$ when $\omega \rightarrow \infty$.

With these high frequency approximations, we can numerically calculate $\tilde{E}_s(\omega, r)$ at any frequency. Figure 3 shows plots of the monopole results for a 600 MeV electron in a pipe of radius 0.01 m at different transverse positions.

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{(-)^n (\nu, n)}{(2z)^n} + \frac{e^{-z+(\nu+\frac{1}{2})\pi i}}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{(\nu, n)}{(2z)^n}, \quad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}, \quad (\text{C5})$$

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{(-)^n (\nu, n)}{(2z)^n} + \frac{e^{-z-(\nu+\frac{1}{2})\pi i}}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{(\nu, n)}{(2z)^n}, \quad -\frac{3\pi}{2} < \arg z < \frac{\pi}{2}. \quad (\text{C6})$$

$$\begin{aligned} \tilde{E}_s(0^+, r) &= \lim_{\omega \rightarrow 0^+} -Qv \frac{iZ_0 c k_r^2}{w\pi\omega} \left\{ \omega \left[-\ln \left(\frac{r}{2c\beta\gamma} \omega \right) \right] \frac{\omega^2 \lambda \frac{c\beta\gamma}{b\omega} \left[-\ln \left(\frac{b\lambda}{2} \right) \right] + k_r c^2 (\lambda^2 - k^2) \left[-\ln \left(\frac{bk_r}{2} \right) \right] \frac{1}{\lambda b}}{\omega^2 \lambda \frac{bk_r}{2} \left[-\ln \left(\frac{b\lambda}{2} \right) \right] - k_r c^2 (\lambda^2 - k^2) \frac{1}{\lambda b}} \right\}, \\ &= \lim_{\omega \rightarrow 0^+} -Qv \frac{iZ_0}{2\pi c \beta^2 \gamma^2} \left[\omega + \omega \frac{\omega \lambda \frac{c\beta\gamma}{b} + k_r c^2 (\lambda^2 - k^2) \frac{1}{\lambda b}}{\omega^2 \lambda \frac{bk_r}{2} - k_r c^2 (\lambda^2 - k^2) \frac{1}{\lambda b}} \right] = \lim_{\omega \rightarrow 0^+} -Qv \frac{iZ_0 \omega}{2\pi c} = 0. \end{aligned} \quad (\text{C7})$$

For the sake of mathematical completeness, we should note that the modified Bessel function has a branch point that is zero when we compute the numerical integration. It is important for us to understand whether there are any issues when frequency ω gets close to zero, even though Fig. 3 looks fine. When $\omega \rightarrow 0$ from the real axis, we must examine the limit. We see that for the input $z \ll 0$, the modified Bessel function has the asymptotic features,

$$I_\nu(z) = \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu, \quad (\text{C8})$$

$$K_\nu(z) = \begin{cases} -[\ln(\frac{z}{2}) + 0.5772 + \dots], & \nu = 0 \\ \frac{\Gamma(\nu)}{2} \left(\frac{z}{2}\right)^\nu, & \nu \neq 0 \end{cases}. \quad (\text{C9})$$

The limit of $\tilde{E}_s(\omega, r)$ satisfies Eq. (C7). When we derive Eq. (C7), we use L'Hospital's rule. Similarly, we can obtain

$$\tilde{E}_s(0^-, r) = \lim_{\omega \rightarrow 0^-} Qv \frac{iZ_0\omega}{2\pi c} = 0. \quad (\text{C10})$$

Therefore, when numerically integrating along the real axis, the integrand should not have any singular points, it is unnecessary to densely interpolate points near some positions to capture its behavior.

By counting the number of points that need to be calculated, we can determine the computational complexity of numerical integration using the trapezoidal rule.

Take the upper right subplot of Fig. 3, for example, the function can be truncated at -10^{15} and 10^{15} Hz. We can evaluate the integral between the truncated points instead of full axis. If we have 2×10^6 points in total, the interval between the points should be 10^9 Hz. When we set $z \sim 1 \times 10^{-4}$ m, the oscillation component $e^{i\omega z/v}$ of Eq. (36) has the period as

$$\frac{2\pi v}{z} \sim \frac{6 \times 3 \times 10^8}{1 \times 10^{-4}} \text{ Hz} \sim 2 \times 10^{13} \text{ Hz}. \quad (\text{C11})$$

Because Fig. 3 is on the log scale, the variation of the function awaiting inverse Fourier transform is slow. Combined $e^{i\omega z/v}$ together, we may claim that there are roughly 10^3 points within a period of significant variation in function values. Thus, the precision of the numerical calculation is on the order of 1/1000.

APPENDIX D: BOUNDARY CONDITION OF RING MODEL

The source boundary and the wall boundary are the two boundaries shared by the monopole and dipole models. Each boundary can determine half of the coefficients. For a monopole, there are a total of four undetermined coefficients, whereas for a dipole, there are eight undetermined coefficients.

1. $r=a$ source boundary

There are two methods to give constraints on the source boundary. One method is to determine the coefficient based on the continuity condition of the electromagnetic field. Next, we will proceed with solving the dipole as a demonstration.

$$(\tilde{\mathbf{B}}_2 - \tilde{\mathbf{B}}_1) \cdot \hat{\mathbf{n}} = 0, \quad (\text{D1})$$

$$\hat{\mathbf{n}} \times (\tilde{\mathbf{H}}_2 - \tilde{\mathbf{H}}_1) = \tilde{\mathbf{K}}, \quad (\text{D2})$$

$$(\tilde{\mathbf{D}}_2 - \tilde{\mathbf{D}}_1) \cdot \hat{\mathbf{n}} = \tilde{\sigma}, \quad (\text{D3})$$

$$\tilde{\mathbf{E}}_2 - \tilde{\mathbf{E}}_1 \times \hat{\mathbf{n}} = \mathbf{0}. \quad (\text{D4})$$

Take B_r , for example,

$$\begin{aligned} B_r &= \left(-\frac{1}{r} \tilde{A}_s - ik \tilde{A}_\phi \right) \\ &\quad - \frac{1}{a} p_s^c I_1(k_r a) - ik \frac{1}{2} p_+^c [I_2(k_r a) + I_0(k_r a)] \\ &= -\frac{1}{a} [p_s I_1(k_r a) + q_s K_1(k_r a)] \\ &\quad - ik \frac{1}{2} \{ p_+ [I_2(k_r a) + I_0(k_r a)] \\ &\quad + q_+ [K_2(k_r a) + K_0(k_r a)] \}. \end{aligned} \quad (\text{D5})$$

We have the linear equation between $p_s^c, p_+^c, p_s, q_s, p_+, q_+$.

Another way is to directly obtain the coefficient relationships through the integration of equations involving the δ functions. Take dipole, for example, that function is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{\phi}}{\partial r} \right) - \left(\frac{1}{r^2} + k_r^2 \right) \tilde{\phi} = -\frac{\rho}{\epsilon_0} = \frac{Q_1}{\epsilon_0 a} \delta(r-a), \quad (\text{D6})$$

$$\tilde{\phi}(a^-) = \frac{c^2 k}{\omega} p_s^c I_1(k_r a),$$

$$\tilde{\phi}(a^+) = \frac{c^2 k}{\omega} [p_s I_1(k_r a) + q_s K_1(k_r a)]. \quad (\text{D7})$$

We integrate Eq. (D6) on both sides, then we get

$$\begin{aligned} &\int_{a^-}^{a^+} \frac{\partial}{\partial r} \left(r \frac{\partial \tilde{\phi}}{\partial r} \right) dr - \int_{a^-}^{a^+} \left(\frac{1}{r^2} + k_r^2 \right) r \tilde{\phi} dr \\ &= \int_{a^-}^{a^+} \frac{Q_1}{\epsilon_0 a} r \delta(r-a) dr. \end{aligned} \quad (\text{D8})$$

Then,

$$r \frac{\partial \tilde{\phi}}{\partial r} \Big|_{a^-}^{a^+} = \frac{Q_1}{\epsilon_0}. \quad (\text{D9})$$

So, we can get the relation between p_s, p_s^c, q_s ,

$$\begin{aligned} & p_s \left\{ \frac{1}{2} a k_r [I_0(k_r a) + I_2(k_r a)] \right\} \\ & + p_s^c \left\{ -\frac{1}{2} a k_r [I_0(k_r a) + I_2(k_r a)] \right\} \\ & + q_s \left\{ -\frac{1}{2} a k_r [K_0(k_r a) + K_2(k_r a)] \right\} = \frac{\mu_0 Q_1 \omega}{k}. \end{aligned} \quad (\text{D10})$$

Likewise, we get the relation between p_+^c, p_+, q_+ ,

$$\begin{aligned} & r \frac{\partial \tilde{A}_r}{\partial r} \Big|_{a^-}^{a^+} = 0, \\ & p_+^c I_2(k_r a) + p_+ [-I_2(k_r a)] - q_+ K_2(k_r a) = 0. \end{aligned} \quad (\text{D11})$$

And the relation between p_+^c, p_+, q_+ ,

$$\begin{aligned} & r \frac{\partial \tilde{A}_\varphi}{\partial r} \Big|_{a^-}^{a^+} = 0, \\ & p_+^c [I_2(k_r a) - k_r a I_1(k_r a)] \\ & + p_+ [k_r a I_1(k_r a) - I_2(k_r a)] \\ & + q_+ [-k_r a K_1(k_r a) - K_2(k_r a)] = 0. \end{aligned} \quad (\text{D12})$$

Totally, we have four independent linear equations for the coefficients of the dipole case.

As for monopole case, we have the following relation, for instance,

$$\begin{aligned} E_s &= (-ik\tilde{\phi} + i\omega\tilde{A}_s) \\ &\times \left[-i\omega \frac{k_r^2}{k_0^2} I_0(k_r a) \right] p_s + \left[-i\omega \frac{k_r^2}{k_0^2} K_0(k_r a) \right] q_s \\ &= \left[-i\omega \frac{k_r^2}{k_0^2} I_0(k_r a) \right] p_s^c. \end{aligned} \quad (\text{D13})$$

We have the linear equation of p_s, q_s, p_s^c . And the relation between p_s, q_s, p_s^c comes from Eq. (D9),

$$\begin{aligned} & [a k_r I_1(k_r a)] p_s + [-a k_r K_1(k_r a)] q_s \\ & - [a k_r I_1(k_r a)] p_s^c = \frac{\mu_0 Q_0 \omega}{k}. \end{aligned} \quad (\text{D14})$$

We have a total of two independent linear equations for monopole coefficients.

2. $r=b$ wall boundary

we can give the equations by Eqs. (D1), (D3), and (D4), and mention that Eq. (D2) will be changed to Eq. (D15),

$$\mathbf{n} \times (\tilde{\mathbf{H}}_2 - \tilde{\mathbf{H}}_1) = \mathbf{0}, \quad (\text{D15})$$

because there is no sufficient current in the conductor.

In the end, we have another four independent equations for dipole and another two for monopole.

APPENDIX E: SPACE CHARGE FOURIER TRANSFORM COUNTERPARTS VERIFICATION

Take the electric field, for example, in the longitudinal component, we have

$$E_z(\theta, r, z) = \frac{1}{4\pi\epsilon_0} \frac{Q \left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \left(\frac{v^2}{c^2}\right) \sin^2 \theta\right)^{\frac{3}{2}}} \left(\frac{\sin \theta}{r}\right)^2 \cos \theta. \quad (\text{E1})$$

To verify the result at different energy, we want to pick up the most special point in every situation. We choose the position that is related to the maximum electric field. We have θ_0 when E_z reaches its maximum

$$\theta_0 = \frac{1}{2} \arccos \left(\frac{1 + \beta^2}{\beta^2 - 3} \right). \quad (\text{E2})$$

So, the z_0 when E_z reaches its maximum is

$$z_0 = \frac{r}{\tan \theta_0}.$$

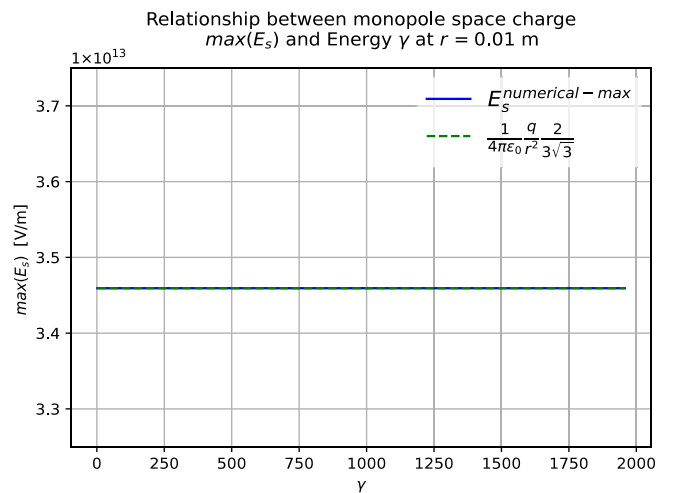


FIG. 20. Maximum longitudinal electric field at different energies. Blue curve is the result calculated by numerical inverse Fourier transform of Eq. (87), while green curve is from analytical results of Eq. (E3).

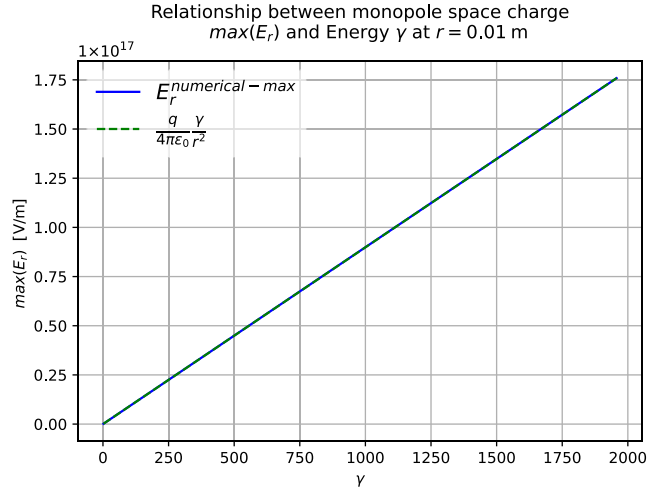


FIG. 21. Maximum transverse electric field at different energies. Blue curve is the result calculated by numerical inverse Fourier transform of Eq. (87), while green curve is from analytical results of Eq. (E5).

Maximum E_z equals

$$\max [E_z(z)] = \frac{1}{4\pi\epsilon_0} \frac{2Q}{3\sqrt{3}r^2}. \quad (\text{E3})$$

From Fig. 20, we see that the two approaches agree well. Quantitatively, $\max [E_z(z)]$ calculated by Eq. (E3) is 3.4593068×10^{13} V/m and numerical integration gives 3.4593102×10^{13} V/m.

Similarly, we have the transverse electric field,

$$E_r(\theta, r, z) = \frac{1}{4\pi\epsilon_0} \frac{Q \left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \left(\frac{v^2}{c^2}\right) \sin^2 \theta\right)^{\frac{3}{2}}} \left(\frac{\sin \theta}{r}\right)^2 \sin \theta. \quad (\text{E4})$$

Still, we choose the maximum point as the most significant point at different energies. When $\theta_0 = 0$, i.e., $z_0 = 0$, E_r reaches its maximum,

$$\max [E_r(z)] = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \gamma. \quad (\text{E5})$$

The plot has good linear properties with respect to energy γ , as Fig. 21 illustrates. Quantitatively, we find that $\max [E_r(z)]$ when $E = 500$ MeV calculated by Eq. (E5) is $8.79408198 \times 10^{13}$ V/m and numerical integration gives $8.79408704 \times 10^{13}$ V/m.

APPENDIX F: MONOPOLE SEPARATION METHOD

In this appendix, we are repeating the calculation steps as we did in Sec. VI C 2 for the monopole case. First, we calculate

the space charge of a monopole ring in the frequency domain. Next, we directly calculate the result in the time domain by Lorentz transform. Through the above calculations, we have obtained the ingredients needed to perform the faster numerical algorithm.

1. Frequency domain

Due to the vacuum limitation, the ring monopole divides the area into two sections.

$$\rho_0 = -\frac{Q_0}{a} \delta(s - vt) \delta(r - a). \quad (\text{F1})$$

We have

$$r < a,$$

$$A_r = 0,$$

$$A_\phi = 0,$$

$$A_s = p_{s-\text{free}}^c I_0(k_r r) e^{ikz},$$

$$\phi = \frac{c^2 k}{\omega} p_{s-\text{free}}^c I_0(k_r r) e^{ikz}, \quad (\text{F2})$$

and

$$r > a,$$

$$A_r = 0,$$

$$A_\phi = 0,$$

$$A_s = q_{s-\text{free}} K_0(k_r r) e^{ikz},$$

$$\phi = \frac{c^2 k}{\omega} q_{s-\text{free}} K_0(k_r r) e^{ikz}, \quad (\text{F3})$$

where

$$p_{s-\text{free}}^c = -\frac{\mu_0 Q_0 \omega K_0(k_r a)}{k},$$

$$q_{s-\text{free}} = -\frac{\mu_0 Q_0 \omega I_0(k_r a)}{k}. \quad (\text{F4})$$

So, the residual resistive wall should have the form

$$r < a,$$

$$A_r = 0,$$

$$A_\phi = 0,$$

$$A_s = (p_s^c - p_{s-\text{free}}^c) I_0(k_r r) e^{ikz},$$

$$\phi = \frac{c^2 k}{\omega} (p_s^c - p_{s-\text{free}}^c) I_0(k_r r) e^{ikz}. \quad (\text{F5})$$

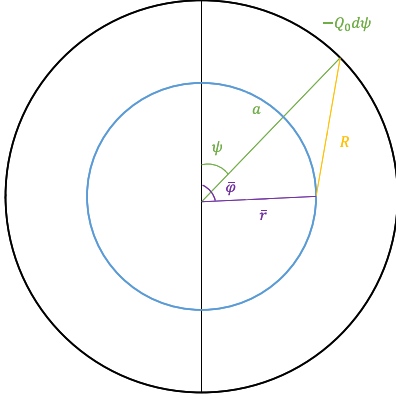


FIG. 22. Relevant parameters of the monopole in the rest frame. The black ring is the monopole ring. We will derive the electric field at $(\bar{r}, \bar{\varphi}, \bar{z})$, which can be projected onto the blue ring along z axis.

Then

$$\begin{aligned}
 a &< r < b, \\
 A_r &= 0, \\
 A_\varphi &= 0, \\
 A_s &= [p_s I_0(k_r r) + (q_s - q_{s\text{-free}}) K_0(k_r r)] e^{ikz}, \\
 \phi &= \frac{c^2 k}{\omega} [p_s I_0(k_r r) + (q_s - q_{s\text{-free}}) K_0(k_r r)] e^{ikz}. \quad (\text{F6})
 \end{aligned}$$

2. Time domain

As previously, we need to solve the space charge of the ring model in the time domain after extracting the space charge component in the frequency domain.

Similarly, we study this problem through Lorentz transform. The settings of relevant parameters are shown in Fig. 22. We set the source term as

$$q = -Q_0 d\psi. \quad (\text{F7})$$

According to cosine theorem, the potential $\Phi(\bar{r}, \bar{\varphi}, \bar{z})$ in the rest frame can be expressed by integration

$$\begin{aligned}
 \bar{\Phi}(\bar{r}, \bar{\varphi}, \bar{z}) &= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \frac{q}{R} d\psi \\
 &= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \frac{-Q_0 d\psi}{(\bar{r}^2 + a^2 - 2\bar{r}a \cos(\bar{\varphi} - \psi) + \bar{z}^2)^{\frac{1}{2}}}. \quad (\text{F8})
 \end{aligned}$$

Further, by letting $\theta = \bar{\varphi} - \psi$, we have

$$\begin{aligned}
 \bar{\Phi}(\bar{r}, \bar{\varphi}, \bar{z}) &= \int_{\bar{\varphi}-2\pi}^{\bar{\varphi}} \frac{1}{4\pi\epsilon_0} \frac{-Q_0 d\theta}{(\bar{r}^2 + a^2 - 2\bar{r}a \cos\theta + \bar{z}^2)^{\frac{1}{2}}} \\
 &= -\frac{Q_0}{2\pi\epsilon_0} \left(\frac{K\left(-\frac{4a\bar{r}}{(a-\bar{r})^2 + \bar{z}^2}\right)}{\sqrt{(a-\bar{r})^2 + \bar{z}^2}} + \frac{K\left(\frac{4a\bar{r}}{(a+\bar{r})^2 + \bar{z}^2}\right)}{\sqrt{(a+\bar{r})^2 + \bar{z}^2}} \right), \quad (\text{F9})
 \end{aligned}$$

where $K(\bar{x})$ are the first kind complete elliptic integrals.

The derived electric field from the potential is

$$\bar{E}(\bar{r}, \bar{\varphi}, \bar{z}) = (\bar{E}_r, \bar{E}_\varphi, \bar{E}_s) = \left(-\frac{\partial \bar{\Phi}}{\partial \bar{r}}, 0, -\frac{\partial \bar{\Phi}}{\partial \bar{z}} \right). \quad (\text{F10})$$

The Lorentz transform in the cylindrical coordinates of this case is

$$\begin{aligned}
 E_r &= \gamma \bar{E}_r, \quad E_\varphi = 0, \quad E_s = \bar{E}_s, \\
 B_r &= 0, \quad B_\varphi = \frac{1}{c^2} v \gamma \bar{E}_r, \quad B_s = 0. \quad (\text{F11})
 \end{aligned}$$

Because of the coordinate transformation,

$$\bar{r} = r, \quad \bar{\varphi} = \varphi, \quad \bar{z} = \gamma z,$$

the electromagnetic field in the laboratory frame can be expressed as

$$\begin{aligned}
 B_\varphi &= \frac{1}{c^2} v E_r = -\frac{1}{c^2} v \gamma \frac{Q_0}{2\pi\epsilon_0} \\
 &\times \frac{N_{F_r}}{2r[(a-r)^2] + (\gamma z)^2[(a+r)^2 + (\gamma z)^2]}, \quad (\text{F12})
 \end{aligned}$$

$$\begin{aligned}
 N_{F_r} &= (r^2 - a^2 - (\gamma z)^2) \sqrt{(a-r)^2 + (\gamma z)^2} \\
 &\times E \left(-\frac{4ar}{(a-r)^2 + (\gamma z)^2} \right) + (r^2 - a^2 - (\gamma z)^2) \\
 &\times \sqrt{(a+r)^2 + (\gamma z)^2} E \left(\frac{4ar}{(a+r)^2 + (\gamma z)^2} \right) \\
 &+ \sqrt{(a-r)^2 + (\gamma z)^2} [(a+r)^2 + (\gamma z)^2] \\
 &\times K \left(-\frac{4ar}{(a-r)^2 + (\gamma z)^2} \right) + [(a-r)^2 + (\gamma z)^2] \\
 &\times \sqrt{(a+r)^2 + (\gamma z)^2} K \left(\frac{4ar}{(a+r)^2 + (\gamma z)^2} \right),
 \end{aligned}$$

$$B_s = 0, \quad E_\varphi = 0, \quad B_r = 0, \quad (\text{F13})$$

$$E_s = -\frac{Q_0 \gamma z}{2\pi\epsilon_0} \left(\frac{\sqrt{(a-r)^2 + (\gamma z)^2} E\left(-\frac{4ar}{(a-r)^2 + (\gamma z)^2}\right)}{((a-r)^2 + (\gamma z)^2)((a+r)^2 + (\gamma z)^2)} + \frac{\sqrt{(a+r)^2 + (\gamma z)^2} E\left(\frac{4ar}{(a+r)^2 + (\gamma z)^2}\right)}{[(a-r)^2 + (\gamma z)^2][(a+r)^2 + (\gamma z)^2]} \right). \quad (\text{F14})$$

We provide the original integral forms for reference.

$$B_\varphi = \frac{1}{c^2} v E_r = -\frac{1}{c^2} v \frac{\gamma}{4\pi\epsilon_0} \times \int_0^{2\pi} \frac{Q_0(r - a \cos \theta) d\theta}{(a^2 + r^2 + (\gamma z)^2 - 2ar \cos \theta)^{\frac{3}{2}}}, \quad (\text{F15})$$

$$E_s = -\frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \frac{Q_0 \gamma z d\theta}{(a^2 + r^2 + (\gamma z)^2 - 2ar \cos \theta)^{\frac{3}{2}}}. \quad (\text{F16})$$

Again, if we are interested in the space charge between the bunches, under the condition $z \gg r \sim a$, we truncate the series from Eqs. (F12) to (F14) to the lowest order. Finally, we have the asymptotic form

$$E_s \approx -\frac{1}{4\pi\epsilon_0} 2\pi Q_0 \left(\frac{\text{sgn}(\gamma z)}{(\gamma z)^2} \right), \quad (\text{F17})$$

$$B_\varphi = \frac{1}{c^2} v E_r \approx -\frac{1}{c^2} v \gamma \frac{1}{4\pi\epsilon_0} 2\pi Q_0 \left(\frac{r}{|\gamma z|^3} \right). \quad (\text{F18})$$

These results are consistent with Eqs. (88) and (89) if we set

$$Q = -2\pi Q_0.$$

APPENDIX G: DERIVATION OF SPACE CHARGE IN RING MODEL

Here we give the derivation process from Eqs. (97) to (100), using the settings of the relevant parameters given by Fig. 10. According to cosine theorem, we have the length of R as

$$R^2 = \bar{r}^2 + a^2 + 2\bar{r}a \cos(\bar{\varphi} - \psi). \quad (\text{G1})$$

The Potential $\Phi(\bar{r}, \bar{\varphi}, \bar{z})$ in the rest frame can be expressed by integration

$$\begin{aligned} \bar{\Phi}(\bar{r}, \bar{\varphi}, \bar{z}) &= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0 R} q d\psi \\ &= \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \frac{-Q_1 \cos \psi d\psi}{(\bar{r}^2 + a^2 - 2\bar{r}a \cos(\bar{\varphi} - \psi) + \bar{z}^2)^{\frac{1}{2}}}. \end{aligned} \quad (\text{G2})$$

Further, by letting $\theta = \bar{\varphi} - \psi$, we have

$$\begin{aligned} \bar{\Phi}(\bar{r}, \bar{\varphi}, \bar{z}) &= \int_{\bar{\varphi}-2\pi}^{\bar{\varphi}} \frac{1}{4\pi\epsilon_0} \frac{-Q_1 \cos(\bar{\varphi} - \theta) d\theta}{(\bar{r}^2 + a^2 - 2\bar{r}a \cos \theta + \bar{z}^2)^{\frac{1}{2}}} \\ &= \int_{\bar{\varphi}-2\pi}^{\bar{\varphi}} \frac{1}{4\pi\epsilon_0} \frac{-Q_1 (\cos \bar{\varphi} \cos \theta + \sin \bar{\varphi} \sin \theta) d\theta}{(\bar{r}^2 + a^2 - 2\bar{r}a \cos \theta + \bar{z}^2)^{\frac{1}{2}}} \\ &= \left(\int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \frac{-Q_1 \cos \theta d\theta}{(\bar{r}^2 + a^2 - 2\bar{r}a \cos \theta + \bar{z}^2)^{\frac{1}{2}}} \right) \cos \bar{\varphi} \\ &= \frac{Q_1}{2\pi\epsilon_0} \left\{ \frac{[(a + \bar{r})^2 + \bar{z}^2] E(\bar{x})}{a\bar{r} \sqrt{(a + \bar{r})^2 + \bar{z}^2}} - \frac{(a^2 + \bar{r}^2 + \bar{z}^2) K(\bar{x})}{a\bar{r} \sqrt{(a + \bar{r})^2 + \bar{z}^2}} \right\} \cos \bar{\varphi}, \end{aligned} \quad (\text{G3})$$

where $E(\bar{x})$ and $K(\bar{x})$ are the second kind and first kind complete elliptic integrals, respectively. And

$$\bar{x} = \frac{4a\bar{r}}{(a + \bar{r})^2 + \bar{z}^2}. \quad (\text{G4})$$

The derived electric field from the potential is given by the following expression

$$\mathbf{E}(\bar{r}, \bar{\varphi}, \bar{z}) = (\bar{E}_r, \bar{E}_\varphi, \bar{E}_s) = \left(-\frac{\partial \bar{\Phi}}{\partial \bar{r}}, -\frac{1}{\bar{r}} \frac{\partial \bar{\Phi}}{\partial \bar{\varphi}}, -\frac{\partial \bar{\Phi}}{\partial \bar{z}} \right). \quad (\text{G5})$$

We explicitly give the Lorentz transform in the cylindrical coordinates of our case

$$\begin{aligned} E_r &= \gamma \bar{E}_r, \quad E_\varphi = \gamma \bar{E}_\varphi, \quad E_s = \bar{E}_s, \\ B_r &= -\frac{1}{c^2} v \gamma \bar{E}_\varphi, \quad B_\varphi = \frac{1}{c^2} v \gamma \bar{E}_r, \quad B_s = 0. \end{aligned} \quad (\text{G6})$$

Because of the coordinate transformation,

$$\bar{r} = r, \quad \bar{\varphi} = \varphi, \quad \bar{z} = \gamma z,$$

the electromagnetic field in the laboratory frame can be expressed by Eqs. (97) to (100).

Certainly, in addition to the closed-form solutions expressed by the complete elliptic integrals, we also present the results of the electromagnetic field expressed in integral form. As described in the main text, they are sometimes more useful and brief in analysis.

$$E_s = -\frac{1}{4\pi\epsilon_0} \times \left(\int_0^{2\pi} \frac{Q_1 \gamma z \cos \theta d\theta}{(r^2 + a^2 - 2ra \cos \theta + (\gamma z)^2)^{\frac{3}{2}}} \right) \cos \varphi, \quad (\text{G7})$$

$$B_r = -\frac{1}{c^2} v E_\varphi = -\frac{v}{c^2} \frac{\gamma}{4\pi\epsilon_0} \times \left(\int_0^{2\pi} \frac{Q_1 \cos \theta d\theta}{r(r^2 + a^2 - 2ra \cos \theta + (\gamma z)^2)^{\frac{3}{2}}} \right) \sin \varphi, \quad (\text{G8})$$

$$B_\varphi = \frac{1}{c^2} v E_r = -\frac{v}{c^2} \frac{\gamma}{4\pi\epsilon_0} \times \left(\int_0^{2\pi} \frac{Q_1 \cos \theta (r - a \cos \theta) d\theta}{(r^2 + a^2 - 2ra \cos \theta + (\gamma z)^2)^{\frac{3}{2}}} \right) \cos \varphi. \quad (\text{G9})$$

APPENDIX H: REDUCED RESISTIVITY METHOD

One may take the limitation, which is $\sigma \rightarrow +\infty$, to reduce the resistivity of the pipe wall. Physically, this model implies that the beam will travel through a pipe composed of perfect conductors at a nonultrarelativistic velocity. Moreover, obtaining the corresponding results through frequency domain analysis is not difficult either. We can perform the procedure, for example, take a point source of the monopole in the frequency domain,

$$\begin{aligned} \tilde{E}_{s,RC}(\omega, r) &= \lim_{\sigma \rightarrow +\infty} -Qv \frac{iZ_0 c k_r^2}{2\pi\omega} \left[K_0(k_r r) + I_0(k_r r) \frac{\omega^2 \lambda K_1(bk_r) K_0(b\lambda) + k_r c^2 (\lambda^2 - k^2) K_0(bk_r) K_1(\lambda b)}{\omega^2 \lambda I_1(bk_r) K_0(b\lambda) - k_r c^2 (\lambda^2 - k^2) I_0(bk_r) K_1(\lambda b)} \right] \\ &= \lim_{\sigma \rightarrow +\infty} -Qv \frac{iZ_0 c k_r^2}{2\pi\omega} \left[K_0(k_r r) + I_0(k_r r) \frac{\omega^2 \lambda K_1(bk_r) + k_r c^2 (\lambda^2 - k^2) K_0(bk_r)}{\omega^2 \lambda I_1(bk_r) - k_r c^2 (\lambda^2 - k^2) I_0(bk_r)} \right] \\ &= \lim_{\sigma \rightarrow +\infty} -Qv \frac{iZ_0 c k_r^2}{2\pi\omega} \left[K_0(k_r r) - I_0(k_r r) \frac{K_0(bk_r)}{I_0(bk_r)} \right]. \end{aligned} \quad (\text{H1})$$

However, there is a drawback when we deal with the time domain. Here we give the reason why we do not use the reduced resistivity method to construct our algorithm. If we want to handle perfect conductor boundaries in the time

domain, one effective approach is the method of image charges. Nevertheless, it is hard to get the image charges reflected by the cylindrical perfect conductor pipe. The Green's function inside a cylinder ($r' < b$) is [9]

$$G(\mathbf{r}, \mathbf{r}') = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} e^{im\phi - \phi'} \int_0^\infty \frac{I_m(k\rho_{<}) [I_m(kR) K_m(k\rho_{>}) - K_m(kR) I_m(k\rho_{>})]}{I_m(kR)} \cos[k(z - z')] dk, \quad (\text{H2})$$

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') d^3x'. \quad (\text{H3})$$

We cannot give the analytical result of this Green's function integration. The complexity of the numerical integration is still too high, which depends on the subtraction of the ring's size from the position r interested. The typical cutoff k for the function being integrated can be achieved to be about 10^9 m^{-1} by reducing the difference between the above two sizes. However, the period of the function remains unchanged whose typical value is $2\pi/z10^3 \text{ m}^{-1}$. To maintain the

precision of calculation, we must assure that there are enough interpolation points within one period.

On the other hand, when we want to use the Lorentz transform to solve for the electromagnetic field in the rest frame, the boundaries are moving with the opposite velocity in fact. Although our boundaries are infinitely long and uniform, there is still a risk involved, and it is important to be aware of this in the analysis.

Therefore, the reduced resistivity method is more complicated than the space charge extrapolation. And it is inevitable that we encounter complex integration processes. For this reason, we will not use this limitation for the optimization of the algorithm.

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