

Dependence of the recoherence times and recoherence increments on the state of phonon bath in a single-qubit dephasing model

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The recoherence times t^* and the maximum values of the recoherence increments γ_{extr} are studied as functions of the bath parameters for a single qubit dephasing model, prepared initially by a special kind of the nonselective measurements. The recoherence-decoherence events (RDE), occurring at the initial stage of the system evolution, are found to be both similar and different from the system dynamics at large times. For instance, in contrast to the RDE observed on large timescales, the sub-Ohmic and Ohmic coupling regimes are more favorable for the short-time recoherence than the super-Ohmic one. On the other hand, the short-time behavior of the recoherence and the long-time dynamics of the decoherence are closely related: The domain of the ohmicity indexes, where the decoherence changes its type (from the complete to incomplete one), is, simultaneously, that of the weakest recoherence. The obtained results give us some hints about the basic characteristics of the environment, which might provide the most optimal values of t^* and γ_{extr} in some sense.

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I. INTRODUCTION

The quantum coherence (QC) along with the superposition of states and the entanglement is one of the most fascinating phenomena of quantum nature, which have been attracting a close attention of scientists for a long time [1]. Usually, when talking about QC, we mean two opposite tendencies in its manifestation. The unavoidable environmentally induced *decoherence* phenomenon is reasonably believed to play a major destructive role in the dynamics of two-state systems of qubits, which are the elementary carriers of quantum information [2,3].

On the other hand, the *recoherence* phenomenon (also referred to as the coherence enhancement [4] or coherence trapping [5]) is another pathway of the QC evolution, generally of a constructive nature. It occurs when a decay of QC is terminated at a certain instant, and the system can partly retain coherence for a finite time period or in the long-time limit, or even revive it. This manifestation of the QC duality was thoroughly studied for a variety of quantum systems such as the charge donor qubits [6], the electrons coupled to the quantized electromagnetic field [7], in the pigment-protein complexes used in the light harvesting technologies [8], in the quantum computer simulations [9], and many others. The recoherence also underlies the quantum error correction and the related techniques [10], thus having a significant practical application in quantum computing.

The recoherence is often studied from the viewpoint of the memory effects in the open quantum systems [11–13], invoking the notion of the so-called non-Markovianity [14,15]. The non-Markovian dynamics implies an information flow from the environment back to the open quantum system [16], yielding a growth of the trace distance between two initial quantum states and increasing their relative distinguishability [17]. Here the keyword is *the information backflow*, since in many cases (including the studies of the present paper), an energy exchange between the quantum open system and its environment is absent, and one can talk only about the entropy exchange expressed in terms of the Bloch vector length [18]. Within the master equation approach, it corresponds to the emergence of negative values of the dephasing rates [19–23], which, in its turn, is directly related to the above mentioned non-Markovian dynamics.

The recoherence is widely recognized to depend on the nature of the qubit-environment interaction [12,15,16]. It becomes noticeable if the ohmicity index s exceeds the value 2. Within the dephasing model [18], it corresponds to the incomplete (partial) decoherence at the long times $t \rightarrow \infty$. Usually, a typical hierarchy in the ohmicity indexes is observed: The bigger is the value of s , the more pronounced is the recoherence induced by the environment [15]. In a general case, a magnitude of the recoherence depends also on the bath temperature T . For instance, at high temperatures, the threshold value of s of the recoherence emergence increases up to $s = 3$. Even more remarkable phenomenon was reported in Ref. [16], where the decoherence-recoherence processes had been found to be governed both by the shape of the spectral weight function and by temperature. Therefore, a possibility to control the system coherence and to switch the direction of

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information fluxes at different times appears, offering a huge prospect for application in quantum information technologies.

All the above-mentioned conclusions concern the recoherence dynamics, when the open quantum system and its environment were chosen to be uncorrelated at the initial instant of system evolution, while the coherence enhancement was measured at large times. It means that at $t = 0$, the density matrix of the total system $\varrho(0)$ is chosen as a direct product, $\varrho(0) = \varrho_S \otimes \varrho_B$, of those of the open quantum (sub)system ϱ_S and the environment (the thermal bath) ϱ_B . However, back in the early 2000s a much more interesting behavior of the recoherence in the chain of quantum registers was reported [11] with more general initial conditions, where the initial state of the composite system is allowed to contain some correlations between the surrounding environment and the qubits. As for the dephasing model used in the present paper, the correlated initial states were studied in Ref. [24] in the context of the system purification, which is a direct consequence of the coherence enhancement.

These studies were continued in Ref. [4] by selection of a more physical initial conditions, imposed on the composite $(S + B)$ system by the *nonselective measurements* [25]. In Ref. [4], there was observed a sequence of recoherence-decoherence events (RDE) occurring at the initial stage of the system evolution. A special kind of the nonselective measurements has been proposed, which ensures an indubitable coherence enhancement regardless of the qubit state vectors, as long as the “qubit-bath” interaction strength is large enough. Obviously, at larger times the vacuum and thermal fluctuations will overcome the impact of initial correlations, and one comes back to the cases described above, when RDE are possible only in the advanced super-Ohmic coupling regimes with $s > 2$. It is also worthwhile to mention that the initial qubit-bath correlations can also suppress the QC [18], as happens at the *selective measurements* [25], when an additional channel of the decoherence appears.

Though in most cases the recoherence processes are studied at long times, when the impact of initial correlations has already finished, the short-time dynamics of RDE is also important if one deals, e.g., with a dynamic decoupling technique to freeze the quantum evolution of the system [26] or performs the repeated measurements on it [27]. The short-time RDE of a qubit subjected to the nonselective measurements were studied in detail in Refs. [28,29] by the time-series expansion that allowed also to count the total number of RDE as the function of coupling strength, ohmicity index, and bath temperature. However, even though one deals with a comparatively short times (as compared to the inverse rate of decoherence) when studying an influence of the correlation contribution on the coherence enhancement [4,27], a total period of the QC growth can be large enough to use a reliable truncation of the above series. Therefore, no analytical expression for the dephasing rates (as those derived in Ref. [29]) can be obtained, and only numerical calculation of the recoherence times and increments is plausible. At the same time, a study of dependence of the recoherence times and increments, caused by the initial correlations, on the bath characteristics remains a relevant problem in many cases.

In this paper, the recoherence times t^* and the maximum values of the recoherence increments γ_{extr} are studied as

functions of coupling strengths, ohmicity indexes, and bath temperature. We found that in contrast to the RDE observed at long times, the sub-Ohmic ($s < 1$) and Ohmic ($s = 1$) regimes are more favorable for the recoherence than the super-Ohmic one ($s > 1$). The most interesting observation is that the short-time behavior of the recoherence and the long-time dynamics of the decoherence can be interrelated: the domain $s \geq 2$, where the decoherence changes its type (from the complete to incomplete one [18]), is, simultaneously, that of the weakest recoherence. The obtained results give us some insights about the basic characteristics of the environment, which can provide the most optimal values of t^* and γ_{extr} for an application in the quantum measurement experiments.

The paper is structured as follows. In Sec. II, we recall the basic relations corresponding to the nonselective quantum measurements. A special attention is paid to the measurement states, which provide a diagonal form of the Gram operator [4,27]. In Sec. III, starting from the Hamiltonian of the one qubit dephasing model, we present the expressions for the generalized decoherence function $\gamma_{\text{tot}}(t)$ and its constituent $\gamma_{\text{cor}}(t)$ related to initial correlations in the composite system, which is essentially simplified in the case of the diagonal Gram operator. In Sec. IV, we perform the numerical calculation of the recoherence times and the recoherence increments depending on the bath parameters: The coupling strength, the ohmicity index, and the temperature. The obtained results are discussed in detail in their relation to a critical regime of the recoherence emergence. In the last section, we conclude our studies by discussing the potential applications of the presented scheme in quantum computing and measurements and outline some problems for future investigation.

II. QUBIT INITIAL STATE PREPARATION BY THE NONSELECTIVE MEASUREMENTS

Suppose that at all times $t < 0$ an open system S was in the thermal equilibrium with a heat bath B , and at time zero one makes a measurement on the system S only. According to the general principles of quantum measurement theory [25,30,31], the initial state of the composite system $(S + B)$ after the measurement is described by the density matrix

$$\varrho_{SB}(0) = \sum_m \Omega_m \varrho_{\text{eq}} \Omega_m^\dagger, \quad (1)$$

where ϱ_{eq} means the equilibrium density matrix at temperature T . Operators Ω_m act in the Hilbert space of the system S and correspond to possible outcomes m of the measurement. For the spin-1/2 system, the dimensionality of the Hilbert space is $N = 2$ and thus $m = \{1, 2\}$. However, the nonselective measurements on the overcomplete basis with $m > N$ are also possible [27], and this more general scheme is usually associated with the notion of the positive operator-valued measure [32].

In order to the initial density matrix (1) be normalized to unity,

$$\text{Tr} \varrho_{SB}(0) = \sum_m \text{Tr} \{ \varrho_{\text{eq}} \Omega_m^\dagger \Omega_m \} = 1, \quad (2)$$

the measurement operators Ω_m have to obey the following relation:

$$\sum_m \Omega_m^\dagger \Omega_m = I, \quad (3)$$

where I denotes the identity operator. Hereafter, Tr denotes the trace over the Hilbert space of the composite ($S + B$) system, while the symbols Tr_S and Tr_B are used to denote the partial traces over the Hilbert spaces of the system S and the heat bath, respectively.

The precise form of Ω_m is determined by the details of the measuring device [25]. In some particular cases, the measurement operators Ω_m can be defined [4,27] as

$$\Omega_m = |\varphi_m\rangle\langle\psi_m|. \quad (4)$$

From a mathematical viewpoint, Ω_m can be treated as some kind of the “skew-projecting” operators defined on the pure states $|\varphi_m\rangle$ and $|\psi_m\rangle$. Obviously, these sets of the measurement states can be related to each other via some unitary transformations,

$$|\varphi_m\rangle = U_m |\psi_m\rangle. \quad (5)$$

Hence the nonselective measurement scheme (1) can be considered as a combined effect of the selective (projective) measurements [18,25], the unitary transformations (5), and the subsequent mixing of the obtained quantum states with the probabilities $w(m) = \text{Tr}\{\Omega_m^\dagger \Omega_m \rho_{\text{eq}}\}$.

One essential remark is in order: Although the states $|\psi_m\rangle$ form an orthonormal complete basis according to (3), this condition holds true for the transformed states $|\varphi_m\rangle$ only if the unitary operators U_m are identical for all outcomes m (i.e., $U_m = U$). In such cases, taking into account that $\Omega_m \Omega_m^\dagger = |\varphi_m\rangle\langle\varphi_m|$ and introducing the Gram operator $G = \sum_m \Omega_m \Omega_m^\dagger$ [4,27], we have another resolution of the unity in addition to Eq. (3),

$$G = \sum_m \Omega_m \Omega_m^\dagger = I. \quad (6)$$

If the form of U_m depends on the outcome m , then the transformed states $|\varphi_m\rangle$ are not orthogonal in general. Some of these states may even be identical [4]. In our research, we will consider the special measurement schemes, when the corresponding Gram operator either satisfies condition (6) or is constructed to be diagonal.

In Ref. [4] we studied in detail various choices of the measurement vectors, namely:

(i) the simplest scheme $|\psi_m\rangle = |\varphi_m\rangle$ corresponding to $U_m = I$ in the expression (5). It occurs when the measuring device does not disturb the basis states during this kind of the nonselective measurement;

(ii) the scheme defined by Eq. (6), when measurement vectors $|\varphi_m\rangle$ also form the complete basic set. In this context, scheme (i) can be viewed as a particular realization of the present case with $U_m = U = I$;

(iii) the most general measuring scheme with an arbitrary Gram operator, when the unitary operators U_m depend on the outcomes m .

As a particular choice of the measurement scheme (iii) that ensures a diagonal form of the Gram operator, we considered the pair of two quantum states (in a standard spin-1/2

representation),

$$|\varphi_1\rangle = c_0|0\rangle + c_1|1\rangle, \quad |\varphi_2\rangle = i(c_0|0\rangle - c_1|1\rangle), \quad (7)$$

with an arbitrary amplitudes c_0 and c_1 . It can be easily shown that for the chosen basic set (7), the corresponding Gram operator reads as

$$G = 2 \begin{pmatrix} |c_1|^2 & 0 \\ 0 & |c_0|^2 \end{pmatrix}. \quad (8)$$

At real amplitudes c_i , for the unitary operators relating the sets $|\psi_i\rangle$ and $|\varphi_i\rangle$ an explicit form for U_m can be found as

$$U_1 = I, \quad U_2 = \sigma_1. \quad (9)$$

In a similar way to (7), one can consider another pair of the state vectors $|\psi_i\rangle$ forming the diagonal Gram operator, which are related to the subset $|\psi_i\rangle$ via the unitary transformations

$$U_1 = \sigma_3, \quad U_2 = \sigma_2. \quad (10)$$

It is seen from Eqs. (9) and (10) that the unitary transformations can be constructed on the basis of the identity matrix and the Dirac operators σ_i , $i = \{1, 2, 3\}$. In the general case, the explicit form of the unitary operators U_i , yielding the diagonal Gram matrix $G = \text{diag}$, can be derived from the following relation:

$$U_1 |\psi_1\rangle\langle\psi_1| U_1^\dagger - U_2 |\psi_1\rangle\langle\psi_1| U_2^\dagger = G - I, \quad (11)$$

where $|\psi_1\rangle$ means an arbitrary pure quantum state.

At the end of this section we have to note that the projective measurements with $|\varphi_m\rangle = |\psi_m\rangle$ (or “skew-projective” ones (4) and (5) with $|\varphi_m\rangle \neq |\psi_m\rangle$) can be considered only as a certain subset of more general measurements performed for preparation of the composite system in a particular initial state. In Ref. [26], the state preparation via some kind of the unitary operation was studied in detail. It was shown that at this initial conditions, the dynamics of the scaled observable $\langle\sigma_1\rangle/N$ for large spin number $N \gg 1$ was close to that obtained by the projective measurement protocol. Obviously, in this case the condition $\Omega\Omega^\dagger = \Omega^\dagger\Omega \equiv I$ is satisfied by definition and can also be applied to obtain a diagonal Gram operator.

III. THE DEPHASING MODEL: DYNAMICS OF DECOHERENCE AT THE NONSELECTIVE MEASUREMENTS

In this section, we briefly analyze the simple *dephasing model* describing the main decoherence mechanism for certain types of system-environment interactions [18,24,33–35]. In this model, the two-state system (qubit) (S) is coupled to a bath (B) of harmonic oscillators. The total Hamiltonian is taken to be (in our units $\hbar = 1$)

$$H = H_S + H_B + H_{\text{int}} \\ = \frac{\omega_0}{2} \sigma_3 + \sum_k \omega_k b_k^\dagger b_k + \sigma_3 \sum_k (g_k b_k^\dagger + g_k^* b_k), \quad (12)$$

where ω_0 is the energy difference between the excited state $|1\rangle$ and the ground state $|0\rangle$ of the qubit. Bosonic creation and annihilation operators $b_k^\dagger$ and b_k correspond to the k th bath mode with frequency ω_k , and g_k are the coupling constants.

Suppose that at time $t = 0$ the state of the composite system ($S+B$) is characterized by some density matrix $\varrho_{SB}(0)$. Then at time t , the average value of a Heisenberg picture operator $A(t)$ is given by

$$\langle A(t) \rangle = \text{Tr}\{\exp(iHt)A\exp(-iHt)\varrho_{SB}(0)\}. \quad (13)$$

Hereafter, the notation $\langle A \rangle$ is used for the averages at $t = 0$.

The quantities of principal interest are the *coherences* $\langle \sigma_{\pm}(t) \rangle$, where $\sigma_{\pm} = (\sigma_1 \pm i\sigma_2)/2$. They are related directly to the off-diagonal elements of the reduced density matrix of the qubit:

$$\langle \sigma_+(t) \rangle = \langle 0|\varrho_S(t)|1 \rangle, \quad \langle \sigma_-(t) \rangle = \langle 1|\varrho_S(t)|0 \rangle, \quad (14)$$

where

$$\varrho_S(t) = \text{Tr}_B\{\exp(-iHt)\varrho_{SB}(0)\exp(iHt)\}. \quad (15)$$

Within the dephasing model (12), the equations of motion for all relevant operators are solvable exactly [18]. This allows one to study the time evolution of the coherences for different initial preparation conditions. We leave out some less essential details for which we refer the reader to Refs. [4,18] and quote the major results.

If the initial state is prepared by the nonselective measurement, then, taking the initial density matrix of the composite system in the form (1), we get the following expression for the coherences (14):

$$\langle \sigma_{\pm}(t) \rangle = \frac{1}{Z} \sum_m \text{Tr}[\Omega_m^\dagger \sigma_{\pm}(t) \Omega_m e^{-\beta H}], \quad (16)$$

where $\beta = 1/T$ (we put the Boltzmann constant to be equal to 1) and $Z = \text{Tr}\{\exp(-\beta H)\}$ is the equilibrium partition function.

The trace over the bath degrees of freedom in Eq. (16) is carried out straightforwardly, yielding the expression

$$\langle \sigma_{\pm}(t) \rangle = \langle \sigma_{\pm} \rangle e^{\pm i\omega_0 t} e^{-\gamma(t)} \frac{\sum_m \{\langle 0|\Omega_m^\dagger \sigma_{\pm} \Omega_m|0 \rangle e^{\beta\omega_0/2 \pm i\Phi(t)} + \langle 1|\Omega_m^\dagger \sigma_{\pm} \Omega_m|1 \rangle e^{-\beta\omega_0/2 \mp i\Phi(t)}\}}{\sum_m \{\langle 0|\Omega_m^\dagger \sigma_{\pm} \Omega_m|0 \rangle e^{\beta\omega_0/2} + \langle 1|\Omega_m^\dagger \sigma_{\pm} \Omega_m|1 \rangle e^{-\beta\omega_0/2}\}} \quad (17)$$

with the initial coherences

$$\langle \sigma_{\pm} \rangle = \frac{1}{2 \cosh(\beta\omega_0/2)} \sum_m \{\langle 0|\Omega_m^\dagger \sigma_{\pm} \Omega_m|0 \rangle e^{\beta\omega_0/2} + \langle 1|\Omega_m^\dagger \sigma_{\pm} \Omega_m|1 \rangle e^{-\beta\omega_0/2}\}. \quad (18)$$

Equation (17) contains two relevant functions. The so-called *generalized decoherence function* $\gamma(t)$ is defined as

$$\gamma(t) = \int_0^\infty d\omega J(\omega) \coth(\beta\omega/2) \frac{1 - \cos \omega t}{\omega^2}, \quad (19)$$

where the spectral density $J(\omega)$, which characterizes the qubit-bath interaction, is introduced by the rule

$$\sum_k 4|g_k|^2 f(\omega_k) = \int_0^\infty d\omega J(\omega) f(\omega). \quad (20)$$

In (20), the continuum limit of the bath modes is performed. Usually, the spectral density function is taken in the form

$$J(\omega) = \lambda \omega_c^{1-s} \omega^s e^{-\omega/\omega_c}, \quad (21)$$

which is most commonly used in the theory of spin-boson systems [24,25,33,36]. Here $\lambda \sim |g_k|^2$ is a dimensionless coupling constant; s denotes the ohmicity index, and ω_c denotes some “cut-off” frequency. In other words, the constant λ defines the interaction strength; the index s determines the nature of interaction ($0 < s < 1$ corresponds to the sub-Ohmic case, $s = 1$ to the Ohmic coupling regime, and $s > 1$ to the super-Ohmic one), while ω_c defines the shape of the spectral density at large frequencies and ensure the finiteness of all the frequency moments. The spectral density function can also be modified by introducing the logarithmic perturbations of the power-law profiles [16].

The generalized decoherence function $\gamma(t)$ is precisely the quantity which determines the relaxation of the off-diagonals (14) due to vacuum and thermal fluctuations in the bath

[18]. It follows from Eq. (19) that the generalized decoherence function is always positive. In the absence of initial correlations [18], when the density matrix of the composite system at $t = 0$ can be written as a direct product $\varrho(0) = \varrho_S \otimes \varrho_B$ of the spin contribution ϱ_S and the bath density matrix $\varrho_B = \exp(-\beta H_B)/\text{Tr}_B \exp(-\beta H_B)$ taken at the thermal equilibrium, the fraction in (17) converts to the unity. In such a case

$$\frac{d|\langle \sigma_{\pm}(t) \rangle|}{dt} = -\frac{d\gamma(t)}{dt} |\langle \sigma_{\pm}(t) \rangle|. \quad (22)$$

The time derivative of $\gamma(t)$ can change its sign, being proportional to sine function in the integrand of (19). Usually [12,15,16], the emergence of RDE is associated with the non-monotonic behavior of $|\langle \sigma_{\pm}(t) \rangle|$ and occurs in the advanced super-Ohmic regimes. As mentioned in the Introduction, the number of RDE increases with s , and they begin to occur at shorter times.

The other function entering Eq. (17), $\Phi(t)$, is given by

$$\Phi(t) = \int_0^\infty d\omega J(\omega) \frac{\sin \omega t}{\omega^2}. \quad (23)$$

As discussed in Ref. [18], this function accounts for the influence of the initial qubit-environment correlations on the dynamics of decoherence. These correlations are inherited from the postmeasurement initial state due to the presence of the interaction term in the Hamiltonian (12).

In general, the expression (17) can be rewritten more transparently as

$$\langle \sigma_{\pm}(t) \rangle = \langle \sigma_{\pm} \rangle \exp[\pm i(\omega_0 t + \chi(t))] \exp[-\gamma_{\text{tot}}(t)], \quad (24)$$

where

$$\gamma_{\text{tot}}(t) = \gamma(t) + \gamma_{\text{cor}}(t) \quad (25)$$

is the total decoherence function including the correlation contribution $\gamma_{\text{cor}}(t)$. The expressions for $\gamma_{\text{cor}}(t)$ and the phase

shift $\chi(t)$ can be obtained explicitly by performing of a polar decomposition in the fraction of (17) [see Eqs. (47)–(49) in Ref. [4]] and are expressed in quite a complicated way via the Euler's angles, which define orientations of the measurement state vectors $|\psi_m\rangle, |\varphi_m\rangle$ on the Bloch sphere.

Two important conclusions have to be made. While the combined action of the vacuum and thermal fluctuations is of the first order in the coupling constant λ [see Eq. (19)], the correlation contribution $\gamma_{\text{cor}}(t)$ contains all orders in λ , being a nonlinear function of $\Phi(t)$, as it clearly follows from Eq. (17). The same is true for the phase shift $\chi(t)$, which defines the renormalization of the qubit energy due to the initial correlations with the thermal bath [4,18].

On the other hand, the correlation contribution $\gamma_{\text{cor}}(t)$ to the generalized decoherence function (25) oscillates with time. Depending on the initial qubit preparation (expressed, e.g., via the above mentioned Euler's angles), it can change its sign at some particular times. It means that the periods of decoherence and recoherence can change each other [28,29]. The first scenario occurs when the total decoherence function is positive, $\gamma_{\text{tot}}(t) > 0$. In the opposite case, if $\gamma_{\text{tot}}(t) < 0$, then the coherence enhancement takes place [4,27,29] at small times, and one can speak about calculation of the coherence increment rather than decrement. At larger times, the joint contribution of the vacuum and thermal fluctuations expressed by (19) definitely exceeds [4,18] the effect of $\gamma_{\text{cor}}(t)$, making the total decoherence function (25) positive and leading to either complete or partial system decoherence [37].

Rather cumbersome expressions for $\gamma_{\text{cor}}(t)$, obtained in Ref. [4] for the general nonselective measurements, in a particular case can be simplified considerably using the following reasoning. Using the polar decomposition, the correlation contribution in Eq. (17) can be rewritten as

$$\exp[\pm i\chi(t) - \gamma_{\text{cor}}(t)] = \cos \Phi(t) \pm i\mathcal{A} \sin \Phi(t), \quad (26)$$

where

$$\mathcal{A} = \frac{\sum_m \{ \langle 0 | \Omega_m^\dagger \sigma_\pm \Omega_m | 0 \rangle e^{\beta\omega_0/2} - \langle 1 | \Omega_m^\dagger \sigma_\pm \Omega_m | 1 \rangle e^{-\beta\omega_0/2} \}}{\sum_m \{ \langle 0 | \Omega_m^\dagger \sigma_\pm \Omega_m | 0 \rangle e^{\beta\omega_0/2} + \langle 1 | \Omega_m^\dagger \sigma_\pm \Omega_m | 1 \rangle e^{-\beta\omega_0/2} \}}. \quad (27)$$

It is seen from Eq. (27) that the equality

$$\begin{aligned} & \sum_m \{ \langle 0 | \Omega_m^\dagger \sigma_\pm \Omega_m | 0 \rangle + \langle 1 | \Omega_m^\dagger \sigma_\pm \Omega_m | 1 \rangle \} \\ &= \text{Tr}_S \left(\sigma_\pm \sum_m \Omega_m \Omega_m^\dagger \right) = 0, \end{aligned} \quad (28)$$

which means a requirement for the Gram operator to be of the diagonal form, gives the quite simple value $\mathcal{A} = \coth(\beta\omega_0/2)$. At such a condition, the correlation contribution to the generalized decoherence function can be written down as follows:

$$\gamma_{\text{cor}}(t) = -\frac{1}{2} \ln \left[1 + \frac{\sin^2 \Phi(t)}{\sinh^2(\beta\omega_0/2)} \right]. \quad (29)$$

Since the values of (29) are nonpositive at any time, this condition ensures the coherence enhancement at the initial stage of the system evolution if $\gamma_{\text{tot}}(t) < 0$, until the vacuum and thermal fluctuations of the bath terminate the growth of $\langle \sigma_\pm(t) \rangle$ and give rise to the decoherence processes with

$\gamma_{\text{tot}}(t) > 0$. If condition (28) is not fulfilled, then the form of $\mathcal{A}$ is much more complicated. In such a case, the domains of eventual coherence enhancement at some times can be interchanged with the regions of the intensified decoherence at other times [28,29], depending on the values of the measurement state vectors, the qubit-bath interaction strength, the ohmicity index and temperature.

To conclude this section, let us make two essential remarks. First, unlike the commonly accepted approach [15], we will associate the RDE with termination of the negative branch of $\gamma_{\text{tot}}(t)$ rather than with a nonmonotonic behavior of $|\langle \sigma_\pm(t) \rangle|$. In other words, we are interested in the periods of time, when the running value of $|\langle \sigma_\pm(t) \rangle|$ exceeds its initial value at $t = 0$; we call this effect the recoherence (or coherence enhancement). More precisely, such a definition is suitable for treating the recoherence phenomenon as *the relative coherence enhancement* at short times.

Second, it is seen that under the condition (28), the recoherence increment (29) does not depend on the initial state of the qubit. It differs from both the general case of nonselective measurements [4] and the initial preparation of the system by the selective measurement (see Eq. (31) in Ref. [18]), when the qubit-bath initial correlation opens an additional channel of the system decoherence, which depends on the mean value $\langle \sigma_3 \rangle$. Since at the condition (28) both components of the function (25) depend only on the state of the thermal bath, an interesting problem arises: to study how the parameters of the qubit environment (the interaction strength, the ohmicity, and the bath temperature) influence the coherence enhancement at the initial state of the system evolution. An answer to this question is the topic of the Sec. IV.

IV. AN INFLUENCE OF THE THERMAL BATH CHARACTERISTICS ON THE QUBIT COHERENCE ENHANCEMENT

Before presenting results for the recoherence times and recoherence increments, we explain our motivation to focus on the nonselective measurements with the diagonal Gram operator [including the case $G = I$, see Eq. (6)]. Though the general case exhibits much richer RDE dynamics [4,28,29], it is the choice (29) that allows one to exclude the influence of the spin state on the time behavior of the open quantum system at the initial stage of its evolution and to study solely the action of environment. In the case under study, the spin state defines only the initial values of the coherence, which are considerably simplified at the measurement scheme with $G = \text{diag}$ (see, e.g., Eqs. (61) and (67) in Ref. [4]).

However, there are also some other cases, when it becomes crucial to minimize an influence of the initial spin state on the open system dynamics, allowing it to be governed mainly by the bath degrees of freedoms. For instance, at the study of time behavior of the quantum concurrence of two noninteracting spins within the identical dephasing model [38], the pure nonentangled initial spin state was chosen as $\varrho_S(0) = |+, +\rangle\langle +, +|$, where $|+\rangle$ denotes the eigenstate of the first Pauli matrix, $\sigma_1|+\rangle = |+\rangle$. Under such a choice, all elements of the initial spin density matrix $\varrho_S(0)$ are equal to 1/4. Since the eigenvalue calculation, arising in such a problem [39,40], is very sensitive to the coefficients at the elements of the

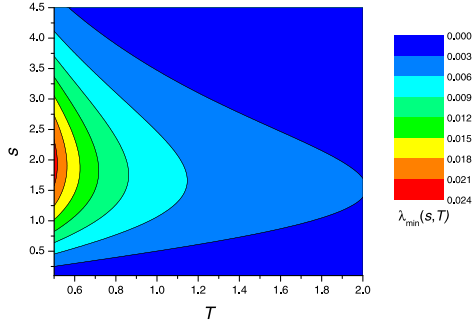


FIG. 1. The contour plot of the critical coupling constant $\lambda_{\min}(s, T)$ as the function of temperature T and ohmicity parameter s . The moderate temperature regime is displayed.

reduced density matrix, and it is essential for the initial two-qubit state not to be part of a decoherence-free subspace, the above-mentioned choice looks optimal if one focuses solely on the study of the bath influence on the qubits entanglement.

The interplay between recoherence and decoherence in the general case of an arbitrary Gram operator was thoroughly studied in Ref. [28]: The total number of RDE depending on the qubit and bath parameters was calculated, and the numerical and analytic results for the recoherence times at the Ohmic coupling were obtained. However, an analytical study of RDE is limited to essentially short times [29], when the recoherence times can be expressed via the frequency moments $\langle \omega^n \rangle \equiv \int_0^\infty \omega^n J(\omega) d\omega$ with $n = 1, 2$. But even then, the huge number of possible initial qubit states does not allow one to systemize the obtained results with respect to the influence of the thermal bath on the coherence dynamics.

To begin with the study of the recoherence processes, let us remind that the joint vacuum and thermal contribution (19) to the total decoherence function (25) is linear in the coupling strength, while the series expansion of its correlation counterpart (29) starts from the quadratic term and contains all even powers in λ terms. Thus the coherence enhancement is a threshold phenomenon: At small qubit-bath interaction, the linear term associated with (19) prevails, and we have pure decoherence at all times. At larger coupling strengths, $\gamma_{\text{cor}}(t)$ dominates, ensuring the coherence enhancement at small times. The threshold value $\lambda_{\min}$, which separates the pure decoherence behavior at $\lambda < \lambda_{\min}$ from the coherence enhancement at $\lambda > \lambda_{\min}$, can be found analytically by equating to zero the second derivative of the function (25) with respect to time. After some algebra, one finds

$$\lambda_{\min}(s, T) = \frac{(T/\omega_c)^{s+1} \Gamma(s+1) \sinh^2(\omega_0/2T)}{(s-1)^2 \Gamma(s-1)^2} \times \{\zeta(s+1, 1 + T/\omega_c) + \zeta(s+1, T/\omega_c)\}, \quad (30)$$

where $\zeta(s, a)$ denotes the generalized Riemann zeta function.

In Fig. 1, the contour plot describing the $(s - T)$ dependence of the critical coupling constant $\lambda_{\min}$ is displayed [41]. Hereafter, all the parameters are presented in the dimensionless form with respect to the cut-off frequency ω_c , namely $\omega_0 \rightarrow \omega_0/\omega_c$, $T \rightarrow T/\omega_c$, and $t \rightarrow \omega_c t$. Besides, we take $\omega_c = 1$ and $\omega_0 = 0.1$ in all our calculations.

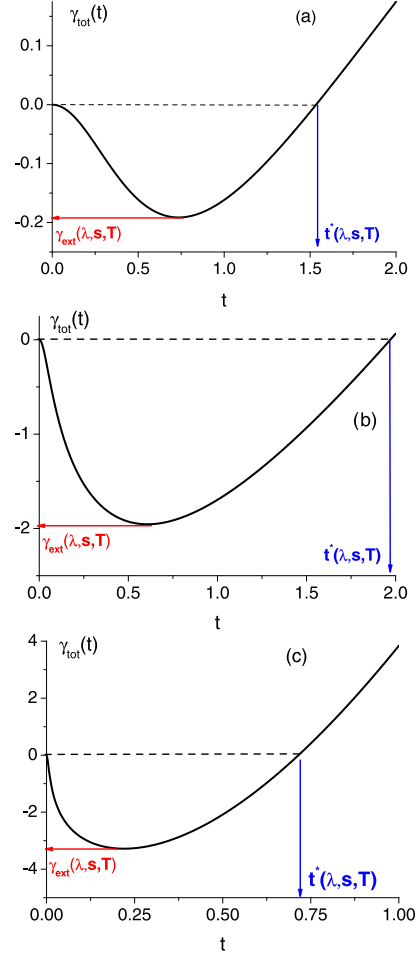


FIG. 2. Time dependence of $\gamma_{\text{tot}}(t)$ at the Ohmic regime $s = 1$ and intermediate coupling $\lambda = 1$ at three different temperatures: (a) $T = 0.1$, (b) $T = 1$, and (c) $T = 10$.

One can see that the maximum values of $\lambda_{\min}$ shift with temperature from the advanced super-Ohmic regime with $s > 2$ to about $s = 1.5$. It is obvious that high temperatures are more favorable for smaller values of $\lambda_{\min}$, while at low T the recoherence occurs at the strong coupling only. It is quite expected, since according to Eq. (29) the correlation contribution $\gamma_{\text{cor}}(t)$ decreases at low temperatures, and there is no other option but to increase the qubit-bath interaction to maintain the eventual recoherence.

Now let us focus our attention on the time dependence on the total decoherence function (25). As has been already said, the negative values of $\gamma_{\text{tot}}(t)$ correspond to the coherence enhancement and define the coherence increment. On the contrary, the case $\gamma_{\text{tot}}(t) > 0$ is related to the decoherence and defines its decrement.

In Fig. 2, we show the time behavior of $\gamma_{\text{tot}}(t)$ at three different temperatures in the Ohmic regime and intermediate coupling. Since the coupling strength $\lambda = 1$ exceeds the critical value $\lambda_{\min}$, one observes the initial recoherence, lasting until the curve $\gamma_{\text{tot}}(t)$ intersects the abscissa axis. We marked this coordinate, indicated by the blue arrow, as $t^*(\lambda, s, T)$ and defined this value as the *recoherence time*. Correspondingly, the minimal value of (25), indicated by the red arrow, is labeled as γ_{ext} . This value is a gradually decreasing function of

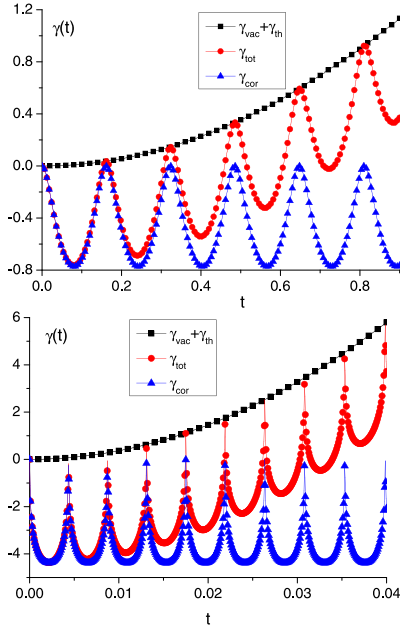


FIG. 3. Time dependence of different contributions to the generalized decoherence function, calculated at $\lambda = 1$, $s = 0.05$, $T = 0.1$ (top panel), and $\lambda = 1$, $s = 7$, $T = 4$ (bottom panel). Red circles correspond to the total decoherence function (25), black squares to the sum of the vacuum and thermal contributions (19), and blue triangles to the correlation contribution (29).

temperature, as it directly follows from Eq. (29). It defines the maximum coherence increment $\Gamma_{\max} = |\gamma_{\text{extr}}|$, also gradually increasing with T .

It has to be emphasized that within the usual definition of the RDE [15,16], the first recoherence-decoherence event

begins at the moment, when the time derivative of $\gamma_{\text{tot}}(t)$ changes its sign. To comply with this widely accepted definition, we would have to redefine the recoherence time as the time indicated by the red arrow rather than blue one in Fig. 2. With this redefinition, the general results for t^* shown later in the top panels of Figs. 4 and 5 would not change essentially, and all the basic qualitative trends would remain the same.

At the special choice of the bath parameters, there appears a sequence of the RDE at short times, as it is clearly seen in Fig. 3. The oscillations of $\gamma_{\text{cor}}(t)$ are strongly enhanced both in the advanced sub-Ohmic and super-Ohmic regimes, since the function $\Phi(t)$ [the argument of the sine in Eq. (29)] behaves as $\lambda t/s$ at small ohmicity indexes and grows as $(s-1)!$ at large integer s . It can be easily verified by expanding the Γ functions entering the expressions for $\Phi(t)$; see, e.g., Ref. [18]. Thus the correlation contribution to the total decoherence function is proven to be strictly periodic in time at the advanced sub-Ohmic regime, whereas at large s this is true only at small t . At large t , its behavior becomes more complicated and the periods of an apparent periodicity can be interspersed with those of clear aperiodicity.

The vacuum and thermal fluctuations of the bath yield the gradually increasing generalized decoherence function (19), which forms the envelope curve touching $\gamma_{\text{tot}}(t)$ at its maxima points. These maxima are smooth (top panel) or sharp (bottom panel) depending on the values of s and the bath temperature. However, in both cases a strongly oscillating correlation contribution $\gamma_{\text{cor}}(t)$ to the total decoherence function forms a sequence of the RDE. In our definition, the recoherence time t^* corresponds to the first intersection of the abscissa axis by the red line.

One can also introduce a notion of the *total recoherence time*. Let us consider the time intervals $\tau_i = [t_{i-1}, t_i]$

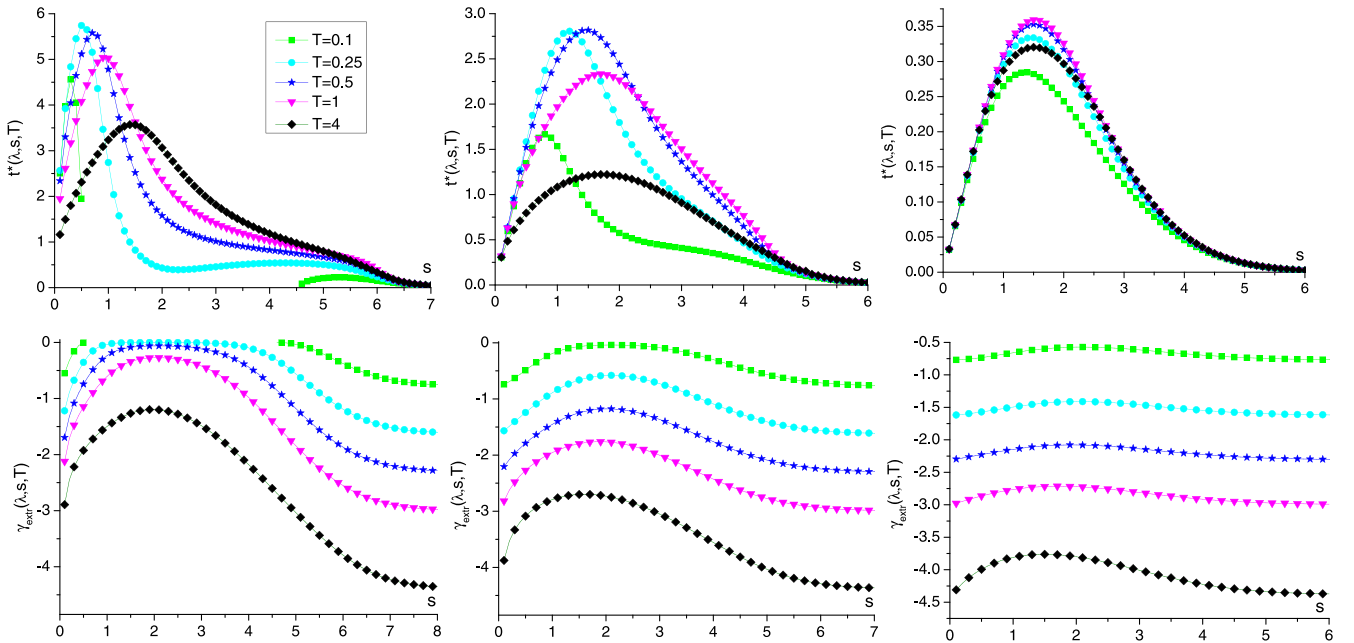


FIG. 4. Top row: The recoherence times $t^*(\lambda, s, T)$ in the low- (left panel, $\lambda = 0.1$), moderate- (central panel, $\lambda = 1$), and high- (right panel, $\lambda = 10$) coupling regimes at different temperatures T as functions of the ohmicity index s . Bottom row: The same for the minimum values $\gamma_{\text{extr}}(\lambda, s, T)$ of the total decoherence function $\gamma_{\text{tot}}(t)$.

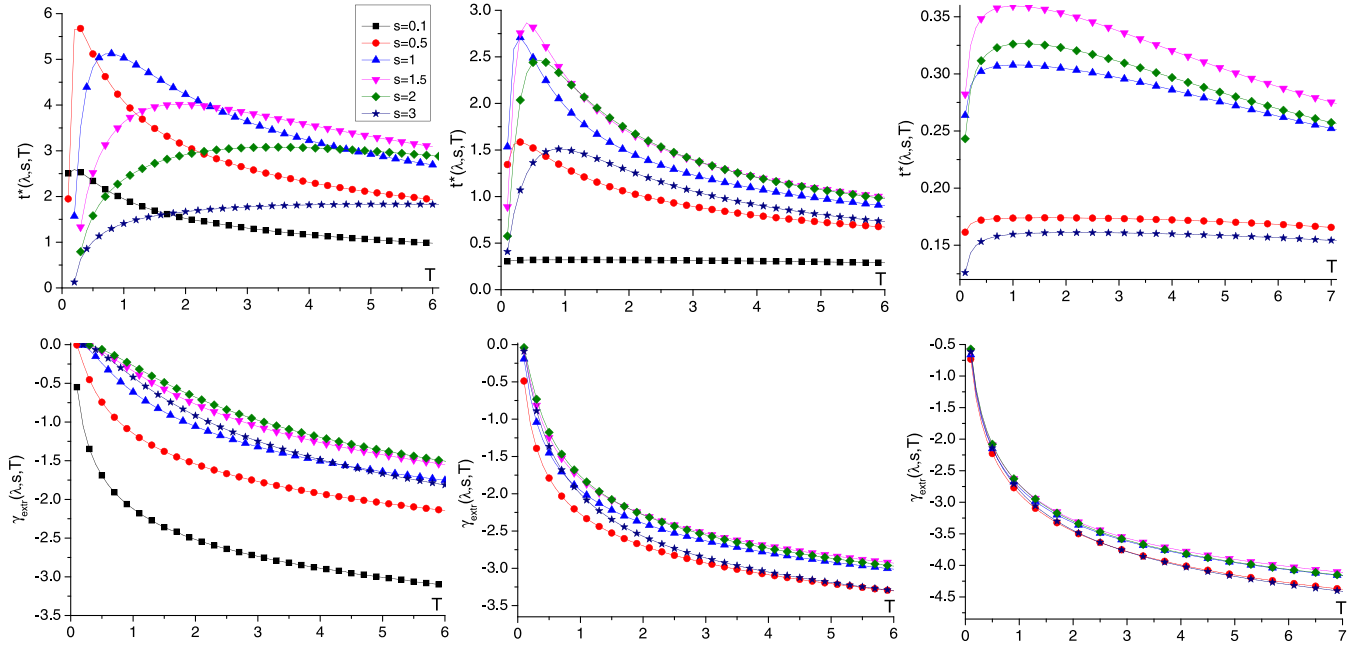


FIG. 5. Top row: The recoherence times $t^*(\lambda, s, T)$ in the low- (left panel, $\lambda = 0.1$), moderate- (central panel, $\lambda = 1$), and high- (right panel, $\lambda = 10$) coupling regimes at different ohmicity indexes s as functions of temperature T . Bottom row: The same for the minimum values $\gamma_{\text{extr}}(\lambda, s, T)$ of the total decoherence function $\gamma_{\text{tot}}(t)$.

formed by two successive instants t_{i-1} and t_i , between which $\gamma_{\text{tot}}(\tau_i) < 0$. The function $\gamma_{\text{tot}}(t)$ can be shown to have a negative branch only if

$$\gamma(t) \leq \ln[\coth(\beta\omega_0)/2]. \quad (31)$$

The expression in the right-hand side of the inequality (31) corresponds to the maximum value of $|\gamma_{\text{cor}}(t)|$, as it can be easily derived from Eq. (29). In other words, the number of recoherence events equals the maximum value i_{max} of the integer i , at which Eq. (31) still holds true. Indeed, it occurs if the (positive) values of $\gamma(t)$ are small enough to “drag” the whole blue curve (including its minima points, which do not depend on the ohmicity index s) upward until the curve is in the upper semiplane completely.

Now it is natural to introduce the total recoherence time as $t_{\text{tot}}^* = \sum_{i=1}^{i_{\text{max}}} \tau_i$. It is seen from the bottom panel of Fig. 3 that in the advanced super-Ohmic regime at high temperature, the total recoherence time is approximately equal to $(i_{\text{max}} - 1)$ times the period of $\gamma_{\text{cor}}(t)$, since the total decoherence function $\gamma_{\text{tot}}(t)$ changes sharply at its upper part. In the advanced sub-Ohmic regime and at low temperatures (top panel in Fig. 3), the behavior of $\gamma_{\text{cor}}(t)$ is rather smooth. The total recoherence time has to be calculated numerically as the sum of all τ_i .

In Fig. 4, we plot the dependence of the recoherence times $t^*(\lambda, s, T)$ and the minimum values $\gamma_{\text{extr}}(\lambda, s, T)$ of the total decoherence function $\gamma_{\text{tot}}(t)$ on the ohmicity index s at different temperatures starting from $T = 0.1$ up to $T = 4$. The results are presented for three coupling regimes with interaction strengths $\lambda = 0.1, 1$, and 10 , respectively. In the advanced sub-Ohmic ($s \ll 1$) and super-Ohmic ($s > 5$) regimes, we used the adopted earlier definition of t^* : The recoherence times are determined as the first intersection point of the corresponding curves with the abscissa axis; see Fig. 3. Hence no t_{tot}^* has been considered. At $0.2 < s < 4$ there is always a

single intersection point, so there is no need to talk about the total recoherence time.

It is seen in Fig. 4 that the maximum recoherence times are shifted to larger ohmicity indexes with temperature, whereas the minimum values $\gamma_{\text{extr}}(\lambda, s, T)$ of the total decoherence function show an opposite tendency. In the high-coupling regime, $\gamma_{\text{extr}}(\lambda, s, T)$ weakly depends on s . One can conclude that at the small coupling strength, the maxima of t^* occur mainly in the sub-Ohmic and Ohmic regimes, whereas at the moderate-to-high coupling, the maximum recoherence times are observed in the super-Ohmic regime with $s \sim 3/2$.

It is also essential to emphasize that in the low-coupling–low-temperature regime there is quite a broad domain of the ohmicity indexes, where $\gamma_{\text{extr}}(\lambda, s, T) \geq 0$, and this is accompanied by a corresponding break of the green curve for $t^*(\lambda, s, T)$. In this domain of $(s - T)$ values, the chosen coupling strength $\lambda = 0.1$ can be shown to be well below λ_{min} ; see Eq. (30). At temperatures $T = 0.25$ and 0.5 , λ_{min} is slightly below the chosen coupling strength $\lambda = 0.1$; the top points of the corresponding cyan and blue curves for $\gamma_{\text{extr}}(\lambda, s, T)$ lie just below zero, and we have a very weak qubit recoherence in this case.

Note also that the maxima of almost all the curves in the bottom row of Fig. 4 (except the black one) lie in the $s \sim 2$ region. In this domain of the ohmicity indexes, the incomplete decoherence at long times begins to occur [18]. Therefore, one can suppose that the short-time system recoherence and the long-time dynamics—the decoherence at large t —can sometimes be closely related in such an interesting manner. Namely, the domain of s , where the decoherence changes its type (from the complete to incomplete one) is, simultaneously, that of the weakest recoherence.

One more remark to be made when inspecting Fig. 4: The minima $\gamma_{\text{extr}}(\lambda, s, T)$ at both small and large s are defined by

the expression $-\ln[\coth(\beta\omega_0)/2]$, see also (31). It is most clearly visible in the moderate and high-coupling regimes, where all the curves are starting and terminating at the same horizontal level. In the advanced sub- and super-Ohmic regimes, the correlation contribution to the total decoherence function solely defines the dynamics of $\gamma_{\text{tot}}(t)$. It is clearly seen in Fig. 3, where during the first recoherence event, the blue and red curves completely coincide and begin to separate only at $i > 2$. Thus the minimum value of $\gamma_{\text{cor}}(t)$ defines the asymptotics of the total decoherence function at low and high s , while γ_{extr} appears as an interplay between $\gamma(t)$ and $\gamma_{\text{cor}}(t)$; according to numerical calculations, its maximum is located in the vicinity of $s = 2$.

A similar behavior of t^* as function of T (at the fixed ohmicity indexes) can be observed in Fig. 5, where the maxima of the recoherence times are shifted with s to higher temperatures. Expectedly, the most pronounced recoherence occurs at high temperatures, as can be seen in the bottom row of Fig. 5. Remarkably, the curves for γ_{extr} in all the panels are ordered from the bottom to the top, when the ohmicity indexes increase from 0.5 to 2. It means that the sub-Ohmic and Ohmic couplings are more preferable for the recoherence than the super-Ohmic ones. However, in the advanced super-Ohmic regime at $s = 3$, the corresponding curve (marked by stars) shows an opposite trend and goes below the blue and pink symbols.

To conclude this section, let us note that the large values of $\exp[-\gamma_{\text{extr}}(\lambda, s, T)] \sim 50 \div 70$ in the expression (24) for coherence, arising at high temperatures (see the bottom row in Fig. 5), should not confuse the readers. A considerable enhancement of the coherence with time is always preceded by a large decrease of its initial value, as it is clearly seen from Eqs. (61) and (67) of Ref. [4]. Thus, the length of the Bloch vector remains smaller than 1, and the spin system remains in a mixed state, even though its purity can increase significantly. Besides, as follows from the general physical reasoning, low temperatures keep the quantum systems in more coherent (purer) states. Thus though high temperatures favor an increase of $\gamma_{\text{extr}}(\lambda, s, T)$, it is low T that enhances an evident “purification” of the system (see lines with blue triangles in Fig. 5 of Ref. [4]).

V. CONCLUSIONS AND OUTLOOK

In this paper, we have examined the RDE for a single qubit interacting with a phonon bath within the dephasing model. The initial state of the composite system (the qubit plus the environment) is prepared through the nonselective quantum measurements of the special kind, when the Gram operator constructed on the spin state vectors takes a diagonal form. In such a case, the correlation contribution to the total decoherence function (i) is a nonpositive oscillating function, nonlinear in the coupling strength and (ii) does not depend on the qubit state.

According to (i), when the coupling strengths are large enough, $\gamma_{\text{cor}}(t)$ can outweigh the joint influence of the vacuum and thermal fluctuations of the bath, yielding a single (or multiple) RDE. We defined the recoherence time t^* as the instant, when $\gamma_{\text{tot}}(t)$ changes its sign from negative to positive, what corresponds to the transition from the coherence en-

hancement to pure decoherence. We also pay attention to the value of γ_{extr} , at which the total decoherence function becomes minimal. We study the dependence of t^* and γ_{extr} on the bath characteristics, namely (a) the coupling constant λ (which defines the strength of the interactions between the qubit and its environment), (b) the ohmicity parameter s (it describes the nature of interactions), and (c) the bath temperature T .

Due to (ii), we can eliminate the influence of the qubit state on the recoherence dynamics occurring at short times and concentrate our attention solely on the influence of the qubit environment. The qubit state determines only the initial value of the system coherence [4].

We study the $(s - T)$ dependence of the recoherence times t^* and of γ_{extr} for weak, intermediate, and strong coupling. Remarkably, it turns out that the weakest recoherence occurs at $s \sim 2$, where there is a transition from the total to the incomplete decoherence at large times. As a result, the short-time dynamics (the coherence enhancement) and the long-time behavior (the decoherence) can be interrelated in such an interesting way. Besides, the obtained results give us some hints about the basic characteristics of the bath, which provide the most optimal values of t^* and γ_{extr} in some sense.

We have also calculated dependence of the critical value λ_{min} of the coupling strength, which separates the pure decoherence processes (at $\lambda < \lambda_{\text{min}}$) from the coherence enhancement (at $\lambda > \lambda_{\text{min}}$), on the ohmicity index and the bath temperature. It is interesting to note, that the domain of the environment parameters, which makes the open quantum system most resistant to the coherence enhancement, to a certain extent overlaps with the domain most favorable for the maximal entanglement of *two spins* in the same dephasing model [38]. This is another quite interesting interrelation, which combines two quantum effects of a different nature. However, our preliminary estimations show that a diagonality of the Gram operator is not a sufficient condition for $\gamma_{\text{cor}}(t)$ to be independent of the spin states, and a more sophisticated initial state preparation is necessary to exploit the short-time recoherence in this case.

Therefore, it would be interesting to investigate how the eventual coherence enhancement in a two qubit system could be related to the dynamics of the quantum concurrence. There is a common viewpoint [42] that an increase of the system-environment correlations of the quantum nature favors the entanglement. It would be interesting to find additional evidence supporting or questioning this statement for the case of the correlated thermal bath interacting with two spins.

Though at the first glance the obtained results could be considered as purely theoretical in nature, the elaborated measurement scheme has evident applications as well. Let us briefly mention some of them.

(a) When studying the quantum memory effects based on recoherence trapping, one can store quantum information in a qubit that is engineered to recover coherence at predictable times. For this case, the following algorithm can be proposed: (i) prepare qubits coupled to the structured bosonic baths; (ii) tune the ohmicity index s and the temperature T in a way the recoherence time t^* and the maximum increment $|\gamma_{\text{extr}}|$ are optimized; (iii) encode the quantum states at $t = 0$, wait until $t = t^*$ when the coherence is restored; and (iv) read out the information at $t = t^*$, exploiting the maximum recoherence before a decoherence sets in.

(b) When designing quantum registers with an environment-induced coherence boost, one needs quantum circuits that actively leverage short-time recoherence to refresh particular qubits. In fact, the basic idea is to use the controlled non-Markovianity as a coherence reservoir. The following algorithm can be applied in this case: (i) construct the multiqubit registers, where each qubit is subject to synchronized nonselective measurements that induce short-time correlations; (ii) between operations, allow the system to evolve into a recoherence zone; and (iii) synchronize quantum gates to times t^* to maximize fidelity.

(c) When designing quantum protocols, where the decoherence plays a functional role (in quantum metrology, quantum stochastic processing, etc.) In this case, one has (i) to treat the bath as a “control register” that modulates system dynamics and (ii) to control the reservoir parameters for stimulation

of the Markov-to-non-Markov transitions and corresponding information backflow.

In all these cases, one is supposed to design a quantum system (a hardware platform) that supports tunable qubit-environment coupling λ , control over bath structure, and fast and controllable nonselective measurements to initialize qubit-bath correlations. A common point is that the designed gates (e.g., rotations, entangling ops) should operate mainly before or during recoherence windows, while avoiding periods of rapid initial decoherence. An elaborated scheme provides a solid theoretical ground for these technological applications.

DATA AVAILABILITY

The data that support the findings of this article are openly available [41].

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