

Resonance gaps, discrepancies, and lines

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The sequence formed by the fractional tune resonances up to order n exactly coincides with the Farey sequence, F_n . In circular accelerators, it is important to place the tunes in regions free of resonances. From numerical evaluations, it is well known that the largest resonance-free gaps happen next to low order resonances. In this report, we provide a mathematical demonstration of this statement for any order $n \in \mathbb{N}$ showing that the resonance gaps are strictly ordered according to the order of the adjacent lowest order resonance up to some given value. We present a connection between the resonance sequence and one of the most important unsolved problems in mathematics, the Riemann hypothesis (RH). If RH is true, it implies that the resonance sequence is significantly more regularly spaced than if it were built at random; however, this has not yet been demonstrated. A new analytical estimate of the number of resonance lines up to order n in the two-dimensional tune diagram is also derived and compared to previous estimates.

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I. INTRODUCTION AND STATEMENT OF MAIN RESULTS

The Farey sequence F_n of order $n \in \mathbb{N}$ is an ascending sequence of irreducible fractions between 0 and 1 whose denominators do not exceed n . These fractions are referred to as Farey fractions. For example, the Farey sequence of order 6 is given by

$$F_6 = \left\{ \frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1} \right\}.$$

For any two neighboring fractions $h/k < h'/k'$ in F_n the gap between them is simply given by $1/(kk')$ as $kh' - hk' = 1$. This property and many other aspects of the theory of Farey sequences can be found in [1–6].

Farey sequences are used in the study of dynamical systems [7–18], as they provide ways to characterize resonances and approximate irrational numbers, appearing as preferential frequencies when considering the stability of systems. For example, in [8], it is conjectured that for typical one-parameter families with orbit frequencies in the interval $[h/k, h'/k']$, with $h/k, h'/k' \in F_n$, the most stable orbit has the frequency of

$$\frac{h + \phi h'}{k + \phi k'},$$

where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. This appears to be the case for sufficiently short intervals, and it originates from the fact that ϕ is *the most irrational number* (meaning that its continued fraction approximation, $\phi = 1 + 1/[1 + 1/(1 + \dots)]$, is the slowest to converge).

In circular accelerators, the particles within the circulating beam feature a tune distribution with some width. This requires a sufficiently large resonance-free interval to avoid emittance growth or particle loss. An illustration of tunes used by some circular colliders is shown in Fig. 1 along with the resonance lines up to order 5. The size of the tune spread within the beam changes between accelerators, e.g., for LHC and SuperKEKB, tune spreads of about 0.015 and 0.036 have been achieved in collisions, respectively [19,20]. The tune of the reference particle is typically chosen via computer simulations or experimentally, e.g., [21–23]. While it is well known that the largest resonance-free gaps happen next to low order resonances, no formal proof of this statement is available in the literature. As an illustration, let us construct the gap sequence in F_6 as

$$G_{F_6} = \left\{ \frac{1}{6}, \frac{1}{30}, \frac{1}{20}, \frac{1}{12}, \frac{1}{15}, \frac{1}{10}, \frac{1}{10}, \frac{1}{15}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \frac{1}{6} \right\},$$

and display it in Fig. 2.

We confirm that the largest gaps with value $1/6$ are placed next to the integer resonances $0/1$ and $1/1$, followed by gaps of $1/10$ placed next to the half integer resonance, etc. We define $g_n(k)$ as the set of gaps in F_n next to a Farey fraction of denominator k . For illustration,

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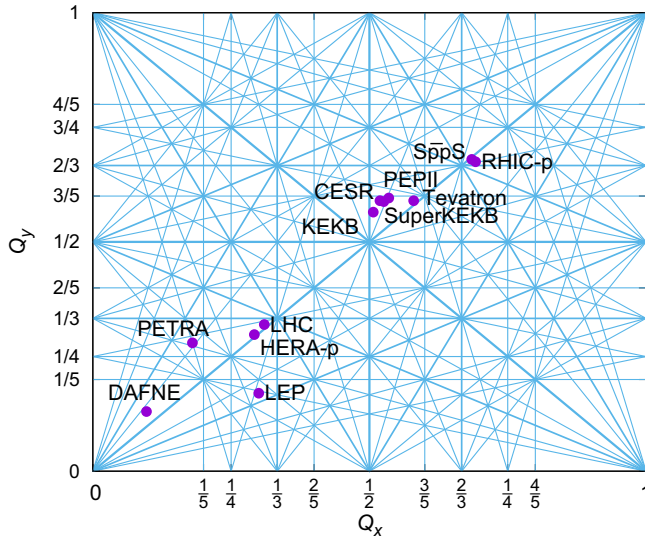


FIG. 1. Fractional tunes used in some current or past particle colliders shown within the two-dimensional resonance diagram up to order 5. Data are taken from [19–21,24–31].

$$\begin{aligned} g_6(1) &= \left\{\frac{1}{6}\right\}, & g_6(2) &= \left\{\frac{1}{10}\right\}, & g_6(3) &= \left\{\frac{1}{15}, \frac{1}{12}\right\}, \\ g_6(4) &= \left\{\frac{1}{20}, \frac{1}{12}\right\}, & g_6(5) &= \left\{\frac{1}{30}, \frac{1}{20}, \frac{1}{15}, \frac{1}{10}\right\}, \\ g_6(6) &= \left\{\frac{1}{30}, \frac{1}{6}\right\}. \end{aligned}$$

In Corollary 1, Sec. II, we demonstrate that the resonance gaps in $g_n(k)$ strictly decrease in size with the order of the resonance, k , for $k \leq (\sqrt{4n+1}+1)/2$. For F_6 , we have $(\sqrt{4 \times 6 + 1} + 1)/2 = 3$, which implies that the gaps of order 3 are strictly smaller than the gap of order 2, which is smaller than the gap of order 1. However, the gap $1/12$ is present both in $g_6(3)$ and $g_6(4)$. Other properties of F_n are also illustrated in Sec. II.

For given $n \in \mathbb{N}$ and $x \in [0, 1]$, $I_n(x)$ is defined as the number of elements in F_n within $[0, x]$. The local

discrepancy $\hat{r}_n(h/k)$ of the Farey fraction h/k in F_n is defined as [32,33]

$$\hat{r}_n(h/k) = \frac{h}{k}N - I_n(h/k),$$

where N denotes the number of elements in F_n minus one, or equivalently, $N = I_n(1)$. The local discrepancy normalized by N , i.e., $\hat{r}_n(h/k)/N$, gives the separation between the i th element in F_n and the i th element in an evenly spaced sequence with the same number of points as F_n (defining $i = I_n(h/k)$). This discrepancy is therefore a measure of the regularity of the resonances in the tune space. The regularity of resonances can be observed in, e.g., frequency maps [13] as regions with higher diffusion rates appearing, eventually, with some periodicity. Hence, understanding the behavior of the discrepancies in F_n could potentially be of use in accelerator beam dynamics. The blue line in Fig. 2 gives the normalized discrepancies of F_6 . The local discrepancy is of great importance in the study of sequences. The local discrepancy appears in general formulas of $I_n(h/k)$ [34,35], which are used in Sec. IV to estimate the number of resonance lines in the two-dimensional tune resonance diagram. Furthermore, the local discrepancy is used in a formulation of the Riemann hypothesis (RH), one of the most important unsolved problems in mathematics. The Franel-Landau formulation [36,37] is expressed as

$$\langle |\hat{r}_n(h/k)| \rangle = \frac{1}{N} \sum_{h/k \in F_n} |\hat{r}_n(h/k)| = O(n^{\frac{1}{2}+\epsilon}) \quad \forall \epsilon > 0 \Leftrightarrow \text{RH}.$$

This hypothesis implies that the average local discrepancy in F_n , taken with the absolute value, $\langle |\hat{r}_n(h/k)| \rangle$, grows at most with a power law almost equal to the square root of n . To quantify this growth, we recall that the best absolute known bound for the normalized local discrepancy is given by Dress [38] as $|\hat{r}_n(h/k)|/N \leq 1/n$, for any value of h/k . The best known estimate of $\langle |\hat{r}_n(h/k)| \rangle$ remains as

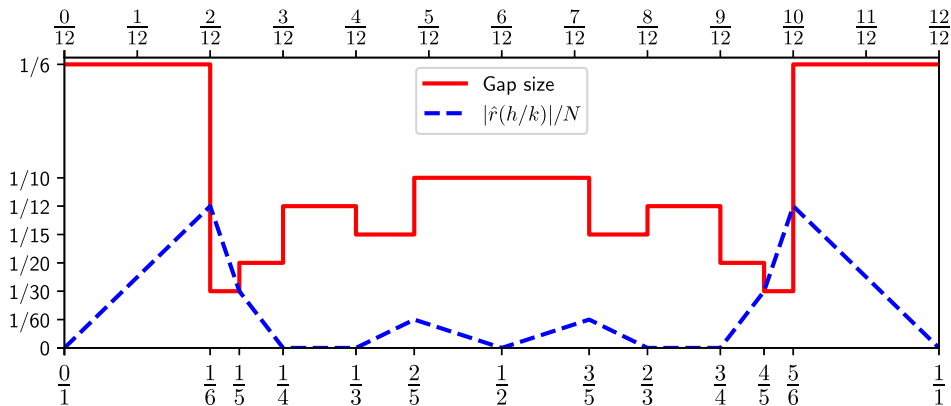


FIG. 2. Gap sizes in F_6 (red) and normalized local discrepancy (blue). Bottom horizontal ticks represent the Farey fractions in F_6 , while top ticks represent the corresponding evenly spaced sequence with the same number of elements as F_6 .

$O(n)$, far from RH. To have a grasp of the size of this growth, we can compare to the random sequence, R_n . We define R_n as the sequence containing the following $N + 1$ points: 0, 1, and $N - 1$ uniform random variables between 0 and 1. The sequence is sorted in ascending order and has the same number of points as F_n . In Sec. III, we make a close comparison between F_n and R_n , summarized in Table I. In particular, we estimate the expected value of the average local discrepancy of R_n , obtaining $O(n)$. Hence, if RH is true, it implies that the tune resonances would have a much more regular distribution than that expected in the random sequence, but best analytical estimates to date cannot exclude a comparable level of regularity in F_n and R_n .

In [14], a closed expression as a sum over Farey fractions is derived to evaluate the number of resonance lines in the two-dimensional resonance diagram of order n , denoted by T_n . For example, $T_5 = 80$ is the number of resonance lines shown on Fig. 1. In Sec. IV, we perform further analytical developments to estimate T_n in the following simplified form,

$$T_n = \frac{2}{3\zeta(3)}n^3 + O(n^{5/2}), \quad (1)$$

where $\zeta(x)$ is the Riemann zeta function. A numerical truncated value of the cubic coefficient is given by $2/(3\zeta(3)) \approx 0.5546049$, where the first 3 digits coincide with the fit value given in [14]. Figure 3 compares this estimate to the actual number of resonance lines up to order 30 considering also the correction term derived from a fit in [14], which enables a 10% accuracy at low orders.

II. RESONANCE GAPS

The gaps between resonances, or equivalently the gaps of the Farey sequence, are studied in [6] by proving the following lemma.

Lemma 1. We write $k \triangleleft k'(n)$ if k' appears immediately to the right of k as denominators in F_n . Then, $k \triangleleft k'(n)$ if and only if $k \leq n$, $n - k < k' \leq n$, and $\gcd(k, k') = 1$, and this occurs only once [6].

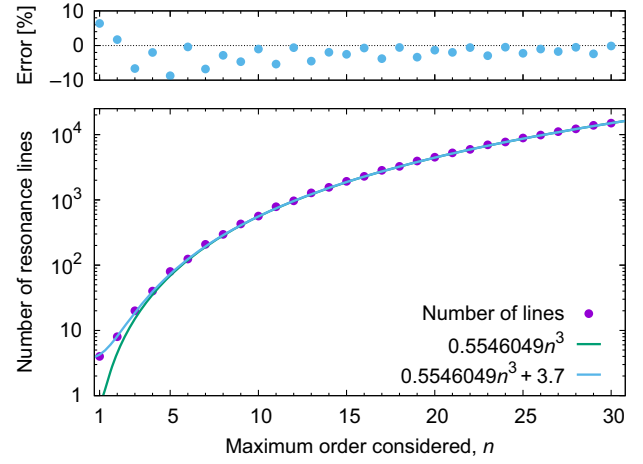


FIG. 3. Number of resonance lines in the two-dimensional tune diagram versus maximum order considered. Two numerical expressions are also shown. The first one includes only the leading cubic order term of T_n in Eq. (1) and the second one adds a correction term already derived from a numerical fit in [14].

The mean value of the Farey gaps is $\mu_{F_n} = 1/N \approx \pi^2/(3n^2)$ [6]. The second moment of the Farey gaps distribution is computed in [6] as

$$m_2 \equiv \sum_{g \in G_{F_n}} g^2 = \frac{12}{\pi^2 n^2} \left(\log n + \frac{1}{2} + \gamma - \frac{\zeta'(2)}{\zeta(2)} \right) + O\left(\frac{\log^2 n}{n^3}\right),$$

where γ is the Euler-Mascheroni constant and $\zeta'(x)$ is the derivative of the Riemann zeta function. The standard deviation of the gaps is, therefore, given by

$$\sigma_{F_n} = \sqrt{\frac{m_2}{N} - \mu_{F_n}^2} = \frac{2\sqrt{\log n - c}}{n^2} + O\left(\frac{\log^{3/2} n}{n^3}\right),$$

$$c = -\frac{1}{2} - \gamma + \frac{\zeta'(2)}{\zeta(2)} + \frac{\pi^4}{36} \approx 1.058.$$

An analytical estimate of the density function of Farey gaps, $d(x)$, is given in [6,39] as

$$d(z) = \begin{cases} 0, & \text{if } z \leq 3/\pi^2; \\ 6 \log(\pi^2 z/3)/(\pi^2 z^2), & \text{if } 3/\pi^2 < z \leq 12/\pi^2; \\ -12 \log(1/2 + \sqrt{1/4 - 3/(\pi^2 z)})/(\pi^2 z^2), & \text{if } 12/\pi^2 < z, \end{cases} \quad (2)$$

where z represents the gap value multiplied by the factor $3n^2/\pi^2$. Figure 4 shows an illustration of the distribution of Farey gaps together with the numerical evaluation for F_{400} . These analytical estimates cannot be applied to the largest gaps in F_n , which are the matter of this work, as the density distribution of the

largest gaps is not continuous (as it will be illustrated in Fig. 5).

We recall that $g_n(k)$ is a set containing all gaps next to the Farey fractions with denominator k in F_n . In the following corollary, we formally establish an order relation between the largest gaps in F_n .

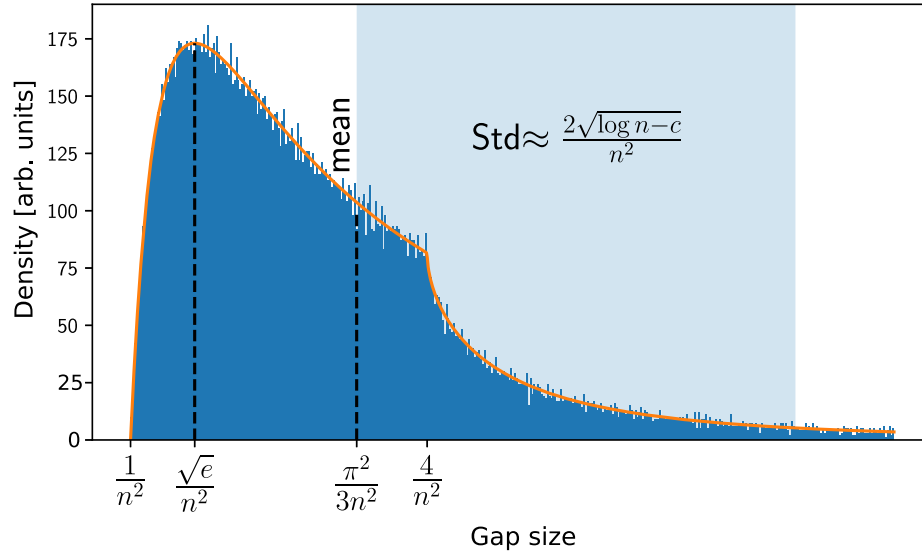


FIG. 4. Distribution of the Farey gaps shown for the analytical estimate [39] in orange and an illustration for F_{400} in blue along with the key parameters of the analytical estimate. The first dashed line marks the mode of the distribution at $\sqrt{e}/n^2$ and the second dashed line marks the mean at $\pi^2/(3n^2)$. The shaded area denotes one standard deviation starting from the mean of the distribution.

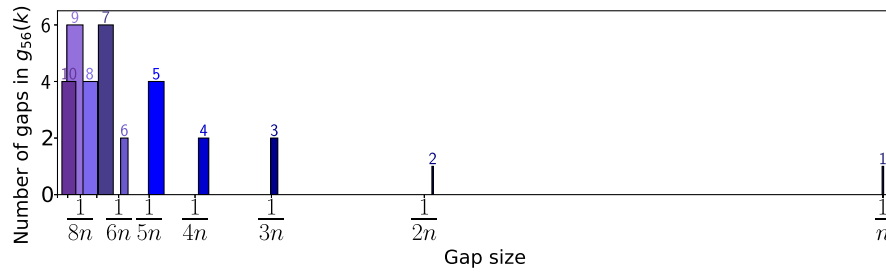


FIG. 5. Number of gaps next to resonances of order $k \leq 10$, considering all resonances up to order $n = 56$, i.e., $g_{56}(k)$, versus gap size. The resonance order k is displayed on top of the bar. The bar width extends between the smallest and the largest gap in $g_{56}(k)$. Note that bars do not overlap for $k \leq \hat{k} = (\sqrt{4 \times 56 + 1} + 1)/2 = 8$.

Corollary 1. Let $n \in \mathbb{N}$, $n > 2$, and $\hat{k} \in \mathbb{R}^+$ be defined as

$$\hat{k} = \frac{1}{2}(\sqrt{4n+1} + 1).$$

Then, the largest gaps in F_n are strictly ordered as follows,

$$g_n(1) > g_n(2) > \dots > g_n(\lfloor \hat{k} \rfloor),$$

where the binary relation “ $>$ ” between sets applies to any of the elements within the sets and $\lfloor x \rfloor$ denotes the integer floor of x .

Proof. Let us recall that for any two neighboring fractions $h/k < h'/k'$ in F_n , the gap between them is simply given by $1/(kk')$ as $kh' - hk' = 1$. For a given integer k such that $1 \leq k \leq n$, Lemma 1 implies that $1/(kk')$ is an element of $g_n(k)$ if and only if $n-k < k' \leq n$ and $\gcd(k, k') = 1$. This is equivalently written as follows,

$$g_n(k) = \left\{ \frac{1}{kk'} : n-k < k' \leq n, \text{ and } \gcd(k, k') = 1 \right\}. \quad (3)$$

The bounds for k' in this definition are used to bound all elements in $g_n(k)$ as follows,

$$\frac{1}{k(n-k+1)} \geq g_n(k) \geq \frac{1}{kn}.$$

Similarly, for $g_n(k-1)$, assuming $k > 1$, we have

$$\frac{1}{(k-1)(n-k+2)} \geq g_n(k-1) \geq \frac{1}{(k-1)n}.$$

To find which k values guarantee that $g_n(k-1) > g_n(k)$, we have to solve the following equation,

$$\frac{1}{(\hat{k}-1)n} = \frac{1}{\hat{k}(n-\hat{k}+1)},$$

that is simply rewritten as

$$n = \hat{k}(\hat{k}-1). \quad (4)$$

The general solution for $\hat{k} \in \mathbb{R}^+$ is given by

$$\hat{k} = \frac{1}{2}(\sqrt{4n+1} + 1).$$

Therefore, for $k < \hat{k}$, we have the following inequality

$$g_n(k-1) \geq \frac{1}{(k-1)n} > \frac{1}{k(n-k+1)} \geq g_n(k), \quad \text{for } k < \hat{k}.$$

In the case that $k = \hat{k} \in \mathbb{N}$ and $n > 2$, by virtue of Eq. (4), n is a multiple of $\hat{k}-1$, implying that $\gcd(n, \hat{k}-1) \neq 1$ and $1/(n(\hat{k}-1)) \notin g_n(\hat{k}-1)$, according to the definition of $g_n(k)$ in Eq. (3), hence

$$g_n(\hat{k}-1) > \frac{1}{(\hat{k}-1)n} \geq g_n(\hat{k}), \quad \text{for } n > 2.$$

For the case $n = 2$, we have that $\hat{k} = 2$ and $g_2(1) = g_2(2)$, which does not fulfill the strict decreasing order and therefore it is excluded from the statement of the corollary. ■

The number of elements in $g_n(k)$ is the same as the number of Farey fractions with denominators equal to k in F_n , that is given by the Euler totient function of k , $\varphi(k)$. Figure 5 illustrates the statement in Corollary 1 for the resonances up to order $n = 56$ and corresponding $\hat{k} = 8$. The plot illustrates that all gaps next to resonances of order k are larger or equal than $1/(kn)$ and that the gap sizes remain strictly ordered by k for $k \leq \hat{k}$.

III. COMPARING RANDOM AND FAREY SEQUENCES

The estimates of the gaps and the discrepancies of the Farey sequences may appear unrelatable in the absence of a reference to compare. The random sequence seems to serve as a solid relatable reference as it appears in various physical and mathematical contexts. Let the random sequence R_n contain the following $N+1$ points: 0, 1, and $N-1$ uniform random variables between 0 and 1. The sequence is sorted in ascending order as F_n . The probability density function of the gaps in the random sequence is given by

$$P_n(x) = (N-1)(1-x)^{N-2}.$$

The expected or mean value of the gap is obtained with the following integral,

$$\mu_{R_n} = \int_0^1 x P_n(x) dx = \frac{1}{N},$$

and the standard deviation of the gap distribution is given by

$$\begin{aligned} \sigma_{R_n} &= \sqrt{\int_0^1 (N-1) \left(x - \frac{1}{N}\right)^2 (1-x)^{N-2} dx} \\ &= \frac{1}{N} \sqrt{\frac{N-1}{N+1}} \approx \mu_{R_n}. \end{aligned}$$

The probability of all gaps being below x in R_n is given by

$$\left(\int_0^x P_n(x') dx'\right)^N \approx e^{-Ne^{-Nx}},$$

where we have used that, for large n , $P_n(x) \approx Ne^{-Nx}$. To estimate the expected largest gap, we need to differentiate this expression and compute the expected value of the resulting distribution. This gives approximately $\log N/N$ as shown in Table I (the derivation is spared to the reader). Similarly, the expected minimum gap is obtained to be approximately $1/N^2$.

To estimate the expected value of the discrepancies of R_n , we need to introduce the probability distribution of the i th random variable in R_n , denoted as $X_i(x)$ and given by

$$X_i(x) = \frac{(N-1)!}{(N-1-i)!(i-1)!} x^{i-1} (1-x)^{N-1-i},$$

where the factor arranged as a fraction represents the number of different combinations of $N-1$ identical elements in three sets, one with just one element (the i th variable), another one with $i-1$ elements, and the last set with $N-1-i$ elements. Hence, the expected value of the

TABLE I. Comparison of the Farey and the random sequences in terms of their gaps and discrepancies. For the random sequence, expected values are shown.

	F_n	R_n
Cardinality	$N = \frac{3}{\pi^2} n^2 + O(n \log n)$	
Minimum gap	$\frac{1}{n^2}$	$\frac{1}{N^2}$
Mean gap (μ)		$\frac{1}{N}$
Gap std (σ)	$\frac{2\sqrt{\log n - c}}{n^2}$	$\frac{1}{N}$
Maximum gap	$\frac{1}{n}$	$\frac{\log N}{N}$
Minimum normalized discrepancy	0	$\frac{2}{eN}$
Maximum normalized discrepancy	$\frac{1}{n}$	$\sqrt{\frac{\pi}{6}} \frac{1}{n}$
$\langle \hat{r}_n(h/k) \rangle$:		
Unconditional	$O(n)$	$\frac{1}{4} \sqrt{\frac{3}{2\pi}} n$
Riemann hypothesis	$O(n^{1/2+\epsilon})$	

normalized discrepancy of x_i is given by

$$E\left(\left|x_i - \frac{i}{N}\right|\right) = 2 \int_0^{\frac{i}{N}} \left(\frac{i}{N} - x\right) X_i(x) dx.$$

The following indefinite integral is used,

$$\int x^{i-1} \left(\frac{i}{N} - x\right) (1-x)^{N-1-i} = \frac{1}{N} x^i (1-x)^{N-i} + C$$

to obtain

$$E(|r_{n,i}|) \equiv E\left(\left|x_i - \frac{i}{N}\right|\right) = 2 \frac{(N-1)!}{(N-1-i)!(i-1)!} \frac{i^i (N-i)^{N-i}}{N^{N+1}}.$$

For $i = 1$, we have

$$E(|r_{n,1}|) = \frac{2}{N} \left(\frac{N-1}{N}\right)^N \approx \frac{2}{e} \frac{1}{N},$$

where the approximation holds for large N . After some algebra, using the Stirling approximation,

$$N! \approx \sqrt{2\pi N} (N/e)^N [1 + 1/(12N)],$$

and going from i to x as $x = i/N$, we obtain the following approximation,

$$E(|r_n(x)|) \approx \frac{2}{\sqrt{2\pi N}} x \sqrt{\frac{1}{x} - 1}.$$

The expected maximum normalized discrepancy of R_n is $\sqrt{\pi/6}/n$ and occurs at $x = 1/2$ compared to Farey's maximum normalized discrepancy of $1/n$ that occurs at $x = 1/n$ [38]. The average of the expected discrepancies of R_n is approximated as follows,

$$\begin{aligned} \langle |\hat{r}_n(x)| \rangle &\approx N \times 2 \int_0^{1/2} \frac{2dx}{\sqrt{2\pi N}} x \sqrt{\frac{1}{x} - 1} \\ &\approx \frac{1}{4} \sqrt{\frac{\pi}{2}} \sqrt{N} \approx \frac{1}{4} \sqrt{\frac{3}{2\pi}} n. \end{aligned}$$

All the above results are collected in Table I to directly compare F_n and R_n . By construction, R_n has the same mean gap as F_n . Both the expected minimum and maximum gaps in R_n are significantly smaller than in F_n and the standard deviation of the gaps in F_n is larger than in R_n due to the factor $\sqrt{\log n - c}$. The minimum discrepancy in F_n reaches 0 at $1/2$ while for R_n it is never zero excluding 0 and 1. The maximum normalized discrepancy in F_n is larger than in R_n but both are $O(1/n)$. If RH is true, it implies that the average discrepancy of the resonance

sequence is $O(n^{1/2+\epsilon}) \forall \epsilon > 0$; however, it has only been demonstrated that it is $O(n)$, compatible with the expected value of the average discrepancy in R_n .

IV. NUMBER OF RESONANCE LINES

The two-dimensional resonance diagram up to order n is a fundamental instrument in accelerator physics which shares similarities with other mathematical structures as the Farey diagram of order (n, n) [5] or the Sòs permutations [40]. A closed expression for the number of resonance lines in the two-dimensional resonance diagram is given in [14] as

$$T_n = 4 + \sum_{h/k \in F_n \cap [0, 1/2]} 2(4I_n(1/k) - 6),$$

where the first monomial 4 accounts for the resonance lines on the sides of the unitary square ($T_1 = 4$) and the sum extends over all the Farey fractions h/k in $F_n \cap [0, 1/2]$. Equivalently,

$$\begin{aligned} T_n &= 4 + \sum_{k=2}^n \varphi(k) (4I_n(1/k) - 6) \\ &= 4 \sum_{k=2}^n \varphi(k) I_n(1/k) + O(n^2), \end{aligned} \quad (5)$$

where we have used that there are $\varphi(k)$ fractions in F_n with denominators equal to k and that the number of elements in F_n is $O(n^2)$. From [34], we have a general expression for $I_n(h/k)$,

$$\begin{aligned} I_n(h/k) &= n \sum_{j=1}^{\lfloor nh/k \rfloor} \frac{\varphi(j)}{j} - \frac{k}{h} \Phi(\lfloor nh/k \rfloor) + \hat{r}_{\lfloor nh/k \rfloor}(\{k/h\}) \\ &\quad - \sum_{j=1}^{\lfloor nh/k \rfloor} \sum_{d|j} \mu(d) \left\{ \frac{n}{d} \right\}, \end{aligned}$$

where $\Phi(p) = \sum_{j=1}^p \varphi(j)$ is the summatory Euler totient function, $\mu(p)$ is the Möbius function and $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x . For h/k being a unit fraction, the previous expression simplifies to [35]

$$I_n(1/k) = n \sum_{j=1}^{\lfloor n/k \rfloor} \frac{\varphi(j)}{j} - k \Phi(\lfloor n/k \rfloor) + O(n/k \log n). \quad (6)$$

The summation in Eq. (5) is split into two parts, the first part being for $k \leq \sqrt{n}$. The property $I_n(1/k) = O(n^2/k)$, see, e.g., [41], is used to give an upper bound to the summation corresponding to this first part. In the second part, $k > \sqrt{n}$, Eq. (6) is used. The two contributions are

given by

$$\begin{aligned} \sum_{k=1}^{\sqrt{n}} \varphi(k) I_n(1/k) &= O\left(n^2 \sum_{k=1}^{\sqrt{n}} \frac{\varphi(k)}{k}\right) = O(n^{5/2}), \\ \sum_{k > \sqrt{n}}^n \varphi(k) I_n(1/k) &= \sum_{i=1}^{\sqrt{n}} \sum_{k=\frac{n}{i+1}+1}^{n/i} \varphi(k) \left(n \sum_{j=1}^i \frac{\varphi(j)}{j} - k\Phi(i) + O(n/k \log n) \right), \end{aligned} \quad (7)$$

where we have used that $\sum_{k=1}^n \frac{\varphi(k)}{k} = 6n/\pi^2 + O[(\log n)^{2/3}(\log \log n)^{4/3}]$ [42]. Analyzing the first term inside the brackets gives

$$\begin{aligned} n \sum_{i=1}^{\sqrt{n}} \sum_{k=\frac{n}{i+1}+1}^{n/i} \varphi(k) \sum_{j=1}^i \frac{\varphi(j)}{j} &= n^3 \frac{3}{\pi^2} \sum_{i=1}^{\sqrt{n}} \left(\frac{1}{i^2} - \frac{1}{(i+1)^2} \right) \sum_{j=1}^i \frac{\varphi(j)}{j} = n^3 \frac{3}{\pi^2} \left(1 - \frac{\sum_{j=1}^{\sqrt{n}+1} \frac{\varphi(j)}{j}}{(\sqrt{n}+1)^2} + \sum_{i=2}^{\sqrt{n}+1} \frac{\varphi(i)}{i^3} \right) \\ &= n^3 \frac{3}{\pi^2} \sum_{i=1}^{\sqrt{n}+1} \frac{\varphi(i)}{i^3} + O(n^{5/2}), \end{aligned} \quad (8)$$

where we have used that $\sum_{k=1}^n \varphi(k) = 3n^2/\pi^2 + O(n \log n)$ together with telescoping sum and we have omitted a remainder term as $O(n^2 \log n)$. Following similar steps for the second term in the brackets of Eq. (7),

$$\begin{aligned} - \sum_{i=1}^{\sqrt{n}} \sum_{k=\frac{n}{i+1}+1}^{n/i} \varphi(k) k \Phi(i) &= -n^3 \frac{2}{\pi^2} \sum_{i=1}^{\sqrt{n}} \left(\frac{1}{i^3} - \frac{1}{(i+1)^3} \right) \Phi(i) = -n^3 \frac{2}{\pi^2} \left(\sum_{i=1}^{\sqrt{n}} \frac{\Phi(i)}{i^3} - \sum_{i=2}^{\sqrt{n}+1} \frac{\Phi(i) - \varphi(i)}{i^3} \right) \\ &= -n^3 \frac{2}{\pi^2} \left(1 - \frac{\Phi(\sqrt{n}+1)}{(\sqrt{n}+1)^3} + \sum_{i=2}^{\sqrt{n}+1} \frac{\varphi(i)}{i^3} \right) = -n^3 \frac{2}{\pi^2} \sum_{i=1}^{\sqrt{n}+1} \frac{\varphi(i)}{i^3} + O(n^{5/2}), \end{aligned} \quad (9)$$

where we have used that $\sum_{k=1}^n \varphi(k)k = 2n^3/\pi^2 + O(n^2 \log n)$ [42] and we have again omitted a remainder as $O(n^2 \log n)$. Recalling the following identity from [42],

$$\sum_{k=1}^{\infty} \frac{\varphi(k)}{k^3} = \frac{\zeta(2)}{\zeta(3)} = \frac{\pi^2}{6\zeta(3)},$$

we can put together the above results obtaining

$$T_n = \frac{2}{3\zeta(3)} n^3 + O(n^{5/2}).$$

The cubic coefficient agrees within 3 digits with the result of a fit presented in [14]. It is interesting to note that the resonance diagram has a factor 3 lower number of lines than the Farey diagram of order (n, n) [5].

V. SUMMARY

We have given a mathematical proof to the well known property that the largest resonance-free gaps happen next to low order resonances and that these are inversely ordered according to the resonance order (up to some given value). If the Riemann hypothesis is true, it implies that the

resonance sequence is significantly more regularly spaced than the expected behavior of the random sequence; however, this remains unproven and current results cannot even exclude similar regularity between the two sequences. An analytical estimate of the number of lines in the two-dimensional resonance diagram has been given in agreement with previous expectations.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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