

Non-Abelian geometric phases in triangular structures and universal SU(2) control in shape space

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We construct holonomic quantum gates for qubits that are encoded in the near-degenerate vibrational E doublet of a deformable three-body system. Using Kendall's shape theory, we derive the Wilczek-Zee connection that governs adiabatic transport within the E manifold. We show that its restricted holonomy group is SU(2), implying universal single-qubit control by closed loops in shape space. We provide explicit loops implementing a $\pi/2$ phase gate and a Hadamard-type gate. For two-qubit operations, we outline how linked holonomic cycles in arrays generate a controlled Chern-Simons phase, enabling an entangling controlled- X (CNOT) gate. We present a Ramsey-echo interferometric protocol that measures the Wilson loop trace of the Wilczek-Zee connection for a control cycle, providing a gauge-invariant signature of the non-Abelian holonomy. As a physically realizable demonstrator, we propose bond-length modulations of a Cs(6s)-Cs(6s)-Cs($nd_{3/2}$) Rydberg trimer in optical tweezers and specify operating conditions that suppress leakage out of the E manifold.

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I. INTRODUCTION

Non-Abelian geometric phases [1] provide one of the conceptual foundations of holonomic quantum computation. In its original form, the approach was formulated in terms of adiabatic loops of degenerate eigenspaces [2]. The ensuing gate is holonomic: It is determined by the closed path traced by the eigenspace and by the associated Wilczek-Zee connection. It is therefore independent of the detailed timing with which the path is traversed, provided the evolution remains adiabatic. This geometric character can provide robustness against certain classes of control imperfections. The framework has since been extended beyond the original adiabatic setting to nonadiabatic schemes [3], including experimental implementations [4]. In recent years, geometric-gate control has developed rapidly across several quantum computing platforms. For example, shortened-path nonadiabatic schemes have been proposed and experimentally implemented in superconducting circuits [5,6]. Noncyclic geometric gates have been shown in trapped-ion settings to combine speed with

noise resilience [7]. Semiconductor-spin experiments have demonstrated high-fidelity geometric manipulation in germanium quantum dots [8]. Together, these advances show that geometric phases have become a practical control resource for quantum information processing [9].

As quantum information science advances across multiple hardware platforms [10–13], there is growing interest in implementing geometric gates in scalable architectures with long coherence times. Programmable arrays of neutral Rydberg atoms are particularly promising. They combine uniform atomic qubits, high-fidelity site-resolved laser control, and engineered interactions [14,15]. In this setting, ultralong-range Rydberg excitations can bind one Rydberg atom to two ground-state atoms, forming triatomic molecular states [16,17]. These Rydberg trimers realize genuinely three-body, nonadditive interactions, which have been established spectroscopically [18]. Trimer spectral lines have also been used to probe nonlocal three-body correlations and to identify new, tunable trimer families [19–21]. These results indicate that molecular trimers can serve as experimentally accessible units for assembly into programmable arrays. This motivates our investigation of qubit encodings in vibrational geometry and holonomic control through non-Abelian geometric phases [1,3,4].

Complementing existing holonomic-gate proposals, we develop a framework in which a qubit is encoded directly in the shape degrees of freedom of a physical triangular system, such as a molecular trimer. We identify the logical two-level system with a near-degenerate vibrational E doublet of the trimer. Kendall's shape sphere [22] then provides the geometric space of triangular shapes on which this doublet is

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transported. Our central result is the construction of the corresponding Wilczek-Zee connection [1,2], which exposes the $SU(2)$ gauge structure generated by vibrational shape dynamics of triangular systems. Although our construction builds on adiabatic Wilczek-Zee holonomy, its geometric structure also connects naturally to nonadiabatic holonomic-gate schemes, where finite-time cyclic evolution of a computational subspace is engineered using pulse-controlled few-level systems [3,4,23,24].

With the connection in hand, we show that the restricted holonomy group is $SU(2)$. Hence, suitable closed loops in shape space can generate universal single-qubit holonomic operations on the E doublet. We introduce the gauge-invariant trace of the Wilson loop as a compact and experimentally accessible diagnostic of the non-Abelian shape-space holonomy. Finally, we outline how linked deformation cycles in arrays of triangular units can generate controlled geometric phases with a Chern-Simons description. When the acquired phase depends on the joint logical state of two units, such linked cycles provide a possible route to two-qubit entangling operations.

Triangular Rydberg-perturber configurations may realize the required E doublet through anisotropic electron-atom scattering and tunable axial confinement. They therefore offer a pathway toward micrometer-scale demonstrations in neutral-atom arrays. As a concrete reference platform, we propose the spectroscopically established $Cs(6s)-Cs(6s)-Cs(nd_{3/2})$ Rydberg trimer [18]. Its near-degenerate vibrational E doublet can serve as a candidate logical qubit. In an optical-tweezer array, phase-controlled bond-length modulations can steer the triangle through closed cycles on Kendall's shape sphere. These cycles realize the Wilson loops of the shape-space connection. The resulting non-Abelian holonomy can then be read out interferometrically through the trace of the Wilczek-Zee Wilson loop. In the Appendix, we outline a concrete control protocol together with platform parameters and operating conditions that suppress leakage from the E manifold.

More broadly, vibrational control structures are not limited to Rydberg trimers. They can arise whenever the internal motion of a deformable triangle supports a two-level system akin to a gapped, near-degenerate vibrational E doublet. Potential additional examples include Efimov-like trimers in ultracold gases [25]. Thus, the shape-space connection framework developed here may apply beyond Rydberg trimers to deformable three-body systems with a suitable two-level computational subspace. In this way, it points to a broad route toward geometric and topological control in few-body quantum systems.

The outline of this article is as follows. We begin by reviewing the topology and geometry of Kendall's shape space for general triangular structures [22]. Building on this framework, we construct the Wilczek-Zee connection that governs the non-Abelian holonomy generated by adiabatic shape changes. We then identify the associated Wilson loop as the holonomic quantum gate acting on a two-level subspace such as a vibrational E doublet and outline a perturbative scheme for evaluating its trace. For arrays of triangular units, we show how entanglement between trimers can be encoded in multitrace Wilson loops and how the resulting relative phases admit a Chern-Simons description for the

Wilczek-Zee gauge field. As explicit examples, we construct a $\pi/2$ phase gate and a Hadamard-type gate for a single triangle and combine them with the Chern-Simons linkage mechanism to outline a two-qubit entangling CNOT gate for two trimers. Finally, in the Appendix we first demonstrate how the Wilczek-Zee connection evades Painlevé's theorem, which would otherwise preclude shape-space holonomy in time-reversal-symmetric settings. We then analyze a $Cs(6s)-Cs(6s)-Cs(nd_{3/2})$ Rydberg trimer in optical tweezers as a concrete candidate platform. The present work focuses on geometric and holonomic considerations together with general estimates, and detailed Hamiltonian-based studies of specific platforms will be presented elsewhere.

II. THE WILCZEK-ZEE CONNECTION

We start by constructing the (adiabatic) $SU(2)$ Wilczek-Zee connection [1] for a generic three-body system. We label by $\mathbf{r}_a(t)$ ($a = 1, 2, 3$) the time-dependent positions of the three objects with masses m_a viewed as the vertices of a (virtual) time-dependent triangle $\Delta(t)$, and we assume that the center of mass is at rest at the origin:

$$m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + m_3 \mathbf{r}_3 = 0.$$

We invoke Guichardet's theorem [26], which states that when the total orbital angular momentum vanishes, $\mathbf{r}_a(t)$ are classically restricted to vibrations on the plane of the triangle. The geometric rotation induced by these vibrations is encoded in a $U(1)$ connection A , which determines the holonomy in the corresponding shape space of triangles [26–29]. Notably in the semiclassical Born-Oppenheimer regime this $U(1)$ connection manifests spectroscopically as a Berry phase producing an observable sign change of the electronic wave function over a closed Jahn-Teller vibrational cycle [30,31].

Classically, Guichardet's $U(1)$ connection A encodes the geometric rotation generated by planar vibrations of the triangle about its normal. But in a quantum three-body system with total angular momentum $L_{\text{tot}} = 0$ the ground state is rotationally invariant. Consequently, the overall spatial orientation, including the direction of the normal, is a gauge redundancy rather than a physical observable. For this, after the center of mass is removed, the internal configuration space is $\mathbb{R}^6 \simeq \mathbb{R}^+ \times S^5$, where $\mathbb{R}^+$ gives the size and S^5 is the preshape sphere of centered unit-size triangles. Quotienting by global rotations identifies preshapes related by $SO(3)$ and yields Kendall's shape sphere $S^5/SO(3) \simeq S_K^2$ [22]. The resulting configuration manifold is $\mathbb{R}^+ \times S_K^2$, with an overall size (breathing) coordinate along $\mathbb{R}^+$ and two independent shape coordinates on S_K^2 . In many triangular molecular systems, the lowest excitations at fixed size form a gapped, near-degenerate vibrational (i.e., shape changing) E doublet associated with these shape degrees of freedom. The adiabatic transport of the E modes in shape space is governed by the non-Abelian Wilczek-Zee connection as an $SU(2)$ gauge field over S_K^2 , and the associated kinematic frame bundle for closed vibrational loops is the Hopf fibration $S_K^3 \rightarrow S_K^2$ with a $U(1)$ fiber.

We describe $\mathbb{R}^+ \times S_K^3 \sim \mathbb{R}^+ \times S_K^2 \times S_K^1$ by two complex Jacobi coordinates (z_1, z_2) and introduce a regular map $Z = (Z_1(z_1, z_2), Z_2(z_1, z_2))$ from $\mathbb{R}^+ \times S_K^3$ onto itself, with

coordinates

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = |Z| \begin{pmatrix} \cos \frac{\vartheta}{2} e^{i\phi_1} \\ \sin \frac{\vartheta}{2} e^{i\phi_2} \end{pmatrix}, \quad |Z| = \sqrt{|Z_1|^2 + |Z_2|^2}, \quad (1)$$

where

$$\phi = \phi_2 - \phi_1$$

is identified as an internal coordinate and

$$\chi = -\frac{1}{2}(\phi_1 + \phi_2)$$

is an external coordinate. Guichardet's U(1) connection [26–29] that governs parallel transport on Kendall's shape sphere via the Hopf fibration of the preshape three-sphere with circle fibers over $\mathbb{S}_K^2$ can be written as

$$A \equiv \frac{i}{2} \frac{Z^\dagger dZ - (dZ^\dagger)Z}{|Z|^2} = d\chi + \frac{1}{2} \cos \vartheta d\phi. \quad (2)$$

Notably, this coincides with the connection of a single Dirac monopole. The Bloch vector $\mathbf{n}(\theta, \phi)$ of the vibrational E doublet is a map from Kendall's shape sphere $\mathbb{S}_K^2$ to the Bloch sphere, and it serves as the local quantization axis of the doublet. By a suitable, generally local gauge choice this axis may be aligned with the triangle's body-frame normal. With $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ being the three Pauli matrices, the full SU(2) Wilczek-Zee connection then reads [32–35]

$$\mathcal{A} = (A \mathbf{n} + d\mathbf{n} \times \mathbf{n}) \cdot \frac{\boldsymbol{\sigma}}{2i} + (\rho d\mathbf{n} + \eta d\mathbf{n} \times \mathbf{n}) \cdot \frac{\boldsymbol{\sigma}}{2i}. \quad (3)$$

Here $d\mathbf{n}$ and $\mathbf{n} \times d\mathbf{n}$ encode the geometric variation of the Bloch vector over $\mathbb{S}_K^2$, and (ρ, η) control the transverse twisting of the E -doublet basis as the shape moves on $\mathbb{S}_K^2$. With

$$\psi = \rho + i\eta, \quad (4)$$

the pair (A, ψ) forms an Abelian Higgs multiplet under local U(1) rotations about $\mathbf{n}$. Together $(A, \mathbf{n}, \rho, \sigma)$ gives six internal variables, as expected for three points in $\mathbb{R}^3$ after the center of mass is removed. Since (3) transforms covariantly under such U(1) rotations [32], any local phase generated by shape motion can be absorbed into a gauge rotation about the quantization axis.

III. HOLONOMIC GATES

Continuous shape changes of the triangle $\Delta(t)$ induce parallel transport along a trajectory $\Gamma(t)$ on Kendall's shape space. In the present case, the relevant connection is the pullback of the Wilczek-Zee connection (3) to $\Gamma(t)$. The corresponding transport within the vibrational E manifold that defines the qubit is described by the Wilson line $U_\Gamma(t)$, which solves

$$\frac{dU_\Gamma}{dt} = -q \mathcal{A}(t) U_\Gamma(t), \quad U_\Gamma(0) = \mathbf{1}, \quad (5)$$

with q being a charge and $\mathcal{A}(t)$ denoting the connection (3) evaluated along the trajectory. In particular, for a closed trajectory $\Gamma(0) = \Gamma(T)$ and with $\mathcal{P}$ denoting path ordering, the

resulting holonomy is the end-point value

$$\begin{aligned} W_\Gamma &\equiv U_\Gamma(T) = \mathcal{P} \exp \left[-q \int_0^T \mathcal{A}(t) dt \right] \\ &= \mathcal{P} \exp \left(-q \oint_\Gamma \mathcal{A} \right). \end{aligned} \quad (6)$$

This holonomy is the unitary quantum gate acting on the logical subspace formed by the instantaneous twofold-degenerate E manifold. By construction it is gauge covariant, while physical observables derived from it are gauge invariant and geometrically robust. We also identify q as the effective Cartan weight in units where the minimal nonzero weight is $1/2$ due to Dirac quantization, which characterizes how the chosen two-level subspace couples to the diagonal (Cartan) component of the Wilczek-Zee connection. The specific value of q is fixed by the microscopic realization of the underlying E doublet and can be determined by calibration. With generic $(A, \psi, \mathbf{n})$ and denoting $D = d + iA$, the curvature two-form of (3) is

$$\begin{aligned} \mathcal{F} &= (d\mathcal{A}^c + \epsilon^{abc} \mathcal{A}^a \wedge \mathcal{A}^b) \frac{\boldsymbol{\sigma}_c}{2i} \\ &= \left[dA + \frac{1}{2}(|\psi|^2 - 1)\omega \right] \mathbf{n} \cdot \frac{\boldsymbol{\sigma}}{2i} \\ &\quad + \frac{1}{2}[D\psi \times \mathbf{e}^* + D^*\psi^* \times \mathbf{e}] \cdot \frac{\boldsymbol{\sigma}}{2i}, \\ d\omega &= \frac{1}{2} \mathbf{n} \cdot d\mathbf{n} \wedge d\mathbf{n}, \quad \mathbf{e} = d\mathbf{n} + i\mathbf{n} \times d\mathbf{n}. \end{aligned} \quad (7)$$

It spans the entire SU(2) Lie algebra, so that by the Ambrose-Singer theorem [36] the connection (3) has restricted holonomy group SU(2). This ensures that (6) provides universal qubit control.

The adiabatic character of (6) is therefore an assumption of the present construction, not a claim of optimal gate speed. Existing nonadiabatic holonomic gates remove this adiabatic requirement by engineering cyclic subspace evolution directly. In the present work, we instead use the adiabatic limit to expose the geometric gauge structure of deformable triangular systems. Possible nonadiabatic or shortcut implementations of the same shape-space holonomies are left as a natural extension.

To evaluate (6) we introduce local orthonormal frames $\{|n_\pm(t)\rangle\}$ along Γ :

$$\mathbf{n} \cdot \boldsymbol{\sigma} |n_\pm\rangle = \pm |n_\pm\rangle, \quad |n_+\rangle\langle n_+| + |n_-\rangle\langle n_-| = \mathbf{1}. \quad (8)$$

We discretize Γ into infinitesimal segments of duration Δt . Setting $t_k = k\Delta t$, with

$$U_{\alpha_{k+1}\alpha_k}(t_k) = \langle n_{\alpha_{k+1}}(t_{k+1}) | e^{-q \mathcal{A}(t_k) \Delta t} | n_{\alpha_k}(t_k) \rangle$$

and $\alpha_k = \pm$ labeling the instantaneous eigenbasis, we write the path-ordered exponential (6) as the product

$$W_\Gamma = \lim_{N \rightarrow \infty} \sum_{\alpha_k = \pm} |n_{\alpha_N}(t_N)\rangle \prod_{k=0}^{N-1} U_{\alpha_{k+1}\alpha_k}(t_k) \langle n_{\alpha_0}(t_0) |. \quad (9)$$

As a product of 2×2 matrices, this representation is particularly suited for actual computations.

The trace of the Wilson loop (6) measures how much the qubit is twisted when transported once around a closed loop Γ .

By gauge invariance this trace can depend on only the charged scalar $\psi = \rho + i\eta$ and the transverse one-form $\mathbf{e}$ introduced in (7) together with their complex conjugates in gauge-covariant combinations. In the instantaneous eigenbasis (8) of $\mathbf{n} \cdot \boldsymbol{\sigma}$ we decompose the connection

$$\mathcal{A} \stackrel{\text{def}}{=} C \frac{\sigma_z}{2i} + \mathcal{A}^{\text{off}} \equiv (A + \omega) \frac{\sigma_z}{2i} + J \sigma_+ + J^* \sigma_-, \quad (10)$$

where $C = A + \omega$ is the diagonal abelian part and $\mathcal{A}^{\text{off}}$ denotes the off-diagonal part: With spherical coordinates

$$\mathbf{n} = \begin{pmatrix} \cos \lambda \sin \mu \\ \sin \lambda \sin \mu \\ \cos \mu \end{pmatrix}, \quad (11)$$

we have [37]

$$\begin{aligned} dC &= dA + \frac{1}{2} \mathbf{n} \cdot d\mathbf{n} \wedge d\mathbf{n} \\ &= \frac{1}{2} \sin \vartheta d\phi \wedge d\vartheta + \sin \mu d\mu \wedge d\lambda, \end{aligned} \quad (12)$$

and

$$\begin{aligned} \mathcal{A}^{\text{off}} &= J \sigma_+ + J^* \sigma_- \\ &= \psi (d\mu - i \sin \mu d\lambda) \sigma_+ + \psi^* (d\mu + i \sin \mu d\lambda) \sigma_- \end{aligned} \quad (13)$$

for the off-diagonal part. The Wilson loop can then be evaluated by treating the diagonal part exactly and expanding systematically in the transverse components using a time-ordered Dyson expansion. In this way we obtain from (9) in the continuum limit

$$\text{Tr } W_\Gamma = 2 \cos \left[\frac{q}{2} \oint_\Gamma C \right] [1 - \mathcal{I}_2 + \mathcal{I}_4 + \mathcal{O}(|\psi|^6)]. \quad (14)$$

Here $\mathcal{I}_2(t)$, $\mathcal{I}_4(t)$, $\dots$ are gauge-invariant quantities in powers of $J(t)$ and $J^*(t)$ that we construct iteratively in powers of $|\psi|^2$ as solutions of the Dyson equation [38]

$$F(t) = 1 - \frac{q^2}{4} \int_0^t dt_1 \int_0^{t_1} dt_2 J(t_1) e^{q \int_{t_2}^{t_1} C} J^*(t_2) F(t_2).$$

The result confirms ψ is a geometric control knob for the holonomic gate.

Remarkably, with T being the period of Γ , the nonvanishing rotation angle Θ of the qubit,

$$\Theta(T) = 2 \arccos \left(\frac{1}{2} \text{Tr } W_\Gamma \right) = q \oint_\Gamma (A + \omega) + \mathcal{O}(|\psi|^2), \quad (15)$$

gives rise to an effective geometric angular momentum contribution $\mathbb{L}_{\text{eff}}$ even in the absence of any dynamical angular momentum, with magnitude

$$\mathbb{L}_{\text{eff}} \approx 2 \frac{\mathbb{I}_\Delta}{T} \arccos \left(\frac{1}{2} \text{Tr } W_\Gamma \right). \quad (16)$$

Here $\mathbb{I}_\Delta$ is the average moment of inertia of the three points $\mathbf{r}_a(t)$ around the average axis of the central angle over the period T .

Finally, in extended architectures we encounter multiple triangular units such as arrays of Rydberg trimers in optical-tweezer geometries. In that setting, entanglement between distinct triangular qubits becomes essential. The

three-dimensional Chern-Simons functional provides a compact, gauge-invariant way to encode this entanglement. For the gauge field (3) and up to an exact three-form, it takes the form

$$\begin{aligned} F_{\text{ChS}}[\mathcal{A}] &= \frac{k}{4\pi} \int \text{Tr} \left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) \\ &= -\frac{k}{4\pi} \int \left\{ \frac{1}{2} C \wedge dC - i[\psi^* D\psi - \psi \bar{D}\psi^*] \wedge \omega \right\}, \end{aligned}$$

where $k \in \mathbb{Z}$ denotes the level and the integral is over the pullback of $\mathbb{S}_K^3$ defined by the time-periodic triangle world volume. On the $\psi = 0$ truncation, the averaged multitrace Wilson loop for several triangular structures yields [39,40]

$$\begin{aligned} \int [dC] \exp \left\{ i \frac{k}{8\pi} \int C \wedge dC + i \sum_i q_i \oint_{\Gamma_i} C \right\} \\ = \exp \left\{ \frac{4\pi i}{k} \sum_{i \neq j} q_i q_j L(\Gamma_i, \Gamma_j) + \frac{2\pi i}{k} \sum_i q_i^2 L_S(\Gamma_i) \right\}, \end{aligned} \quad (17)$$

where the linking L and self-linking L_S quantify the topological contribution to two-qubit entangling phases. The result mirrors entanglement of Abelian anyons with charges q_i and statistics $4\pi q_i q_j / k$ [41].

IV. EXAMPLES OF GATES

A. Single-qubit gates

As concrete examples of the general theory, we proceed to demonstrate how to construct loops on Kendall's shape sphere $\mathbb{S}_K^2$ that realize a $\pi/2$ phase rotation gate and a Hadamard-type gate, respectively, on the E doublet. To this end, we take the equilibrium trimer's initial shape to have polar coordinates (θ_0, ϕ_0) on $\mathbb{S}_K^2$ and introduce a small elliptical loop:

$$\Gamma : \begin{cases} \theta(s) = \theta_0 + a \cos s \\ \phi(s) = \phi_0 + \frac{b}{\sin \theta_0} \sin s \end{cases} \quad s \in [0, 2\pi), \quad (18)$$

with $a, b \ll 1$ setting the loop size. To leading order in a and b , this loop encloses the solid angle

$$\Omega_\Gamma \simeq \pi ab. \quad (19)$$

For the computational basis, we choose the E -doublet eigenstates at the base point,

$$|0\rangle \equiv |E^{(1)}\rangle, \quad |1\rangle \equiv |E^{(2)}\rangle. \quad (20)$$

This choice fixes a frame for the degenerate E manifold at (θ_0, ϕ_0) . In using it to describe the evolution around Γ , we assume the adiabatic approximation: The state remains within the instantaneous E doublet along the loop, and the resulting action on the computational basis is given by the Wilczek-Zee holonomy.

1. $\pi/2$ phase gate about the z axis

We work in the pinned-normal regime, where the normal vector $\mathbf{n}$ is locked to the bias field. In this case, the

leading-order contribution in $|\psi|$ to the Wilczek-Zee connection coincides with the Guichardet connection along a fixed Pauli axis, which we take to be σ_z in the basis (20). The holonomy (6) for the loop Γ becomes

$$W_\Gamma = \mathcal{P} \exp \left(-q \oint_\Gamma \mathcal{A} \right) \simeq \exp \left(-\frac{i}{2} \Theta_\Gamma \sigma_z \right).$$

Here

$$\Theta_\Gamma = \frac{q}{2} \Omega_\Gamma + \mathcal{O}(|\psi|^2),$$

with q being the holonomy charge, in line with the general relation (14) for the Wilson-loop trace. The resulting gate is the single-qubit phase rotation

$$U_\Gamma \equiv W_\Gamma \simeq \exp \left(-\frac{i}{2} \Theta_\Gamma \sigma_z \right) = \begin{pmatrix} e^{-i\Theta_\Gamma/2} & 0 \\ 0 & e^{+i\Theta_\Gamma/2} \end{pmatrix}.$$

To obtain a $\pi/2$ rotation, we choose the loop area such that

$$\Theta_\Gamma = \frac{\pi}{2} \implies \Omega_\Gamma = \frac{\pi}{q} \implies ab = \frac{1}{q}.$$

With this choice the holonomy takes the explicit form

$$U_\Gamma^{(\pi/2)} = \exp \left(-\frac{i}{4} \pi \sigma_z \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}, \quad (21)$$

which is the standard $\pi/2$ phase gate about the z axis.

2. Hadamard gate from a rotation about the y axis

For a Hadamard-type gate we require a loop whose holonomy is a $\pi/2$ rotation about an axis in the equatorial plane of the Bloch sphere. In the representation (3) of the Wilczek-Zee connection and after an appropriate gauge choice and rescaling of ψ , we write along the loop Γ_H

$$\mathcal{A}(s) = C(s) \frac{\sigma_z}{2i} + \mathcal{A}_\perp(s),$$

with

$$\mathcal{A}_\perp(s) = \text{Re } \psi(s) \frac{\sigma_x}{2i} + \text{Im } \psi(s) \frac{\sigma_y}{2i},$$

where $s \in [0, 2\pi)$ parametrizes the loop Γ_H . Solving for the Wilson line $U(s)$ from (5), we obtain for the holonomy

$$W_{\Gamma_H} \equiv U(2\pi) = \mathcal{P} \exp \left[-q \int_0^{2\pi} \mathcal{A}(s) ds \right]. \quad (22)$$

We start with the Abelian evolution

$$\begin{aligned} U_z(s) &= \mathcal{P} \exp \left[-q \int_0^s C(s') \frac{\sigma_z}{2i} ds' \right] \\ &= \exp \left[-q \frac{\sigma_z}{2i} \int_0^s C(s') ds' \right] \end{aligned}$$

since the path ordering is redundant. We then write

$$U(s) = U_z(s) V(s)$$

and substitute this equation into (5),

$$\begin{aligned} \frac{d}{ds} (U_z V) &= \left(\frac{dU_z}{ds} \right) V + U_z \frac{dV}{ds} \\ &= -q \left(C \frac{\sigma_z}{2i} + \mathcal{A}_\perp \right) U_z V. \end{aligned}$$

Since U_z itself satisfies

$$\frac{dU_z}{ds} = -q C(s) \frac{\sigma_z}{2i} U_z(s), \quad (23)$$

the diagonal terms cancel, and we are left with

$$U_z \frac{dV}{ds} = -q \mathcal{A}_\perp U_z V. \quad (24)$$

Multiplying from the left by $U_z^{-1}(s)$ yields the interaction-picture equation

$$\frac{dV}{ds} = -q \mathcal{A}_\perp^{(I)}(s) V(s), \quad V(0) = \mathbf{1},$$

where we have defined

$$\mathcal{A}_\perp^{(I)}(s) := U_z^{-1}(s) \mathcal{A}_\perp(s) U_z(s). \quad (25)$$

The solution is

$$\begin{aligned} V(2\pi) &= \mathcal{P} \exp \left[-q \int_0^{2\pi} \mathcal{A}_\perp^{(I)}(s) ds \right] \\ &= \mathcal{P} \exp \left[-q \int_0^{2\pi} U_z^{-1}(s) \mathcal{A}_\perp(s) U_z(s) ds \right]. \end{aligned}$$

Thus, we have the factorization

$$W_{\Gamma_H} = U(2\pi) = U_z(2\pi) V(2\pi).$$

Since $U_z(s)$ is a rotation around σ_z , the transformed transverse generator $\mathcal{A}_\perp^{(I)}(s)$ remains in the x - y plane and can be written as

$$\mathcal{A}_\perp^{(I)}(s) = |\psi(s)| \left[\cos \Theta(s) \frac{\sigma_x}{2i} + \sin \Theta(s) \frac{\sigma_y}{2i} \right],$$

where the angle $\Theta(s)$ depends on the phase of $\psi(s)$ and on the accumulated z phase,

$$r(s) = q \int_0^s A(s') ds'.$$

By steering the control phase of $\psi(s)$ along the loop, we can impose

$$\Theta(s) = \frac{\pi}{2} \quad \forall s \in [0, 2\pi), \quad (26)$$

so that in the interaction picture the non-Abelian part of the connection is aligned along σ_y at all parameter values:

$$\mathcal{A}_\perp^{(I)}(s) = |\psi(s)| \frac{\sigma_y}{2i}. \quad (27)$$

In the small-loop limit, where the curvature is approximately constant over the enclosed area and higher-order path-ordering effects can be neglected, the interaction-picture holonomy becomes

$$V(2\pi) \simeq \exp \left(-i \frac{q}{2} \Omega_H \sigma_y \right) + \mathcal{O}(\Omega_H^2, |\psi|^2), \quad (28)$$

where Ω_H is the oriented area (solid angle) associated with the loop in control space; for the elliptical loop (18) it is given by (19). Writing

$$V(2\pi) \simeq \exp \left(-\frac{i}{2} \Theta_H \sigma_y \right)$$

and

$$\Theta_H = \frac{q}{2} \Omega_H + \mathcal{O}(\Omega_H^2, |\psi|^2),$$

a Hadamard-type transformation corresponds to a rotation by $\Theta_H = \pi/2$, so we choose

$$\Theta_H = \frac{\pi}{2} \implies \Omega_H = \frac{\pi}{q}. \quad (29)$$

The associated non-Abelian gate is then

$$U_H = \exp\left(-\frac{i\pi}{4}\sigma_y\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad (30)$$

which equals the standard Hadamard gate composed on the right with a Pauli σ_z and is therefore locally equivalent to the Hadamard gate in the computational basis (20). The full holonomy is

$$W_{\Gamma_H} = U_z(2\pi) U_H,$$

where the Abelian factor $U_z(2\pi)$ is a z -phase gate that can be calibrated or compensated using the phase-gate protocol in the previous section.

The corresponding loop Γ_H can be chosen to have the form (18), with (a, b) selected to satisfy (29). Using (19), the Hadamard condition fixes

$$ab = \frac{1}{q}.$$

Thus, the $\pi/2$ phase gate (21) and the Hadamard-type gate (30) can both be implemented by loops with the same enclosed solid angle $\Omega = \pi/q$, but with different control of the phase of $\psi(s)$: The phase-gate protocol in the previous section yields a pure z rotation, whereas enforcing $\Theta(s) = \pi/2$ along the loop yields the Hadamard-type y rotation U_H .

B. Two-qubit holonomic CNOT gate

We proceed to outline how (17) can be used to build a two-qubit entangling gate between two trimers and how, in combination with the above single-qubit gates, this yields a CNOT gate. For this we consider two triangular structures labeled u and v , each encoding a logical qubit (20) in its vibrational E doublet. For a two-qubit gate we assume that each triangle couples to the Abelian connection C with a holonomy charge that depends on its logical state:

$$Q_u = \begin{cases} 0, & |0\rangle_u \\ Q, & |1\rangle_u \end{cases}, \quad Q_v = \begin{cases} 0, & |0\rangle_v \\ Q, & |1\rangle_v \end{cases}. \quad (31)$$

Physically, this corresponds to a coupling of the Abelian part of the Wilczek-Zee connection only to one branch of the E doublet, e.g., via a state-dependent light shift or auxiliary level, while the other branch is effectively neutral. Here the charges Q_u and Q_v should be viewed as platform-dependent but quantized Cartan weights assigned to the logical states of the i th ($i = u, v$) trimer during the gate cycle. In particular, one branch can be engineered to be effectively neutral ($Q_i = 0$) while the other carries charge $Q_i = Q$. For the gate cycle, we drive both trimers simultaneously through loops Γ_u and Γ_v in their respective shape spaces, such that on the shape space

$$L(\Gamma_u, \Gamma_v) = 1, \quad L_S(\Gamma_u) = L_S(\Gamma_v) = 0.$$

For each computational basis state $|\alpha\beta\rangle$ the charges (Q_u, Q_v) then take the eigenvalues given by (31), so that the Chern-Simons path integral (17) contributes a state-dependent phase

$$\exp\{i\Phi_{\alpha\beta}\} = \exp\left\{i\frac{4\pi}{k} Q_u Q_v\right\}.$$

This defines a diagonal unitary matrix U_{CS} that acts on the computational basis with eigenvalues given by these phases. Specifically, on the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ we have

$$U_{CS} = \text{diag}(1, 1, 1, e^{i\phi}), \quad \phi = \frac{4\pi Q^2}{k}. \quad (32)$$

Choosing k such that

$$\phi = \pi \iff \frac{4\pi Q^2}{k} = \pi \Rightarrow k = 4Q^2,$$

we obtain the standard controlled- Z gate

$$U_{CZ} \equiv U_{CS}(\phi = \pi) = \text{diag}(1, 1, 1, -1) \\ = |0\rangle\langle 0|_u \otimes \mathbb{1}_v + |1\rangle\langle 1|_u \otimes \sigma_z^v. \quad (33)$$

Any additional single-qubit phases accumulated during the joint loop can be absorbed into the local phase gates constructed in the previous section.

Now let the first trimer u be the control and the second trimer v be the target. We denote by U_H the holonomic Hadamard-type gate (30) acting on v . Since it differs from the standard Hadamard gate by the action of Pauli σ_z on the right, the canonical Hadamard on v is

$$H_v = U_H \sigma_z^v,$$

and on the two-qubit Hilbert space the corresponding operator is $\mathbb{1}_A \otimes H_v$. The composite sequence

$$U_{\text{CNOT}} = (\mathbb{1}_u \otimes H_v) U_{CZ} (\mathbb{1}_u \otimes H_v^\dagger)$$

realizes a CNOT gate with u as the control and v as the target. Indeed,

$$H_v \sigma_z^v H_v^\dagger = \sigma_x^v \equiv X_v,$$

so that using (33), we obtain

$$U_{\text{CNOT}} = |0\rangle\langle 0|_u \otimes \mathbb{1}_v + |1\rangle\langle 1|_u \otimes X_v.$$

Operationally, the CNOT is then obtained by (1) applying the holonomic Hadamard-type gate H_v to the E doublet of trimer v , (2) executing a joint gate cycle in which trimers u and v traverse linked loops (Γ_u, Γ_v) in configuration space, producing the Chern-Simons controlled phase (32), and (3) applying the holonomic gate $H_v^\dagger$ on v (e.g., by running the loop corresponding to H_v in reverse).

In combination with the single-qubit gates above, we may now construct a universal set of holonomic gates for arrays of triangular qubits.

V. OUTLOOK

We have established the concept of shape space as a natural and experimentally accessible control manifold for non-Abelian holonomies in deformable three-body systems.

For platforms whose low-energy vibrational spectrum contains a gapped, nearly degenerate E doublet, we have derived the Wilczek-Zee connection governing adiabatic transport on Kendall's shape sphere of triangles. The resulting restricted holonomy group is $SU(2)$, so that closed loops in shape space implement universal single-qubit holonomic gates within the E manifold. Moreover, the gauge-invariant trace of the associated Wilson loop provides a compact and experimentally accessible diagnostic of these gates, offering a direct signature of non-Abelian geometric transport.

Among the future theoretical challenges is the development of platform-specific mappings from explicit microscopic trimer Hamiltonians to the effective gauge data. This includes dynamical control of the knobs $(\mathbf{n}, \psi)$ that enter the Wilczek-Zee connection and a quantitative characterization of the resulting control landscape. To proceed from a few-body molecular structure to robust geometric gates, a systematic treatment of imperfections is also needed. This includes the effects of symmetry breaking within the E manifold, stray-field shifts, trap anisotropies, and coupling to spectator vibrational modes. Such detailed platform-specific investigations will clarify the parameter regimes in which shape-space holonomies provide a practical advantage for quantum control.

Beyond a single qubit, we have also proposed holonomic control in trimer arrays as a theoretically and experimentally challenging direction. The entangling mechanism based on linked holonomic cycles in shape space provides a framework for exploring multitramer architectures with two-qubit phases governed primarily by linking data. This suggests a pathway toward more elaborate topological control primitives in programmable molecular arrays.

As a future experimental objective, we propose a proof-of-principle demonstration on a bond-length-modulated $Cs(6s)$ - $Cs(6s)$ - $Cs(nd_{3/2})$ Rydberg trimer trapped in optical tweezers. With details given in the Appendix, we propose to (1) prepare and spectroscopically characterize an E doublet in a single triangular unit, (2) implement phase-controlled bond-length modulations that execute calibrated closed cycles on Kendall's shape sphere, and (3) read out the resulting non-Abelian holonomy interferometrically using a Ramsey-echo protocol that refocuses dynamical phases while preserving the geometric signal. By varying the loop area, orientation, and repetition number, the measured Wilson-loop trace can be benchmarked against its predicted dependence on the control cycle, enabling a quantitative and gauge-invariant characterization of the effective Wilczek-Zee connection. In addition, we propose the design of nonadiabatic holonomies as a potential route to faster gates without compromising the underlying geometric structure.

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DATA AVAILABILITY

The data that support the findings of this article are openly available from the authors.

APPENDIX

1. Painlevé's theorem

The Schrödinger equation governing a trimer as an isolated, conservative many-body system is time reversal invariant, provided magnetic fields are absent and weak interaction effects are ignored. In that case, the classical "falling-cat" theorem by Painlevé [42] states that "if the relative motion of an isolated system of particles is governed solely by conservative forces, and if at some instant $t = t_0$ all particles are at rest, then the system can never return to the same internal configuration of relative positions while having a different overall orientation in space."

This suggests that in our physical scenarios the curvature (12) of the Guichardet connection A should, in fact, identically vanish. Nevertheless, we now demonstrate through a broadly applicable example that, although Painlevé's theorem is correct, its assumptions are not generic. For this we consider a (semi)classical trimer with bond lengths

$$r_{ab}(t) = |\mathbf{r}_a - \mathbf{r}_b|$$

that vibrate around their time-averaged equilibrium values

$$d_{ab} \sim \langle r_{ab}(t) \rangle.$$

We Taylor expand the potential

$$V(r_{ab}) \approx V(d_{ab}) + \frac{1}{2} V''(d_{ab})(r_{ab} - d_{ab})^2 + \dots,$$

where we keep only the leading, harmonic contribution. For a time-reversal-invariant trajectory, without loss of generality, we allow bond r_{12} to oscillate according to

$$r_{12}(t) = d_{12} + \epsilon_{12} \cos(\Omega_{12}t) \quad (\epsilon_{12} \ll d_{12}), \quad (A1)$$

and we choose vertex 3 to describe the Rydberg atom of the trimer. Since the Rydberg excitation breaks the equilateral D_{3h} symmetry into the C_{2v} symmetry of an isosceles configuration, for the other two bonds we may choose $d_{13} = d_{23} = d$ and $\epsilon_{13} = \epsilon_{23} = \epsilon$ (with $\epsilon \ll d$) and $\Omega_{13} = \Omega_{23} = \Omega$, so that

$$\begin{pmatrix} r_{13}(t) \\ r_{23}(t) \end{pmatrix} = \begin{pmatrix} d + \epsilon \cos(\Omega t + \phi_{13}) \\ d + \epsilon \cos(\Omega t + \phi_{23}) \end{pmatrix}, \quad (A2)$$

and in general, the bond $r_{12}(t)$ differs from $r_{13}(t)$ and $r_{23}(t)$ in its mean length, amplitude, and frequency. For generic ϕ_{13} and ϕ_{23} time reversal is then not a symmetry of the given trajectory (A2). However, it becomes a symmetry if $\phi_{13} = -\phi_{23}$ and $t \rightarrow -t$ is combined with multiplication of (A2) by Pauli matrix $i\sigma_y$ exchanging vertices 1 and 2. We performed numerical investigations with (A1) and (A2) to confirm that, despite Painlevé's theorem, for a generic time-reversal-symmetric trajectory the vibrational modes induce rotational motion of the triangle around its normal axis, even with zero mechanical angular momentum. Thus, the Guichardet connection A and, in particular, the geometric angular momentum (16) cannot vanish. Figure 1 illustrates this in both shape space [Fig. 1(a)] and physical space [Fig. 1(b)]. Figure 1(c) shows the time evolution of both the rotation angle $\theta(t)$ and the effective angular

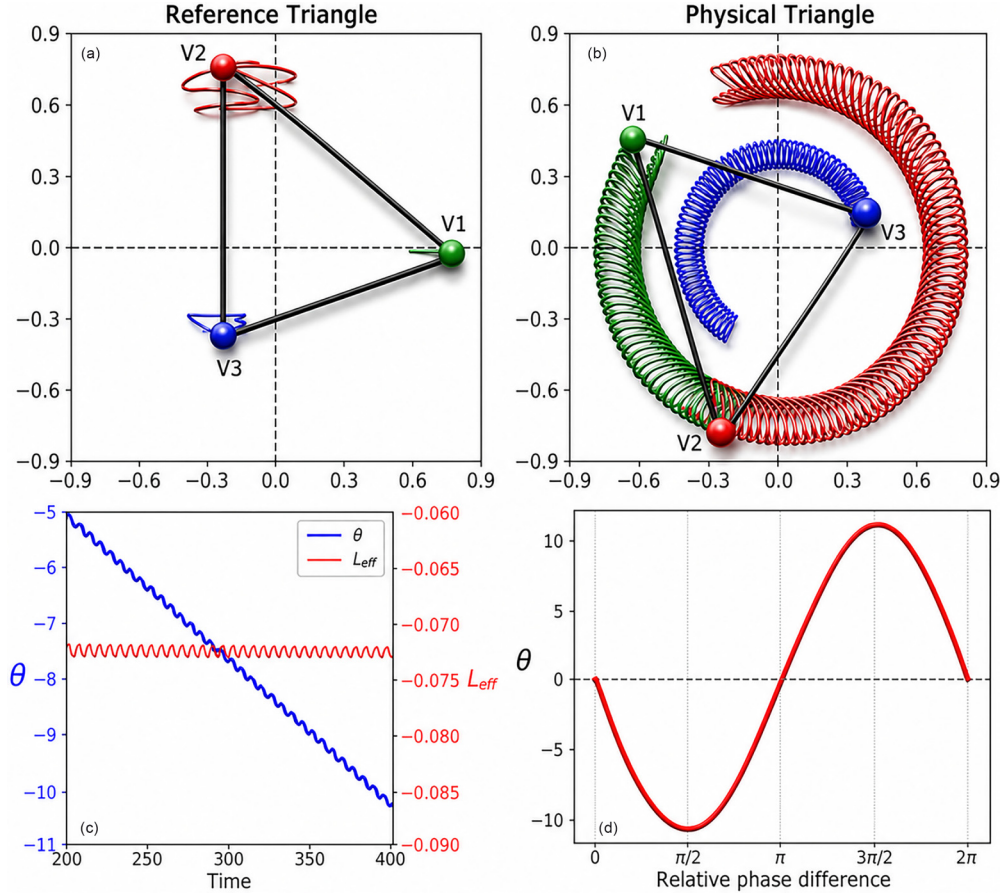


FIG. 1. (a) A representative shape-space time evolution of a triangle $\Delta(t)$ with numerical parameter values $d = 1.0$, $\epsilon = 0.15$, $d_{12} = 1.1$, $\epsilon_{12} = 0.2$, $\Omega/\Omega_{12} = 3$, $m_{1,2} = 2.1$, and $m_3 = 4.7$. (b) Corresponding rotational motion in the physical space. (c) Time evolutions of the angle $\theta(t)$ and the effective angular momentum (16). (d) Comparison of the rotational distance (effective angular velocity) with different values of $\phi_{13} - \phi_{23}$.

momentum (16) over an extended simulation. As the ratio of the sampling time step to the trajectory length decreases, $\theta(t)$ exhibits linear growth in time while the geometric angular momentum $\mathbb{L}_{\text{eff}}(t)$ in (16) approaches a constant value, implying that the motion asymptotically resembles uniform rotation of a rigid trimer, despite the absence of mechanical angular momentum. Finally, Fig. 1(d) demonstrates that for generic $\phi_{13} = -\phi_{23}$ rotational motion is present and vanishes only when $\phi_{13} = \phi_{23} = 0 \bmod \pi$. This is also the sole value where all three bond lengths come to a stop simultaneously, fully in line with Painlevé's theorem. Notably, the effective angular velocity reaches its maximum value at $\phi_{13} = -\phi_{23} = \frac{\pi}{4} + \frac{k\pi}{2}$, with $k \in \mathbb{Z}$. This is also the phase value that emerges from the solution to the equations of motion derived from the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\dot{r}_{13}r_{23} - \dot{r}_{23}r_{13}) - \frac{\Omega}{2}[(r_{13} - d)^2 + (r_{23} - d)^2],$$

describing uniform circular precession around a fixed point. The enclosed phase-space area over a cycle generates a Berry phase of π which is characteristic of the spin- $\frac{1}{2}$ system [43] that has been observed in molecular triangles [30,31].

2. Application to cesium trimer

As a demonstrator we propose a $\text{Cs}(6s)\text{-Cs}(6s)\text{-Cs}(nd_{3/2})$ Rydberg trimer confined within a single site of an optical-tweezer array. We specify a concrete operating point and control protocol that realize a vibrational qubit encoded in the nearly degenerate E doublet, together with an experimentally accessible Wilson-loop trace of the corresponding Wilczek-Zee connection.

a. Setup

Spectroscopic studies have established that triatomic $\text{Cs}(6s)\text{-Cs}(6s)\text{-Cs}(nd_{3/2})$ Rydberg molecules can be formed by binding one excited Cs atom to two ground-state atoms [18], with characteristic bond lengths $R_1 = R_2 \sim 2000a_0$ ($\simeq 0.1 \mu\text{m}$). Typical optical tweezers provide confinement on the micrometer scale and can be arranged in programmable arrays [14,15], allowing the entire trimer to fit comfortably within a single trap volume. In this configuration the two perturbors remain bound within the Rydberg electron's molecular potential, while the tweezer supplies overall center-of-mass confinement (see Fig. 2). The lifetime of a room-temperature Cs Rydberg atom is typically a few to a few tens of microseconds, increasing significantly under cryogenic conditions [14]. This timescale is compatible with the loop durations

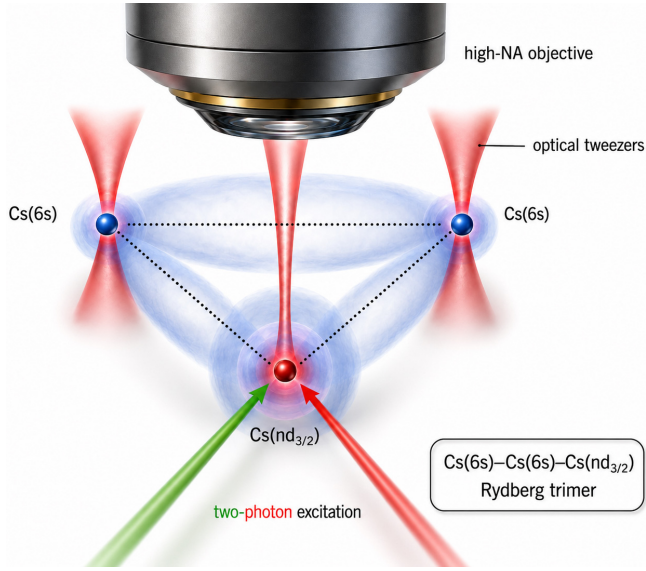


FIG. 2. Outline of proposed Cs Rydberg trimer in optical tweezers. Three optical tweezers, focused by a high numerical aperture objective lens, trap three Cs atoms in a triangular configuration, realizing a Cs(6s)-Cs(6s)-Cs(nd_{3/2}) Rydberg trimer. Rydberg excitation is illustrated schematically by a two-photon beam pair addressing the central site.

required for shape-space control in the present geometry [14,18]. Figure 1 illustrates representative kinematics within a classical vibrational model.

The planar tweezer potential is used to tune the trimer's apex angle such that the equilibrium geometry lies close to D_{3h} symmetry. In this setup, the lowest internal vibrational modes decompose into a breathing A mode with energy $\mathcal{E}_A$ and a nearly degenerate pair of angular modes with energies $\mathcal{E}_{E^{(1)}}$ and $\mathcal{E}_{E^{(2)}}$ that span the logical E manifold. If we define the mean E -manifold energy

$$\mathcal{E}_E \equiv \frac{\mathcal{E}_{E^{(1)}} + \mathcal{E}_{E^{(2)}}}{2}$$

and denote by T_{loop} the duration of a single holonomic loop in shape space, our target is a nearly degenerate but gapped window

$$\Delta_{\text{gap}} \equiv \mathcal{E}_A - \mathcal{E}_E \gg \frac{\hbar}{T_{\text{loop}}}, \quad (\text{A3})$$

and

$$\delta_E \equiv |\mathcal{E}_{E^{(1)}} - \mathcal{E}_{E^{(2)}}| \ll \frac{\hbar}{T_{\text{loop}}}. \quad (\text{A4})$$

These conditions isolate the E doublet adiabatically from higher vibrational modes while keeping it effectively degenerate over the loop time T_{loop} . In this energy regime, non-Abelian transport of the E doublet along a closed loop on Kendall's shape sphere $\mathbb{S}_K^2$ is governed by the SU(2) Wilczek-Zee connection $\mathcal{A}$.

b. Control knobs and mapping to (A, ψ)

The complex control parameter $\psi = \rho + i\eta$ in (4) is controlled by synchronized, low-amplitude modulations of the

two base bond lengths r_{13} and r_{23} realized via acousto-optic control of the tweezer positions and depths. We map each point (θ, ϕ) on Kendall's shape sphere $\mathbb{S}_K^2$ to a Bloch vector $\mathbf{n}(\theta, \phi)$ for the E doublet, with the corresponding instantaneous eigenbasis defined up to local U(1) phase rotations about $\mathbf{n}$. We choose a gauge where $\mathbf{n}$ serves as the local quantization axis, such that it is aligned with the body-frame normal of a canonical representative of each shape. The differentials $d\mathbf{n}$ and $\mathbf{n} \times d\mathbf{n}$ then describe how this Bloch-vector field varies over $\mathbb{S}_K^2$ and encode the shape-dependent evolution of the E doublet along the loop in shape space, in a manner consistent with zero total mechanical angular momentum.

A weak bias field defines a reference quantization axis in the laboratory frame that we align with the equilibrium normal of the trimer. This choice fixes the relation between the laboratory frame and the body frame, so that the Berry-Wilczek-Zee evolution of the E doublet is generated by the driven shape dynamics. To suppress excitation of the breathing mode and to keep the triangle close to fixed size so that its shape remains on $\mathbb{S}_K^2$, we comodulate the apex bond length $r_{12}(t)$ so that

$$\delta r_{13}(t) + \delta r_{23}(t) + \delta r_{12}(t) \simeq 0.$$

In line with (A2), a convenient parametrization of a small elliptical loop in shape space is

$$\begin{aligned} \delta r_{13}(t) &= \epsilon \cos(\Omega t), \\ \delta r_{23}(t) &= \epsilon \sin(\Omega t + \phi), \end{aligned}$$

where $\epsilon \ll 1$, loop frequency $\Omega = 2\pi/T_{\text{loop}}$, and the controllable phase ϕ corresponds to the relative phase $\phi_{13} - \phi_{23}$ [see also Fig. 1(d)].

We introduce the instantaneous eigenbasis of $\mathbf{n} \cdot \boldsymbol{\sigma}$ to decompose the SU(2) connection $\mathcal{A}$ into a diagonal Abelian part $C \equiv A + \omega$ and an off-diagonal transverse part $J\sigma^+ + J^*\sigma^-$ as in (10)–(13). In the present illustrative regime the weak bias field keeps the relevant shapes near a single patch of $\mathbb{S}_K^2$, so that ω is small, and $C \simeq A$, the embedded Guichardet connection. The holonomy is then governed by the loop area on Kendall's $\mathbb{S}_K^2$ through A , together with the complex control parameter ψ through the transverse coupling J . The magnitude $|\psi|$ sets the strength, and the phase $\arg \psi$ sets the direction of the non-Abelian mixing within the E doublet.

c. A demonstrator loop and expected gate

With the normal $\mathbf{n}$ pinned so that $C \simeq A$, we consider a small elliptical gate loop Γ that encircles a solid angle $\Omega_\Gamma \ll 1$ on $\mathbb{S}_K^2$. With (2) and (11), the loop integral (14) evaluates to

$$\oint_\Gamma A = \frac{1}{2} \Omega_\Gamma,$$

so that the geometric gate angle is (7),

$$\Theta \approx \frac{q}{2} \Omega_\Gamma + \mathcal{O}(|\psi|^2),$$

with the $\mathcal{O}(|\psi|^2)$ corrections being due to the transverse off-diagonal components (13). Since a single small elliptical loop produces only a small rotation angle Θ , larger angles such as a $\pi/2$ single-qubit rotation (a $\sqrt{X}$ -type gate) can be realized

by repeating the loop N times. This yields

$$\Theta_{\text{tot}} \approx N \frac{q}{2} \Omega_{\Gamma},$$

provided the adiabaticity condition $\delta_E \ll \hbar/T_{\text{loop}} \ll \Delta_{\text{gap}}$ remains satisfied for each traversal.

d. Timing and adiabaticity

Adiabatic transport within the E manifold requires the separation of scales as specified in (A3) and (A4),

$$\delta_E \ll \frac{\hbar}{T_{\text{loop}}} \ll \Delta_{\text{gap}}. \quad (\text{A5})$$

Here T_{loop} is the duration of a single small loop Γ on Kendall's shape sphere $\mathbb{S}_K^2$. The lower inequality ensures that the two E levels remain effectively degenerate over the course of one loop, while the upper inequality suppresses nonadiabatic transitions out of the E subspace. In this regime, treating the off-diagonal couplings to the breathing mode perturbatively yields, in the leading order, the Dyson estimate

$$P_{\text{leak}} \sim \left(\frac{\hbar}{T_{\text{loop}} \Delta_{\text{gap}}} \right)^2 \quad (\text{A6})$$

for the leakage probability P_{leak} out of the E manifold. When a larger rotation angle is realized by repeating small loops N times, the total gate time becomes $T_{\text{gate}} = NT_{\text{loop}}$. With room-temperature Rydberg lifetimes of a few to a few tens of microseconds [14], choosing T_{loop} in the submicrosecond to few-microsecond range and $N = \mathcal{O}(1-10)$ leads to typical total gate times in the 5–25 μs range. This is compatible with the $\text{Cs}(nd_{3/2})$ Rydberg lifetime, especially under cryogenic conditions. While such gate durations are adequate for a proof-of-principle demonstrator, achieving a high-fidelity quantum-logic operation would require either longer lifetimes, faster loops, or larger single-loop areas that reduce N . In practice, axial confinement and field control, which are standard in tweezer arrays [14,15], provide the primary experimental knobs for tuning Δ_{gap} and δ_E in $\text{Cs}(nd_{3/2})$ trimers.

e. Readout: Gauge-invariant Wilson-loop trace

A minimal readout protocol is a Ramsey-echo sequence within the E manifold that proceeds as follows:

- (1) Prepare an E superposition.

(2) Execute the loop Γ during the first free-precession window.

(3) Apply a π pulse that exchanges $|E^{(1)}\rangle \leftrightarrow |E^{(2)}\rangle$.

(4) Execute the time-reversed loop Γ^{-1} .

(5) Close with a $\pi/2$ pulse.

This sequence cancels dynamical phases associated with the residual splitting δ_E while doubling the geometric contribution of the Wilczek-Zee holonomy. The resulting Ramsey fringe yields a basis-calibrated estimate of the gauge-invariant Wilson-loop trace (9), (14), and (15). The full trace can be reconstructed by preparing two linearly independent initial superpositions within the E manifold. The geometric origin of the signal can be verified by reversing the loop orientation, which flips the sign of the solid angle Ω_{Γ} , and by varying the enclosed area on Kendall's shape sphere $\mathbb{S}_K^2$ while keeping the loop duration T_{loop} fixed.

f. Error channels and operating window

The dominant sources of error are (1) spontaneous decay and dephasing of the Rydberg electron over the total gate time T_{gate} , producing an error $\sim 1 - e^{-T_{\text{gate}}/\tau_R}$, with τ_R being the lifetime of the $\text{Cs}(nd_{3/2})$ Rydberg state, (2) nonadiabatic leakage out of the E subspace during each loop, with a per-loop leakage probability that scales as in Eq. (A6), and (3) the residual splitting δ_E between $|E^{(1)}\rangle$ and $|E^{(2)}\rangle$ causing the two E levels to acquire slightly different dynamical phases during the loop that, if not refocused by the Ramsey-echo protocol, would accumulate an unwanted dynamical phase of order $\sim \delta_E T_{\text{gate}}/\hbar$.

Additional errors may arise from (4) stray electric fields that shift and mix the E doublet, which can be mitigated using standard field-nulling techniques, and (5) trap-intensity and waveform noise that modulate the loop area Ω_{Γ} and hence affect the geometric rotation angle, which can be mitigated using phase-locked control of the tweezer waveforms.

In summary, conditions (A3) and (A4) define the required operating window. Figure 1 illustrates how relative-phase control of the bond oscillations shapes the loop on Kendall's $\mathbb{S}_K^2$ and thereby sets the resulting geometric rotation. Taken together with the natural bond-length scale ($\sim 0.1 \mu\text{m}$), the accessible Rydberg lifetimes, and standard tweezer and field controls [14,15], the present considerations place the proposed demonstrator well within reach of current neutral-atom technology.

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