

Algebraic structure of Fock-state lattices

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We analyze Fock-state lattices (FSLs) from an algebraic viewpoint. Starting from a Lie algebra, we associate a FSL constructed from the action of its generators: diagonal (Cartan) generators define the lattice sites, while off-diagonal (root) generators determine the lattice bonds. This construction reveals that identifying an underlying algebraic structure provides direct physical insight into FSLs, including their dimensionality, connectivity, symmetry constraints, and possible transport and revival phenomena. By examining several common Lie algebras, we identify not only their associated FSLs but also the corresponding Lie phase spaces, thereby establishing a systematic connection between FSL dynamics and phase-space geometry. In many cases, both the phase space and the FSL exhibit nontrivial curvature, opening possibilities for exploring quantum dynamics in curved synthetic spaces. We further address whether every integrable Hamiltonian admits an underlying Lie algebra that reproduces the same FSL structure. We show that this is not generally the case, particularly for Hamiltonians that are nonlinear in the generators, and that for systems combining different types of degrees of freedom the appropriate underlying structure may instead be a Lie superalgebra.

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I. INTRODUCTION

In the past few decades, quantum simulators have become ubiquitous tools for the simulation of quantum phenomena [1,2]. Broadly speaking, a quantum simulator is defined as a controllable quantum-mechanical system that can be used to emulate other quantum systems. Compared to a quantum computer, quantum simulators offer several important practical advantages. First, a quantum simulator can be tailored to a specific problem, thus requiring fewer resources. Second, the high degree of control over the implemented degrees of freedom enables easier preparation of initial states and more efficient readout procedures.

Examples of platforms that exhibit these features include ultracold atoms in optical lattices [3,4], trapped ions [5–7], and linear optical systems [8,9]. Many of these systems, such as atoms and ions, employ subsystems equipped with highly controllable internal degrees of freedom. Motivated by this, researchers have proposed exploiting these internal degrees of freedom to mimic additional spatial dimensions, leading to

the concept of synthetic dimensions [10]. This development has opened new avenues to investigate, for example, topological phenomena and other exotic quantum phases of matter [11–13], particularly in photonic architectures [12,14], while also extending the capabilities of cold-atom [11,15–17] and trapped-ion platforms [18,19].

The flexibility of synthetic dimensions suggests that this idea can be pushed even further in the construction of quantum simulators. Instead of relying on a real-space lattice, one can devise a lattice that exists purely in Hilbert space. This type of lattice is commonly referred to as a Fock-state lattice (FSL) [20–23].

A FSL is constructed by selecting a quantum system whose internal states, typically expressed in the Fock basis, are interpreted as lattice sites, while the transitions between them are engineered through an appropriate choice of Hamiltonian. As synthetic lattices, FSLs allow one to exploit the available number of states as effective sites. For example, a single quantum harmonic oscillator provides, in principle, an infinite number of such sites. However, an important caveat is that, in many instances, the translational symmetry of the FSL is broken due to the underlying algebra of the corresponding creation and annihilation operators. Fortunately, many properties of interest do not rely on translational symmetry, such as topological characteristics defined by local markers [24], or frustration, which instead relies on macroscopic degeneracy [25].

Experimentally, FSLs have been explored in a wide range of contexts, including the realization of topological phases [24,26–29], flat bands [30], synthetic gauge fields and the Aharonov-Bohm effect [31], and many-body dynamics [32].

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Theoretically, FSLs have been connected to Floquet space [29,33], dark states [34], the preparation of highly excited Fock states [35], localization phenomena such as Aharonov-Bohm cages [36–38], and open quantum systems [39].

In this work, we present a conceptually different approach to FSLs that does not rely on Hamiltonians, but is instead based on Lie algebras. Specifically, the generators of a Lie algebra are divided into Cartan generators, those forming a set of mutually commuting and diagonalizable operators, and root generators. From the diagonal Cartan generators, one identifies a weight lattice, which establishes a correspondence to Fock states. The root generators, being nondiagonal in the Fock basis, induce transitions between the sites of the weight lattice. In this manner, a FSL can be constructed directly from the Lie algebra.

This approach has the advantage that algebraic techniques can be applied to FSLs. For example, one can associate a phase space and generalized coherent states with the FSL, providing alternative descriptions of its structure. By considering commonly studied Lie algebras, we identify the corresponding phase spaces and FSLs, and explore their dynamics after assigning a Hamiltonian. For a general algebra, the corresponding phase space is not flat, reflecting the lack of translational invariance of the FSL. This demonstrates that FSLs can simulate dynamics in curved spaces. While one-dimensional lattices can always be embedded in flat space, higher-dimensional lattices, such as those arising from $\mathfrak{su}(3)$ and $\mathfrak{so}(5)$, cannot be embedded in this way.

The Hamiltonian and Lie-algebra frameworks are related: given a Lie algebra, one can construct a Hamiltonian as a linear combination of its generators. However, this relation is not invertible: for a general Hamiltonian, the existence of a corresponding Lie algebra is not guaranteed. For example, obtaining a finite FSL requires that the Hamiltonian be integrable, but integrability alone does not ensure a closed Lie algebra. This consideration holds for Hamiltonians involving the same type of degrees of freedom. For integrable models involving multiple types of degrees of freedom, a conventional Lie algebra may not suffice, though in some cases a correspondence between Hamiltonians and algebras can be restored by considering Lie superalgebras.

The outline of the paper is as follows. In the next section, we review the Hamiltonian formulation of Fock-state lattices and introduce the key concepts of Lie algebras in Sec. II B. In Sec. III, we establish the general correspondence between Lie algebras and Fock-state lattices, addressing important aspects through examples such as the Euclidean algebra, $\mathfrak{su}(2)$, and $\mathfrak{su}(1, 1)$. We explore the relationship between the Hamiltonian and Lie-algebra approaches in Sec. IV. Finally, in Sec. V, we summarize our findings and discuss future directions.

II. BACKGROUND: FOCK-STATE LATTICES AND LIE ALGEBRAS

A. Fock-state lattices: Hamiltonian description

Fock-state lattices (FSLs) provide an example in which physics is realized in synthetic dimensions rather than in real space. In an FSL, lattice sites correspond to physical states of a quantum system instead of spatial positions. The term “Fock

state” should therefore not be interpreted literally; rather, it refers more generally to numberlike basis states relevant to the system under consideration. Examples include bosonic Fock states $|n\rangle$ ($n \in \mathbb{Z}_{\geq 0}$), fermionic Fock states $|n_1, n_2, \dots\rangle$ ($n_i = 0, 1$), spin or angular-momentum states $|S, m\rangle$ ($-S \leq m \leq S$), or site-localized lattice states $|j\rangle$ ($j \in \mathbb{Z}$).

Any Hamiltonian $\hat{H}$ can be represented as a matrix $\bar{H}$ in this basis, with elements $H_{\{nm\}} = \langle\{n\}|\hat{H}|\{m\}\rangle$, where $|\{n\}\rangle$ denotes a basis state labeled by the multi-index $\{n\}$. For a single degree of freedom, $\{n\}$ reduces to a single integer, whereas for multiple degrees of freedom it collects all relevant quantum numbers. Each basis state $|\{n\}\rangle$ is associated with a vertex (lattice site) with onsite energy

$$\varepsilon_{\{n\}} \equiv \langle\{n\}|\hat{H}|\{n\}\rangle. \quad (1)$$

Edges in the lattice are determined by allowed transitions,

$$J_{\{nm\}} \equiv \langle\{n\}|\hat{H}|\{m\}\rangle \neq 0, \quad (2)$$

where $|J_{\{nm\}}|$ plays the role of a tunneling amplitude. By definition, $\varepsilon_{\{n\}} = J_{\{nn\}}$. The Hamiltonian $\bar{H}$, together with the chosen basis, therefore defines a graph that we refer to as a Fock-state lattice. Explicitly,

$$\hat{H} = \sum_{\{n\}} \varepsilon_{\{n\}} |\{n\}\rangle\langle\{n\}| + \sum_{\{n\} \neq \{m\}} J_{\{nm\}} |\{n\}\rangle\langle\{m\}|. \quad (3)$$

Writing the state as

$$|\Psi(t)\rangle = \sum_{\{n\}} c_{\{n\}}(t) |\{n\}\rangle, \quad (4)$$

and collecting the coefficients into the vector $\bar{C}(t)$, the Schrödinger equation becomes

$$i \frac{d\bar{C}(t)}{dt} = \bar{H} \bar{C}(t), \quad (5)$$

with site populations

$$P(\{n\}, t) = |c_{\{n\}}(t)|^2. \quad (6)$$

Within the FSL framework, the physical problem is thus mapped onto that of a single particle hopping on a lattice, whose dimensionality, connectivity, and hopping amplitudes are determined by the underlying physical model. For many-body systems with many degrees of freedom, the resulting lattice is typically complex, high dimensional, and exhibits long-range hopping [40]. The power of FSLs is therefore most evident in systems with few, well-isolated, and controllable degrees of freedom, as commonly realized in quantum-optical settings such as trapped ions, cavity or circuit QED, and nonlinear optics [21,23].

To illustrate these ideas, we consider a simple yet nontrivial example: two degenerate bosonic modes described by the Hamiltonian

$$\hat{H}_{2m} = J_0(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}). \quad (7)$$

This model can be realized, for instance, in a bimodal cavity containing a two-level system dispersively coupled to two degenerate cavity modes [22]. Defining the detuning $\Delta = \Omega - \omega$, where Ω is the atomic transition frequency and ω the cavity frequency, and assuming $|\Delta| \gg g\sqrt{\bar{n}}$ (with $\bar{n}$ the average photon number), second-order perturbation theory

yields an effective coupling of the above form with $J_0 = g^2/\Delta$. Alternatively, we note that (7) has the form of a beam-splitter Hamiltonian frequently appearing in integrated photonics [41].

The total excitation number

$$\hat{N} = \hat{n}_a + \hat{n}_b, \quad (8)$$

with $\hat{n}_a = \hat{a}^\dagger \hat{a}$ and $\hat{n}_b = \hat{b}^\dagger \hat{b}$, is conserved. The Fock states $|j\rangle \equiv |n_a = j, n_b = N - j\rangle$, with $j = 0, 1, \dots, N$, can therefore be labeled by a single integer and grouped into sectors of fixed N . As a result, the FSL consists of a set of one-dimensional chains, one for each excitation sector. Each chain is finite, and the tunneling amplitudes

$$\begin{aligned} J_{j,j+1} &= \langle j+1 | \hat{H}_{2m} | j \rangle = J_0 \sqrt{(j+1)(N-j)}, \\ J_{j,j-1} &= \langle j-1 | \hat{H}_{2m} | j \rangle = J_0 \sqrt{j(N-j+1)} \end{aligned} \quad (9)$$

depend explicitly on the site index j . Tunneling beyond nearest neighbors vanishes, so that the lattice is of tight-binding type.

Further insight is obtained using the Schwinger-boson mapping [42,43]

$$\hat{S}_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}), \quad \hat{S}^+ = \hat{a}^\dagger \hat{b}, \quad \hat{S}^- = \hat{b}^\dagger \hat{a}, \quad (10)$$

which casts the Hamiltonian into the simple form

$$\hat{H}_{2m} = 2J_0 \hat{S}_x = J_0(\hat{S}^+ + \hat{S}^-). \quad (11)$$

Its eigenstates are $|S, m\rangle_x$ with $-S \leq m \leq S$, where $S = N/2$. The spectrum is finite and equally spaced, $\varepsilon_m = 2J_0 m$, implying perfect revivals of any initial state with revival time $T_{\text{rev}} = \pi/J_0$.

It is straightforward to generalize this model to three modes,

$$\hat{H}_{3m} = J_0(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{c} + \hat{a}^\dagger \hat{c} + \text{H.c.}). \quad (12)$$

The total excitation number $\hat{N} = \hat{n}_a + \hat{n}_b + \hat{n}_c$ is again conserved. The corresponding FSL is a finite, tight-binding two-dimensional triangular lattice [21]. The Schwinger-boson construction generalizes naturally to multimode systems [44]. For three modes, the mapping can be expressed as

$$\begin{aligned} \hat{H}_1 &= \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}), \\ \hat{H}_2 &= \frac{1}{2\sqrt{3}}(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} - 2\hat{c}^\dagger \hat{c}), \\ \hat{I}_+ &= \hat{a}^\dagger \hat{b}, \quad \hat{U}_+ = \hat{b}^\dagger \hat{c}, \quad \hat{V}_+ = \hat{a}^\dagger \hat{c}, \end{aligned} \quad (13)$$

with $\hat{I}_-$, $\hat{U}_-$, and $\hat{V}_-$ given by the Hermitian conjugates of the corresponding raising operators. The notation follows standard conventions. Expressed in terms of these operators, the Hamiltonian takes the form

$$\hat{H}_{3m} = J_0(\hat{I}_+ + \hat{I}_- + \hat{U}_+ + \hat{U}_- + \hat{V}_+ + \hat{V}_-). \quad (14)$$

In the two-mode model, the introduction of a complex phase factor $\hat{S}_\pm \rightarrow e^{\pm i\phi} \hat{S}_\pm$ has no physical effect since it can be removed by an appropriate unitary rotation generated by $\hat{S}_z$. This reflects the fact that the phase corresponds to a global gauge choice.

In contrast, for the three-mode model, introducing a phase factor in any one of the operators $\hat{I}_\pm$, $\hat{U}_\pm$, or $\hat{V}_\pm$ cannot, in general, be eliminated by a unitary transformation. From the FSL perspective, such a phase renders the tunneling amplitudes complex and leads to nontrivial phases accumulated around closed loops of the lattice. This is directly analogous to the Peierls substitution for a charged particle hopping on a lattice in the presence of a perpendicular magnetic field, with the phase factors corresponding to an effective magnetic flux piercing the plaquettes of the FSL.

B. General concepts of Lie algebras

To establish the connection between Lie algebras and FSLs in the following section, we briefly review the necessary elements of Lie-algebra theory.

Let $\hat{X}_a$ denote the generators of a Lie algebra $\mathfrak{g}$, obeying the commutation relations [45,46]

$$[\hat{X}_a, \hat{X}_b] = if_{ab}^c \hat{X}_c, \quad (15)$$

where f_{ab}^c are the *structure constants*. Exponentiation of the generators $\hat{X}_a$, or of their linear combinations, yields elements of the associated Lie group G .

The *rank* r of the algebra is defined as the number of mutually commuting generators that can be simultaneously diagonalized. These generators form the *Cartan subalgebra* and are denoted $\hat{C}_a$, $a = 1, \dots, r$. The remaining generators are *root generators* $\hat{E}_\alpha$, labeled by roots α . The roots occur in pairs $\pm\alpha$, and the commutation relations take the general form

$$[\hat{E}_{\alpha_i}, \hat{E}_{\alpha_j}] = \begin{cases} \sum_a (\alpha_i)_a \hat{C}_a, & \text{if } \alpha_i = -\alpha_j \\ N_{\alpha_i, \alpha_j} \hat{E}_{\alpha_i + \alpha_j}, & \text{if } \alpha_i + \alpha_j \text{ is a root} \\ 0, & \text{otherwise} \end{cases} \quad (16)$$

where $(\alpha_i)_a$ denotes the component of the root vector α_i along the a th Cartan direction. The total number of root pairs is therefore $(\dim \mathfrak{g} - r)/2$. In a *semisimple* Lie algebra, the Cartan generators $\hat{C}_a$ are the only mutually commuting elements. All other generators have nontrivial commutation relations that connect them to the rest of the algebra, so that there are no Abelian ideals. An *ideal* is a subspace $\mathcal{I}$ of the algebra such that any commutators involving an element in $\mathcal{I}$ will also be in $\mathcal{I}$. The simplest example would be if the identity $\hat{\mathbb{I}}$ is part of the algebra, then it forms an ideal. Any generator commutes with $\hat{\mathbb{I}}$ such that it is automatically in $\mathcal{I} = \text{span}\{\hat{\mathbb{I}}\}$. An element commuting with every other element of the algebra is also called a *central element*.

Given a Lie group G , one can associate a Lie-algebra phase space (LPS) as follows. Consider a reference state $|\psi_0\rangle$ that is annihilated by all root generators corresponding to positive roots,

$$\hat{E}_{\alpha_i} |\psi_0\rangle = 0, \quad \forall \alpha_i > 0. \quad (17)$$

We define the displacement operators

$$\hat{D}_{\alpha_i}(\beta) = \exp(\beta \hat{E}_{\alpha_i} - \beta^* \hat{E}_{-\alpha_i}), \quad \alpha_i > 0 \quad (18)$$

and the corresponding family of states

$$|\psi_{\alpha_i}(\beta)\rangle = \hat{D}_{\alpha_i}(\beta) |\psi_0\rangle. \quad (19)$$

TABLE I. Summary of key Lie-algebra concepts used in constructing Fock-state lattices (FSLs).

Concept and symbol	Definition and role	Physical and FSL Intuition
Cartan generators $\hat{E}_a$	Mutually commuting generators, $a = 1, \dots, r$	Label lattice directions (vertex coordinates)
Root generators $\hat{E}_{\pm\alpha}$	Nondiagonal generators connecting eigenstates of Cartan generators	Induce hopping between lattice sites
Reference state $ \psi_0\rangle$	Annihilated by all positive root generators highest weight	Defines highest site in FSL center site for noncompact algebras
Displacement operator $\hat{D}_\alpha(\beta)$	$\exp(\beta\hat{E}_\alpha - \beta^*\hat{E}_{-\alpha})$	Moves along lattice edges; generates coherent states
Rank r	Number of Cartan generators	Dimension of the FSL underlying lattice directions
Semisimple Lie algebra	A Lie algebra with no “Abelian” (commuting) ideal parts	Broken translational invariance of the FSL
Lie-algebra phase space (LPS)	Manifold of coherent states G/H	Continuous space embedding the discrete FSL
Perelomov coherent states (PCS)	$ \psi(\beta)\rangle = \hat{\Omega}_\alpha(\beta) \psi_0\rangle$	Points in the LPS; smooth interpolation of lattice dynamics
Husimi function $Q(\beta)$	$Q(\beta) \propto \langle\psi(\beta) \hat{\rho} \psi(\beta)\rangle$	Quasiprobability distribution on LPS; visualizes population and coherence

The set of all such states forms a manifold, referred to as the Lie-algebra phase space, in which each point corresponds to a unique quantum state.

When there is only a single pair of root generators (so that the root label can be omitted), the states $|\psi(\beta)\rangle$ are known as Perelomov coherent states (PCSs) [47,48]. Well-known examples include bosonic coherent states associated with the Heisenberg-Weyl algebra and spin coherent states associated with $\mathfrak{su}(2)$. In general, PCSs are not eigenstates of the root generator $\hat{E}_\alpha$; the Heisenberg-Weyl algebra, for which Perelomov and Glauber coherent states coincide, is a notable exception.

For Lie algebras with more than one pair of root generators (and hence multiple Cartan generators), one constructs a generalized displacement operator $\hat{\Omega}(\beta)$ involving all root generators. Acting with $\hat{\Omega}(\beta)$ on the reference state produces a unique generalized coherent state. Thus, for algebras with a single root pair one has $\hat{D}_\alpha(\beta) = \hat{\Omega}(\beta)$, whereas for higher-rank algebras this identification no longer holds.

It is important to distinguish the LPS from the conventional phase space (RPS) used in quantum optics, which is defined in terms of canonical quadratures and associated quasiprobability distributions [49–51]. The RPS depends on the specific physical realization and on the choice of canonical variables prior to quantization. By contrast, the LPS is an intrinsic object determined solely by the Lie algebra and its representation, defined as the manifold of generalized coherent states.

To connect the two notions, one may equip the LPS with quasiprobability distributions. For example, the Husimi function for Glauber coherent states $|\alpha\rangle$ is defined as [49–51]

$$Q(\alpha) = \frac{1}{\pi} \langle\alpha|\hat{\rho}|\alpha\rangle, \quad (20)$$

where the coherent-state parameter is $\alpha = (x + ip)/\sqrt{2}$, with $x = \sqrt{2} \operatorname{Re}(\alpha)$ and $p = \sqrt{2} \operatorname{Im}(\alpha)$. This construction generalizes naturally to Perelomov coherent states,

$$Q(\beta) = w(\beta) \langle\psi(\beta)|\hat{\rho}|\psi(\beta)\rangle, \quad (21)$$

where $w(\beta)$ is a normalization factor, e.g., $w(\beta) = \pi^{-1}$ for the phase space of a particle on a line and $w(\beta) = (2S +$

$1)/(4\pi)$ for spin coherent states. These Husimi functions are defined on the same manifolds as the corresponding LPSs and play a role analogous to quasiprobability distributions in standard phase space [49–51]. The Husimi function comes natural once the PCSs have been introduced, but more generally, a variety of quasiprobability distributions, including generalized Wigner and Glauber–Sudarshan functions, can be defined on LPSs within the Stratonovich-Weyl framework [52–55].

In Table I we summarize the various quantities introduced in this subsection. In the second column we explain how they are defined, while in the third we give their physical interpretation, and especially their meaning in terms of the FSLs.

III. FROM LIE ALGEBRAS TO FOCK-STATE LATTICES

The previous section presented the basic ingredients needed to construct a FSL from a Lie algebra. This constitutes a conceptually different approach: rather than starting from a specific physical system and its Hamiltonian, we begin with the algebraic structure itself. As we will discuss in the next section, while a Lie algebra can be used to define a FSL, the reverse is generally not true. That is, starting from a Hamiltonian and its associated FSL does not automatically imply the existence of a corresponding Lie algebra. Even for integrable Hamiltonians, one cannot generally expect an underlying Lie algebra. However, when a Lie algebra does exist, its techniques provide a powerful tool for understanding the FSL physics and for describing the lattice in terms of a phase-space picture.

The goal of this section is to demonstrate generically how a Lie algebra defines a FSL, what the resulting lattice geometry is, and pinpoint important details. In Table II we list a set of algebras, their generators, and provide the corresponding phase-space structure and the generated FSL. First, in Sec. III A, to outline the basic idea we employ the Euclidean algebra. Then in connection with the LPS of the $\mathfrak{su}(2)$, in Sec. III B we discuss Casimir operators and how the solution of the Heisenberg equations in the algebra might miss information about the dynamics. Finally, taking the $\mathfrak{su}(1, 1)$

TABLE II. Relation between Lie algebras, their Cartan and root generators, associated phase spaces, and the resulting Fock-state lattices, including semisimplicity.

Algebra	Cartan generators	Root generators	Phase space	Fock-state lattice	Semisimple?
Euclidean $\mathfrak{e}(2)$	$\hat{E}_0$	$\hat{E}^\pm$	Cylinder	1D infinite chain	Nonsemisimple
Heisenberg-Weyl $\mathfrak{hw}$	$\hat{n}$	$\hat{a}, \hat{a}^\dagger$	Plane	Semi-infinite 1D chain	Nonsemisimple
$\mathfrak{su}(2)$	$\hat{S}_z$	$\hat{S}^\pm$	Sphere	Finite 1D chain	Semisimple
$\mathfrak{su}(3)$	$\hat{H}_1, \hat{H}_2$	$\hat{I}_\pm, \hat{U}_\pm, \hat{V}_\pm$	Compact 4D manifold	2D triangular lattice	Semisimple
$\mathfrak{so}(5)$	$\hat{H}_1, \hat{H}_2$	$\hat{\Sigma}_{\pm\alpha}$	Compact 6D manifold	2D square	Semisimple
$\mathfrak{su}(1, 1)$	$\hat{K}_0$	$\hat{K}^\pm$	Hyperboloid	Semi-infinite 1D chain	Nonsemisimple

algebra as an example, in Sec. III C we discuss localization in LPS. The additional algebras listed in Table II are addressed in the Appendix.

A. Fock-state lattices: Lie-algebra description

We are now in a position to introduce the FSL associated with a given Lie algebra. Since the Cartan generators $\hat{C}_a$ are diagonal, they define a set of quantum numbers m_i^a that label the Fock states. In the language of Lie algebras, these m_i^a label the sites of the weight lattice. For an algebra of rank r , the Fock states can thus be written as

$$|m_1^1, m_1^2, \dots, m_1^r\rangle. \quad (22)$$

Consequently, the rank of the Lie algebra determines the dimensionality of the corresponding weight lattice. The FSL is obtained by taking the weight lattice, associating the sites with the corresponding Fock states, and adding the additional structure provided by the root generators. To do this, we note that the root generators $\hat{E}_\alpha$ couple different Fock states, and because they are independent, each site in the FSL has as many nonvanishing tunneling rates as there are root generators. The roots α correspond to these tunneling directions, which in many cases, but not all, coincide with the principal axes of the FSL. Taken together, the Cartan and root generators fully determine the lattice structure, with $\hat{C}_a$ specifying the vertices and $\hat{E}_\alpha$ defining the edges.

As an example, consider the Euclidean algebra $\mathfrak{e}(2)$. The $\mathfrak{e}(2)$ algebra can be linked to a translationally invariant one-dimensional (1D) FSL, i.e., a particle hopping between nearest neighbors along a line. Denoting the lattice site j by $|j\rangle$, the Cartan generator and root generators can be represented as [56], respectively,

$$\hat{E}_0 = \sum_j j |j\rangle\langle j|, \quad \hat{E}^\pm = \sum_j |j \pm 1\rangle\langle j|, \quad (23)$$

with $\hat{E}^- = (\hat{E}^+)^\dagger$ and commutation relations

$$[\hat{E}_0, \hat{E}^\pm] = \pm \hat{E}^\pm, \quad [\hat{E}^-, \hat{E}^+] = 0. \quad (24)$$

Thus, one sees that while $\hat{E}_0$ defines the lattice sites $|j\rangle$, the root generators provide the transitions between them $|j-1\rangle \leftrightarrow |j\rangle \leftrightarrow |j+1\rangle$. In momentum space, the states $|k\rangle$ are eigenstates of the root generators

$$\hat{E}^\pm |k\rangle = e^{\mp ik} |k\rangle, \quad (25)$$

while the Cartan generator $\hat{E}_0$ acts as the position operator and can be written as $\hat{E}_0 = i\partial_k$.

One must notice that although the shift operators $\hat{E}^\pm$ commute and have well-defined eigenvalues in momentum space, they are not Cartan generators because they do not label states in the original lattice basis: they move states between sites rather than measuring a property like $\hat{E}_0$ does.

As previously mentioned, the displacement operators determine the structure of the phase space. Considering the root generators $\hat{E}_\pm$, it is defined as, following the prescription given by Eq. (18),

$$\hat{D}(\beta) = e^{\beta \hat{E}^+ - \beta^* \hat{E}^-}. \quad (26)$$

Its action on a lattice site $|j\rangle$ (see Appendix) becomes

$$\begin{aligned} \hat{D}(\beta)|j\rangle &= e^{\beta \hat{E}^+ - \beta^* \hat{E}^-}|j\rangle \\ &= \sum_{l=-\infty}^{+\infty} J_l(2|\beta|) e^{il \arg \beta} |j+l\rangle, \end{aligned} \quad (27)$$

where $l \in \mathbb{Z}$ and $J_l(x)$ denotes the Bessel function of the first kind.

The reference state $|\psi_0\rangle$ plays a special role as it is annihilated by all positive root generators (note that the distinction between positive and negative root generators is sometimes arbitrary). Often, the reference state is clear from the physical system: for bosons or fermions, the vacuum serves as $|\psi_0\rangle = |0\rangle$, while for a spin S it is $|\psi_0\rangle = |S, S\rangle$. For a single particle on an infinite one-dimensional lattice, i.e., the $\mathfrak{e}(2)$ algebra, no reference state exists, which is related to the structure of the LPS. Nevertheless, if we choose $|\psi_0\rangle = |0\rangle$ we obtain

$$|\beta\rangle = \hat{D}(\beta)|0\rangle = \sum_{l=-\infty}^{+\infty} J_l(2|\beta|) e^{il \arg \beta} |l\rangle, \quad (28)$$

which may be interpreted as a set of Euclidean PCSs.

Turning to the LPS of the $\mathfrak{e}(2)$ algebra, the displacement operator $\hat{D}(\beta)$ induces translations along the lattice, with a displacement proportional to $|\beta|$. The Cartan generator, on the other hand, acts diagonally in the Fock basis and generates phase factors, corresponding to translations in a conjugate angular variable. Therefore, the LPS is periodic in one direction and flat in the other, forming a cylindrical geometry. This cylinder reflects the underlying periodicity of the algebra and gives rise to a corresponding Brillouin zone. The FSL can be thought of as lying along the symmetry axis of the cylinder. The probability distribution $P(j) = |\langle j|\psi\rangle|^2$ can then be ob-

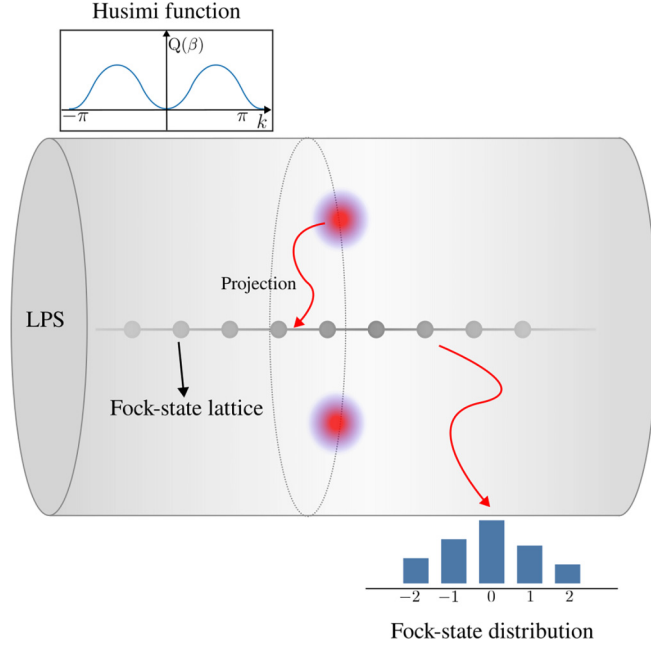


FIG. 1. Demonstration of the Lie-algebra phase space (LPS, gray cylinder) and the corresponding FSL, black dots. The Husimi function $Q(\beta)$ is illustrated as the bright structure encircling the LPS cylinder, while the Fock-state distribution $P(j) = \langle j|\hat{\rho}|j\rangle$ is shown as the bar diagram at the bottom. Heuristically, the Fock-state distribution can be viewed as the result of “projecting” the Husimi distribution onto the symmetry axis of the cylinder, corresponding to the FSL.

tained from a Husimi function $Q(\beta)$ by projecting onto this axis. This is schematically shown in Fig. 1, where the LPS is the gray cylinder and the sites of the FSL are the black dots along the vertical axis.

When describing the FSL in terms of the LPS, it is useful to clarify the action of the displacement operator on PCSs. Analogous to regular phase space in quantum optics (the x - p plane for a particle, or the Bloch sphere for a spin S), the reference coherent state is displaced along geodesics, preserving their local shape. In the LPS, successive applications of $\hat{D}_\alpha(\beta)$ also follow geodesics (note that here we keep α and β fixed during consecutive displacements), and the phase-space distance between states remains constant. PCSs are isotropic minimum-uncertainty states, saturating the Robertson inequality

$$\Delta\hat{X}_a \Delta\hat{X}_b \geq \frac{1}{2} | \langle [\hat{X}_a, \hat{X}_b] \rangle |, \quad (29)$$

and are locally Gaussian when expressed in local canonical (*Darboux*) coordinates on the Lie-algebra phase space.

We also note that the Cartan generators $\hat{C}_a$ may also generate displacements along geodesics in certain cases. For example, for a spin coherent state initialized at the equator of the $\mathfrak{su}(2)$ LPS, a rotation around the z axis, i.e., displaced with the Cartan generator, will displace the state along the equator. However, since they are diagonal in the Fock basis, they do not induce transitions in the FSL, but they add local phase shift akin onsite energy shifts.

B. Casimir operators and Lie-algebra solutions

One operator not yet mentioned is the Casimir operator $\hat{\Gamma}$, which can be expressed in terms of the generators $\hat{X}_a$ and is defined by the property that it commutes with all $\hat{X}_a$. Casimir operators are essential for determining the phase-space structure: as “constants of motion,” they define invariant surfaces that form the underlying phase space for the coherent states. In order to exemplify its role we consider the $\mathfrak{su}(2)$ algebra, related to the description of a spin- S particle.

A spin- S particle has a finite Hilbert space with Fock states $|S, m\rangle$ and a compact phase space forming a coherent-state sphere of radius S . For the regular phase space, we consider distributions such as the Husimi function, living on this sphere. In the LPS, coherent states correspond to points on the sphere, often referred to as the Bloch sphere. Extending this to distributions, the LPS can be identified with the standard phase-space description for a spin- S particle.

The $\mathfrak{su}(2)$ algebra is then generated by the Cartan generator $\hat{S}_z$, whose eigenstates are $|S, m\rangle$ with $m = -S, \dots, S$, defining the FSL sites, and the root generators $\hat{S}_\pm$ establishing the transitions $|S, m-1\rangle \leftrightarrow |S, m\rangle \leftrightarrow |S, m+1\rangle$ following Eq. (A20). The generators satisfy the standard angular momentum commutation relations

$$[\hat{S}^+, \hat{S}^-] = 2\hat{S}_z, \quad [\hat{S}_z, \hat{S}^\pm] = \pm\hat{S}^\pm. \quad (30)$$

For the $\mathfrak{su}(2)$ algebra, the Casimir operator is defined as $\hat{S}^2$:

$$\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2, \quad (31)$$

which commutes with all generators, $[\hat{S}^2, \hat{S}_i] = 0$. Its eigenvalues and eigenvectors are given by

$$\hat{S}^2 |S, m\rangle = S(S+1) |S, m\rangle \quad (\hbar = 1), \quad (32)$$

with $S \in \{0, \frac{1}{2}, 1, \dots\}$. One can see then that the Casimir operator sets the total conserved spin S , that in turn sets the representation of the algebra. From Eq. (31) it also follows that the LPS structure is a sphere.

Let us take the reference state (vacuum) to be $|\psi_0\rangle = |S, S\rangle$, localized at the north pole. Introducing polar and azimuthal angles $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$, we define the complex displacement parameter

$$\varsigma = -\frac{\theta}{2} e^{i\phi}, \quad (33)$$

and the spin displacement operator [57]

$$\hat{D}(\varsigma) = e^{\varsigma\hat{S}^+ - \varsigma^*\hat{S}^-}. \quad (34)$$

The resulting Perelomov coherent states (which coincide with standard spin coherent states) are

$$|\theta, \phi\rangle = \hat{D}(\varsigma) |S, S\rangle. \quad (35)$$

The LPS of the $\mathfrak{su}(2)$ algebra, together with the FSL, are depicted in Fig. 2.

Because the Casimir operator is invariant under the group action, its action on the coherent states results in

$$\hat{S}^2 |\theta, \phi\rangle = S(S+1) |\theta, \phi\rangle. \quad (36)$$

Therefore, the whole set $|\theta, \phi\rangle$ lives inside the same spin- S phase space and each allowed value of the Casimir eigenvalue

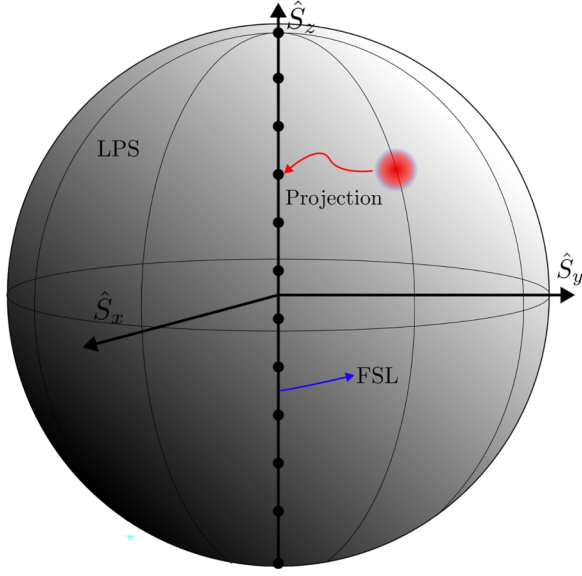


FIG. 2. The LPS of the $\mathfrak{su}(2)$ algebra takes the form of a sphere. Each point on the sphere (for example, the red dot) represents a coherent state $|\theta, \phi\rangle = \hat{D}(\zeta)|S, S\rangle$ and Husimi functions are localized positive functions. The FSL lies on the $\hat{S}_z$ axis and the projection of the coherent state distribution on it gives the FSL lattice distribution. Notice that the only states in the spin $\hat{S}_z$ basis that lies on the surface are the reference state $|S, S\rangle$ and the state $|S, -S\rangle$, i.e., the projection of these PCS onto the FSL give exact Fock states, while any other PCS results in a distribution over many Fock states.

$S(S+1)$ determines a distinct invariant sphere. Moreover, the Casimir operator also plays a role in a fundamental property of the coherent states, namely, the resolution of identity [see Eq. (A22) of the Appendix], where the prefactor $2S+1$ is directly related to the operator eigenvalue $S(S+1)$.

For completeness, let us mention that since $\hat{S}_z$ is the Cartan generator, the FSL population can be visualized as the projection of the phase-space distribution onto the z axis of the Bloch sphere. The corresponding FSL distribution is [derived by taking the norm square of the basis expansion coefficient Eq. (A21) found in the Appendix]

$$P(m) = |\langle S, m | \theta, \phi \rangle|^2 = \binom{2S}{S+m} \left(\cos \frac{\theta}{2} \right)^{2(S+m)} \left(\sin \frac{\theta}{2} \right)^{2(S-m)} \quad (37)$$

with variance

$$\Delta m = \sqrt{\frac{S}{2}} |\sin \theta|, \quad (38)$$

which vanishes at the poles and reaches a maximum at the equator ($\theta = \pi/2$). This reflects the broken translational invariance of the FSL and highlights that localization properties in the LPS might not scale the same as the localization in the FSL, a point we address further in Sec. III C.

Without having to involve FSLs, we can directly see applications of the Lie-algebra methods by noting the following. Consider a Hamiltonian $\hat{H}$ that can be expressed as a linear combination of the elements $\hat{X}_a$ of some finite Lie algebra,

i.e.,

$$\hat{H} = \sum_{a=1}^{\dim \mathfrak{g}} k_a \hat{X}_a. \quad (39)$$

The corresponding Heisenberg equations become

$$\frac{d}{dt} \hat{X}_b = -i[\hat{X}_b, \hat{H}] = i \sum_{a=1}^{\dim \mathfrak{g}} k_a \sum_{c=1}^{\dim \mathfrak{g}} f_{ab}^c \hat{X}_c, \quad (40)$$

implying that the algebra closes and the Hamiltonian is integrable. Nonetheless, even if the above result is general, one should be aware that a closed set of Heisenberg equations might not provide full information of the dynamics, but only about the evolution contained within the span of the Lie algebra.

To address this, let us return to the $\mathfrak{su}(2)$ algebra. For a more flexible FSL representation, a semisimple algebra like $\mathfrak{su}(2)$ can be realized using two bosonic modes via the Schwinger-boson mapping [42]:

$$\hat{S}_z = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}), \quad \hat{S}^+ = \hat{a}^\dagger \hat{b}, \quad \hat{S}^- = \hat{b}^\dagger \hat{a}. \quad (41)$$

The Casimir operator maps to the total boson number

$$\hat{N} = \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}, \quad (42)$$

and the spin Fock states $|S, m\rangle$ correspond to bosonic number states $|n_a, n_b\rangle$ with $n_a + n_b = 2S$.

Now, consider the Hamiltonian

$$\hat{H}_{\mathfrak{su}(2)} = \delta \hat{S}_z + 2J \hat{S}_x$$

or its two-mode boson representation

$$\hat{H}_{\mathfrak{su}(2)} = \frac{\delta}{2}(\hat{n}_a - \hat{n}_b) + J(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}). \quad (43)$$

We can solve for either the spin operators or the boson operators (e.g., via a Bogoliubov diagonalization). The first yields solutions like $\hat{S}_x(t)$ or equivalently $(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a})(t)$, while the latter gives $\hat{a}(t)$ and $\hat{b}(t)$. Specifically, assuming the initial state $|S, S\rangle$, the analytical solutions are, according to the derivation in the Appendix, Sec. 3,

$$\begin{aligned} \langle \hat{S}_x(t) \rangle &= \frac{2J\delta}{\Omega^2} S [1 - \cos(\Omega t)], \\ \langle \hat{S}_y(t) \rangle &= -\frac{2J}{\Omega} S \sin(\Omega t), \\ \langle \hat{S}_z(t) \rangle &= S \frac{\delta^2 + (2J)^2 \cos(\Omega t)}{\Omega^2}, \\ \langle \hat{a}(t) \rangle &= \sqrt{2S} \left(\cos \frac{\Omega t}{2} - i \frac{\delta}{\Omega} \sin \frac{\Omega t}{2} \right), \\ \langle \hat{b}(t) \rangle &= -i\sqrt{2S} \frac{2J}{\Omega} \sin \frac{\Omega t}{2}, \end{aligned}$$

with $\Omega = \sqrt{\delta^2 + (2J)^2}$.

Since the boson operators $\hat{a}$ and $\hat{b}$ are not elements of the Lie algebra, the last two equations contain information not captured by the Lie-algebra solution (the first three equations). In other words, while the Heisenberg equations (40) for the $\mathfrak{su}(2)$ generators close and yield exact dynamics for all observables within the algebra, the Bogoliubov approach

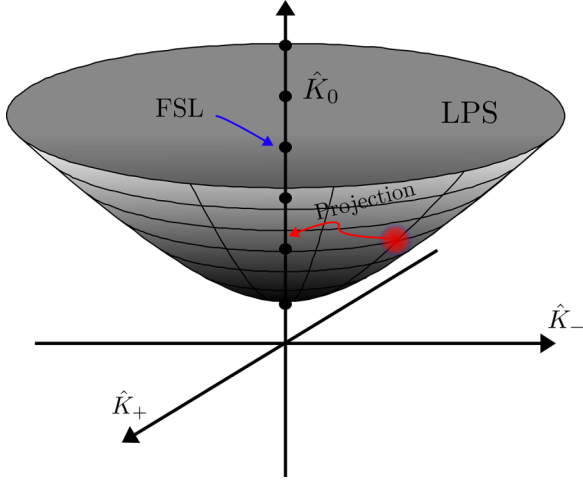


FIG. 3. The hyperboloid shape of the LPS of the $\mathfrak{su}(1, 1)$ algebra. Points on its surface are given by coherent states generated by the displacement operator (47). The FSL is given by the $\hat{K}_0$ axis, whose vertices are its eigenstates. With the bosonic representation (46) one sees that the true LPS is actually composed of two hyperboloids, one with Bargmann index $k = \frac{1}{4}$, for Fock states with even parity $|2n\rangle$, and another with $k = \frac{3}{4}$ for states with odd parity $|2n+1\rangle$ $n \in \mathbb{N}$.

provides a finer-grained description at the level of the underlying modes. The Lie-algebraic solution captures the collective degrees of freedom, whereas the Bogoliubov method resolves the microscopic mode structure.

C. Localization correspondence: LPS vs FSL

As we saw, PCS are locally Gaussian in LPS. Nonetheless, localization in the LPS does *not* automatically imply localization in the FSL, though in many examples it does. Similarly, semisimple Lie algebras (and some others) admit bosonic representations, allowing a description in multidimensional canonical RPSs; here too, LPS localization does not guarantee RPS localization. We address this point in connection with the $\mathfrak{su}(1, 1)$ algebra.

To the best of our knowledge, the FSL of the $\mathfrak{su}(1, 1)$ algebra does not seem to be discussed in the past. As we shall see below, it is related to Hamiltonians describing various types of field squeezing. It is conceptually different from say, $\mathfrak{e}(2)$ and $\mathfrak{su}(2)$, where there is a mapping between local states in the LPS and local states in the FSL. Here, instead, locality in LPS does not automatically imply locality in the FSL.

The $\mathfrak{su}(1, 1)$ algebra has commutation relations similar to $\mathfrak{su}(2)$ up to a sign change,

$$[\hat{K}^+, \hat{K}^-] = -2\hat{K}_0, \quad [\hat{K}_0, \hat{K}^\pm] = \pm\hat{K}^\pm. \quad (44)$$

This seemingly minor change has profound consequences: the algebra becomes noncompact. The Casimir operator

$$\hat{\Gamma} = \hat{K}_0^2 - \hat{K}_1^2 - \hat{K}_2^2, \quad (45)$$

where $\hat{K}^\pm = \hat{K}_1 \pm i\hat{K}_2$. Notably, $\mathfrak{su}(1, 1)$ is isomorphic to the Lorentz algebra $\mathfrak{so}(2, 1)$, reflecting the form of the Casimir operator. The form (45) implies that the associated LPS has a hyperboloid geometry, as depicted in Fig. 3. Let $K_+ =$

$r \sin \theta$ and $K_- = r \cos \theta$, then according to Eq. (45) $K_0 = \pm\sqrt{\Gamma + r^2}$.

With a single Cartan generator $\hat{K}_0$, the corresponding FSL is one dimensional, with Fock states $|k, m\rangle$ where $m \in \mathbb{N}$ is a non-negative integer. The Bargmann index k , related to $\hat{\Gamma}$, depends on the chosen representation. The root generators $\hat{K}_\pm$ dictate the transitions $\dots \leftrightarrow |k, m-1\rangle \leftrightarrow |k, m\rangle \leftrightarrow |k, m+1\rangle \leftrightarrow \dots$ [see Eq. (A52) and its derivation in the Appendix].

A standard boson representation of the $\mathfrak{su}(1, 1)$ algebra is

$$\hat{K}_0 = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \frac{1}{2}), \quad \hat{K}^+ = \frac{1}{2}(\hat{a}^\dagger)^2, \quad \hat{K}^- = \frac{1}{2}\hat{a}^2. \quad (46)$$

Both the vacuum $|0\rangle$ and the single excitation $|1\rangle$ are annihilated by $\hat{K}^-$, allowing either to serve as a reference state, $|\psi_0^{(0,1)}\rangle$.

The possibility of choice of two distinct reference states, $|0\rangle$ or $|1\rangle$, that arise when using the single-mode bosonic realization, is connected to the existence of two distinct representations (46). Choosing $|0\rangle$ yields a Bargmann index $k = \frac{1}{4}$, while $|1\rangle$ gives $k = \frac{3}{4}$, corresponding to even- and odd-parity FSLs, respectively. As we saw, the associated FSL is the set of non-negative integers, but the choice of reference state, either $|0\rangle$ or $|1\rangle$, restricts the dynamics to even or odd Fock states, respectively. For $k = \frac{1}{4}$ the allowed transitions in the FSL are $|0\rangle \leftrightarrow |2\rangle \leftrightarrow |4\rangle \leftrightarrow \dots$, while for $k = \frac{3}{4}$ one has $|1\rangle \leftrightarrow |3\rangle \leftrightarrow |5\rangle \leftrightarrow \dots$. In the Hamiltonian language, this is reflected in parity symmetry, $\hat{a} \rightarrow -\hat{a}$ and $\hat{a}^\dagger \rightarrow -\hat{a}^\dagger$, leaving the generators (46) invariant. Thus, the FSL is actually two 1D chains, and similarly the LPS will also be two hyperboloid surfaces, one for each Bargmann index k .

The displacement operator

$$\hat{D}(\beta) = e^{\beta(\hat{a}^\dagger)^2 - \beta^* \hat{a}^2} \quad (47)$$

is the familiar single-mode squeezing operator $\hat{S}(\xi)$ with $\xi = \beta/2$. The corresponding PCS is $|k, \xi\rangle$, depending on the reference state. For the vacuum ($k = \frac{1}{4}$), it expands in the Fock basis as [58]

$$|\xi\rangle \equiv \hat{S}(\xi)|0\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} (-e^{i\theta} \tanh r)^n |2n\rangle, \quad (48)$$

where we parametrize $\xi = re^{i\theta}$. From it we can extract the boson distribution

$$p(n, \xi) = |\langle n|\xi\rangle|^2 = \frac{1}{\cosh r} \frac{(2n)!}{4^n (n!)^2} \tanh^{2n} r. \quad (49)$$

Now suppose the state evolves under the Hamiltonian $\hat{H} = \omega \hat{a}^\dagger \hat{a}$ for a time $T = 2\pi/\omega$. The quadrature operators $\hat{x}(t)$ and $\hat{p}(t)$ then undergo a full 2π rotation in phase space, and accordingly the Husimi distribution $Q(\alpha)$ completes one full revolution. The generalized Husimi distribution in the LPS, however, performs two circuits around the hyperboloid. This reflects the fact that the generators $\hat{K}^\pm$ are quadratic in the bosonic operators and therefore return to themselves already after $T = \pi/\omega$. Thus, one sees that the necessity of a doubled LPS, as previously pointed out, follows directly from the quadratic structure of the algebra. As a side note, upon completing a full 2π rotation, the state acquires a geometric phase

associated with the curvature of the LPS manifold [59,60]. In the FSL, such a geometric contribution will not, however, manifest in the distribution $p(n, \xi)$.

Let us now address the additional subtlety of the relation between localization in LPS and FSL. For both the $\mathfrak{e}(2)$ and $\mathfrak{su}(2)$ algebras, the mapping can be visualized as perpendicular projections once the appropriate surfaces are identified. This construction relies on the specific parametrization of those phase spaces. In the present case, however, the LPS is a hyperboloid while the FSL corresponds to the $\hat{K}_0$ (“ z ”) axis. The parametrization $\xi = re^{i\theta}$, together with the form of the Casimir operator (45), implies that the projection from the LPS onto the FSL is not perpendicular.

This is evident when considering the squeezed vacuum. Its FSL distribution $p(n, \xi)$, Eq. (49), is such that its average and standard deviation at large squeezing parameter $r \gg 1$, i.e., at long distances from the center of the hyperboloid, goes as (following the derivation in the Appendix, Sec. 6)

$$\langle n \rangle \sim \frac{e^{2r}}{4}, \quad (50)$$

$$\Delta n \sim \frac{e^{2r}}{2\sqrt{2}}. \quad (51)$$

We can see that as one moves away from the center of the hyperboloid, the FSL distribution will not only move exponentially far away but also have a wave packet that exponentially delocalizes. Consequently, the intuitive geometric correspondence between localization in the LPS and population along the FSL differs qualitatively from the cases discussed earlier. Since the mapping from the LPS to the FSL is nonlinear and shaped by the curvature of the underlying group manifold, localization properties are not preserved under projection. Thus, states that are well localized in the LPS may correspond to broad distributions on the FSL.

An alternative view is to draw parallels to the geometry of Lorentz boosts. The coherent-state manifold is the upper sheet of a two-sheeted hyperboloid, analogous to the constant-proper-time hyperboloid in Minkowski space [61]. The coherent-state parameter plays the role of the Lorentz rapidity, while the Cartan coordinate satisfies $\langle \hat{K}_0 \rangle = k \cosh(2r)$, where r is the squeezing parameter and k is the Bargmann index. Consequently, $\frac{d\langle \hat{K}_0 \rangle}{dr} = 2k \sinh(2r)$, which grows exponentially for large r . Therefore, a coherent state whose intrinsic width Δr remains approximately constant on the hyperboloid is mapped onto a distribution in the FSL with width

$$\Delta K_0 \simeq 2k \sinh(2r) \Delta r,$$

which also grows exponentially with distance from the origin. Since the FSL is precisely the eigenbasis of $\hat{K}_0$, this implies an exponentially broad occupation of lattice sites despite the state remaining localized on the LPS.

To conclude, the localization correspondence is strictly preserved in flat or compact, linear representations, such as the aforementioned Euclidean $\mathfrak{e}(2)$ and $\mathfrak{su}(2)$ algebras. For noncompact algebras such as $\mathfrak{su}(1,1)$ one can expect this correspondence to break down. In noncompact algebras, the intrinsic negative curvature of the phase-space manifold induces nonlinear projections onto the linear weight space of

the Cartan generators. Consequently, a state that remains perfectly localized as a coherent wave packet when displaced along a hyperbolic phase-space trajectory will undergo high broadening when expanded in the discrete Fock basis. Thus, localization in LPS does not guarantee lattice-site localization in the presence of algebraic noncompactness.

IV. DISCUSSIONS AND FURTHER ASPECTS

We have seen how a Lie algebra $\mathfrak{g}$ defines a FSL by assigning Fock states to the sites of the weight lattice, and using the root generators to define the nonvanishing tunneling rates. Moreover, from the elements $\hat{X}_a$ of $\mathfrak{g}$ we can form a Hamiltonian via a linear combination as in Sec. III B. In this case, the Lie-algebra and the Hamiltonian FSLs are identical up to factors multiplying the tunneling rates, i.e., assuming all $k_a = 1$ in Sec. III B) reproduces the Lie-algebra FSL. Thus, starting from a Lie algebra we constructed its FSL and a family of Hamiltonians.

Historically, FSLs have been discussed where the Hamiltonian alone defines the lattice, without reference to an underlying Lie algebra. Yet, if such an algebra exists, it can provide further insight, for example, by identifying a LPS or simplifying analytical derivations. We therefore ask whether there is a reverse connection: Given $\hat{H}$, does there exist an underlying Lie algebra? It is clear that this is a strong statement, it is not enough for the Hamiltonian to be symmetric under a Lie group; their FSLs must coincide. We focus on finite algebras, i.e., the Hamiltonian has a finite number of degrees of freedom. If the Hamiltonian is nonintegrable, we typically have no closed set of Heisenberg equations, and we cannot expect a finite underlying algebra. Hence, we consider integrable cases, the simplest being

$$\hat{H}_{2b} = \sum_{i,j=1}^N \left(A_{ij} \hat{a}_i^\dagger \hat{a}_j + \frac{1}{2} B_{ij} \hat{a}_i^\dagger \hat{a}_j^\dagger + \frac{1}{2} B_{ij}^* \hat{a}_i \hat{a}_j \right), \quad (52)$$

for bosonic modes $\hat{a}_i$ with A Hermitian and B symmetric. The elements of the underlying Lie algebra are

$$\{\hat{a}_i^\dagger \hat{a}_j, \hat{a}_i^\dagger \hat{a}_j^\dagger, \hat{a}_i \hat{a}_j\}_{i,j=1}^N. \quad (53)$$

It is straightforward to show that this forms a closed set under commutation; this is the bosonic representation of the *symplectic algebra* $\mathfrak{sp}(2N)$. The Cartan generators are $\hat{a}_i^\dagger \hat{a}_i$, and the root generators can be grouped into hopping terms $\hat{a}_i^\dagger \hat{a}_j$ ($i \neq j$), pair creation $\hat{a}_i^\dagger \hat{a}_j^\dagger$, and pair annihilation $\hat{a}_i \hat{a}_j$. The FSL is then a hypercube, possibly with tunneling beyond nearest neighbors. Adding symmetries reduces the set of elements and we can find subalgebras of $\mathfrak{sp}(2N)$, like many of those discussed in the previous section.

Adding pump terms linear in creation and annihilation operators does not break integrability. From the Appendix, Sec. 2, the set $\{\hat{a}, \hat{a}^\dagger, \hat{\mathbb{I}}\}$ forms the (nonextended) $\mathfrak{hw}$ algebra. Indeed, the set

$$\{\hat{a}_i, \hat{a}_i^\dagger, \hat{a}_i^\dagger \hat{a}_j, \hat{a}_i^\dagger \hat{a}_j^\dagger, \hat{a}_i \hat{a}_j, \hat{\mathbb{I}}\}_{i,j=1}^N \quad (54)$$

forms a closed algebra called the inhomogeneous symplectic Lie algebra $\mathfrak{isp}(2N)$,

$$\mathfrak{isp}(2N) = \mathfrak{sp}(2N) \ltimes \mathfrak{hw}_N. \quad (55)$$

Here, $\ltimes$ denotes the semidirect product, meaning that the N -mode Heisenberg-Weyl algebra $\mathfrak{hw}_N$ is a normal ideal of $\mathfrak{isp}(2N)$; that is, $[g, h] \in \mathfrak{hw}_N$ if $h \in \mathfrak{hw}_N$ and $g \in \mathfrak{sp}(2N)$. Since the algebra contains the central element $\mathbb{I}$, it is not semisimple.

For fermions $\hat{c}_i$ and $\hat{c}_i^\dagger$ with antisymmetric B , the corresponding algebra

$$\{\hat{c}_i^\dagger \hat{c}_j, \hat{c}_i^\dagger \hat{c}_j^\dagger, \hat{c}_i \hat{c}_j\}_{i,j=1}^N \quad (56)$$

is $\mathfrak{so}(2N)$, and the FSL is a finite hypercube $\{0, 1\}^N$. Extending as in the bosonic case,

$$\{\hat{c}_i, \hat{c}_i^\dagger, \hat{c}_i^\dagger \hat{c}_j, \hat{c}_i^\dagger \hat{c}_j^\dagger, \hat{c}_i \hat{c}_j, \mathbb{I}\}_{i,j=1}^N \quad (57)$$

also forms a closed algebra, though without a standard name.

We now consider examples where different degrees of freedom combine. For instance, the Jaynes-Cummings model [22]

$$\hat{H}_{\text{JC}} = \omega \hat{a}^\dagger \hat{a} + \frac{\Omega}{2} \hat{\sigma}_z + g(\hat{a}^\dagger \hat{\sigma}^- + \hat{\sigma}^+ \hat{a}) \quad (58)$$

has Pauli operators $\hat{\sigma}$. Due to particle conservation, the FSL is an infinite set of two-site lattices, one for each conserved excitation number. Although integrable, the Heisenberg equations do not close, so no finite-dimensional underlying Lie algebra exists. Constructing an infinite-dimensional algebra from all commutators would yield an infinite rank and thus an infinite-dimensional FSL. Hence, integrability alone does not guarantee a finite Lie algebra. A similar argument applies to the Lipkin-Meshkov-Glick model [22]

$$\hat{H}_{\text{LMG}} = \Omega \hat{S}_z + \frac{g}{S} \hat{S}_x^2, \quad (59)$$

which is solvable, but its Heisenberg equations do not close. Its FSL consists of two 1D finite chains, which cannot be reproduced from an infinite Lie algebra.

In quantum optics, models like Jaynes-Cummings combine bosonic and spin degrees of freedom. Using a Jordan-Wigner transformation [43], a single spin $\frac{1}{2}$ can be mapped to a fermion: $\hat{\sigma}_z = 2\hat{c}^\dagger \hat{c} - 1$, $\hat{\sigma}^+ = \hat{c}^\dagger$, $\hat{\sigma}^- = \hat{c}$. This motivates Lie superalgebras, where the algebra is $\mathfrak{g} = \mathfrak{g}_{\text{even}} \oplus \mathfrak{g}_{\text{odd}}$ with bosonic (even) and fermionic (odd) generators. The supercommutator is defined as

$$[\hat{x}, \hat{y}] = \begin{cases} [\hat{x}, \hat{y}] & x \text{ and } y \text{ even,} \\ [\hat{x}, \hat{y}], & x \text{ even, } y \text{ odd} \\ \{\hat{x}, \hat{y}\}, & x \text{ odd, } y \text{ odd.} \end{cases} \quad (60)$$

For the Jaynes-Cummings model, the generators

$$\{\hat{a}^\dagger \hat{a}, \hat{c}^\dagger \hat{c}, \hat{a}^\dagger \hat{c}, \hat{c}^\dagger \hat{a}\}, \quad (61)$$

with Cartan generators $\hat{a}^\dagger \hat{a}$ and $\hat{c}^\dagger \hat{c}$ (even) and root generators $\hat{a}^\dagger \hat{c}$ and $\hat{c}^\dagger \hat{a}$ (odd), reproduce the FSL from $\hat{H}_{\text{JC}}$. The corresponding Lie superalgebra is $\mathfrak{su}(1|1)$: one bosonic and one fermionic mode. If we try the same to the quantum Rabi model (Jaynes-Cummings including the counter-rotating terms $\hat{a}^\dagger \hat{\sigma}^+$ and $\hat{a} \hat{\sigma}^-$), we do not find a finite Lie superalgebra, despite the fact that the model is integrable [62].

The “fermionization” can be generalized to the Tavis-Cummings model. The Jordan-Wigner transformation of spin operators $\hat{S}^\pm$ produces fermionic strings, so the transformed Hamiltonian $\hat{H}_{\text{TC}}$ is no longer quadratic. Nevertheless, identifying Cartan generators $\hat{a}^\dagger \hat{a}$ and $\hat{S}_z$ (even) and root generators

$\hat{a}^\dagger \hat{S}^-$ and $\hat{a} \hat{S}^+$ (odd), we see the superalgebra $\mathfrak{su}(1|N)$, with $N = 2S$ fermionic modes. The corresponding fermionic Hilbert space dimension is 2^N , typically much larger than $2S + 1$, as the Jordan-Wigner mapping involves all spin sectors. Also here, the Hamiltonian and Lie superalgebra FSLs agree.

Extending to superalgebras thus allows identification of underlying algebras for a larger set of integrable Hamiltonians. However, like the quantum Rabi model, mapping the Lipkin-Meshkov-Glick Hamiltonian (59) to a fermionic model does not yield a finite Lie (super)algebra, so the inversion procedure is not generally possible.

In the Appendix, Sec. 5, we present some details on the $\mathfrak{so}(5)$ model. Before doing so, let us conclude this section by discussing one peculiarity of the corresponding FSL. The general rule governing the dimensionality of Hamiltonian FSLs is that each additional bosonic degree of freedom increases the dimension of the lattice by one, while a spin S contributes only a quasidimension, i.e., a finite direction. Conversely, every continuous symmetry partitions the FSL into sectors labeled by the conserved quantity, thereby reducing the effective dimensionality by one. Hence, for generic Hamiltonians [21],

$$\text{Dim(FSL)} = \text{No. boson modes} - \text{No. continuous symmetries.} \quad (62)$$

At first sight, the $\mathfrak{so}(5)$ model appears to violate this rule. Its Schwinger-boson realization employs four bosonic modes, while boson-number conservation provides only a single $U(1)$ symmetry, suggesting a three-dimensional FSL according to Eq. (62). Instead, the FSL is two dimensional. The reason is that the four-mode bosonic Fock space constitutes an enlarged representation space: restricting the dynamics to operators belonging to the $\mathfrak{so}(5)$ algebra confines the system to a single irreducible representation, thereby eliminating one additional occupation-number direction. This reduction is not associated with an extra continuous symmetry, but with the algebraic constraints defining the irreducible representation. Consequently, the remaining independent coordinates are the eigenvalues of the two commuting Cartan generators of $\mathfrak{so}(5)$, and the FSL is the corresponding two-dimensional weight lattice. The additional Casimir operators merely label the irreducible representation and determine its geometry; they do not generate independent symmetries or conserved directions.

Thus, while Eq. (62) is valid for generic Hamiltonian FSLs, Lie-algebraic constructions require a refinement. In particular, when the dynamics is generated entirely by a Lie algebra $\mathfrak{g}$ and restricted to a single irreducible representation, the dimensionality of the FSL is determined by the rank of the algebra,

$$\text{Dim(FSL)} = \text{rank}(\mathfrak{g}). \quad (63)$$

V. CONCLUSION

In this work, rather than analyzing FSLs solely as structures induced by specific Hamiltonians, we have taken an alternative approach by introducing them from the perspective of Lie algebras. More precisely, a Lie algebra defines a FSL through its generators: the eigenvalues of the Cartan generators label the sites of the weight lattice, and by identifying

these sites with eigenvectors (Fock states) and introducing tunneling processes governed by the root generators, one obtains a Lie-algebra FSL. Once such a FSL is defined, it can naturally be associated with a phase space intrinsic to the Lie algebra. In particular, phase-space distributions such as the Husimi function $Q(\beta)$ live on the manifold of generalized coherent states. This Lie phase space can differ significantly from the phase spaces commonly encountered in textbooks on quantum optics. For example, while the $\mathfrak{su}(1, 1)$ algebra (relevant for squeezing dynamics) can be represented using a single bosonic mode and described in the usual flat x - p phase space, it may alternatively be represented on its LPS, which is a curved two-dimensional hyperboloid.

When an FSL admits an underlying Lie-algebra structure, the dynamics can therefore be analyzed equivalently in the LPS. As in standard phase-space formulations of quantum mechanics, this representation can offer additional physical insight by encoding quantum states in terms of real-valued distributions. In particular, the curvature of the LPS can give rise to nontrivial geometric effects, such as geometric phases familiar from spin systems. It can be shown [57,63,64] that when the Hamiltonian is linear in the generators, the Husimi function (21) evolves according to

$$\partial_t Q = \{Q, H_{\text{cl}}\}_{\text{LP}}, \quad (64)$$

where H_{cl} is the corresponding classical Hamiltonian and the Lie-Poisson bracket is defined as

$$\{F, G\}_{\text{LP}} = f_{ab}^c x_c \frac{\partial F}{\partial x_a} \frac{\partial G}{\partial x_b}, \quad (65)$$

with $x_a = \langle \hat{X}_a \rangle$ and structure constants f_{ab}^c identical to those in Eq. (15). This formulation generalizes the familiar phase-space evolution equations encountered in conventional quantum mechanics.

Beyond establishing a link between FSLs and phase space, we have highlighted several key insights. For example, the relation between localization in the LPS and localization in the FSL is generally nontrivial, particularly for noncompact algebras such as $\mathfrak{su}(1, 1)$, where curvature induces nonlinear projections between the two descriptions.

We have also examined the inverse problem: starting from an integrable Hamiltonian, constructing its FSL, and asking whether a Lie algebra exists that reproduces the same lattice structure. For quadratic bosonic and fermionic Hamiltonians, such underlying algebras can indeed be identified. However, for integrable models combining different types of degrees of freedom, such as the Jaynes-Cummings and Tavis-Cummings models, no finite Lie algebra exists. In these cases, Lie superalgebras provide a natural extension, allowing one to recover the correct FSL structure and offering an algebraic interpretation of the dynamics.

Importantly, Hamiltonian integrability alone does not guarantee the existence of an underlying Lie (super)algebra, as illustrated by the Lipkin-Meshkov-Glick model. Thus, while the algebraic construction of FSLs provides a powerful framework for understanding and visualizing quantum dynamics, it is not universally invertible starting from an arbitrary integrable Hamiltonian. Overall, our results establish a systematic foundation for exploring FSLs in both Lie-algebraic and superalgebraic settings, and open avenues for analytical and

numerical investigations of quantum systems with structured Hilbert spaces.

DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX: LIE ALGEBRA FOCK-STATE LATTICES: EXAMPLES

In this Appendix we provide the details on the algebras mentioned in Table II, their corresponding FSLs, and possible physical realizations.

1. Euclidean algebra: $\mathfrak{e}(2)$

The Euclidean algebra $\mathfrak{e}(2)$ is spanned by the operators $\{\hat{E}_0, \hat{E}_+, \hat{E}_-\}$, where

$$\hat{E}_0 = \sum_j j |j\rangle\langle j|, \quad \hat{E}^+ = \sum_j |j+1\rangle\langle j|, \quad (A1)$$

and $\hat{E}^- = (\hat{E}^+)^\dagger$. One can easily check that they satisfy the following commutation relations:

$$[\hat{E}_0, \hat{E}^\pm] = \pm \hat{E}^\pm, \quad [\hat{E}^-, \hat{E}^+] = 0. \quad (A2)$$

As mentioned in the main text, despite $\hat{E}_\pm$ commuting, the Cartan generator of the algebra is $\hat{E}_0$ since it is diagonal in the j basis, defining the FSL sites $\{|j\rangle, j \in \mathbb{Z}\}$. In this way, the operators $\hat{E}_\pm$ are the root generators, determining the transitions between the lattice sites

$$\hat{E}_+ |j\rangle = |j+1\rangle, \quad (A3)$$

$$\hat{E}_- |j\rangle = |j-1\rangle. \quad (A4)$$

The displacement operator that generates coherent states in this algebra is defined according to

$$\hat{D}(\beta) = e^{\beta \hat{E}^+ - \beta^* \hat{E}^-}. \quad (A5)$$

Its action on a basis state $|j\rangle$ can be easily derived as follows:

$$\begin{aligned} \hat{D}(\beta) |j\rangle &= \sum_{m,n=0}^{\infty} \frac{\beta^m (-\beta^*)^n}{m! n!} (\hat{E}^+)^m (\hat{E}^-)^n |j\rangle \\ &= \sum_{m,n=0}^{\infty} \frac{\beta^m (-\beta^*)^n}{m! n!} |j+m-n\rangle \\ &= \sum_{l=-\infty}^{+\infty} J_l(2|\beta|) e^{il \arg \beta} |j+l\rangle, \end{aligned} \quad (A6)$$

where $l = m - n$ and $J_l(x)$ denotes the Bessel function of the first kind. In the second step, we used the commutativity of the root operators, and in the final step we employed the standard series expansion of the Bessel functions. Since $J_{-l}(x) = (-1)^l J_l(x)$, the probability distribution $|J_l(x)|^2$ is symmetric about $l = 0$, and it is predominantly supported within $|l| \lesssim 2|\beta|$, outside of which it decays superexponentially.

With the choice of $|0\rangle$ as a reference state, the action of the displacement operator defines the set of Euclidean PCS

[Eq. (28)] $\hat{D}(\beta)|0\rangle = |\beta\rangle$, and by using the identity

$$\int_0^\infty J_m(2r)J_n(2r)r\,dr = \frac{1}{2}\delta_{mn}, \quad (\text{A7})$$

one obtains the resolution of the identity

$$\int \frac{d^2\beta}{\pi} |\beta\rangle\langle\beta| = \mathbb{I}. \quad (\text{A8})$$

In order to determine the underlying LPS geometry, consider the following Casimir operator:

$$\hat{\Gamma} = \hat{E}_x^2 + \hat{E}_y^2 = \hat{\mathbb{I}}, \quad (\text{A9})$$

where $\hat{E}_x = (\hat{E}_+ + \hat{E}_-)/2$ and $\hat{E}_y = (\hat{E}_+ - \hat{E}_-)/2$. As a constant of motion, one sees that $\hat{\Gamma}$ restricts a state on a circle of unit radius S^1 . However, since $\hat{E}_0$ does not appear on Eq. (A9) a coherent state can assume any value for $\hat{E}_0$, i.e., $\langle\hat{E}_0\rangle_\beta \in \mathbb{R}$. Therefore, the LPS geometry is $S^1 \times \mathbb{R}$, i.e., a cylinder.

Physical realization and FSL dynamics

To illustrate the connection between the LPS and the FSL, consider the Hamiltonian

$$\hat{H}_{\text{WS}} = \omega\hat{E}_0 - J(\hat{E}^+ + \hat{E}^-). \quad (\text{A10})$$

This is the famous Wannier-Stark Hamiltonian, describing a particle in a 1D tilted tight-binding lattice, where ω is the tilt strength (corresponding to a constant force) and J is the tunneling rate [65]. For nonzero ω , the spectrum is the Wannier-Stark ladder $\varepsilon_j = \omega j$ for integers j . While the spectrum is independent of J , the eigenvectors $|\phi_j\rangle$ are exponentially localized around site j . The equidistant spectrum implies perfect revivals with period $T_{\text{rev}} = 2\pi/\omega$. Two characteristic types of evolution are observed: for initial states localized in real space, e.g., $|\psi(0)\rangle = |j\rangle$, the evolved state $|\psi(t)\rangle$ displays a breathing motion; for initial states localized in momentum space, $|\psi(t)\rangle$ exhibits the typical Bloch-oscillation dynamics.

On the LPS, for an initial Fock state $|j\rangle$, the Husimi distribution is $Q(|\beta|) = |J_l(2|\beta|)|^2$, independent of $\arg\beta$ and hence of k , so the Fock-state Husimi function encircles the cylinder. During evolution, $Q(\beta, t) = Q(|\beta|, t)$ remains independent of k , while its width along the cylinder oscillates with frequency ω . If, instead, the initial state is localized in k with a small width Δk , then $Q(\beta)$ is stretched along the cylinder (width $\sim 1/\Delta k$), and during evolution it both moves up and down the cylinder and encircles it simultaneously.

For $\omega = 0$, Eq. (25) tells us that the spectrum forms a band $\varepsilon(k) = 2J \cos(k)$ with the quasimomentum $k \in (-\pi, +\pi]$, and the eigenstates are the extended Bloch states [42].

2. Heisenberg-Weyl algebra

The Heisenberg-Weyl algebra $\mathfrak{hw}$ shares certain similarities with the Euclidean algebra, but differs in several important aspects. Mathematically, $\mathfrak{hw}$ is spanned by the generators $\{\hat{a}, \hat{a}^\dagger, \mathbb{I}\}$, where $\mathbb{I}$ denotes the identity operator. Since $\mathbb{I}$ commutes with all other elements of the algebra, it does not belong to the Cartan subalgebra but instead generates a central extension of the algebra. In physical applications it is often convenient to enlarge the set of generators to $\{\hat{a}, \hat{a}^\dagger, \hat{n}, \mathbb{I}\}$,

where the number operator $\hat{n} = \hat{a}^\dagger\hat{a}$ may be viewed as a Cartan generator.

In contrast to the $\mathfrak{e}(2)$ algebra, the root generators of $\mathfrak{hw}$ include normalization factors

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad (\text{A11})$$

and the existence of a vacuum state $|0\rangle$ imposes a lower boundary on the FSL, which therefore admits a natural reference state $|\psi_0\rangle = |0\rangle$. The nonvanishing commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \mathbb{I} \quad (\text{A12})$$

implies the absence of a Hermitian phase operator $\hat{\phi}$ canonically conjugates to $\hat{n}$, i.e., $[\hat{n}, \hat{\phi}] = i\mathbb{I}$, unless the Hilbert space is truncated to finite dimension [66,67].

Despite this, coherent states $|\alpha\rangle$, defined via the displacement operator

$$\hat{D}_b(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}, \quad |\alpha\rangle = \hat{D}_b(\alpha)|0\rangle, \quad (\text{A13})$$

are well behaved. They are Gaussian minimum-uncertainty states and form an overcomplete basis, with overlap

$$|\langle\beta|\alpha\rangle|^2 = e^{-|\alpha-\beta|^2}. \quad (\text{A14})$$

In this case, the Lie-algebra phase space coincides with the regular phase space of a bosonic mode or a particle on a line, namely the x - p plane. Unlike the $\mathfrak{e}(2)$ algebra, the displacement operators do not commute, but instead satisfy

$$\hat{D}_b(\alpha)\hat{D}_b(\beta) = e^{i\text{Im}(\alpha\beta^*)}\hat{D}_b(\alpha+\beta). \quad (\text{A15})$$

The Cartan generator $\hat{n}$ is diagonal in the bosonic Fock basis $|n\rangle$, while the root generators $\hat{a}$ and $\hat{a}^\dagger$ connect neighboring Fock states in the FSL. Consequently, the FSL is an infinite one-dimensional tight-binding chain with sites labeled by $n = 0, 1, \dots$. The normalization factors in Eq. (A11) imply that the tunneling amplitudes are site dependent, scaling as $\sqrt{n}$.

To draw an analogy between the LPS and the FSL similar to Fig. 1 for the $\mathfrak{e}(2)$ algebra, we note that the LPS is the x - p plane, while the Cartan generator $\hat{n}$ may be viewed as defining an axis perpendicular to this plane. Introducing the parabolic surface

$$n(x, p) = \frac{p^2}{2} + \frac{x^2}{2} + \frac{1}{2}, \quad (\text{A16})$$

one may relate the Husimi function $Q(x, p)$ to the Fock-state distribution by first projecting $Q(x, p)$ onto this surface and subsequently projecting onto the n axis.

Physical realization and dynamics

As a physical realization, consider the driven harmonic oscillator,

$$\hat{H}_{\text{dHO}} = \Delta\hat{n} + \eta(\hat{a} + \hat{a}^\dagger), \quad (\text{A17})$$

where Δ is the detuning between the oscillator and the drive, and η is the drive amplitude [21,49]. Since the Heisenberg-Weyl algebra is finite dimensional as a Lie algebra, the Heisenberg equations of motion close and the system is analytically solvable. Here it is clear that as long as $\Delta \neq 0$ we need the extended algebra $\{\hat{a}, \hat{a}^\dagger, \hat{n}, \mathbb{I}\}$ instead of $\{\hat{a}, \hat{a}^\dagger, \mathbb{I}\}$. We note that the Hamiltonian can alternatively

be diagonalized using the displacement $\hat{D}_b(-\eta/\Delta)$, yielding eigenenergies $\varepsilon_n = \Delta n - \eta^2/\Delta$ and eigenstates given by displaced Fock states.

3. $\mathfrak{su}(2)$ algebra

The $\mathfrak{su}(2)$ algebra is generated by the spin operators $\{\hat{S}_z, \hat{S}_+, \hat{S}_-\}$, satisfying the commutation relations

$$[\hat{S}^+, \hat{S}^-] = 2\hat{S}_z, \quad [\hat{S}_z, \hat{S}^\pm] = \pm\hat{S}^\pm. \quad (\text{A18})$$

The Cartan generator $\hat{S}_z$ defines the FSL sites

$$\hat{S}_z |S, m\rangle = m |S, m\rangle, \quad (\text{A19})$$

where $m = -S, \dots, S$. The root generators $\hat{S}_\pm$, on the other hand, establish the connection between the sites

$$\hat{S}_\pm |S, m\rangle = \sqrt{S(S+1) - m(m \pm 1)} |S, m \pm 1\rangle. \quad (\text{A20})$$

In this way, it is possible to see that the resulting FSL is a finite one-dimensional lattice that, due to the form of the action of $\hat{S}_\pm$, does not have translational invariance.

Perelomov coherent states $|\theta, \phi\rangle$ in this algebra are defined through the action of the displacement operators (34) and (35). In the spin Fock basis, they expand as

$$|\theta, \phi\rangle = \sum_{m=-S}^S \binom{2S}{S+m}^{1/2} \left(\cos \frac{\theta}{2}\right)^{S+m} \left(\sin \frac{\theta}{2}\right)^{S-m} \times e^{-i(S-m)\phi} |S, m\rangle. \quad (\text{A21})$$

After a laborious calculation one verifies that they satisfy the completeness relation

$$\frac{2S+1}{4\pi} \int \sin \theta d\theta d\phi |\theta, \phi\rangle \langle \theta, \phi| = \mathbb{I}. \quad (\text{A22})$$

Moreover, if $\mathbf{n}$ denotes the unit vector along the coherent-state direction and $\hat{S}_\perp$ is the spin component perpendicular to $\mathbf{n}$, then

$$(\Delta \hat{S}_\perp)^2 = \frac{\hbar^2 S}{2}. \quad (\text{A23})$$

Going back to the FSL, the probability distribution of a PCS $|\theta, \phi\rangle$ in the lattice is given by

$$P(m) = \binom{2S}{S+m} \left(\cos \frac{\theta}{2}\right)^{2(S+m)} \left(\sin \frac{\theta}{2}\right)^{2(S-m)}, \quad (\text{A24})$$

which is clearly a binomial distribution $\text{Bin}(k = S + m)$. If we identify $p = \cos(\theta/2)$, then the distribution mean easily follows:

$$\langle k \rangle = 2Sp \rightarrow \langle m \rangle = 2Sp - S = S \cos \theta. \quad (\text{A25})$$

Similarly for the variance

$$\begin{aligned} \text{Var}(k) &= 2Sp(1-p) \\ \rightarrow \text{Var}(m) &= 2Sp(1-p) = \frac{S}{2} \sin^2 \theta, \end{aligned}$$

resulting in the standard deviation $\Delta(m) = \sqrt{\frac{S}{2}} |\sin \theta|$.

a. Heisenberg equation solutions

Here we derive the solution of the Hamiltonian (43) in the Schwinger-boson representation mentioned in the main text. Writing the Heisenberg equations of motion

$$i \frac{d}{dt} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} \delta/2 & J \\ J & -\delta/2 \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}, \quad (\text{A26})$$

whose matrix has eigenvalues

$$\lambda_\pm = \pm \frac{\Omega}{2}, \quad \Omega = \sqrt{\delta^2 + (2J)^2}. \quad (\text{A27})$$

Exponentiating the evolution matrix gives

$$\begin{pmatrix} \hat{a}(t) \\ \hat{b}(t) \end{pmatrix} = e^{-iMt} \begin{pmatrix} \hat{a}(0) \\ \hat{b}(0) \end{pmatrix}, \quad (\text{A28})$$

where

$$e^{-iMt} = \cos \frac{\Omega t}{2} \mathbb{I} - \frac{2i}{\Omega} \sin \frac{\Omega t}{2} M. \quad (\text{A29})$$

Explicitly,

$$\begin{aligned} \hat{a}(t) &= \left(\cos \frac{\Omega t}{2} - i \frac{\delta}{\Omega} \sin \frac{\Omega t}{2} \right) \hat{a}(0) - i \frac{2J}{\Omega} \sin \frac{\Omega t}{2} \hat{b}(0), \\ \hat{b}(t) &= -i \frac{2J}{\Omega} \sin \frac{\Omega t}{2} \hat{a}(0) + \left(\cos \frac{\Omega t}{2} + i \frac{\delta}{\Omega} \sin \frac{\Omega t}{2} \right) \hat{b}(0). \end{aligned} \quad (\text{A30})$$

For the highest-weight state $|S, S\rangle$, corresponding to the bosonic Fock state

$$|S, S\rangle = |2S, 0\rangle, \quad (\text{A31})$$

one has

$$\langle \hat{a}(0) \rangle = \sqrt{2S}, \quad \langle \hat{b}(0) \rangle = 0, \quad (\text{A32})$$

yielding

$$\begin{aligned} \langle \hat{a}(t) \rangle &= \sqrt{2S} \left(\cos \frac{\Omega t}{2} - i \frac{\delta}{\Omega} \sin \frac{\Omega t}{2} \right), \\ \langle \hat{b}(t) \rangle &= -i \sqrt{2S} \frac{2J}{\Omega} \sin \frac{\Omega t}{2}. \end{aligned} \quad (\text{A33})$$

The spin expectation values follow directly from the inverse Schwinger map,

$$\langle \hat{S}_x \rangle = \frac{1}{2} (\langle \hat{a}^\dagger \rangle \langle \hat{b} \rangle + \langle \hat{b}^\dagger \rangle \langle \hat{a} \rangle), \quad (\text{A34})$$

$$\langle \hat{S}_y \rangle = \frac{1}{2i} (\langle \hat{a}^\dagger \rangle \langle \hat{b} \rangle - \langle \hat{b}^\dagger \rangle \langle \hat{a} \rangle), \quad (\text{A35})$$

and

$$\langle \hat{S}_z \rangle = \frac{1}{2} (|\langle \hat{a} \rangle|^2 - |\langle \hat{b} \rangle|^2), \quad (\text{A36})$$

from which one obtains

$$\begin{aligned} \langle \hat{S}_x(t) \rangle &= \frac{2J\delta}{\Omega^2} S [1 - \cos(\Omega t)], \\ \langle \hat{S}_y(t) \rangle &= -\frac{2J}{\Omega} S \sin(\Omega t), \\ \langle \hat{S}_z(t) \rangle &= S \frac{\delta^2 + (2J)^2 \cos(\Omega t)}{\Omega^2}. \end{aligned} \quad (\text{A37})$$

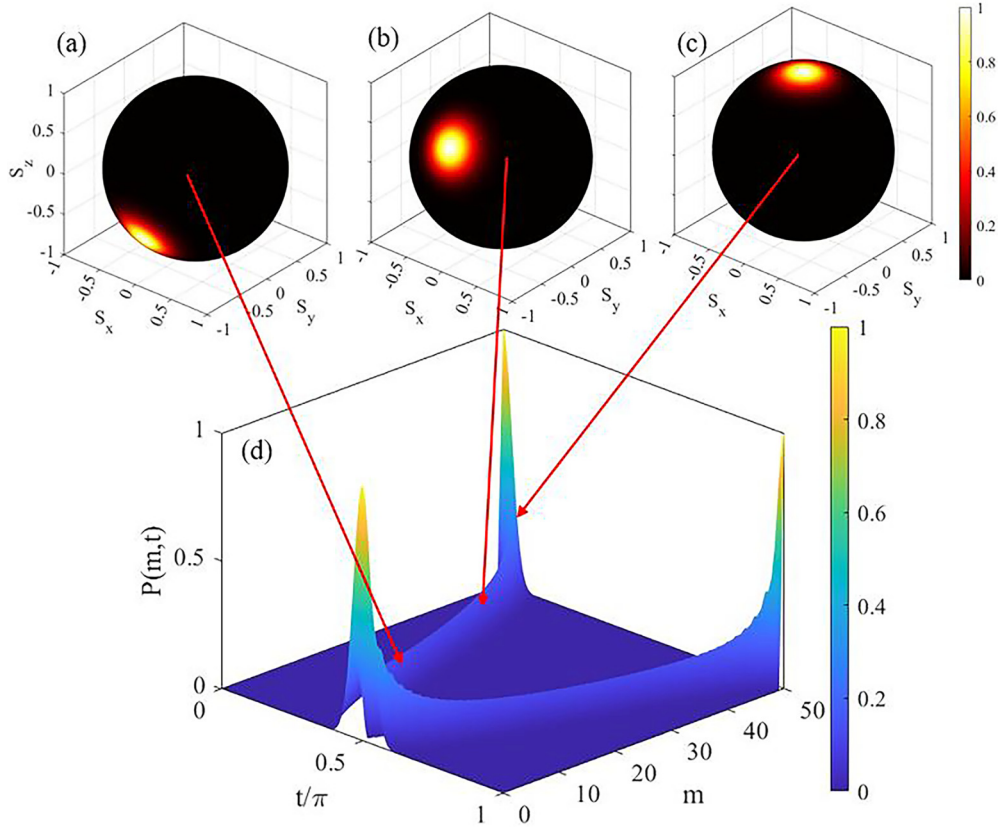


FIG. 4. Demonstration of the time evolution in the FSL (d) and the LPS (a)–(c) for the $\mathfrak{su}(2)$ model. (d) Shows the time evolution of the FSL population $P(m, t)$, while (a)–(c) display the Husimi function $Q(\theta, \phi)$ on the Bloch sphere. The initial state is $|\psi(0)\rangle = |S, S\rangle$ with $S = 50$, corresponding to a Perelomov coherent state, and the dynamics are governed by the Hamiltonian (7) with tunneling rate $J = 1$. In the FSL, the state starts at one edge of the lattice and undergoes oscillatory dynamics with revival time $T_{\text{rev}} = \pi$, such that at half the revival time the population is completely transferred to the opposite edge. On the LPS, since the Hamiltonian generates a displacement, the state remains coherent throughout the evolution and follows a geodesic (great circle) on the Bloch sphere. (a)–(c) Correspond to times $t = 0, \pi/6$, and $\pi/3$, respectively, with red arrows indicating the corresponding FSL distributions.

b. Physical realization and dynamics

Considering the two-mode Hamiltonian $\hat{H}_{2m}$ of Eq. (7) as a realization of the $\mathfrak{su}(2)$ FSL, the system exhibits perfect state transfer between the two modes [21,68]. For an initial state $|\psi_a\rangle \otimes |0\rangle$, the state is perfectly transferred, up to phase factors, to mode b after half the revival time $T_{\text{rev}}/2$, reflecting the equidistant spectrum of the FSL. This behavior is illustrated in Fig. 4 for the initial Fock state $|N, 0\rangle$. In addition to the time evolution of the FSL population $P(m, t)$, the figure also shows three snapshots of the corresponding Husimi function, which follows a geodesic on the Bloch sphere.

4. $\mathfrak{su}(3)$ algebra

The $\mathfrak{su}(3)$ algebra differs from the previous examples in that it possesses two Cartan generators, resulting in a two-dimensional FSL. In addition, there are three root pairs, giving rise to six root generators, where each pair defines a displacement operator $\hat{D}_{\alpha_i}(\beta_i)$ [Eq. (18)]. Consequently, each lattice site in the FSL has six nearest neighbors, forming a triangular lattice which was already mentioned in Sec. II A.

For $\mathfrak{su}(2)$ the minimal matrix representation is given by the Pauli matrices, while for $\mathfrak{su}(3)$ it is the 3×3 Gell-Mann matrices $\hat{\lambda}_k$ ($k = 1, 2, \dots, 8$). Analogous to the Pauli ma-

trices, the Gell-Mann matrices can be generalized to larger representations. More relevant for FSLs, as in $\mathfrak{su}(2)$, a Schwinger-boson mapping allows the Cartan and root generators to be expressed in terms of bosonic modes [44], as introduced in Eq. (12). The operators $\hat{H}_1$ and $\hat{H}_2$ act as the Cartan generators and are diagonal in the Fock basis. The remaining operators, $\hat{I}_{\pm}$, $\hat{U}_{\pm}$, and $\hat{V}_{\pm}$, are root generators, generally denoted $\hat{E}_{\alpha}$ and labeled by roots α . In the lattice plane, not all three roots are independent; rather, $\alpha_3 = \alpha_1 + \alpha_2$, consistent with the three primitive directions of a triangular lattice. As in $\mathfrak{su}(2)$, the sum of squared generators defines the quadratic Casimir operator $\hat{F}_2$, while a second, cubic Casimir operator $\hat{F}_3$ involves products of three generators. The operator $\hat{F}_2$ enforces conservation of the total boson number, so the Fock states can be written as

$$|n_a, n_b, n_c\rangle = |n_a, n_b, N - n_a - n_b\rangle, \quad (\text{A38})$$

with $N = n_a + n_b + n_c$ the total boson number. The physical interpretation of $\hat{F}_3$ is less direct, but it is associated with a chirality in the FSL connecting the three modes.

In the previous examples, we could visualize the mapping from a two-dimensional LPS to a one-dimensional FSL. Here, the LPS is four dimensional while the FSL is two dimensional, rendering such visualization impractical. Instead, we

focus on understanding the FSL structure through the bosonic representation [21]. For fixed N , the triangular FSL contains $(N+1)(N+2)/2$ sites corresponding to the Fock states in Eq. (A38). The three corners correspond to single-mode occupation, e.g., $|N, 0, 0\rangle$, while sites along an edge have exactly one empty mode, e.g., $|n_a, N - n_a, 0\rangle$. More generally, along any primitive lattice direction one bosonic mode remains fixed, and the corresponding root-generator pair, e.g., $(\hat{L}_-, \hat{L}_+)$, induces nearest-neighbor hopping along that direction. Accordingly, the displacement operator

$$\hat{D}_I(\beta_I) = e^{\beta_I \hat{L}_+ - \beta_I^* \hat{L}_-}, \quad (\text{A39})$$

and analogously for the $\hat{U}$ and $\hat{V}$ generators, generates translations along the lattice directions associated with α_1 , α_2 , and α_3 , respectively.

Any of the corner states may be chosen as the reference state $|\psi_0\rangle$, and all $\mathfrak{su}(3)$ PCSs can be generated from it using the operator

$$\hat{\Omega}(\xi) = e^{\xi_1 \hat{a}^\dagger \hat{b} + \xi_2 \hat{b}^\dagger \hat{c} + \xi_3 \hat{a}^\dagger \hat{c} + \text{H.c.}}, \quad \xi_i \in \mathbb{C}, \quad (\text{A40})$$

such that $|\xi\rangle = \hat{\Omega}(\xi)|N, 0, 0\rangle$. Because the displacement operators $\hat{D}_{\alpha_i}(\beta_i)$ do not mutually commute, $\hat{\Omega}(\xi)$ cannot be factorized into a simple product of displacement operators, reflecting the noncommutativity of the algebra (rotations in higher dimensions).

An alternative representation of the $\mathfrak{su}(3)$ PCSs is given by [47,48,69]

$$|\xi\rangle = \frac{1}{\sqrt{N!}} (\zeta_1 \hat{a}^\dagger + \zeta_2 \hat{b}^\dagger + \zeta_3 \hat{c}^\dagger)^N |0, 0, 0\rangle. \quad (\text{A41})$$

For the four-dimensional LPS, the coefficients ζ_i may be parametrized by four angles as

$$\begin{aligned} \zeta_1 &= \cos \theta_1, \\ \zeta_2 &= e^{i\phi_1} \sin \theta_1 \cos \theta_2, \\ \zeta_3 &= e^{i\phi_2} \sin \theta_1 \sin \theta_2, \end{aligned} \quad (\text{A42})$$

with $0 \leq \theta_1, \theta_2 \leq \pi/2$ and $0 \leq \phi_1, \phi_2 \leq 2\pi$. In principle, one can also derive an explicit mapping between the parameter sets $\{\zeta_i\}$ and $\{\xi_i\}$.

Physical realization and dynamics

We now turn to the evolution in the FSL under the Hamiltonian

$$\hat{H}_{\mathfrak{su}(3)} = J(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{c} + e^{i\phi} \hat{a}^\dagger \hat{c} + \text{H.c.}). \quad (\text{A43})$$

The phase factor $e^{i\phi}$ introduces a staggered synthetic magnetic flux through the triangular FSL, breaking time-reversal symmetry and inducing chiral motion, consistent with a non-vanishing local topological marker. The dynamics of this model, including chiral edge currents, have been studied in detail in Ref. [21].

Figure 5 shows two snapshots of the distribution $P(n_a, n_b, N - n_a - n_b, t)$ evolved under Hamiltonian (A43). In both panels, the initial state is $|30, 30, 30\rangle$, corresponding to the central lattice site. In Fig. 5(a), the dynamics are time-reversal symmetric ($\phi = 0$), while in Fig. 5(b) time-reversal symmetry is broken by choosing $\phi = \pi/3$, resulting in a

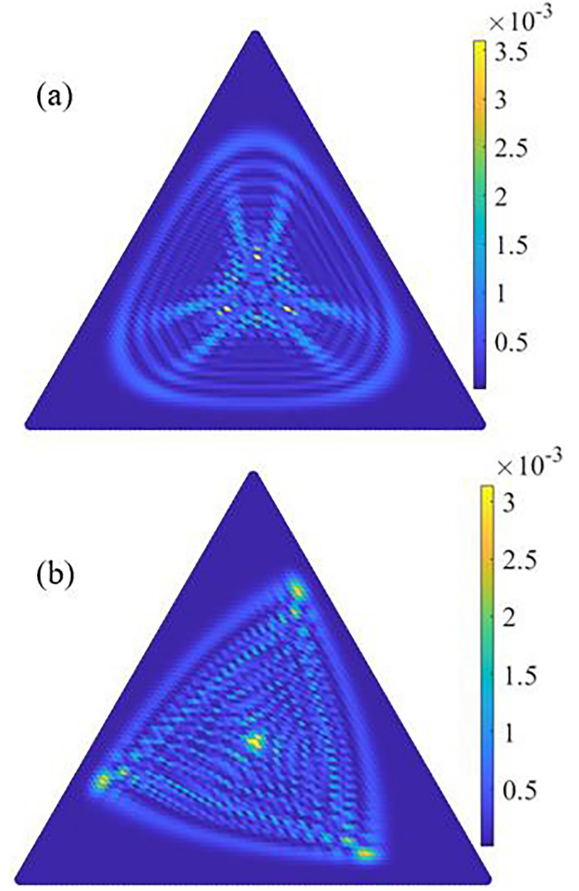


FIG. 5. Two snapshots of the distribution $P(n_a, n_b, N - n_a - n_b, t)$ under the $\mathfrak{su}(3)$ Hamiltonian (A43). The initial state is $|30, 30, 30\rangle$, i.e., $n_a = n_b = n_c = 30$, corresponding to the center site of the lattice. (a) Shows the time-reversal symmetric case ($\phi = 0$), while (b) has $\phi = \pi/3$, introducing a synthetic magnetic flux in the FSL and breaking the symmetry seen in (a). Both snapshots are taken at $t = 0.3J$ with total boson number $N = 90$.

synthetic magnetic field in the FSL. The presence of a gauge field clearly breaks the symmetry observed in Fig. 5(a). In both cases, $t = 0.3J$ and the total boson number is $N = 90$.

5. $\mathfrak{so}(5)$ algebra

Similarly to the $\mathfrak{su}(1, 1)$ algebra, the FSL emerging from the $\mathfrak{so}(5)$ algebra has not been discussed previously. As it turns out, it exhibits several interesting properties, some of which we discuss here. The minimal representation of the $\mathfrak{so}(5)$ algebra consists of 5×5 real antisymmetric matrices, ten in total. The generators are associated with rotations in the different coordinate planes, analogous to how the generators of $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ realize complex rotations. Like $\mathfrak{su}(3)$, the $\mathfrak{so}(5)$ algebra has two Cartan generators, $\hat{H}_{1,2}$, resulting in a two-dimensional FSL. There are four pairs of root generators $\hat{\Sigma}_{\pm\alpha}$, which can be divided into *short* roots (α_1, α_2) and *long* roots ($\alpha_1 + \alpha_2, \alpha_1 - \alpha_2$). These generate tunneling between nearest- and next-nearest-neighbor sites in the FSL, respectively. Consequently, the FSL forms a square lattice with both horizontal and vertical and diagonal tunneling terms. Since the LPS is six dimensional and compact, the corresponding FSL

is also compact. In addition to the quadratic Casimir operator $\hat{\Gamma}_2$ (boson conservation), the algebra admits a quartic Casimir operator $\hat{\Gamma}_4$, which involves products of four generators.

The bosonic representation of the $\mathfrak{so}(5)$ algebra involves four modes $\hat{a}_{\uparrow,\downarrow}$ and $\hat{b}_{\uparrow,\downarrow}$. One might naively expect that two bosonic modes $\hat{a}$ and $\hat{b}$ would suffice to construct a square FSL with both nearest- and next-nearest-neighbor tunneling. However, such a construction would necessarily involve generators that do not conserve the total boson number, such as $\hat{a}^\dagger \hat{b}^\dagger$ and $\hat{a} \hat{b}$, rendering the corresponding FSL noncompact. This situation instead corresponds to the algebra $\mathfrak{su}(1, 1) \oplus \mathfrak{su}(1, 1)$.

In the bosonic realization of $\mathfrak{so}(5)$, the Cartan generators may be chosen as

$$\begin{aligned}\hat{H}_1 &= \frac{1}{2}(\hat{a}_\uparrow^\dagger \hat{a}_\uparrow - \hat{a}_\downarrow^\dagger \hat{a}_\downarrow), \\ \hat{H}_2 &= \frac{1}{2}(\hat{b}_\uparrow^\dagger \hat{b}_\uparrow - \hat{b}_\downarrow^\dagger \hat{b}_\downarrow),\end{aligned}\quad (\text{A44})$$

with root generators

$$\begin{aligned}\hat{\Sigma}_{\alpha_1} &= \hat{a}_\uparrow^\dagger \hat{a}_\downarrow, & \hat{\Sigma}_{\alpha_2} &= \hat{b}_\uparrow^\dagger \hat{b}_\downarrow, \\ \hat{\Sigma}_{\alpha_1+\alpha_2} &= \hat{a}_\uparrow^\dagger \hat{b}_\downarrow, & \hat{\Sigma}_{\alpha_1-\alpha_2} &= \hat{a}_\downarrow^\dagger \hat{b}_\uparrow,\end{aligned}\quad (\text{A45})$$

together with their Hermitian conjugates corresponding to the negative roots. This construction, involving four bosonic modes divided into two a and two b modes, is standard: the labels a and b denote orbital degrees of freedom, while the subscripts indicate an internal spin.

The Fock states are of the form $|n_{a_\uparrow}, n_{a_\downarrow}, n_{b_\uparrow}, n_{b_\downarrow}\rangle$, with the total boson number $N = n_{a_\uparrow} + n_{a_\downarrow} + n_{b_\uparrow} + n_{b_\downarrow}$ being conserved. The lattice-site indices (i, j) of the FSL are determined by the eigenvalues of the Cartan generators, $m_1 = n_{a_\uparrow} - n_{a_\downarrow}$ and $m_2 = n_{b_\uparrow} - n_{b_\downarrow}$. Fixing the total boson number restricts the lattice to a finite square with a diamond-shaped boundary in the (i, j) plane. The corner sites of this diamond correspond to extremal Fock states such as $|N, 0, 0, 0\rangle$, $|0, N, 0, 0\rangle$.

It is instructive to note that $(\hat{H}_1, \hat{\Sigma}_{\pm\alpha_1})$ and $(\hat{H}_2, \hat{\Sigma}_{\pm\alpha_2})$ each form an $\mathfrak{su}(2)$ subalgebra. The remaining root generators, which correspond to diagonal tunneling processes in the FSL, are essential for closing the full $\mathfrak{so}(5)$ algebra. Their presence also implies that the FSL cannot be decomposed into independent lower-dimensional lattices. Algebraically, this reflects the fact that $\mathfrak{so}(5) \not\cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

Physical realization and dynamics

To discuss FSL dynamics, we consider the Hamiltonian

$$\begin{aligned}\hat{H}_{\mathfrak{so}(5)} &= J_1(\hat{a}_\uparrow^\dagger \hat{a}_\downarrow + \hat{b}_\uparrow^\dagger \hat{b}_\downarrow + \text{H.c.}) \\ &+ J_2(e^{i\phi} \hat{a}_\uparrow^\dagger \hat{b}_\uparrow + \hat{a}_\uparrow^\dagger \hat{b}_\downarrow + \hat{a}_\downarrow^\dagger \hat{b}_\uparrow + \hat{a}_\downarrow^\dagger \hat{b}_\downarrow + \text{H.c.}),\end{aligned}\quad (\text{A46})$$

where J_1 and J_2 are the tunneling amplitudes for nearest- and next-nearest-neighbor processes, respectively, and the phase factor ϕ generates a synthetic flux in the FSL. Although several closed loops can be drawn within a square plaquette containing vertical, horizontal, and diagonal bonds, these loops are not independent. The square loop decomposes into sums of triangular loops, and gauge freedom removes all but a single gauge-invariant phase. Consequently, the dynamics is

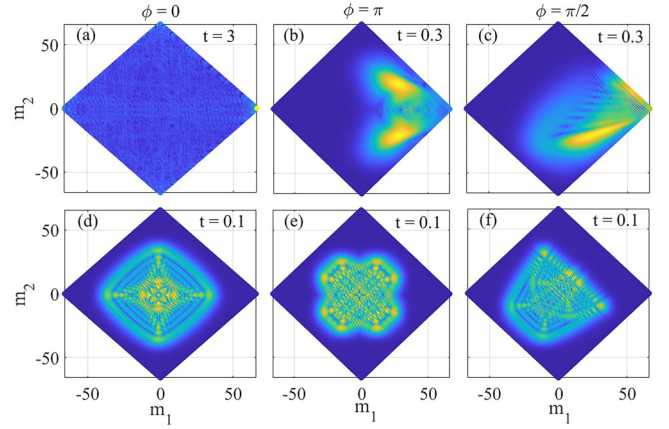


FIG. 6. Snapshots of the fourth root of the distribution $P(m_1, m_2, t)$ in the FSL corresponding to the $\mathfrak{so}(5)$ model, evolved under the Hamiltonian (A46). The upper row corresponds to the initial state $|n_{a_\uparrow} = 132, n_{a_\downarrow} = 0, n_{b_\uparrow} = 0, n_{b_\downarrow} = 0\rangle$, while the lower row shows evolution from the central lattice site $|n_{a_\uparrow} = 33, n_{a_\downarrow} = 33, n_{b_\uparrow} = 33, n_{b_\downarrow} = 33\rangle$. The phase is $\phi = 0$ in (a) and (d), $\phi = \pi$ in (b) and (e), and $\phi = \pi/2$ in (c) and (f). For $\phi = \pi/2$, time-reversal symmetry is broken and the distribution exhibits asymmetric chiral evolution. All plots are shown for the balanced case $J_1 = J_2 = 1$ with total boson number $N = 132$, and the time t for each example is given in the panels.

governed by a single independent flux parameter, even though some closed paths accumulate zero net phase.

For fixed N , the Hilbert-space dimension is $\text{Dim}(\mathcal{H}) = N^2/2 + N + 1$, which equals the number of sites in the FSL. Since $\hat{H}_{\mathfrak{so}(5)}$ is quadratic, it can be diagonalized by first diagonalizing the corresponding single-particle Hamiltonian

$$\hat{h}_{\mathfrak{so}(5)} = \begin{pmatrix} 0 & J_1 & J_1 & J_2 e^{i\phi} \\ J_1 & 0 & J_2 e^{i\phi} & J_1 \\ J_1 & J_2 e^{-i\phi} & 0 & J_1 \\ J_2 e^{-i\phi} & J_1 & J_1 & 0 \end{pmatrix}. \quad (\text{A47})$$

The eigenvalues are roots of a quartic polynomial and are generally cumbersome. However, for $J_1 = J_2 = J$ one finds

$$\epsilon_{1,2,3,4} = \pm J \sqrt{2 \pm 2 \cos\left(\frac{\phi}{2}\right)}, \quad (\text{A48})$$

with the full many-body spectrum given by

$$E_{\{n_i\}} = \sum_{i=1}^4 \epsilon_i n_i, \quad n_i \in \{0, 1, \dots, N\} \quad (\text{A49})$$

subject to the constraint $\sum_{i=1}^4 n_i = N$. If

$$\frac{\epsilon_1}{\epsilon_3} = \sqrt{\frac{2 + 2 \cos(\phi/2)}{2 - 2 \cos(\phi/2)}} = \cot\left(\frac{\phi}{4}\right) \quad (\text{A50})$$

is a rational number m/n , the system exhibits perfect revivals after a time $T_{\text{rev}} = \pi m/[J \cos(\phi/4)]$. This revival behavior does not persist for generic values of J_1 and J_2 .

Examples of the FSL distribution for various initial states and values of ϕ are shown in Fig. 6. For visual clarity, we plot the fourth root of the distribution. For $\phi = 0$ and $\phi =$

π , the Hamiltonians (A46) and (A47) are real and preserve time-reversal symmetry, leading to symmetric evolution. For $\phi = \pi/2$, by contrast, the distribution preferentially flows in one direction, giving rise to a chiral current in the lattice.

6. $\mathfrak{su}(1, 1)$ algebra

The $\mathfrak{su}(1, 1)$ algebra generator set is given by three operators $\{\hat{K}_0, \hat{K}_+, \hat{K}_-\}$, whose commutation relations follow

$$[\hat{K}^+, \hat{K}^-] = -2\hat{K}_0, \quad [\hat{K}_0, \hat{K}^\pm] = \pm\hat{K}^\pm. \quad (\text{A51})$$

The Cartan generator $\hat{K}_0$ determines the FSL lattice sites $|k, m\rangle$ while the root generators, governing the state transitions, give the FSL edges. The action of the generators on these states is [70]

$$\begin{aligned} \hat{K}_0|k, m\rangle &= (k + m)|k, m\rangle, \\ \hat{K}^+|k, m\rangle &= \sqrt{(m+1)(m+2k)}|k, m+1\rangle, \\ \hat{K}^-|k, m\rangle &= \sqrt{m(m+2k-1)}|k, m-1\rangle, \end{aligned} \quad (\text{A52})$$

with $m \in \mathbb{Z}$ and $k = \frac{1}{4}$ or $\frac{3}{4}$ as discussed in the main text.

In the boson representation (46), the displacement operator is given by the squeezing operator

$$\hat{D}(\xi) = e^{2\xi(\hat{a}^\dagger)^2 - \beta^* \hat{a}^2}, \quad (\text{A53})$$

and it generates PCS of the form $|k, \xi\rangle$, where k depends on the choice of a reference state. More precisely, with the vacuum as the reference state ($k = \frac{1}{4}$), the PCS is given by [58]

$$|\xi\rangle \equiv \hat{D}(\xi)|0\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} (-e^{i\theta} \tanh r)^n |2n\rangle \quad (\text{A54})$$

in the Fock basis, and with the parametrization $\xi = re^{i\theta}$.

As mentioned in the main text, the LPS geometry can be determined by considering the Casimir operator (45). Since it is a constant of motion, by parametrizing $K_+ = r \sin \theta$ and $K_- = r \cos \theta$, then K_0 must satisfy $K_0 = \pm\sqrt{\Gamma + r^2}$. Together with the angular variable θ one sees that the LPS consists of a hyperbolic sheet (see Fig. 3), in which the bosonic representation enforces the occupation of the sheet in which $K_0 = +\sqrt{\Gamma + r^2} > 0$, due to the existence of the vacuum state.

a. FSL distribution asymptotics

The FSL distribution is obtained from the expansion coefficients of the squeezed vacuum in the Fock basis,

$$p(n, \xi) = |\langle 2n|\xi\rangle|^2 = \frac{1}{\cosh r} \frac{(2n)!}{4^n (n!)^2} \tanh^{2n} r, \quad (\text{A55})$$

where only even Fock states contribute because the squeezed vacuum preserves photon-number parity.

To understand how the wave packet spreads along the FSL as the squeezing increases, let us examine the asymptotic form of the distribution for $r \gg 1$. Using

$$\frac{1}{\cosh r} = 2e^{-r} + O(e^{-3r}) \quad (\text{A56})$$

and

$$\tanh r = 1 - 2e^{-2r} + O(e^{-4r}), \quad (\text{A57})$$

one finds

$$\ln(\tanh r) = -2e^{-2r} + O(e^{-4r}), \quad (\text{A58})$$

so that

$$\tanh^{2n} r = \exp[2n \ln(\tanh r)] \simeq \exp(-4ne^{-2r}). \quad (\text{A59})$$

Therefore,

$$p(n, \xi) \sim \frac{2e^{-r}(2n)!}{4^n (n!)^2} \exp(-4ne^{-2r}). \quad (\text{A60})$$

For large n , Stirling's approximation gives

$$\frac{(2n)!}{4^n (n!)^2} = \frac{\binom{2n}{n}}{4^n} \simeq \frac{1}{\sqrt{\pi n}}, \quad (\text{A61})$$

so that the asymptotic distribution becomes

$$p(n, \xi) \simeq \frac{2e^{-r}}{\sqrt{\pi n}} \exp(-4ne^{-2r}), \quad (\text{A62})$$

whose characteristic width grows exponentially with the squeezing parameter.

The first and second moments of the distribution can be obtained exactly from the generating function

$$S(x) = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2} x^n = \frac{1}{\sqrt{1-x}}, \quad (\text{A63})$$

with

$$x = \tanh^2 r. \quad (\text{A64})$$

Since the normalized probability distribution is

$$p_n = \frac{x^n}{S(x)} \frac{(2n)!}{4^n (n!)^2}, \quad (\text{A65})$$

its moments are

$$\langle n \rangle = \frac{xS'(x)}{S(x)} \quad (\text{A66})$$

and

$$\langle n^2 \rangle = \frac{xS'(x) + x^2 S''(x)}{S(x)}. \quad (\text{A67})$$

Evaluating the derivatives yields

$$\langle n \rangle = \frac{\sinh^2 r}{2} \quad (\text{A68})$$

and

$$\langle n^2 \rangle = \frac{3}{4} \sinh^4 r + \frac{1}{2} \sinh^2 r. \quad (\text{A69})$$

The variance is therefore

$$(\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2 = \frac{\sinh^2 r}{2} (\sinh^2 r + 1). \quad (\text{A70})$$

Finally, using

$$\sinh r \simeq \frac{e^r}{2}, \quad r \gg 1 \quad (\text{A71})$$

one obtains

$$\langle n \rangle \simeq \frac{e^{2r}}{8} \quad (\text{A72})$$

and

$$\Delta n = \sqrt{\frac{\sinh^2 r}{2}(\sinh^2 r + 1)} \simeq \frac{e^{2r}}{4\sqrt{2}}. \quad (\text{A73})$$

Thus, both the mean occupation number and its standard deviation increase exponentially with the squeezing parameter, i.e.,

$$\langle n \rangle, \quad \Delta n \propto e^{2r}, \quad (\text{A74})$$

showing that, although the state remains localized on the LPS, its representation in the FSL becomes exponentially delocalized.

b. Physical realization and dynamics

As for the other algebras, let us consider the evolution within the FSL under a Hamiltonian

$$\hat{H}_{\text{su}(1,1)} = \omega \hat{a}^\dagger \hat{a} + \frac{1}{2}(\xi(\hat{a}^\dagger)^2 + \xi^* \hat{a}^2). \quad (\text{A75})$$

As an initial state we consider the vacuum, such that the evolved state is a squeezed vacuum as in Eq. (48) but with time-dependent parameters and an overall phase factor. More precisely, the evolved state will be

$$|\psi(t)\rangle = e^{i\chi(t)}|\zeta(t)\rangle, \quad (\text{A76})$$

where

$$\chi = \frac{\omega t}{2} + \frac{1}{2} \arctan\left(\frac{\omega}{\Omega} \tan(\Omega t)\right) \quad (\text{A77})$$

and if we parametrize $\zeta(t) = r(t)e^{i\theta(t)}$, we have [58]

$$\tanh r(t) = \frac{|\xi|}{\sqrt{\omega^2 + \Omega^2 \cot^2(\Omega t)}}, \quad (\text{A78})$$

$$\theta(t) = \theta - \frac{\pi}{2} - \arctan\left(\frac{\omega}{\Omega} \tan(\Omega t)\right),$$

with $\Omega = \sqrt{\omega^2 - |\xi|^2}$. It is clear that there are two types of solutions, whether $\omega^2 - |\xi|^2$ is positive or negative, leading to stable oscillatory solutions or unstable exponential runaway solutions. The average boson number

$$\langle \hat{n} \rangle_t = \frac{|\xi|^2}{\Omega^2} \sin^2(\Omega t), \quad (\text{A79})$$

with a variance $\text{Var}(n)_t = 2\langle \hat{n} \rangle_t(\langle \hat{n} \rangle_t + 1)$.

For completeness, we note two other bosonic realizations of $\text{su}(1, 1)$. First, the single-mode representation [71]

$$\hat{K}_0 = \hat{a}^\dagger \hat{a} + \frac{1}{2}, \quad \hat{K}^+ = \sqrt{\hat{a}^\dagger} \hat{a}^\dagger, \quad \hat{K}^- = \hat{a} \sqrt{\hat{a}}, \quad (\text{A80})$$

with $k = \frac{1}{2}$, relevant for Jaynes-Cummings models with intensity-dependent couplings [72], and related to more general Holstein-Primakoff transformations [73]. Second, the two-mode squeezing representation [74,75]

$$\hat{K}_0 = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + 1), \quad \hat{K}^+ = \hat{a}^\dagger \hat{b}^\dagger, \quad \hat{K}^- = \hat{a} \hat{b}, \quad (\text{A81})$$

with $k = (|n_a - n_b| + 1)/2$. Also for these representations, the corresponding FSL distributions $p(n, \xi)$ will exponentially decay for increasing n 's, and with widths Δn growing exponentially with $|\xi|$.

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