Radiation from a longitudinal bunch with variable charge value

Denis D. Alekseev and Andrey V. Tyukhtin*

Saint Petersburg State University, 7/9 Universitetskaya nab., St. Petersburg, 199034 Russia

(Received 20 January 2026; accepted 29 June 2026; published 29 July 2026)

The work studies the electromagnetic field of a thin extended (filament) bunch of charged particles formed over some finite time. The main attention is paid to the analysis of the radiation field. We show that the radiation field is the main part of the total field in the relatively thin spherical layer with the thickness determined by the time interval of the bunch formation. The energy characteristics of the radiation, including the angular distribution of the power, the power and the total energy of the radiation, as well as the spectral energy characteristics, are calculated and analyzed. The radiation under consideration is compared with the radiation of a variable point charge moving with a constant velocity, as well as with the bremsstrahlung of a point charge having constant magnitude.

DOI: 10.1103/37hr-m4kl

I. INTRODUCTION

The problems of electromagnetic radiation of charged particle bunches are studied in many monographs and in a huge number of journal papers (see, for instance, [1–4]). Relatively simple analytical results in such problems are usually obtained for bunches having a constant charge value. The study of radiation of particle bunches taking into account their complex dynamics is usually carried out by numerical methods. However, it is possible to obtain results in a relatively simple analytical form not only for charges with a constant value but also for some other models.

One of such models was considered in [5–7], where the bunch is assumed to be a point but variation of its charge magnitude with time is taken into account. Such a model can be used in different situations. One of them is realized when the bunch moves in a relatively dense medium. In this situation, the particles of the bunch interact with the medium particles, which leads to certain changes in the bunch characteristics. Depending on the particle mass and velocity, the medium density, and other factors, it is possible to have both a rapid deviation of particles from a rectilinear trajectory (leading to the bunch scattering) and an almost rectilinear motion of the bunch (see, for instance, [8–11]). The latter variant is typical for bunches of heavy particles (protons and ions). It should also be noted that these particles usually lose the majority of their energy in a relatively small section of the trajectory (this effect is referred to as the "Bragg peak") [8–10]. Because of this feature, bunches of heavy particles are widely used in medicine (proton and ion therapy). In such a situation, considering radiation at relatively low frequencies (i.e., at wavelengths larger than the bunch size), one can consider the bunch to be practically pointlike, but it should be taken into account that the charge value is variable with time.

Similar problems have been studied in our works [5–7]. In Refs. [5,6], the radiation from a point bunch moving with constant velocity is investigated. We considered both the case of a homogeneous medium [5] and the case where the bunch crosses a flat interface between two media [6]. In these problems, the radiation fields of different types are studied: radiation arising from the charge variability, Cherenkov radiation generated at a sufficiently high charge velocity, and transition radiation arising in the presence of an interface between media.

Reference [7] examines radiation from a point bunch that has a variable charge and moves with a variable velocity. For arbitrary charge magnitudes and velocities, the expressions for the radiation field components have been obtained. The energy characteristics of radiation of a variable charge with a constant velocity are compared with the characteristics of bremsstrahlung and synchrotron radiation.

In the present work, we will abandon the point bunch model and take into account both its finite longitudinal size and variation in the charge magnitude over time. The model under consideration represents a simplified source-formation scenario, for example, for laser-driven emission or a photocathode-type bunch formation, where the bunch is created in a finite time. Such a situation takes place, in particular, when a pulse of laser radiation acts on a target

*Contact author: a.tyuhtin@spbu.ru

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI.

and dislodges electrons. In the simplest model, the electrons instantly acquire a finite velocity, which then practically does not change. The thickness of the bunch is assumed to be small compared with its finite length, which is equal to the product of the laser pulse time interval and the velocity of the ejected electrons.

II. POTENTIALS AND FIELD COMPONENTS

It is assumed that the bunch begins to form at the point "O" at the moment $t = 0$. After departure from the target, the bunch particles move with constant velocity $\vec{v} = v\vec{e}_z = c\beta\vec{e}_z$ along the z axis (c is the light speed in vacuum). The formation time is equal to τ so that the final length of the bunch is $l = v\tau$. It is assumed that the charge is uniformly distributed along the bunch length, and the linear charge density is equal to $\eta = q/l = q/(c\beta\tau)$, where q is the final charge value of the bunch (after finishing the bunch formation process). The bunch thickness is assumed to be negligible. The process takes place in vacuum.

It should be noted that the model under consideration contains a number of idealizations. Specifically, we assume that the bunch particles accelerate over a negligible time, and the transverse dimension of the bunch is zero. Surrounding conductors and other objects are excluded from our consideration. We also assume the absence of incident radiation (only the field of the source being described is analyzed).

Under these assumptions, the volume charge density ρ_b and the volume current density $\vec{j}_\mathrm{b}$ of the bunch can be written in the form

$$\begin{aligned} &\rho_\mathrm{b} = \eta\delta(x)\delta(y)\Theta(t)\Theta(z)\Theta[z - v(t-\tau)]\Theta(vt - z), \\ &\vec{j}_\mathrm{b} = \rho_\mathrm{b}\vec{v}, \end{aligned} \tag{1}$$

where $\delta(\xi)$ is δ function, $\Theta(\xi) = 0$ for $\xi < 0$, and $\Theta(\xi) = 1$ for $\xi > 0$. Note that, according to the law of charge conservation, a point charge is formed at the origin "O." It is characterized by the charge density

$$\rho_\mathrm{q} = -q\delta(x)\delta(y)\delta(z)\Theta(t)\left\{\begin{matrix} t/\tau & \text{for } t < \tau \\ 1 & \text{for } t > \tau \end{matrix}\right\}. \tag{2}$$

It should be noted that this charge can change due to propagation across the target, recombination, and other processes. However, if the target is not a good conductor, the characteristic time of such processes is much longer than the time of the bunch formation. Therefore, formula (2) can be considered as valid for the time interval of interest. The total charge density is $\rho = \rho_\mathrm{b} + \rho_\mathrm{q}$. Since the point charge is stationary, $\vec{j}_\mathrm{q} = 0$, the total current density is equal to $\vec{j} = \vec{j}_\mathrm{b} = v\rho\vec{e}_z$. Note that due to the geometry of the source (a thread and a point), the field is axially symmetric and has only components E_r, E_z, and H_φ (in cylindrical coordinates).

Using the Lorentz gauge condition, we have the D'Alembert equation for the vector and scalar potentials:

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left\{\begin{matrix} A_z \\ \Phi \end{matrix}\right\} = -4\pi\left\{\begin{matrix} j_z/c \\ \rho \end{matrix}\right\} = -4\pi\left\{\begin{matrix} \beta\rho_\mathrm{b} \\ \rho_\mathrm{b} + \rho_\mathrm{q} \end{matrix}\right\}. \tag{3}$$

The potentials can be found using Green's function $G(\vec{R}, t)$, which satisfies the causality principle:

$$G(\vec{R}, t) = -\frac{\delta(t - R/c)}{4\pi R}, \tag{4}$$

where $\vec{R} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$, $R = |\vec{R}|$.

Note that the scalar potential can be split into two parts: $\Phi = \Phi_\mathrm{b} + \Phi_\mathrm{q}$, where Φ_b is determined by the bunch charge (ρ_b), and Φ_q is determined by the stationary charge at the origin (ρ_q). We can see from (3) that $\Phi_\mathrm{b} = A_z/\beta$. The potential Φ_q is calculated elementarily due to the simplicity of the expression (2). Therefore, we calculate further the vector potential and write out the expression for the scalar potential without derivation.

The vector potential is equal to the following convolution integral:

$$A_z(\vec{R}, t) = -\frac{4\pi}{c}\int_{\mathbb{R}^4} G(\vec{R} - \vec{R}', t - t')j_z(\vec{R}', t')dx'dy'dz'dt'. \tag{5}$$

Integration over x', y', and t' is carried out elementarily due to the presence of δ functions, and we obtain

$$A_z(r, z, t) = \beta\Phi_\mathrm{b}(r, z, t) = \frac{q}{c\tau}\int_{-z}^{\infty}\frac{\zeta(\tilde{z})}{\sqrt{r^2 + \tilde{z}^2}}d\tilde{z}, \tag{6}$$

where

$$\zeta(\tilde{z}) = \left\{\begin{matrix} 1 & \text{for } \frac{z+\tilde{z}}{v} < t_r < \frac{z+l+\tilde{z}}{v}, \\ 0 & \text{otherwise,} \end{matrix}\right. \tag{7}$$

$$t_r = t - \sqrt{r^2 + \tilde{z}^2}/c, \tag{8}$$

and $r^2 = x^2 + y^2$. Analysis of the function $\zeta(\tilde{z})$ shows that it can be written in the following form:

$$\zeta(\tilde{z}) = \left\{\begin{matrix} 1 & \text{for } z_\mathrm{b} < \tilde{z} < z_\mathrm{f}, \\ 0 & \text{otherwise,} \end{matrix}\right. \tag{9}$$

where

$$z_{\rm f} = -\frac{(z-vt)+\beta\sqrt{(z-vt)^2+(1-\beta^2)r^2}}{1-\beta^2},$$
$$z_{\rm b} = -\frac{(z+l-vt)+\beta\sqrt{(z+l-vt)^2+(1-\beta^2)r^2}}{1-\beta^2}. \tag{10}$$

We get three different options. (1) The case $z_{\rm f} < -z$. It is easy to show that this inequality can be written in the form $R > ct$. In this case, according to (6), we obtain $A_z = \Phi = 0$. The absence of the field in the region $R > ct$ is explained by the fact that the field begins to be generated at the moment $t = 0$ at the point "O" and propagates at the speed of light c. (2) The case $z_{\rm b} < -z < z_{\rm f}$. This inequality can be written in the form $c(t-\tau) < R < ct$, i.e., this is a spherical layer. In this area

$$A_z = \frac{q}{c\tau}\left[\operatorname{arsinh}\left(\frac{z_{\rm f}}{r}\right) + \operatorname{arsinh}\left(\frac{z}{r}\right)\right],$$
$$\Phi = \frac{A_z}{\beta} - \frac{q}{R}\frac{ct-R}{c\tau}. \tag{11}$$

(3) The case $-z < z_{\rm b}$. This inequality can be written in the form $R < c(t-\tau)$ (the spherical area). In this area

$$A_z = \frac{q}{c\tau}\left[\operatorname{arsinh}\left(\frac{z_{\rm f}}{r}\right) - \operatorname{arsinh}\left(\frac{z_{\rm b}}{r}\right)\right],$$
$$\Phi = \frac{A_z}{\beta} - \frac{q}{R}. \tag{12}$$

Note that the selection of regions (2) and (3) has a simple physical meaning: an observer located in the spherical layer (2) "sees" the bunch in the process of its formation, and an observer in ball (3) sees the fully formed bunch having a length $l = v\tau$.

Electric and magnetic fields are determined by the formulas

$$\vec{H} = \operatorname{rot}\vec{A}, \qquad \vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}. \tag{13}$$

It is easy to show that, in the cylindrical coordinates r, φ, z, there are only three nonzero components of the field: E_r, E_z, H_φ. It is advisable to move to the spherical coordinates R, θ, φ using the formulas

$$R = \sqrt{r^2+z^2}, \qquad \theta = \arccos(z/R),$$
$$E_R = E_r\sin\theta + E_z\cos\theta,$$
$$E_\theta = E_r\cos\theta - E_z\sin\theta. \tag{14}$$

Omitting intermediate transformations (see the Appendix), we present below the formulas for the field components. In the spherical layer $c(t-\tau) < R < ct$, the following results can be obtained:

$$H_\varphi = \frac{q}{Rc\tau\sin\theta}\left[\frac{\beta ct/R - \cos\theta}{F} + \cos\theta\right],$$
$$E_R = -\frac{q\beta\cos\theta}{Rc\tau F} + \frac{qt}{R^2\tau}\left[\frac{1}{F} - 1\right],$$
$$E_\theta = \frac{q}{Rc\beta\tau\sin\theta}\left[1 - F - \frac{\beta ct}{R}\frac{\cos\theta - \beta ct/R}{F}\right], \tag{15}$$

where $F = \sqrt{(\cos\theta - \beta ct/R)^2 + (1-\beta^2)\sin^2\theta}$.

Note that these formulas acquire their simplest form at the leading edge of radiation, where $R = ct$, and $F = 1 - \beta\cos\theta$:

$$E_R = 0, \qquad H_\varphi = E_\theta = \frac{q}{Rc\tau}\frac{\beta\sin\theta}{1-\beta\cos\theta}. \tag{16}$$

As can be seen, at the front $R = ct$ the field depends on the distance as $1/R$ and does not have a longitudinal component, which means that it is a radiation field in its "pure form".

If the observation point is located inside the spherical layer at $R = ct - h$ (where $h < c\tau$) then, strictly speaking, the field depends on the distance in a more complex way and has the longitudinal component. However, if $h \ll R$, then the correction terms have the relative order h/R. Thus, formula (16) are approximately true throughout the entire spherical layer if $c\tau \ll R$. This means that the field is a radiation field in this area.

In the ball $R < c(t-\tau)$, the following formulas can be obtained (see the Appendix):

$$H_\varphi = \frac{q\cot\theta}{Rc\tau}\left[\frac{1}{G} - \frac{1}{F}\right] - \frac{q\beta}{R^2\sin\theta}\left[\frac{t/\tau - 1}{G} - \frac{t/\tau}{F}\right],$$
$$E_R = \frac{q\beta\cos\theta}{Rc\tau}\left[\frac{1}{G} - \frac{1}{F}\right] - \frac{q}{R^2}\left[\frac{t/\tau - 1}{G} - \frac{t/\tau}{F} + 1\right],$$
$$E_\theta = \frac{q(1-\beta^2\sin^2\theta)}{R\beta c\tau\sin\theta}\left[\frac{1}{G} - \frac{1}{F}\right] - \frac{q\cot\theta}{R^2}\left[\frac{t/\tau - 1}{G} - \frac{t/\tau}{F}\right], \tag{17}$$

where $G = \sqrt{[\cos\theta - \beta c(t-\tau)/R]^2 + (1-\beta^2)\sin^2\theta}$. If $R \gg l$, then these formulas lead to the following approximate expressions:

$$E_R \approx \frac{q}{R^2}\cdot\left[\frac{1-\beta^2}{(1-\beta\cos\theta)^2} - 1\right],$$
$$H_\varphi \approx E_\theta \approx \frac{q}{R^2}\cdot\frac{\beta(1-\beta^2)\sin\theta}{(1-\beta\cos\theta)^3}. \tag{18}$$

As we see, in the ball $R < c(t-\tau)$, the field decreases proportionally to $1/R^2$ with increasing R. This means that the field in this area is not a radiation field. This fact is quite clear physically: an observer located inside this ball sees the fully formed bunch, which moves with

the constant velocity in vacuum, and, accordingly, does not radiate.

III. ENERGY CHARACTERISTICS OF RADIATION

The angular distribution of the radiation power is determined by the formula

$$\frac{dP}{d\Omega} = \frac{\partial^2 W}{\partial t \partial \Omega} = \frac{c}{4\pi} R^2 |[\vec{E} \times \vec{H}]|, \tag{19}$$

where W is a radiation energy. At the large distance ($R \gg c\tau$) in the spherical layer ($c(t-\tau) < R < ct$), according to (16), we obtain

$$\frac{dP}{d\Omega} = \frac{c}{4\pi} R^2 E_\theta^2 = \frac{q^2}{4\pi c \tau^2} \frac{\beta^2 \sin^2\theta}{(1-\beta\cos\theta)^2}. \tag{20}$$

It is easy to obtain the following expressions for the angle of the maximum radiation and the corresponding value of $dP/d\Omega$:

$$\theta_{\max} = \arccos\beta, \qquad \left(\frac{dP}{d\Omega}\right)_{\max} = \frac{q^2\beta^2}{4\pi c\tau^2(1-\beta^2)}. \tag{21}$$

The radiation power is obtained by integrating the angular distribution over angles:

$$P = \int_0^{2\pi}\int_0^{\pi} \frac{dP}{d\Omega} \sin\theta d\theta d\varphi = \frac{q^2}{\beta c\tau^2}\left[\ln\left(\frac{1+\beta}{1-\beta}\right) - 2\beta\right]. \tag{22}$$

The full energy of radiation W is determined by the integral of power over time. Since P does not depend on t during the radiation time τ, then $W = P\tau$.

If $\beta \ll 1$, then we have the following approximate formulas:

$$\begin{aligned} &\frac{dP}{d\Omega} \approx \frac{q^2\beta^2\sin^2\theta}{4\pi c\tau^2}, \\ &\theta_{\max} \approx \frac{\pi}{2}, \qquad \left(\frac{dP}{d\Omega}\right)_{\max} \approx \frac{q^2\beta^2}{4\pi c\tau^2}, \\ &P \approx \frac{2}{3}\frac{q^2\beta^2}{c\tau^2}, \qquad W \approx \frac{2}{3}\frac{q^2\beta^2}{c\tau}. \end{aligned} \tag{23}$$

Thus, in the nonrelativistic regime, the radiation power is proportional to β^2, and the radiation pattern is determined by the function $\sin^2\theta$, as for a dipole.

As the velocity of the particles of the bunch increases, the angle $\theta_{\max}$ decreases and $(dP/d\Omega)_{\max}$ increases.

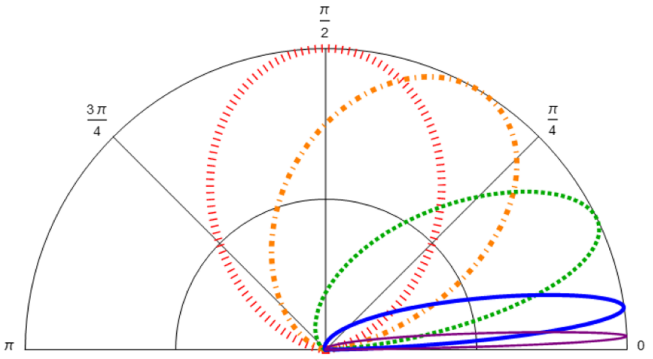

FIG. 1. Directional patterns of the radiation power for different values of β: $\beta = 0.001$ (red dashed line), $\beta = 0.5$ (orange dot-dashed line), $\beta = 0.9$ (green dotted line), $\beta = 0.99$ (blue thick line), and $\beta = 0.999$ (purple solid line).

For the ultrarelativistic mode ($\gamma = 1/\sqrt{1-\beta^2} \gg 1$), one can obtain

$$\begin{aligned} &\theta_{\max} \approx \frac{1}{\gamma}, \qquad \left(\frac{dP}{d\Omega}\right)_{\max} \approx \frac{q^2\gamma^2}{4\pi c\tau^2}, \\ &P \approx \frac{2q^2}{c\tau^2}[\ln(2\gamma) - 1], \qquad W \approx \frac{2q^2}{c\tau}[\ln(2\gamma) - 1]. \end{aligned} \tag{24}$$

It can be shown that the angular width of the radiation pattern at the level $0.5(dP/d\Omega)_{\max}$ is equal to $\Delta\theta = \theta_1 - \theta_2 \approx 2/\gamma$, where $\theta_{1,2} \approx (\sqrt{2} \pm 1)/\gamma$. The power and energy of radiation tend to infinity at $\gamma \to \infty$ but very slowly (logarithmically).

Figure 1 shows the radiation patterns for different velocities of the bunch (each curve is normalized to its maximum).

Let us now obtain the spectral characteristics of the radiation under consideration. The Fourier transform of the radiation field (which exists only in the spherical layer $c(t-\tau) < R < ct$) is equal to

$$\begin{aligned} E_{\omega\theta} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} E_\theta e^{i\omega t} dt \\ &= \frac{q}{2\pi c R}\frac{\beta\sin\theta}{1-\beta\cos\theta}\frac{\sin(\omega\tau/2)}{\omega\tau/2}\exp\left[i\omega\left(\frac{R}{c}+\frac{\tau}{2}\right)\right]. \end{aligned} \tag{25}$$

One can show that the spectral-angular density of the radiation energy is equal to the following expression:

$$\frac{\partial^2 W}{\partial\omega\partial\Omega} = cR^2|E_{\omega\theta}|^2 = \frac{q^2}{4\pi^2 c}\frac{\beta^2\sin^2\theta}{(1-\beta\cos\theta)^2}\frac{\sin^2(\omega\tau/2)}{(\omega\tau/2)^2}. \tag{26}$$

The spectral density of radiation energy is

$$\frac{dW}{d\omega} = 2\pi \int_0^{\pi} \frac{\partial^2 W}{\partial\omega\partial\Omega} \sin\theta d\theta = \frac{q^2}{\pi c\beta}\left[\ln\left(\frac{1+\beta}{1-\beta}\right) - 2\beta\right] \frac{\sin^2(\omega\tau/2)}{(\omega\tau/2)^2}. \quad (27)$$

This quantity, like the total energy $W = P\tau$, is proportional to β^2 at $\beta \ll 1$, and grows logarithmically to infinity if $\beta \to 1$. As we see, the main part of the radiation spectrum occupies the frequency range $\omega < 2\pi/\tau$. Note that in the case of the negligibly small bunch ($\tau \to 0$), formulas (26) and (27) give the known expressions for the radiation from the instantly starting point charge.

The angular density of the radiation energy is

$$\frac{dW}{d\Omega} = \int_0^{+\infty} \frac{\partial^2 W}{\partial\omega\partial\Omega} d\omega = \frac{q^2}{4\pi c\tau} \frac{\beta^2 \sin^2\theta}{(1-\beta\cos\theta)^2} = \tau \frac{\partial P}{\partial\Omega}. \quad (28)$$

The total radiation energy can be obtained by integrating (27) over positive frequencies or integrating (28) over all angles. Of course, both options lead to the formula $W = P\tau$ [see Eq. (22)].

IV. COMPARISON WITH RADIATION FROM POINT CHARGES

It is interesting to compare the radiation under consideration with radiation from various moving point sources. One of them is a point charge with variable magnitude. In Ref. [7], a variable point charge moving along an arbitrary trajectory with an arbitrary dependence of velocity on time is considered. The special case of the motion of a variable point charge with a constant velocity is analyzed in detail. In such a situation, the angular distribution of the radiation power and the total radiation power are equal to [7]

$$\frac{dP_q}{d\Omega} = \frac{\dot{\tilde{q}}^2\beta^2\sin^2\theta}{4\pi c(1-\beta\cos\theta)^3}, \qquad P_q = \frac{\dot{\tilde{q}}^2}{2c\beta}\left[\frac{2\beta}{1-\beta^2} - \ln\left(\frac{1+\beta}{1-\beta}\right)\right], \quad (29)$$

where $\tilde{q}$ is a variable value of charge, and $\dot{\tilde{q}} = d\tilde{q}/dt_r$. Note that here we consider the power as the energy emitted per unit time of radiation t_r: $P_q = dW_q/dt_r$. In the case of a moving point source, the differential of t_r differs from the differential of the observation time t: $dt = (1-\beta\cos\theta)dt_r$ [1,7]. (In the case of the forming bunch considered above, these differentials are the same due to the fact that the radiation is emitted from a fixed point "O.")

In the case of the linear dependence of the charge on time ($\tilde{q} = qt_r/\tau$, $\dot{\tilde{q}} = q/\tau$), the total radiation energy for the time interval $(0,\tau)$ is equal to

$$W_q = \int_0^{\tau} P_q dt_r = \frac{q^2}{2c\beta\tau}\left[\frac{2\beta}{1-\beta^2} - \ln\left(\frac{1+\beta}{1-\beta}\right)\right]. \quad (30)$$

If $\beta \ll 1$, then we have the following approximate formulas:

$$\frac{dP_q}{d\Omega} \approx \frac{\dot{\tilde{q}}^2\beta^2}{4\pi c}\sin^2\theta, \quad P_q \approx \frac{2\dot{\tilde{q}}^2\beta^2}{3c}, \quad W_q \approx \frac{2q^2\beta^2}{3c\tau}. \quad (31)$$

In the ultrarelativistic case ($\gamma \gg 1$), one can obtain

$$\theta_{q\,\max} \approx \frac{1}{\gamma\sqrt{2}}, \qquad \left(\frac{dP_q}{d\Omega}\right)_{\max} \approx \frac{8\dot{\tilde{q}}^2\gamma^4}{9\pi c}, \qquad P_q \approx \frac{\dot{\tilde{q}}^2\gamma^2}{c}, \qquad W_q \approx \frac{q^2\gamma^2}{c\tau}. \quad (32)$$

Let us also write the corresponding formulas for a point bunch, having a constant charge value q, under the condition that a time-dependent velocity $\vec{v} = c\vec{\beta}$ and the acceleration are parallel or antiparallel (the case of bremsstrahlung). In such a situation, the angular distribution of the radiation power and the full radiation power are equal to [1,7]

$$\frac{dP_b}{d\Omega} = \frac{q^2\dot{\tilde{\beta}}^2\sin^2\theta}{4\pi c(1-\tilde{\beta}\cos\theta)^5}, \qquad P_b = \frac{2q^2\dot{\tilde{\beta}}^2}{3c(1-\tilde{\beta}^2)^3}, \quad (33)$$

where $\dot{\tilde{\beta}} = d\tilde{\beta}/dt_r$. In the case of the linear dependence of the velocity on time ($\tilde{\beta} = \beta t_r/\tau$, $\dot{\tilde{\beta}} = \beta/\tau$), the total radiation energy for the time interval $(0,\tau)$ is equal to

$$W_b = \int_0^{\tau} P_b dt_r = \frac{q^2}{c\tau}\left[\frac{\beta^2(5-3\beta^2)}{6(1-\beta^2)^2} + \frac{\beta}{4}\ln\left(\frac{1+\beta}{1-\beta}\right)\right]. \quad (34)$$

If $\beta \ll 1$, then we have the following approximate formulas:

$$\frac{dP_b}{d\Omega} \approx \frac{q^2\dot{\tilde{\beta}}^2}{4\pi c}\sin^2\theta, \quad P_b \approx \frac{2q^2\dot{\tilde{\beta}}^2}{3c}, \quad W_b \approx \frac{2q^2\beta^2}{3c\tau}. \quad (35)$$

In the ultrarelativistic case ($\tilde{\gamma} = (1-\tilde{\beta}^2)^{-1/2} \gg 1$), one can obtain

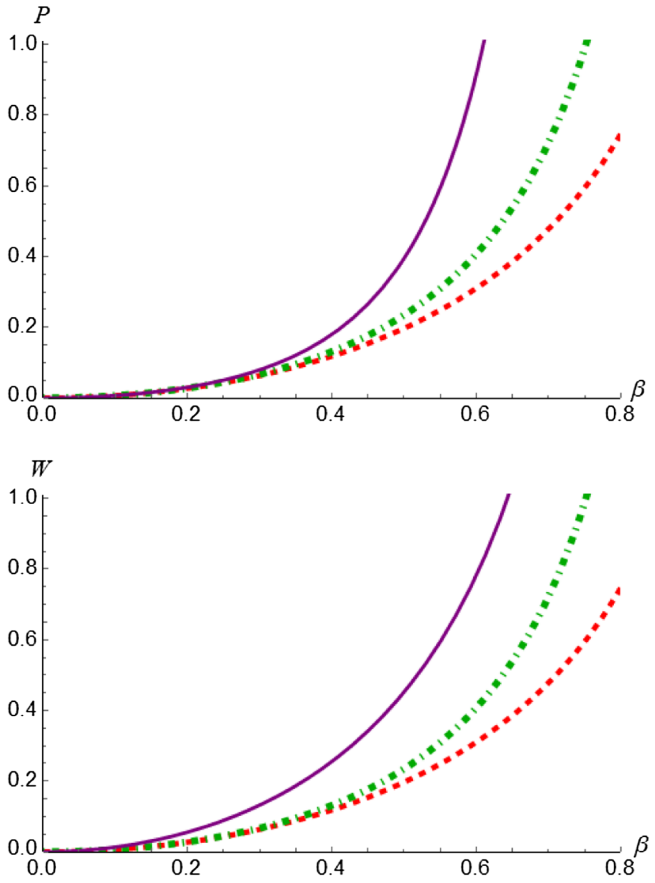

FIG. 2. Dependence of the radiation power in units $q^2/c\tau^2$ (top) and the radiation energy in units $q^2/c\tau$ (bottom) on the velocity in the cases of the forming linear bunch (red dotted lines), the variable point charge (green dashed-dotted lines), and the bremsstrahlung (purple solid lines). For the bremsstrahlung power, the value of $\tilde{\beta}$ is shown on the abscissa.

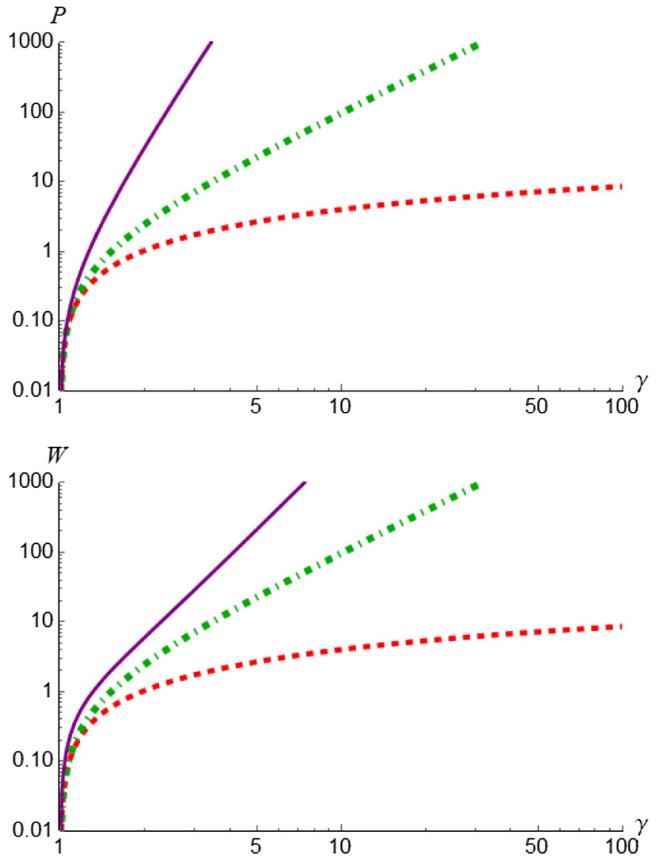

FIG. 3. Dependence of the radiation power in units $q^2/c\tau^2$ (top) and the radiation energy in units $q^2/c\tau$ (bottom) on the Lorentz factor γ in the cases of the forming linear bunch (red dotted lines), the variable point charge (green dashed-dotted lines), and the bremsstrahlung (purple lines). The scale on both axes is logarithmic. For the bremsstrahlung power, the value of $\tilde{\gamma}$ is shown on the abscissa.

$$\theta_{\mathrm{b\,max}} \approx \frac{1}{2\tilde{\gamma}}, \qquad \left(\frac{dP_{\mathrm{b}}}{d\Omega}\right)_{\max} \approx \frac{2^{11} q^2 \dot{\tilde{\beta}}^2 \tilde{\gamma}^8}{5^5 \pi c}, \qquad P_{\mathrm{b}} \approx \frac{2q^2 \dot{\tilde{\beta}}^2 \tilde{\gamma}^6}{3c}, \qquad W_{\mathrm{b}} \approx \frac{q^2 \tilde{\gamma}^4}{3c\tau}. \tag{36}$$

Figures 2 and 3 show the difference in power and energy of different types of radiation, provided that the values of q, β, and τ are the same in all three variants. For the nonrelativistic mode, the radiation powers and the total energies are approximately equal in all cases. If β is not small, then the difference between different types of radiation becomes significant. The bremsstrahlung of a point charge is the most intense, and the radiation of the forming bunch is the least intense. The emission from the variable point charge occupies the intermediate position. This "hierarchy" of radiation becomes very pronounced for the ultrarelativistic regime. Comparing formulas (36), (32), and (24), we see that the maxima of the angular power density for the cases of the accelerating point charge, the variable point charge, and the forming linear bunch are related as $\gamma^8 : \gamma^4 : \gamma^2$ (in order of magnitude). The corresponding ratios for powers have the form $\gamma^6 : \gamma^2 : \ln\gamma$, and the ratios for energies have the form $\gamma^4 : \gamma^2 : \ln\gamma$.

As we see, in all situations, the radiation power and the total radiation energy tend to infinity if $\gamma \to \infty$ ($\beta \to 1$), but according to different laws. The strongest singularity takes place for bremsstrahlung of a point charge, while the weakest (logarithmic) singularity takes place for the radiation of the forming linear bunch. The singularity arises from neglecting the bunch size. Taking into account the finite longitudinal size weakens the singularity but does not eliminate it completely. The singularity will disappear completely if we consider that the transverse size of the bunch is also nonzero.

V. CONCLUSION

We investigated the electromagnetic field of a thin extended bunch of charged particles that is formed over

a finite time interval. It has been shown that the field has a wave character in a certain spherical layer and a quasi-Coulomb character in the ball region. The energy characteristics of the bunch radiation have been calculated and analyzed, namely, the angular distribution of power, the power, and the total energy of the radiation, as well as the spectral energetic characteristics.

The study showed that the radiation of the forming charge is qualitatively similar to the radiation of a variable point charge and the bremsstrahlung of a point charge. In particular, as the velocity increases, the angle between the maximum radiation direction and the particle motion direction decreases, as well as the angular width of the radiation pattern. However, there are significant quantitative differences between these three cases, which increase dramatically as the speed of the charge approaches the speed of light. The study revealed a "hierarchy" of these three types of radiation: the bremsstrahlung is the strongest, the radiation from the forming bunch is the weakest, and the radiation of a variable point charge stands in between. The difference in powers and total energies of radiation of these sources is small for the nonrelativistic mode and very large for the ultrarelativistic regime.

It should be noted that the solution obtained is the simplest variant of radiation generated during the finite time of the bunch formation. This solution can be the natural starting point for a formation-wake formalism in the rigid bunch model. Based on the results obtained, more complex models can be considered that take into account a finite transverse size of the bunch, a nonuniform charge distribution, and so on.

APPENDIX: DERIVATION OF FORMULAS (15) AND (17)

The formulas (15) and (17) contain the following quantities:

$$\begin{aligned} F &= \sqrt{(\cos\theta - vt/R)^2 + (1-\beta^2)\sin^2\theta}, \\ G &= \sqrt{[\cos\theta - v(t-\tau)/R]^2 + (1-\beta^2)\sin^2\theta}. \end{aligned} \tag{A1}$$

It is convenient to first express a number of necessary auxiliary quantities in terms of F and G. We present the corresponding formulas:

$$\begin{aligned} \sqrt{(z-vt)^2 + (1-\beta^2)r^2} &= RF, \\ \sqrt{(z+l-vt)^2 + (1-\beta^2)r^2} &= RG; \end{aligned} \tag{A2}$$

$$\begin{aligned} z_\mathrm{f} &= -R(1-\beta^2)^{-1}(\cos\theta - vt/R + \beta F), \\ z_\mathrm{b} &= -R(1-\beta^2)^{-1}[\cos\theta - v(t-\tau)/R + \beta G]; \end{aligned} \tag{A3}$$

$$\frac{\partial z_\mathrm{f}}{\partial r} = -\frac{\beta\sin\theta}{F}, \quad \frac{\partial z_\mathrm{b}}{\partial r} = -\frac{\beta\sin\theta}{G}; \tag{A4}$$

$$\begin{aligned} \frac{\partial z_\mathrm{f}}{\partial z} &= -\frac{1}{1-\beta^2}\left(1 + \beta\frac{\cos\theta - vt/R}{F}\right), \\ \frac{\partial z_\mathrm{b}}{\partial z} &= -\frac{1}{1-\beta^2}\left[1 + \beta\frac{\cos\theta - v(t-\tau)/R}{F}\right]; \end{aligned} \tag{A5}$$

$$\frac{\partial z_\mathrm{f}}{\partial t} = -v\frac{\partial z_\mathrm{f}}{\partial z}, \qquad \frac{\partial z_\mathrm{b}}{\partial t} = -v\frac{\partial z_\mathrm{b}}{\partial z}; \tag{A6}$$

$$\begin{aligned} \sqrt{r^2 + z_\mathrm{f}^2} &= R\frac{F + \beta(\cos\theta - vt/R)}{1-\beta^2}, \\ \sqrt{r^2 + z_\mathrm{b}^2} &= R\frac{G + \beta[\cos\theta - v(t-\tau)/R]}{1-\beta^2}. \end{aligned} \tag{A7}$$

The components of the field, according to (13), are

$$H_\varphi = -\frac{\partial A_z}{\partial r}, \quad E_r = -\frac{\partial\Phi}{\partial r}, \quad E_z = -\frac{\partial\Phi}{\partial z} - \frac{1}{c}\frac{\partial A_z}{c\partial t}. \tag{A8}$$

Let us consider the expression for the magnetic field. If the observation point is located in a spherical layer $c(t-\tau) < R < ct$, then, differentiating the vector potential (11), we obtain

$$H_\varphi = \frac{q}{c\tau}\left[\frac{z}{rR} + \frac{1}{\sqrt{r^2 + z_\mathrm{f}^2}}\left(\frac{z_\mathrm{f}}{r} - \frac{\partial z_\mathrm{f}}{\partial r}\right)\right]. \tag{A9}$$

Substituting (A3), (A4), and (A7) into (A9), one can obtain

$$\begin{aligned} H_\varphi &= \frac{q}{Rc\tau\sin\theta}\left\{\cos\theta \right. \\ &\quad \left. + \frac{(vt/R - \cos\theta)F - \beta F^2 + \beta(1-\beta^2)\sin^2\theta)}{F[F + \beta(\cos\theta - vt/R)]}\right\} \\ &= \frac{q}{Rc\tau\sin\theta}\left(\cos\theta + \frac{vt/R - \cos\theta}{F}\right). \end{aligned} \tag{A10}$$

The last expression coincides with the first of the formulas (15).

If the observation point is located in the ball $R < c(t-\tau)$, then, after differentiating the vector potential (12), we obtain

$$H_\varphi = \frac{q}{c\tau}\left[\frac{1}{\sqrt{r^2 + z_\mathrm{f}^2}}\left(\frac{z_\mathrm{f}}{r} - \frac{\partial z_\mathrm{f}}{\partial r}\right) - \frac{1}{\sqrt{r^2 + z_\mathrm{b}^2}}\left(\frac{z_\mathrm{b}}{r} - \frac{\partial z_\mathrm{b}}{\partial r}\right)\right]. \tag{A11}$$

Substituting (A3), (A4), and (A7) into (A11), after simple transformations, we obtain the first of the formulas (17).

Similarly, by differentiating the vector and scalar potentials and applying formulas (A2)–(A7), as well as (14), one can obtain expressions for E_R, E_θ presented in (15) and (17).

The results (15) and (17) were also verified by comparison with the computations performed using the original formulas (6) and (13).

[1] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics. Vol. 2: The Classical Theory of Fields* (Butterworth-Heinemann, Oxford, 1980).

[2] V. P. Zrelov, *Vavilov-Cherenkov Radiation in High-Energy Physics* (Israel Program for Scientific Translations, Jerusalem, 1970).

[3] V. L. Ginzburg and V. N. Tsytovich, *Transition Radiation and Transition Scattering* (Hilger, London, 1990).

[4] G. N. Afanasiev, *Vavilov-Cherenkov and Synchrotron Radiation: Foundations and Applications* (Kluwer Academic, Dordrecht, 2004).

[5] A. V. Tyukhtin and X. Fan, Radiation from a moving bunch of particles with a variable charge, Zhurnal Tehnicheskoi Fiziki **67**, 1222 (2022).

[6] X. Fan and A. V. Tyukhtin, Radiation of a variable charge flying into a medium, Phys. Rev. A **108**, 043506 (2023).

[7] A. V. Tyukhtin, Radiation from a particle bunch with both charge and velocity varying in time, Radiophys. Quantum Electron. **68**, 65 (2025).

[8] M. A. Baturitskiy, *Interaction of Ionizing Radiation with Matter* (MGEU, Minsk, 2005).

[9] S. H. Park and J. O. Kang, Basics of particle therapy I: Physics, Radiat. Oncology J. **29**, 135 (2011).

[10] K. W. Jang, W. J. Yoo, S. H. Shin, D. Shin, and B. Lee, Fiber-optic Cerenkov radiation sensor for proton therapy dosimetry, Opt. Express **20**, 13907 (2012).

[11] S. G. Stuchebrov, Y. M. Cherepennikov, A. A. Krasnykh I. A. Miloichikova, and A. V. Vukolov, The method for the electron beam cross section measurement based on the detection of Cherenkov radiation in dielectric fiber, J. Instrumentation **13**, C05020 (2018).