

Convexity of the longitudinal variation of third-order resonance driving terms and its application in dynamic aperture optimization

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The optimization of the dynamic aperture (DA) of a storage ring is typically a nonconvex problem with multiple local optima. Recent studies showed that reducing the variation of resonance driving terms (RDTs) along the longitudinal position improves DA very effectively, as the reduction in the longitudinal variation of lower-order RDTs suppresses higher-order nonlinear terms. Therefore, minimizing the longitudinal variation of third-order RDTs is crucial for DA optimization. In this paper, we prove that the longitudinal variation of third-order RDTs, quantified using their rms value $f_{3,\text{rms}}$ at sextupole locations, is a special convex function. In the space of sextupole strengths, the isosurfaces of $f_{3,\text{rms}}$ are a series of concentric and coaxial ellipsoidal surfaces, with the central position possessing minimum $f_{3,\text{rms}}$. The scanning results of a storage ring lattice show a strong consistency between the distributions of $f_{3,\text{rms}}$ and DA, indicating that the optimization of DA can be regarded as a roughly approximate convex optimization problem. Based on this, a fast DA optimization method based on particle tracking is developed, where a high-quality initial population for an intelligent algorithm is generated with a Gaussian distribution based on the geometric structure of $f_{3,\text{rms}}$.

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I. INTRODUCTION

In the design of a storage ring, dynamic aperture (DA) is a key performance metric that influences beam injection and lifetime. Within the parameter space defined by sextupole strengths, the DA typically exhibits a complex distribution characterized by multiple local optima, representing a typical nonconvex optimization problem. Furthermore, as the natural emittance decreases and the number of sextupole families increases, the optimization of DA becomes increasingly challenging.

The optimization methods for DA can be categorized into numerical and analytical approaches [1,2]. Numerical optimization methods directly optimize DA based on particle tracking using techniques such as the scanning method and intelligent optimization algorithms [3–8]. The numerical methods have the potential to find the global optimum. However, they require high computational costs and offer little physical insight to guide further design and optimization. In contrast, analytical optimization methods

have the advantage of providing physical guidance. Minimizing resonance driving terms (RDTs) [9,10] combined with frequency map analysis (FMA) [11,12] is a widely used analytical approach in which the DA is enlarged by reducing RDTs and controlling amplitude-dependent tune shifts (ADTs) [13–17].

Recently, it was demonstrated that reducing the variation of RDTs along the longitudinal position is much more effective in enlarging DA than reducing the commonly used one-turn RDTs [18,19]. The underlying physics was elucidated through a further analysis of the Baker-Campbell-Hausdorff (BCH) formula [20], which shows that reducing the longitudinal variation of lower-order RDTs effectively reduces higher-order RDTs and controls ADTs [18]. Therefore, minimizing the longitudinal variation of third-order RDTs is crucial for DA optimization.

In this paper, we will further prove that the longitudinal variation of third-order RDTs, quantified using their rms value at sextupole locations, is a special convex function. Specifically, its isosurfaces are a series of concentric and coaxial ellipsoidal surfaces in the sextupole strength space. Then, it will be shown, using lattice examples, that solutions with better DAs are distributed in the central region of the ellipsoid, even though the optimization of DA is generally a nonconvex optimization problem. Building on this convexity, a fast DA optimization method based on particle tracking will be developed.

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The remainder of this paper is organized as follows. The theory for RDTs and reducing their longitudinal variation is revisited in Sec. II, followed by a further demonstration using a simple FODO lattice. In Sec. III, we prove the convexity of the longitudinal variation of third-order RDTs. Using storage ring lattice examples, Sec. IV first demonstrates, via global scanning, the convexity of the third-order RDT variation and compares its distribution with that of DAs; then, a fast numerical method is developed for DA optimization based on this convexity. Finally, the conclusion and outlook are given in Sec. V.

II. CONTROL OF THE LONGITUDINAL VARIATION OF THIRD-ORDER RDTs

A. The longitudinal variation of third-order RDTs

The one-turn map $\mathcal{M}$ of a storage ring with N thin nonlinear kicks is given by [9]

$$\begin{aligned}\mathcal{M}_{0 \rightarrow N+1} &= \mathcal{A}_0^{-1} e^{\hat{v}_1} e^{\hat{v}_2} \cdots e^{\hat{v}_N} \mathcal{R}_{0 \rightarrow N+1} \mathcal{A}_{N+1} \\ &= \mathcal{A}_0^{-1} e^{h} \mathcal{R}_{0 \rightarrow N+1} \mathcal{A}_{N+1},\end{aligned}\quad (1)$$

where $\mathcal{A}$ is a normalizing map, $\mathcal{R}$ is a phase space rotation, and $e^{\hat{v}_n}$ and e^h are the Lie maps of the n -th nonlinear kick and the ring, respectively. Using the BCH formula, e^h can be expanded as

$$\begin{aligned}e^h &= \exp \left(: \sum_{i=1}^N \hat{v}_i + \frac{1}{2} \sum_{1 \leq i < j} \{ \hat{v}_i, \hat{v}_j \} \right. \\ &\quad + \frac{1}{12} \sum_{1 \leq i < j} \left[\{ \hat{v}_i, \{ \hat{v}_i, \hat{v}_j \} \} - \{ \hat{v}_j, \{ \hat{v}_i, \hat{v}_j \} \} \right] \\ &\quad \left. + \frac{1}{4} \sum_k \left\{ \sum_{1 \leq i < j} \{ \hat{v}_i, \hat{v}_j \}, \hat{v}_k \right\} + \cdots : \right),\end{aligned}\quad (2)$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket.

Now we define two quantities to represent the accumulation along the ring of the first two terms in Eq. (2):

$$A_t^{(1)} \equiv \sum_{i=1}^t \hat{v}_i, \quad (3)$$

$$\begin{aligned}A_t^{(2)} &\equiv \frac{1}{2} \sum_{1 \leq i < j} \{ \hat{v}_i, \hat{v}_j \} = \frac{1}{2} \sum_{j=2}^t \left\{ \sum_{i=1}^j \hat{v}_i, \hat{v}_j \right\} \\ &= \frac{1}{2} \sum_{j=2}^t \{ A_j^{(1)}, \hat{v}_j \}.\end{aligned}\quad (4)$$

It is seen that $A_t^{(2)}$ is constructed from $A_t^{(1)}$. Combining Eqs. (2), (3), and (4), it is known that higher-order terms are formed through the accumulation process of lower-order terms. Consequently, minimizing the variation of

lower-order terms along the ring leads to the reduction of higher-order terms [18,21,22]. Using the resonance basis, the terms in the expansion of e^h can be expressed in terms of RDTs [9,10]. Correspondingly, minimizing the longitudinal variation of third-order RDTs can effectively reduce higher-order RDTs and thus enlarge the DA.

In this paper, we only study the on-momentum DA. The geometric RDT $f_{jklm}(z)$ at an observation position z between the n -th and $(n+1)$ -th sextupoles is given by [10]

$$f_{jklm}(z) = \frac{\sum_{w=n+1}^{n+N} h_{w,jklm} e^{i[(j-k)\Delta\phi_{w,x}^{(z)} + (l-m)\Delta\phi_{w,y}^{(z)}]}}{1 - e^{2\pi i[(j-k)\nu_x + (l-m)\nu_y]}}, \quad (5)$$

where $h_{w,jklm}$ is a coefficient dependent on the strength of the w -th sextupole and the beta functions at the sextupole location, $\Delta\phi_{w,x}^{(z)}$ and $\Delta\phi_{w,y}^{(z)}$ are the horizontal and vertical phase advances between the w -th sextupole and the observation position z , and ν_x and ν_y are the horizontal and vertical tunes, respectively. In this paper, the longitudinal variation of RDTs is quantified using the rms value of their magnitudes at sextupole locations, as in Ref. [23]. For the longitudinal variation of third-order RDTs, it is calculated as:

$$\begin{aligned}f_{3,\text{rms}} &= \sqrt{\sum_{j,k,l,m \geq 0}^{j+k+l+m=3} f_{jklm,\text{rms}}^2}, \\ \text{with } f_{jklm,\text{rms}} &= \sqrt{\frac{1}{N} \sum_{i=1}^N |f_{jklm}(z_i)|^2},\end{aligned}\quad (6)$$

where $f_{jklm,\text{rms}}$ is the rms for a single third-order RDT and z_i is the position of the i -th sextupole.

B. Example: A FODO lattice

A simply designed FODO lattice, shown in Fig. 1, was used to illustrate the relationship between reducing the longitudinal variation of RDTs and improving the DA. In this lattice, both the quadrupoles and sextupoles employ a thin-lens model. We randomly generated 500 nonlinear solutions for study, with the horizontal and vertical chromaticities corrected to (1.5, 1.5).

For each nonlinear solution, the DA was tracked through FMA using the elegant code [24], obtaining the number of surviving particles and their frequency diffusion rates. The number of surviving particles, denoted as N_{particle} , represents the size of DA. The frequency diffusion rate, indicating the stability of particle motion, is computed as

$$d_r = \log_{10} \left(\sqrt{\Delta\nu_x^2 + \Delta\nu_y^2} / N_{\text{turn}} \right), \quad (7)$$

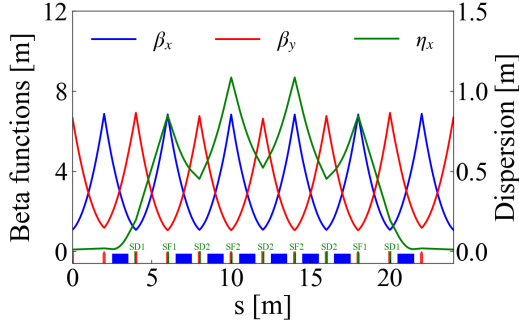


FIG. 1. Magnet layout and optical functions of a FODO lattice cell, one quarter of the simply designed ring with thin-lens quadrupoles and sextupoles. The four families of chromatic sextupoles indicated in the layout are placed at the same locations as quadrupoles.

where N_{turn} is the total number of tracking turns and $\Delta\nu_x$ and $\Delta\nu_y$ are the frequency differences in the x and y planes between the first and second half of tracking turns, respectively. To represent the quality of DA, we define $\bar{d}_r$, which is the average frequency diffusion rate of all surviving particles. Therefore, N_{particle} , together with $\bar{d}_r$, provides a comprehensive characterization of the DA.

Figure 2 illustrates the relationship between $f_{3,\text{rms}}$, N_{particle} , and $\bar{d}_r$ for the nonlinear solutions. It is clearly seen that (1) N_{particle} increases with the reduction of $f_{3,\text{rms}}$ and that (2) for a given N_{particle} , $\bar{d}_r$ decreases with the reduction of $f_{3,\text{rms}}$. This indicates that reducing $f_{3,\text{rms}}$ enlarges the DA and also enhances the stability of particle motion, thereby improving the overall DA performance. In the figure, the blue solutions within the red ellipse are an exception due to smaller ADTSs.

The relationship between the third-order one-turn RDTs, N_{particle} , and $\bar{d}_r$ for the nonlinear solutions is presented in Fig. 3. In the figure, the third-order one-turn RDTs are quantified by their rms value, denoted as $h_{3,\text{ring}}$ [18]. It is seen that (1) the relationship between $h_{3,\text{ring}}$ and N_{particle} is significantly weaker than that between $f_{3,\text{rms}}$ and N_{particle} and that (2) there is no clear relationship between

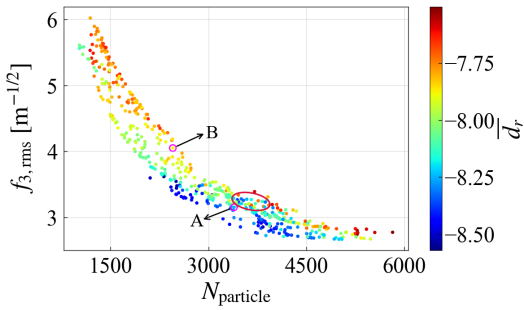


FIG. 2. Relationship between $f_{3,\text{rms}}$, N_{particle} , and $\bar{d}_r$ for the nonlinear solutions of the FODO lattice. The blue solutions within the red ellipse are influenced by the ADTS terms.

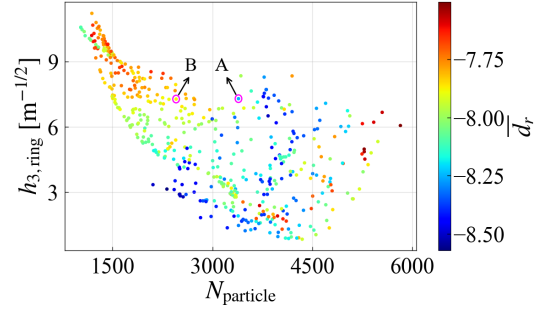


FIG. 3. Relationship between $h_{3,\text{ring}}$, N_{particle} , and $\bar{d}_r$ for the nonlinear solutions of the FODO lattice.

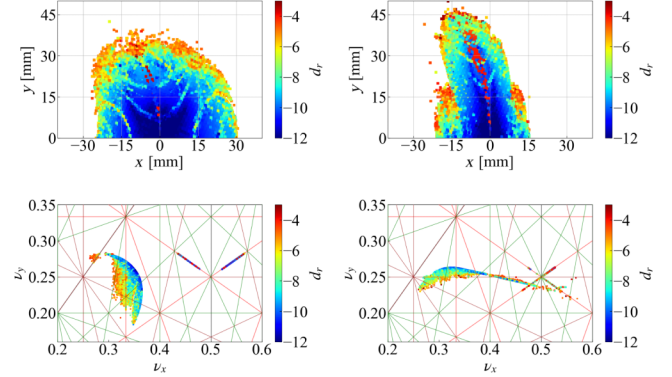


FIG. 4. DAs and frequency maps of two solutions with similar $h_{3,\text{ring}}$ but different $f_{3,\text{rms}}$, which are marked as A and B in Figs. 2 and 3. Left: Solution A with smaller $f_{3,\text{rms}}$; Right: Solution B with larger $f_{3,\text{rms}}$.

$h_{3,\text{ring}}$ and $\bar{d}_r$. We also see that some solutions with larger N_{particle} and lower $\bar{d}_r$ have relatively large $h_{3,\text{ring}}$ values. Actually, the $h_{3,\text{ring}}$ values of these solutions with better DAs are small when placed in a larger solution space. The reason we generated the nonlinear solutions within a relatively small variable space is that this greatly reduces the influence of ADTS differences among solutions on DA, making the analysis of the relationship between RDTs and DA cleaner.

Comparing Figs. 2 and 3 clearly shows that reducing the longitudinal variation of third-order RDTs is significantly more effective in improving the DA performance than reducing the commonly used one-turn RDTs. Figure 4 presents the DAs and frequency maps of two solutions with similar $h_{3,\text{ring}}$ values but different $f_{3,\text{rms}}$ values, which are marked in Figs. 2 and 3. It is clearly seen that the solution with smaller $f_{3,\text{rms}}$ has a larger DA with more stable-motion particles.

III. CONVEXITY OF THE LONGITUDINAL VARIATION OF THIRD-ORDER RDTs

As is well known, the DA optimization is generally a nonconvex optimization problem. However, as will be

proven, $f_{3,\text{rms}}$, quantifying the longitudinal variation of third-order RDTs, is a special convex function.

A. Definition of the convex function

An optimization problem can be formulated in the form [25]:

$$\begin{aligned} & \text{minimize} && f_0(\vec{x}), \\ & \text{subject to} && f_i(\vec{x}) \leq b_i, \quad i = 1, \dots, m. \end{aligned} \quad (8)$$

Here the vector $\vec{x} = (x_1, \dots, x_n)$ is the optimization variable; the functions $f_0(\vec{x})$ and $f_i(\vec{x})$ are the objective and constraint functions, respectively; and the constants $b_1, \dots, b_m$ are the bounds for the constraints. A convex optimization problem is one in which the objective and constraint functions are convex, which means that they satisfy

$$f_i(\alpha\vec{x} + \beta\vec{y}) \leq \alpha f_i(\vec{x}) + \beta f_i(\vec{y}), \quad (9)$$

for all $\vec{x}, \vec{y} \in \mathbf{R}^n$ and all $\alpha, \beta \in \mathbf{R}$ with $\alpha + \beta = 1$, $\alpha \geq 0, \beta \geq 0$ [25].

A function f is convex if and only if its domain is a convex set and its Hessian H or second derivative $\nabla^2 f$ is positive semidefinite, expressed as

$$H = \nabla^2 f(\vec{x}) \geq 0, \quad (10)$$

for all $\vec{x}$ in the domain [25]. As an example, Fig. 5 shows a strictly convex quadratic function. Its Hessian matrix is positive definite (i.e., $H > 0$), which means that it has a unique minimum point within its domain.

B. Proof of the convexity of $f_{3,\text{rms}}$

Now we will first prove that, without considering the constraint of chromaticity correction, $f_{3,\text{rms}}$ is a special convex function defined on the N -dimensional real space of sextupole strengths, characterized by ellipsoidal isosurfaces in the sextupole strength space. Then, when the

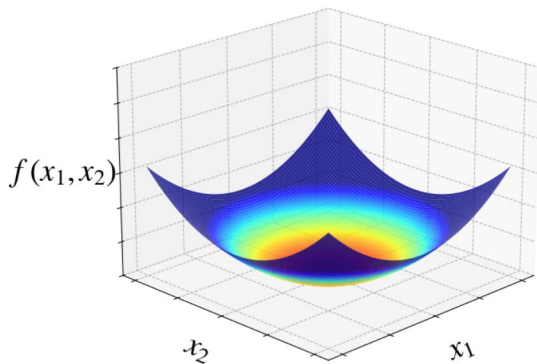


FIG. 5. A three-dimensional plot depicting a simple convex quadratic function $f(x_1, x_2) = x_1^2 + x_2^2$.

constraint of chromaticity correction is considered, $f_{3,\text{rms}}$ preserves convexity on its feasible set, but its isosurfaces become $(N - 2)$ -dimensional ellipsoidal surfaces.

For a third-order geometric RDT generated by normal sextupoles, the coefficient $h_{w,jklm}$ of $f_{jklm}(z)$ in Eq. (5) is given by [10]

$$h_{w,jklm} = -\frac{i^{l+m} K_w (\beta_{w,x})^{\frac{l+k}{2}} (\beta_{w,y})^{\frac{l+m}{2}}}{8 j! k! l! m!}, \quad (11)$$

where K_w is the normalized integrated strength of the w -th sextupole and $\beta_{w,x}$ and $\beta_{w,y}$ are the horizontal and vertical beta functions at the sextupole location, respectively. Combining Eqs. (5) and (11), it is known that $f_{jklm}(z)$ is a linear combination, with complex coefficients, of sextupole strengths. Then, $|f_{jklm}(z)|^2$ can be simplified into the following form:

$$\begin{aligned} |f_{jklm}(z)|^2 &= [\Re f_{jklm}(z)]^2 + [\Im f_{jklm}(z)]^2 \\ &= (A_1 K_1 + \dots + A_n K_n + \dots + A_N K_N)^2 \\ &\quad + (B_1 K_1 + \dots + B_n K_n + \dots + B_N K_N)^2 \\ &= C_{11} K_1^2 + \dots + C_{ij} K_i K_j + \dots + C_{NN} K_N^2, \end{aligned} \quad (12)$$

where A_n , B_n , and C_{ij} are real coefficients and $C_{ij} = C_{ji}$. Further combining Eqs. (6) and (12), $f_{3,\text{rms}}^2$ has the following form:

$$\begin{aligned} f_{3,\text{rms}}^2 &= \sum_{j,k,l,m \geq 0}^{j+k+l+m=3} \left(\frac{1}{N} \sum_{i=1}^N |f_{jklm}(z_i)|^2 \right) \\ &= D_{11} K_1^2 + \dots + D_{ij} K_i K_j + \dots + D_{NN} K_N^2 \\ &= (K_1, K_2, \dots, K_N) \begin{pmatrix} D_{11} & D_{12} & \dots & D_{1N} \\ D_{21} & D_{22} & \dots & D_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ D_{N1} & D_{N2} & \dots & D_{NN} \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_N \end{pmatrix} \\ &= \mathbf{K}^T \mathbf{D} \mathbf{K}, \end{aligned} \quad (13)$$

where $\mathbf{K}$ is an N -dimensional real vector of sextupole strengths without considering the constraint of chromaticity correction and $\mathbf{D}$ is a square matrix whose elements D_{ij} are real coefficients determined by the lattice functions.

Since $C_{ij} = C_{ji}$ in Eq. (12), it follows that $D_{ij} = D_{ji}$ in Eq. (13), which means that $\mathbf{D}$ is a real symmetric matrix. Since $|f_{jklm}(z)|$ is non-negative, as illustrated in Fig. 6 using a nonlinear solution of the FODO lattice as an example, the quadratic form $f_{3,\text{rms}}^2 = \mathbf{K}^T \mathbf{D} \mathbf{K}$ is positive for all nonzero $\mathbf{K}$. Hence, $\mathbf{D}$ is a positive definite matrix [26]. Consequently, $f_{3,\text{rms}}$ can be written as

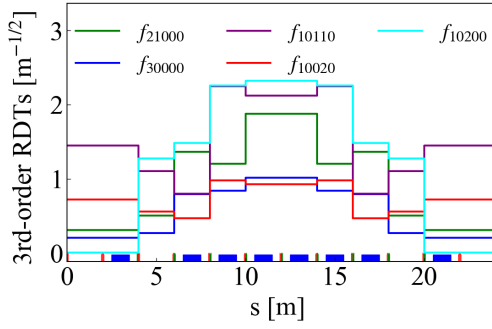


FIG. 6. Magnitude variation of third-order RDTs along the longitudinal position for a nonlinear solution of the FODO lattice.

$$f_{3,\text{rms}} = \sqrt{\mathbf{K}^T \mathbf{D} \mathbf{K}} = \|\mathbf{K}\|_{\mathbf{D}}, \quad (14)$$

where $\|\mathbf{K}\|_{\mathbf{D}}$ is a quadratic norm [25]. Every norm on an N -dimensional real space $\mathbb{R}^N$ is convex, and the unit ball of a quadratic norm is an ellipsoid [25]. Therefore, $f_{3,\text{rms}}$ is a convex function defined on $\mathbb{R}^N$, with its isosurfaces forming a series of concentric and coaxial ellipsoidal surfaces in the variable space.

In the lattice design, chromaticities are generally required to be corrected to small positive values. This requirement imposes constraints on the strengths of sextupoles, which satisfy the chromaticity correction equations in the horizontal and vertical planes:

$$\begin{cases} \Delta \xi_x &= \xi_x^{(1)} - \xi_x^{(0)} = \frac{1}{4\pi} \sum_{i=1}^N \beta_{i,x} \eta_{i,x} K_i \\ &= (E_1, \dots, E_i, \dots, E_N) \mathbf{K} = \mathbf{E}^T \mathbf{K}, \\ \Delta \xi_y &= \xi_y^{(1)} - \xi_y^{(0)} = -\frac{1}{4\pi} \sum_{i=1}^N \beta_{i,y} \eta_{i,x} K_i \\ &= (F_1, \dots, F_i, \dots, F_N) \mathbf{K} = \mathbf{F}^T \mathbf{K}. \end{cases} \quad (15)$$

Here, $\xi_x^{(0)}$ and $\xi_y^{(0)}$ are the horizontal and vertical natural chromaticities, respectively; $\xi_x^{(1)}$ and $\xi_y^{(1)}$ are the corrected chromaticities; and E_i and F_i are real coefficients determined by the dispersion $\eta_{i,x}$ and beta functions at the location of the i -th sextupole. Under the two linear equality constraints above, the feasible set of $f_{3,\text{rms}}$ is

$$\{\mathbf{K} \in \mathbb{R}^N | \Delta \xi_x = \mathbf{E}^T \mathbf{K}, \Delta \xi_y = \mathbf{F}^T \mathbf{K}\},$$

which is an $(N-2)$ -dimensional affine subspace in $\mathbb{R}^N$ and thus is a convex set [25]. Therefore, $f_{3,\text{rms}}$ preserves convexity on its feasible set, and its isosurfaces become $(N-2)$ -dimensional ellipsoidal surfaces.

By eliminating the two equality constraints, $f_{3,\text{rms}}$ can be expressed as a convex function defined on $\mathbb{R}^{N-2}$ [25]:

$$f_{3,\text{rms}} = \sqrt{\frac{1}{2} \mathcal{K}^T \mathcal{D} \mathcal{K} + \mathcal{G}^T \mathcal{K} + \mathcal{H}}, \quad (16)$$

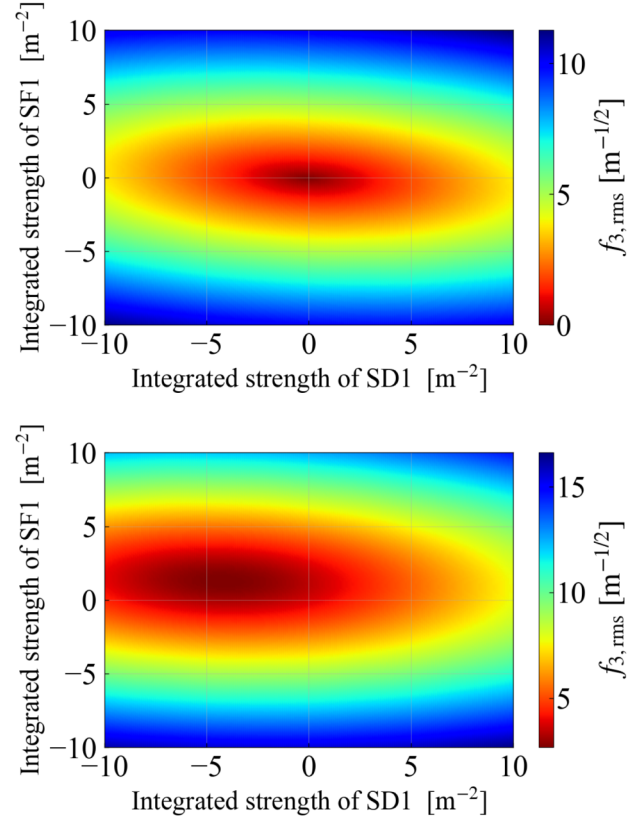


FIG. 7. Distributions of $f_{3,\text{rms}}$ in the normalized integrated strength space of the sextupole families SD1 and SF1 for the FODO lattice with (lower plot) and without (upper plot) the chromaticity correction constraint.

where $\mathcal{K}$ is a vector of dimension $N-2$, $\mathcal{D}$ is a positive definite matrix, $\mathcal{G}$ is a constant vector with the same dimension as $\mathcal{K}$, and $\mathcal{H}$ is a constant. For a given lattice, we can fit the parameters $\mathcal{D}$, $\mathcal{G}$, and $\mathcal{H}$ and then analytically obtain the solution with the minimum $f_{3,\text{rms}}$, i.e., the center of the ellipsoid. This allows us to generate a Gaussian distribution of $f_{3,\text{rms}}$ in the sextupole strength space, which will be used to develop a fast DA optimization method described in the next section. When the N sextupoles are grouped into M families, the analysis above remains applicable in the strength space of sextupole families.

Taking the FODO lattice as an example, we scanned the normalized integrated strengths of the sextupole families SD1 and SF1 to obtain their $f_{3,\text{rms}}$ values. Figure 7 shows the distributions of $f_{3,\text{rms}}$ in the strength space of SD1 and SF1 for both the cases with and without the chromaticity correction constraint. For the constrained case, the horizontal and vertical chromaticities were corrected to (1.5, 1.5) with the other two sextupole families, SD2 and SF2. While for the unconstrained case, SD2 and SF2 were not used. It is clearly seen from Fig. 7 that the contours of $f_{3,\text{rms}}$ in both cases are ellipses, in agreement with the theoretical analysis above. Compared to the unconstrained case, the center of the ellipses in the constrained case

described by Eq. (16) shifts from the origin, and their shapes also change.

In addition, $|f_{jklm}(z)|$, $f_{jklm,\text{rms}}$, and $h_{3,\text{ring}}$ have a similar mathematical form to that of $f_{3,\text{rms}}$, which implies that they are also convex functions with ellipsoidal iso-surfaces. However, we found that the longitudinal variation of fourth-order RDTs, quantified in the same form as Eq. (6), is not a convex function.

IV. APPLICATION

The convex function $f_{3,\text{rms}}$ plays a crucial role in the nonconvex DA optimization. Therefore, the optimization of DA can be regarded as a roughly approximate convex optimization problem, which will be demonstrated through numerical scanning of the HLS-III storage ring lattice. Furthermore, based on the convexity of $f_{3,\text{rms}}$, a fast numerical method for DA optimization will be developed and applied to the SSRF storage ring lattice and a multi-bend achromat (MBA) lattice.

A. Analysis of the HLS-III lattice

To compare the distributions of $f_{3,\text{rms}}$ and DA in the sextupole strength space, scanning of sextupole strengths was performed using the HLS-III storage ring lattice [27]. For each nonlinear solution, $f_{3,\text{rms}}$ was calculated, and DA was tracked through FMA. The total frequency diffusion rate of all surviving particles [28], denoted as $\sum d_r$, was used to quantitatively represent the DA, as it characterizes not only the size of DA but also the stability of particle motion. Compared to using the DA area as the objective in DA optimization, the solution optimized using $\sum d_r$ as the objective exhibits comparable DA size but more stable particle motion [28]. Figure 8 presents the linear optical functions and magnet layout of one lattice cell of the HLS-III storage ring. In order to reduce the computational cost of scanning, the original seven families of sextupoles were regrouped into fewer families.

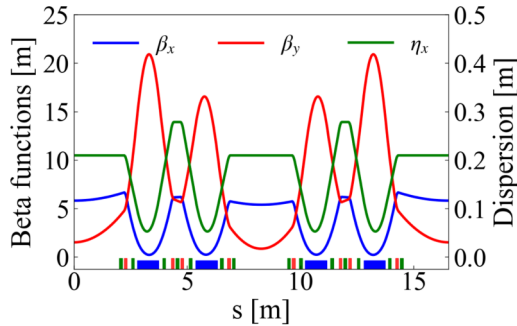


FIG. 8. Linear optical functions and magnet layout of one lattice period of the HLS-III storage ring. In the magnet layout, bends are represented in blue, quadrupoles in red, and sextupoles in green.

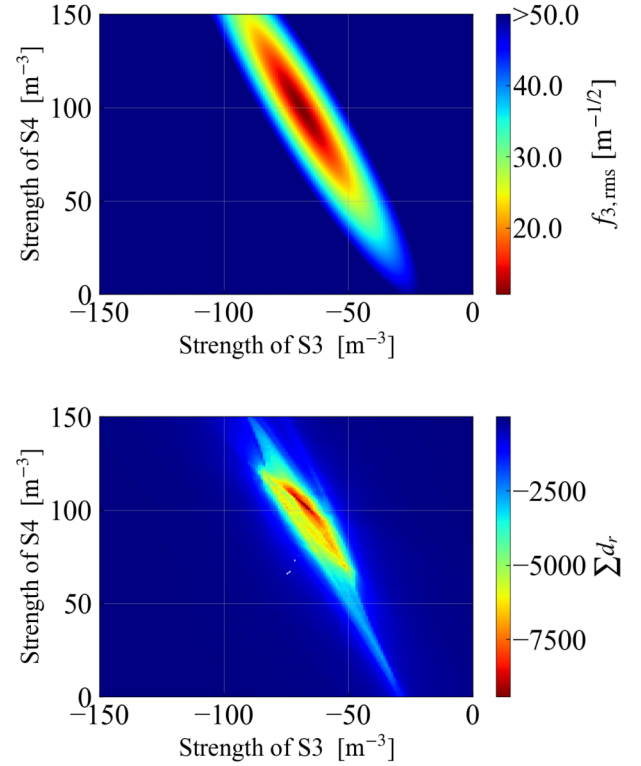


FIG. 9. Distributions of $f_{3,\text{rms}}$ (upper plot) and $\sum d_r$ (lower plot) in the normalized strength space of the sextupole families S3 and S4 for the HLS-III lattice with four families of sextupoles. In the upper plot, points with $f_{3,\text{rms}}$ values greater than 50 are shown in dark blue.

We firstly considered four families of sextupoles: two (S1, S2) located in the straight sections and two (S3, S4) in the arc sections. In the scanning, the sextupoles S3 and S4 served as free variables, with the horizontal and vertical chromaticities corrected to (1.5, 1.5) with the sextupoles S1 and S2. The normalized strengths of S3 and S4 were scanned over the ranges of $(-150, 0) \text{ m}^{-3}$ and $(0, 150) \text{ m}^{-3}$, respectively. Figure 9 shows the distributions of $f_{3,\text{rms}}$ and $\sum d_r$ in the strength space of the sextupoles S3 and S4. It is seen that the contours of $f_{3,\text{rms}}$ are a series of concentric and coaxial ellipses and that the distributions of $f_{3,\text{rms}}$ and $\sum d_r$ exhibit strong consistency. Specifically, regions with smaller $f_{3,\text{rms}}$ values broadly coincide with those with lower $\sum d_r$, and their variation trends are roughly the same.

Then, we studied the case with five sextupole families: three (S3, S4, S5) in the arc sections as free variables and two (S1, S2) in the straight sections for correcting the chromaticities to (1.5, 1.5). The normalized strengths were scanned over the ranges of $(-150, 0) \text{ m}^{-3}$ for S3 and S5 and $(0, 150) \text{ m}^{-3}$ for S4. Figure 10 shows the projected distributions of $f_{3,\text{rms}}$ and $\sum d_r$ in the strength spaces of S3 and S4, S3 and S5, and S4 and S5. It is seen that the consistency between the distributions of $f_{3,\text{rms}}$ and $\sum d_r$

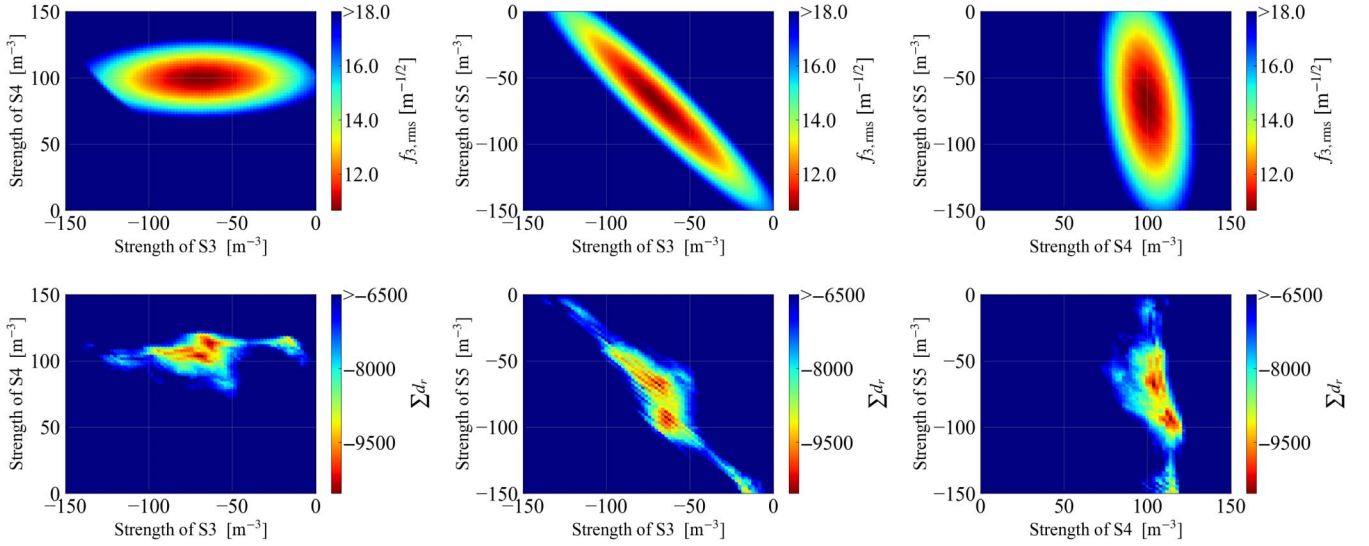


FIG. 10. Projected distributions of $f_{3,\text{rms}}$ and $\sum d_r$ in the normalized strength spaces of the sextupole families S3 and S4 (two left plots), S3 and S5 (two middle plots), and S4 and S5 (two right plots) for the HLS-III lattice with five families of sextupoles. The projection from the three-variable space (S_i, S_j, S_k) onto the two-variable space (S_i, S_j) is formed by taking, for each (S_i, S_j) , the point that has the smallest $f_{3,\text{rms}}$ or lowest $\sum d_r$ value in the S_k dimension. The incompleteness of the outer ellipses of $f_{3,\text{rms}}$, as shown in the upper left plot, is due to the limitation of the variable ranges. In the three upper plots, points with $f_{3,\text{rms}}$ values greater than 18 are shown in dark blue; in the three lower plots, points with $\sum d_r$ values higher than -6500 are shown in dark blue.

remains strong. Although $\sum d_r$ has two extreme regions (i.e., the red regions in the figure), they are not discrete “islands” but distributed on the same “continent,” which is located within the central region of the ellipsoid of $f_{3,\text{rms}}$. This differs from the common understanding that DA may have a distribution of multiple extreme islands.

The presence of the two extreme regions is mainly due to differences in the ADTS values of the nonlinear solutions, as DA is affected by both RDTs and ADTSs. Figure 11 shows the distribution of the three ADTS terms h_{2200} , h_{1111} , and h_{0022} for solutions with lower $\sum d_r$ values. The h_{1111} values of the solutions in one extreme region are all positive, whereas those in the other region are predominantly negative. The differences in ADTSs imply that different nonlinear solutions are affected by

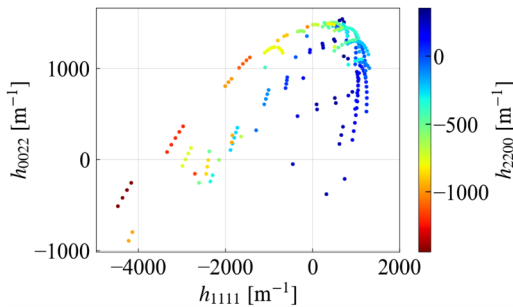


FIG. 11. Distribution of the ADTS terms h_{2200} , h_{1111} , and h_{0022} for the solutions with $\sum d_r$ lower than -8500 . For the two extreme regions of $\sum d_r$, the h_{1111} values are positive in one region and predominantly negative in the other.

different resonance lines, thereby complicating the DA distribution in the variable space. In Sec. II B, we used a FODO lattice as a study example and chose a relatively small solution space precisely in order to reduce the impact of ADTS differences among nonlinear solutions on DA.

Two nonlinear solutions from the two extreme regions were selected, with their sextupole strengths, $f_{3,\text{rms}}$, and $\sum d_r$ listed in Table I. Figure 12 presents the FMA of their DAs and the longitudinal variations of their third-order RDTs. While the sextupole strengths differ between the two solutions, particularly for S2 and S5, they have comparable $f_{3,\text{rms}}$ values and DA performance. Recently, a chaos suppression method was applied to optimize the DA of the NSLS-II storage ring lattice, yielding a solution with significantly different sextupole strengths but a similar DA compared to the reference solution [29]. The longitudinal variations of third-order RDTs for both solutions are also

TABLE I. The normalized strengths of sextupoles, $f_{3,\text{rms}}$, and $\sum d_r$ for the two selected nonlinear solutions from Fig. 10.

Parameter	Solution 1	Solution 2	Unit
Strength of S1	56.84	53.60	m^{-3}
Strength of S2	-76.60	-60.47	m^{-3}
Strength of S3	-64.00	-58.00	m^{-3}
Strength of S4	98.00	102.00	m^{-3}
Strength of S5	-62.00	-92.00	m^{-3}
$f_{3,\text{rms}}$	16.66	16.52	$\text{m}^{-1/2}$
$\sum d_r$	-8826.80	-8919.80	[...]

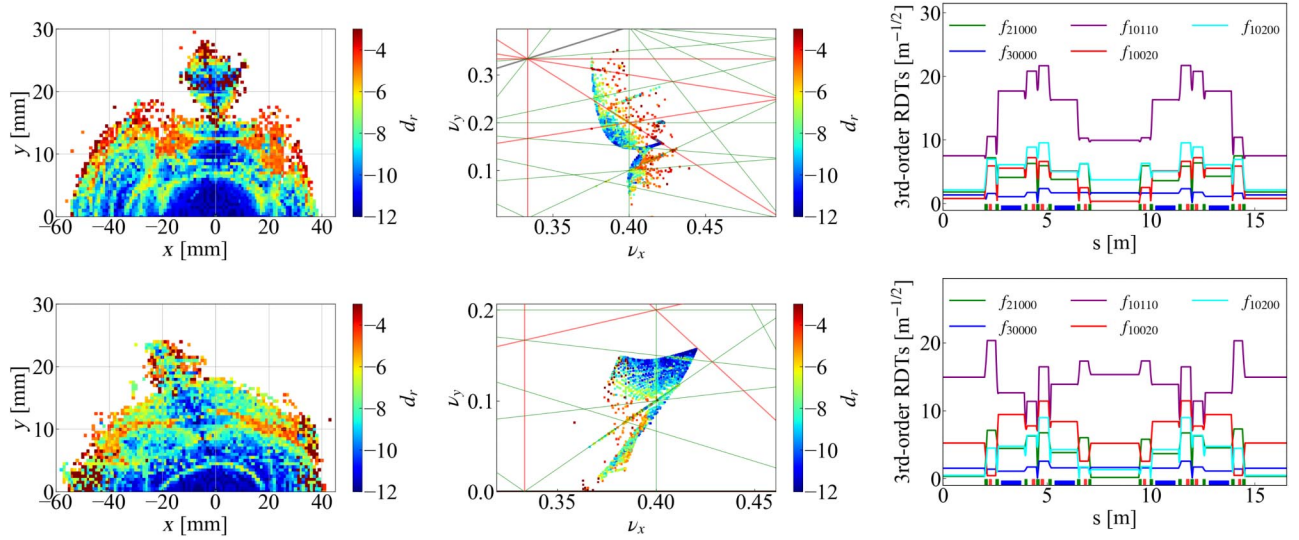


FIG. 12. FMA of DAs (left and middle plots) and longitudinal variations of third-order RDTs (right plots) for the two selected nonlinear solutions from Fig. 10. The three upper plots are for solution 1 in Table I and the lower plots are for solution 2. In the RDT calculation, sextupoles are sliced into thin pieces, and consequently, the magnitude of f_{jklm} can exhibit a valley-like sharp variation inside sextupoles.

comparable [29]. These results indicate that distinct good solutions with comparable DAs are typically connected through the ellipsoidal surface of $f_{3,\text{rms}}$.

A normalization factor κ , defined as the ratio of $f_{3,\text{rms}}$ to the minimum $f_{3,\text{rms}}$, is used to characterize the size of the ellipsoid with an isosurface of $f_{3,\text{rms}}$. The projection of the $f_{3,\text{rms}}$ ellipsoids with different κ values in the normalized strength space of the sextupole families S3 and S5 is shown in Fig. 13. Together with the lower middle plot of Fig. 10, it can be seen that superior solutions with lower $\sum d_r$ values are located inside and around the $f_{3,\text{rms}}$ ellipsoid with $\kappa = 1.10$. For solutions within ellipsoids with different κ , the range of $f_{3,\text{rms}}$, average $\sum d_r$, and the proportion of these solutions to all solutions in the strength

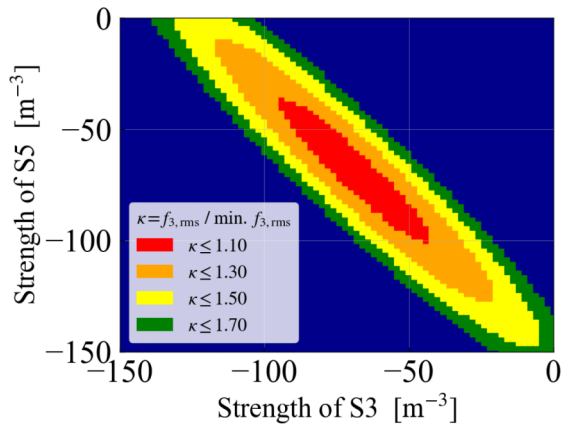


FIG. 13. Projection of the $f_{3,\text{rms}}$ ellipsoids with different κ values in the normalized strength space of the sextupole families S3 and S5.

space are listed in Table II. For $\kappa = 1.10$, the solutions inside the corresponding $f_{3,\text{rms}}$ ellipsoid account for only 0.13% of all solutions, a very small fraction. This indicates that searching for the optimal DA inside and around an ellipsoid with a smaller κ will be much more efficient than searching in the entire space.

Combining the scan results from the two cases with different numbers of sextupole families, it is observed that the distributions of $f_{3,\text{rms}}$ and $\sum d_r$ are largely consistent and that the extreme regions of $\sum d_r$ are enclosed by the ellipsoidal contour with smaller $f_{3,\text{rms}}$. Since $f_{3,\text{rms}}$ is a convex function and has a very strong relationship with DA, it is reasonable to infer that in a higher-dimensional variable space, the distributions of $f_{3,\text{rms}}$ and DA will also exhibit good consistency. That is, better DA solutions are also distributed within the central region of the $f_{3,\text{rms}}$ ellipsoid. Therefore, this suggests that, based on the distribution of $f_{3,\text{rms}}$, more solutions can be generated in the ellipsoid with smaller $f_{3,\text{rms}}$ values to serve as the initial population of an intelligent algorithm for improving the performance of particle tracking based DA optimization.

TABLE II. Range of $f_{3,\text{rms}}$, average $\sum d_r$, and the proportion of these solutions relative to all solutions, for solutions within ellipsoids with different κ .

κ	$f_{3,\text{rms}}$	Avg. $\sum d_r$	Prop. (%)
1.10	$\leq 11.73 \text{ m}^{-1/2}$	-7215	0.13
1.30	$\leq 13.86 \text{ m}^{-1/2}$	-6638	0.74
1.50	$\leq 15.99 \text{ m}^{-1/2}$	-6041	1.76
1.70	$\leq 18.13 \text{ m}^{-1/2}$	-5521	3.03

B. Optimization of the SSRF lattice

In the numerical optimization of DA using intelligent algorithms, the initial population is generally randomly distributed within the variable space. Based on the roughly approximate convexity of DA and the ellipsoidal geometry of $f_{3,\text{rms}}$, a high-quality initial population can be generated to improve optimization performance. It is observed from Eq. (16) that $f_{3,\text{rms}}^2$ is a standard quadratic function, which resembles the exponent part of the multivariate Gaussian probability density function. This leads us to initialize the population with a Gaussian distribution that is geometrically aligned with the ellipsoid of $f_{3,\text{rms}}^2$. The population initialization steps are as follows: (1) Determine the parameters $\mathcal{D}$, $\mathcal{G}$, and $\mathcal{H}$ in Eq. (16) for $f_{3,\text{rms}}^2$ by fitting with some randomly generated nonlinear solutions and then obtain the central position with minimum $f_{3,\text{rms}}^2$; (2) utilize the central position and $\mathcal{D}$ to construct the probability density function for a Gaussian distribution, with 3σ equal to λ times minimum $f_{3,\text{rms}}^2$, where $\lambda = \kappa^2 > 1.0$; and (3) initialize the population using this Gaussian distribution in the variable space.

The SSRF storage ring lattice with eight families of sextupoles [30] was used to conduct the DA optimization with a Gaussian-distributed initial population. The lattice of one superperiod of the SSRF storage ring is presented in Fig. 14. A differential evolution algorithm [7] with a population size of 50 was iterated for 50 generations to perform the optimization. The factor λ was set to 1.25 in the generation of a Gaussian-distributed initial population. The optimization variables are the strengths of the six sextupole families in the straight sections, with the two families of sextupoles in the arc sections correcting the horizontal and vertical chromaticities to (1.0, 1.0). The on-momentum DA was optimized by minimizing $\sum d_r$ as the single optimization objective. As a comparison, the DA optimization with a randomly distributed initial population was also conducted, with the same algorithm settings, optimization variables, and optimization objective. To reduce the impact of the randomness of the algorithm,

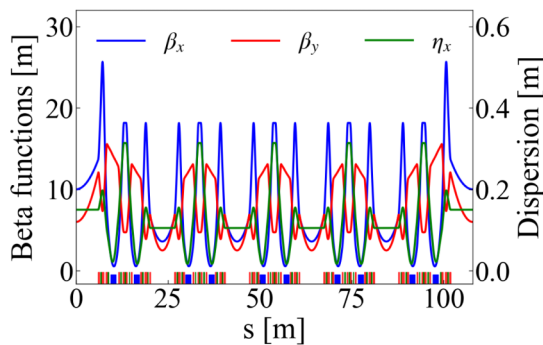


FIG. 14. Linear optical functions and magnet layout of one lattice superperiod of the SSRF storage ring.

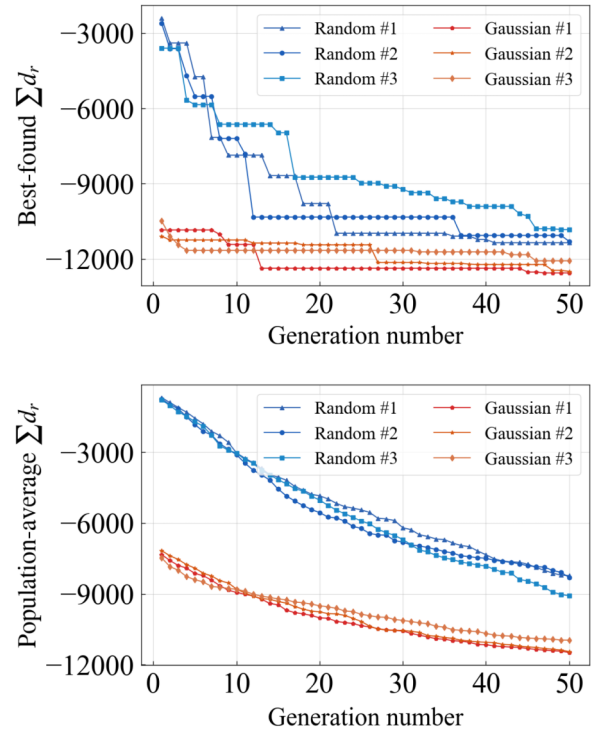


FIG. 15. Evolutions of the best-found $\sum d_r$ (upper plot) and the population-average $\sum d_r$ (lower plot) with the generation number, for Gaussian-distributed and randomly distributed initial populations. Three optimization runs were performed for each kind of distribution.

each of the two optimizations was performed 3 times independently.

Figure 15 presents the convergence curves of the best-found $\sum d_r$ and the population-average $\sum d_r$ for Gaussian-distributed and randomly distributed initial populations. The results show that the optimization with a Gaussian-distributed initial population converges significantly faster than that with a randomly distributed initial population. In the first generation, the Gaussian case shows significantly better best and average results than the random case. Moreover, its results at the 10th generation even outperform those of the random case at the 50th generation. This clearly demonstrates that adopting a Gaussian distribution substantially improves the quality of the initial population, thereby enhancing search efficiency. Furthermore, the optimization results also reflect the strong consistency between the distributions of $f_{3,\text{rms}}$ and DA in a higher-dimensional variable space.

C. Optimization of a 7BA lattice

Then, we applied the proposed DA optimization method to a 7BA lattice [23], as shown in Fig. 16. The storage ring, consisting of 14 such lattice cells and with a circumference of 336 m, achieves a natural emittance of 68.4 pm rad at an energy of 2.2 GeV. This lattice has six families of

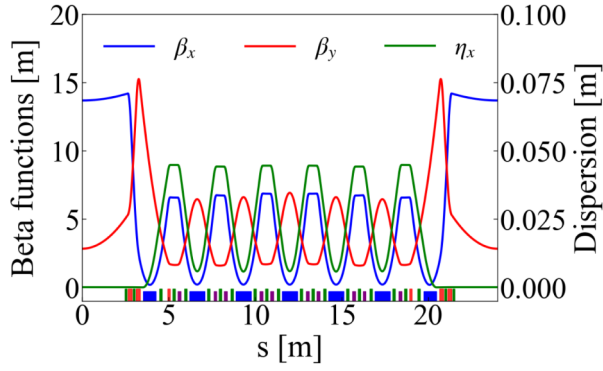


FIG. 16. Linear optical functions and magnet layout of a 7BA lattice cell. In the magnet layout, reverse bends are represented in purple.

sextupoles. The strengths of four were used as optimization variables, and the remaining two were used to correct the horizontal and vertical chromaticities to (2.0, 2.0). In its DA optimization, we imposed a constraint on the DA tracking boundary to exclude islands not useful for beam injection and beam lifetime. The tracking range in the horizontal direction was set to (−12.0, 12.0) mm and that in the vertical was (0.0, 3.5) mm. The optimization objective was to minimize $\sum d_r$ for surviving particles in this bounded area.

The same algorithm and settings as in the DA optimization of the SSRF lattice were used, and two initial populations, Gaussian-distributed and randomly distributed, were generated for this optimization. The convergence curves of the best-found $\sum d_r$ for both initial populations are shown in Fig. 17. It is clearly seen that, for this 7BA lattice, the DA optimization with a Gaussian-distributed initial population still converges much faster than the randomly distributed one. Figure 18 presents the DAs of the optimal solutions obtained with the two initial populations. It can be seen that the two DAs are close and that the DA regions within the bounded area do not include islands and are relatively symmetric. Therefore, the efficiency and effectiveness of the proposed method were

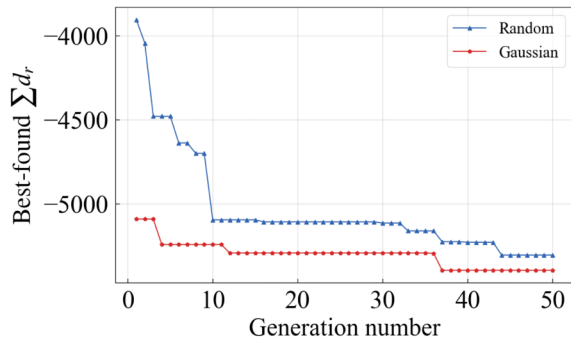


FIG. 17. Evolutions of the best-found $\sum d_r$ with the generation number, for Gaussian-distributed and randomly distributed initial populations.

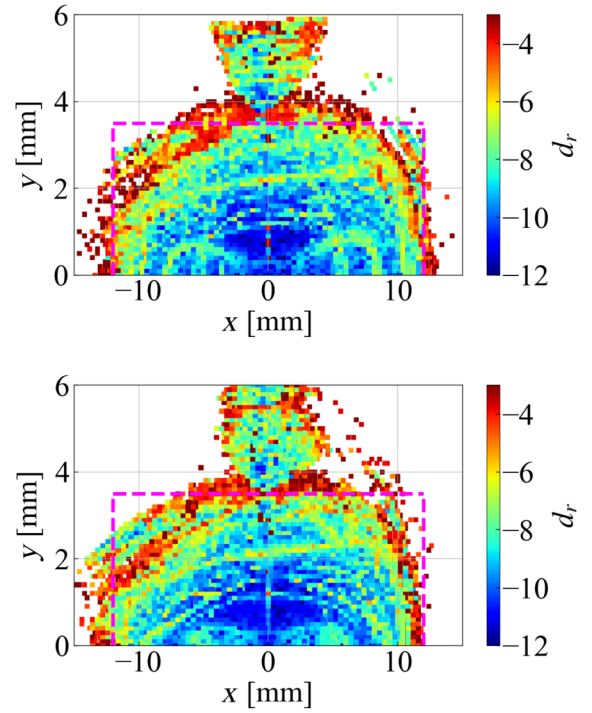


FIG. 18. DAs of two optimal solutions obtained with the Gaussian-distributed (upper plot) and randomly distributed (lower plot) initial populations. The constrained boundaries are shown as magenta dashed lines.

demonstrated across different lattices and DA optimization cases with and without constrained DA boundaries.

V. CONCLUSION AND OUTLOOK

The optimization of DA is crucial for the design of low-emittance storage rings. However, its inherent nonconvex nature makes global optimization very challenging. This paper further investigated the properties of the longitudinal variation of third-order RDTs, which plays a key role in DA optimization. We proved that the longitudinal variation of third-order RDTs, quantified by their rms value $f_{3,rms}$ at sextupole locations, is a special convex function. The isosurfaces of $f_{3,rms}$ in the sextupole strength space are a series of concentric and coaxial ellipsoidal surfaces. This convexity connects the complex nonconvex DA optimization problem with a convex function featuring a well-defined geometric structure, providing a new foundation for understanding and accelerating DA optimization.

The results from scanning the sextupole strength space of the HLS-III storage ring lattice showed a strong consistency between the distributions of $f_{3,rms}$ and DA. Solutions with better DAs are distributed within the central region of the ellipsoid of $f_{3,rms}$, which is different from the common understanding that DA may have multiple isolated extreme regions. Therefore, DA optimization can be regarded as a roughly approximate convex optimization problem. Based on this, we developed a fast DA

optimization method based on particle tracking. In this method, a high-quality initial population for an intelligent algorithm is generated with a Gaussian distribution that follows the geometry of the $f_{3,\text{rms}}^2$ ellipsoid. This significantly accelerates convergence, as demonstrated in the DA optimizations of the SSRF storage ring lattice and a 7BA lattice, compared to the commonly used method with a randomly distributed initial population. In the future, the work will be extended to the optimization of both on- and off-momentum DAs, and we will further explore the simultaneous optimization of linear optics and nonlinear dynamics based on this work.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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