

Energy estimates for wave amplification in quasiperiodic Fibonacci time-modulated media

Ilya Ruchiev  and Bryn Davies ^{*}

Mathematics Institute, [University of Warwick](https://www.warwick.ac.uk), Coventry CV4 7AL, United Kingdom



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We show that Fibonacci time quasicrystals can be approximated by temporal supercells, which capture the nonperiodic but self-similar modulation in an analytically tractable form. This reveals a fractal hierarchy of instability gaps, indexed by the wave number k , in which wave energy is amplified exponentially. These estimates are validated by the observation of super k gaps: instability regions that are independent of the duration of the temporal supercell. This approach predicts the regions of parametric amplification and provides arbitrarily accurate estimates of the energy growth rate.

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I. INTRODUCTION

There has been longstanding interest in understanding wave propagation in media whose properties vary in time [1–4]. This has been renewed by the recent development of space-time metamaterials, which offer unprecedented freedom to manipulate waves [5,6]. Examples of exciting phenomena include nonreciprocal transmission [3,4], Fresnel drag [7], parametric amplification [8,9], exceptional points [10,11], topological localization [12,13], and frequency conversion through the coupling of harmonic modes [14–16]. As well as posing challenging theoretical questions, recent experimental breakthroughs mean these systems are now a reality [17,18]. Although these experimental implementations are still in their infancy, this progress has driven theoretical studies into exotic modulation patterns, setting out exciting future directions for the field.

Time modulation, even of spatially homogeneous materials, has deep implications for wave propagation. For example, a temporal interface leads to a reflected (time-reversed) wave [1,19]. When the modulation is periodic in time, systems often display parametric intervals of stability and instability [1,2,15,20,21]. The dependence on wave number yields alternating intervals of stable propagation and amplification in so-called k gaps [15]. As well as periodic modulation, random [20,22] and quasiperiodic temporal [23,24] modulations have been studied. In both cases, amplification phenomena analogous to k -gap behavior have been observed.

Quasicrystals are exciting phases of matter that exhibit long-range order without translational symmetry [25]. Systems with spatially quasiperiodic modulation often exhibit exotic spectral properties, such as fractal and Cantor-like

spectra [26], unexpected symmetries in reciprocal space [25], unique topological properties [27–29], and transitions between universally localized or propagating modes [30,31]. One of the most widely studied examples of (spatial) quasicrystals is the *Fibonacci quasicrystal* [32]. Defined based on a tiling rule, its self-similar properties lend themselves to concise analysis [33].

The unusual spectral properties of quasicrystals have inspired their use in wave control and metamaterial applications, such as topological waveguides [34–37] and graded frequency filters [38]; see Sec. 11 of Ref. [39] for a summary. These breakthroughs have had to overcome the fact that the exotic spectra of quasicrystals are exceptionally difficult to compute, and they have relied on methods that apply Bloch's theorem through supercell and superspace approximation methods [40–42] as well as associated homogenization techniques [43].

In this work, we reveal how Fibonacci time quasicrystals (FTQs) exhibit parametric regions of stability and amplification. Our main contribution is showing how temporal supercell approximations can be used to model FTQs. We demonstrate the existence of *super k gaps*: momentum gaps that exist for any sufficiently long supercell approximations. This phenomenon shows that supercells give bona fide predictions of amplification in FTQs. Using supercells and super k gaps, we reveal concrete evidence of a fractal pattern of k gaps in FTQs. Our methods reveal that these k gaps, where amplification occurs, resemble the exotic spectral properties observed for spatial Fibonacci quasicrystals [32]. Finally, we can use supercells to forecast the average rate of energy amplification with arbitrarily good accuracy. This improves on existing estimates for random media [20,22], demonstrating that the optimal way to characterize quasiperiodic modulation is through periodic approximation rather than averaging.

II. PROBLEM SETTING

We study the propagation of an electromagnetic wave in a medium with dielectric function $\varepsilon(t)$ that is modulated in time. We restrict our study to one spatial dimension and suppose

^{*}Contact author: bryn.davies@warwick.ac.uk

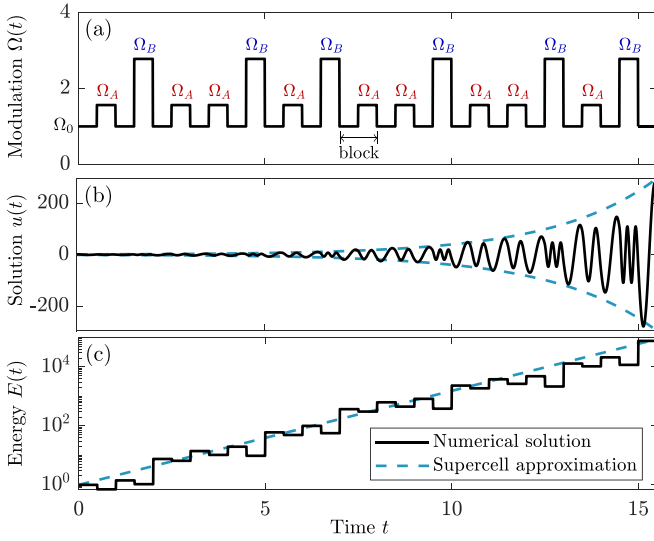


FIG. 1. Piecewise-constant modulation according to a Fibonacci rule. (a) The coefficient $\Omega(t)$ alternates between Ω_0 and either Ω_A or Ω_B , depending on the corresponding letter in the Fibonacci word (here, $\mathcal{F}_5 = ABAABABA$ is shown). (b) For certain parameter values (here, $k = 3.2$), the solution grows exponentially. (c) The (exponential) rate of energy growth can be estimated using a supercell approximation.

that the material is homogenous and nonmagnetic such that the system can be modeled by the linear wave equation

$$\frac{\partial^2 u}{\partial x^2}(x, t) - \frac{\varepsilon(t)}{c^2} \frac{\partial^2 u}{\partial t^2}(x, t) = 0. \quad (1)$$

Taking the spatial Fourier transform leads to an ordinary differential equation in reciprocal space parametrized by the wave number k ,

$$\frac{\partial^2 \hat{u}(k, t)}{\partial t^2} + \Omega^2 \hat{u}(k, t) = 0, \quad (2)$$

where we have defined $\Omega = \Omega(k, t) := c^2 k^2 / \varepsilon(t)^2$.

In this study of FTQs, we will focus on the canonical case where the modulation $\Omega(t)$ is piecewise constant in time, modulated through a series of temporal interfaces [as in Fig. 1(a)]. Periodic [21,44,45], random [22], and quasiperiodic [46] versions of this system have been studied previously as well as piecewise constant space-time modulation [47,48]. This setting is convenient since, on any section where $\Omega(t) = \Omega_j$ is constant, the solution to Eq. (2) can be written as

$$\hat{u}_j(t) = \alpha_j e^{-i\Omega_j t} + \beta_j e^{i\Omega_j t}, \quad (3)$$

for some constants α_j, β_j , which need to be chosen such that the solution is continuous across the jumps in $\Omega(t)$. To achieve a realization of a FTQ, we consider the case where $\Omega(t)$ alternates between a background value Ω_0 and one of two values Ω_A and Ω_B , which are selected according to a Fibonacci tiling rule [32] [this is depicted in Fig. 1(a)]. This tiling rule gives a pattern of A's and B's that is defined sequentially, where each new word in the sequence is formed by combining the previous two. If the first two terms are $\mathcal{F}_0 = B$ and $\mathcal{F}_1 = A$,

then we form subsequent words according to

$$\mathcal{F}_n = \mathcal{F}_{n-1} \cup \mathcal{F}_{n-2}. \quad (4)$$

This gives $\mathcal{F}_2 = AB$, $\mathcal{F}_3 = ABA$, $\mathcal{F}_4 = ABAAB$, and so on. As $n \rightarrow \infty$, the ratio of the number of A's to B's in the sequence $\mathcal{F}_n$ converges to the golden ratio $\varphi = (1 + \sqrt{5})/2$.

III. TRANSFER MATRICES AND SUPERCELLS

We wish to understand under what conditions the interactions between waves in the FTQ and its quasiperiodic temporal modulation lead to stable propagation or amplitude growth. We want to elucidate the dependence on the wave number k and understand the analog of the k gaps observed for periodic temporal modulation [2,15].

We use the methods and analytic framework from Ref. [22] and will exploit the general form in Eq. (3) of solutions to Eq. (2). Successive values of the constants α_j and β_j in Eq. (3) can be related by a transfer matrix $M = M(\Omega_j)$,

$$\begin{bmatrix} \alpha_{j+1} \\ \beta_{j+1} \end{bmatrix} = M(\Omega_j) \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix}. \quad (5)$$

For details, including explicit expressions for the entries of $M(\Omega_j)$, see Ref. [22]. Propagation on time intervals spanning many blocks of modulated $\Omega(t)$ can be described by products of many of the matrices $M(\Omega_j)$. An example of a solution in the FTQ is shown in Fig. 1(b), where the parameters have been chosen such that the amplitude grows exponentially. The parameter values $c = 1$, $\varepsilon_A = 0.8$, and $\varepsilon_B = 0.6$ are taken, along with a normalized background value $\varepsilon_0 = 1$. We will use these values for all subsequent numerical examples; see Appendix for a discussion of other parameter ranges. Understanding which wave numbers experience amplification is a challenging problem (related to the notorious spectral problems for spatial Fibonacci quasicrystals [26]). To characterize this, we will approximate the quasicrystal as being periodic, using a temporal *supercell*. That is, we take sections of the quasiperiodic modulation as unit cells for periodic modulation (allowing us to use Bloch's theorem).

The key to using the supercell method to study quasicrystals is to choose a sequence of successively larger supercells to get increasingly good approximations. Here, we elect to take the unit cells that are generated by the tiling rule in Eq. (4). For each unit cell in this sequence, we have an associated transfer matrix

$$\mathcal{M}_{\mathcal{F}_n} = \prod_{j \in \mathcal{F}_n} M(\Omega_j) M(\Omega_0). \quad (6)$$

We can use Bloch's theorem to compute dispersion curves, as in Ref. [15]. We have the dispersion relation

$$\cos(T\omega) = \frac{1}{2} \text{tr}[\mathcal{M}_{\mathcal{F}_n}(k)], \quad (7)$$

where ω is a quasifrequency that is the temporal Bloch parameter. Deriving Eq. (7), we use the fact that $\det(\mathcal{M}) = 1$, which is equivalent to the condition that $|\alpha_j|^2 - |\beta_j|^2$ is conserved for all j [22].

The crucial quantity that governs whether a periodic supercell solution grows or not is the maximum eigenvalue of the associated transfer matrix $\mathcal{M}$. If $(\lambda_k, \mathbf{v}_k)$ are the eigenpairs of $\mathcal{M}$, then we can write the initial conditions as

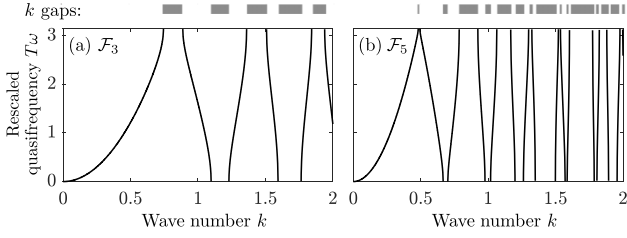


FIG. 2. Dispersion curves for the temporal supercells given by the Fibonacci words (a) $\mathcal{F}_3 = ABA$ and (b) $\mathcal{F}_5 = ABAABABA$. The k gaps are shaded above each plot.

$(\alpha_0, \beta_0)^\top = A\mathbf{v}_1 + B\mathbf{v}_2$ and then

$$\begin{bmatrix} \alpha_{jT} \\ \beta_{jT} \end{bmatrix} = A\lambda_1^j \mathbf{v}_1 + B\lambda_2^j \mathbf{v}_2, \quad (8)$$

where T is the duration of the temporal supercell. Thus, the solution will grow exponentially if one of the eigenvalues is bigger than 1, i.e., if $\max\{|\lambda_1|, |\lambda_2|\} > 1$. Since $\text{tr}[\mathcal{M}_{\mathcal{F}_n}]$ is the sum of the eigenvalues, this condition is equivalent to the dispersion relation in Eq. (7) having a real-valued solution for k and ω .

We can use the dispersion relation in Eq. (7) to study the pattern of k gaps the sequence of supercell approximations. Two examples of dispersion curves are shown in Fig. 2, for the supercells $\mathcal{F}_3 = ABA$ and $\mathcal{F}_5 = ABAABABA$. Whenever k is such that there is no real-valued solution, we say that k is in a k gap; these regions are shown above each plot in Fig. 2. The pattern of k gaps for successive Fibonacci-generated supercell approximations is shown in Fig. 3(a). As the duration of the supercell increases, the pattern of k gaps becomes increasingly intricate, appearing to converge toward a fractal pattern, reminiscent of the fractal, Cantor-like spectra that are often seen for spatial Fibonacci quasicrystals [32].

IV. SUPER k GAPS

One of the striking features of Fig. 3(a) is the occurrence of gaps that persist independent of the index n and the duration of the supercell $\mathcal{F}_n$. This behavior is reminiscent of the super band gaps that are observed when periodic approximations are applied to spatial quasicrystals [49,50]. Similarly, we can

define *super k gaps* as values of k which are in k gaps of the supercells $\mathcal{F}_n$ for all $n \geq n_0$.

The existence of super k gaps can be verified by studying the traces of the matrices $\mathcal{M}_{\mathcal{F}_n}$. From the dispersion relation in Eq. (7), we can see that k is in a k gap if and only if $|\text{tr}[\mathcal{M}_{\mathcal{F}_n}(k)]| > 2$. A very convenient property of the tiling rule in Eq. (4) is that the traces of the associated transfer matrices can be related to one another. In Ref. [33], it was discovered that, if $x_n = \text{tr}[\mathcal{M}_{\mathcal{F}_n}]$, then

$$x_{n+1} = x_n x_{n-1} - x_{n-2}. \quad (9)$$

This uses the fact that $\det(\mathcal{M}_{\mathcal{F}_n}) = 1$. This recursion relation can be used to show that, if k is in a k gap for both $\mathcal{F}_N$ and $\mathcal{F}_{N+1}$, for some N , then it will be in a k gaps for $\mathcal{F}_n$ for all $n \geq N$ [51]. Hence, any such values of k , which we can easily spot in Fig. 3, must be in super k gaps. The reason it is exciting to identify super k gaps is that they are parameter regions where we can be confident that the supercell approximations are accurately predicting the instability.

V. ENERGY GROWTH RATE

As well as predicting stability and instability, supercell approximations can be used to predict the long-term behavior of the energy of the wave. We want to estimate the rate at which the energy grows when the system is unstable. We define the energy as

$$E_j = |\alpha_j|^2 + |\beta_j|^2. \quad (10)$$

This is constant on any interval where $\Omega(t)$ is constant. However, it may grow whenever $\Omega(t)$ has a jump, as seen in Fig. 1(c).

For any periodic modulation, Eq. (8) can be used to estimate the energy growth rate. If $\lambda := \max\{|\lambda_1|, |\lambda_2|\}$, then we have that $\alpha_j, \beta_j \sim \lambda^{j/T}$, which implies that

$$E(t) \sim \exp\left(2 \frac{\ln(\lambda)}{T} t\right), \quad (11)$$

where T is the duration of the supercell. Figures 3(c) and 3(d) show examples of energy profiles for parameters in a k gap and parameters corresponding to stable propagation, respectively. In both cases, the estimate in Eq. (11) based on the $\mathcal{F}_5$ supercell is shown, which gives a good prediction of the behavior. The growth rate $\log(\lambda_{\mathcal{F}_5})/T$ is shown as a

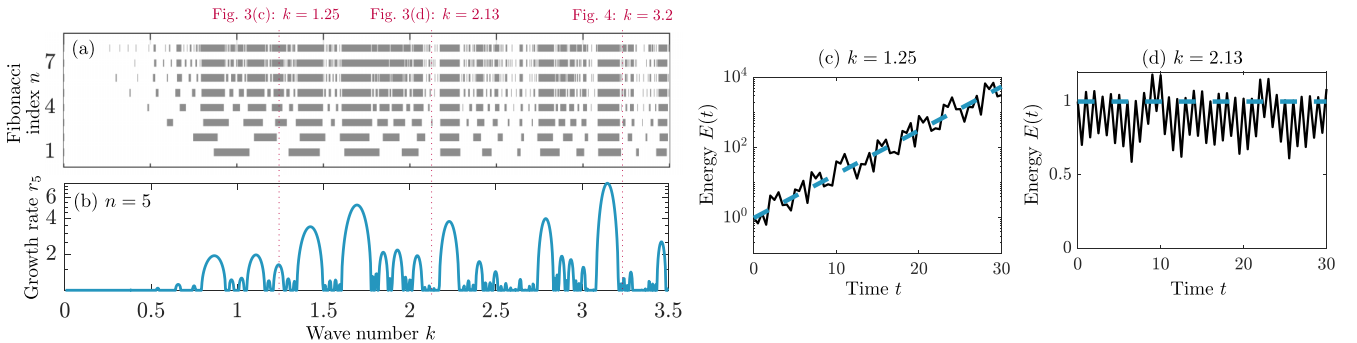


FIG. 3. Supercell approximations can be used to predict stability. (a) The pattern of k gaps for successive supercell approximations. The k gaps are shaded for successive Fibonacci unit cells, indexed by n . (b) The growth rate, estimated using the $\mathcal{F}_5$ supercell. The energy $E(t)$ for (c) $k = 1.25$ and (d) $k = 2.13$. In both cases, the data are plotted alongside the slope predicted by the $\mathcal{F}_5$ supercell approximation.

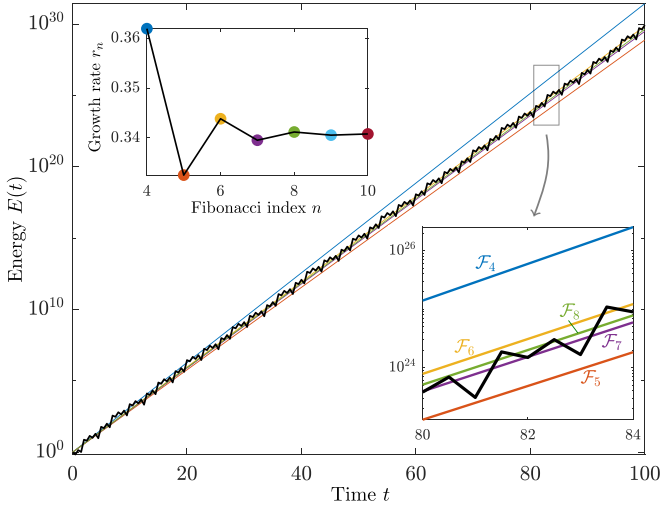


FIG. 4. The supercell estimates for the energy growth rate converge as the duration of the unit cell is increased. Here, $k = 3.2$ is in a super k gap [see Fig. 3(a)]. The growth rate is estimated as $E(t) = \exp(r_n t)$, where the growth rate r_n is based on the $\mathcal{F}_n$ supercell. The convergence of the growth rates r_n is illustrated in the inset.

function of k in Fig. 3(b), which gives a measure of the depth of the super k gaps.

The accuracy of the estimate in Eq. (11) for quasiperiodic modulation can be improved by using longer supercells. In Fig. 4, we show the energy $E(t)$ for parameter values in a k gap, along with growth rate estimates for several different Fibonacci unit cells $\mathcal{F}_n$. In the inset plot, we see that the growth rate estimate converges in the limit of large n . This shows that an arbitrarily good approximation of the average growth rate can be obtained by taking a sufficiently long supercell.

To see why the energy growth rate estimates converge, note that they are given by $r_n = \log(\lambda_{\mathcal{F}_n})/\mathbb{F}_n$, where $\mathbb{F}_n$ is the n th Fibonacci number (which is the duration of the $\mathcal{F}_n$ unit cell). Since $\det(\mathcal{M}_{\mathcal{F}_n}) = 1$, if we are in a k gap, then we have that $x_n = \text{tr}(\mathcal{M}_{\mathcal{F}_n}) = \lambda_{\mathcal{F}_n} + 1/\lambda_{\mathcal{F}_n}$. In a super k gap, we expect $\lambda_{\mathcal{F}_n}$ to be very large when n is large, so we can approximate $x_n \approx \lambda_{\mathcal{F}_n}$ in this regime. Recalling Eq. (9), when the values are large, we have that $x_{n+1} \approx x_n x_{n-1}$, or equivalently $\log(x_{n+1}) \approx \log(x_n) + \log(x_{n-1})$. The limiting behavior

of this recursion relation can be characterized by taking the ansatz $\log(x_n) = \rho^n$. This yields the characteristic polynomial $\rho^2 = \rho + 1$, which shows that $\log(x_n)$ will grow according to $\log(x_n) \sim \varphi^n$ when n is large, where $\varphi = (1 + \sqrt{5})/2$ is the golden ratio. It is also a well-known property of Fibonacci numbers that $\mathbb{F}_n \sim \varphi^n$ when n is large. Putting this together, we have that the supercell growth rates converge

$$r_n = \frac{\log(\lambda_{\mathcal{F}_n})}{\mathbb{F}_n} \sim \frac{\log(x_n)}{\mathbb{F}_n} \xrightarrow{n \rightarrow \infty} R, \quad (12)$$

for finite R , which is the average growth rate of the energy of waves in the FTQ.

VI. CONCLUSIONS

Using supercell approximations, we have revealed the intricate pattern of k gaps exhibited by FTQs. This has analogs with the spectra of equivalent spatially modulated systems. The key difference is the exponential growth of solutions. Hence, the key theoretical insight is that supercell approximations accurately predict the rate of energy growth. Further, the existence of super k gaps guarantees that these approximations will converge, giving arbitrarily good predictions of the energy growth rate for sufficiently long temporal supercells.

Quasiperiodic temporal modulation leads to systems with exotic spectral and topological properties. It is anticipated that the methods developed here, which allow amplification properties to be estimated accurately and efficiently, will accelerate the use of FTQs for applications. There are many potential advantages of unlocking new symmetries and topologies, such as the beneficial robustness properties that have been observed for temporal quasicrystals [52]. As well as the methods developed here, we expect that extending the study of quasinormal modes to quasiperiodic modulation would be a useful tool to realize the full potential of this [53,54].

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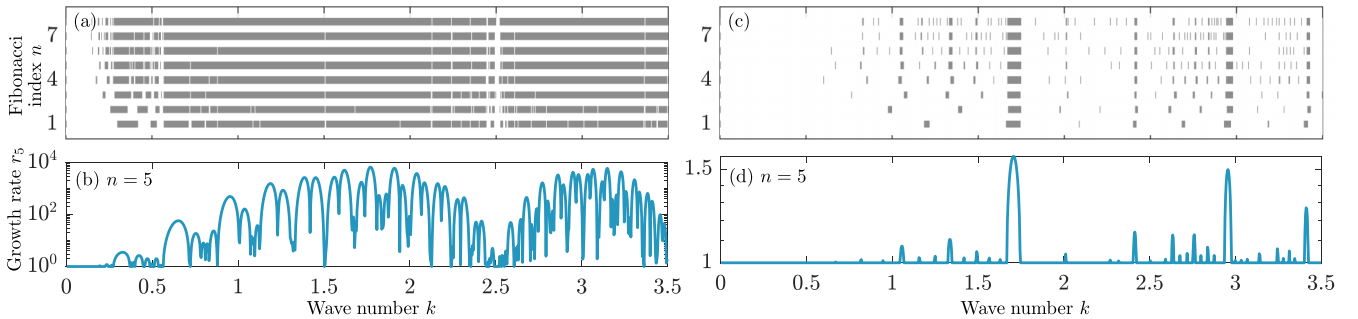


FIG. 5. Stability predictions for high- and low-contrast parameter regimes. Supercell approximations are used to predict the pattern of k gaps for successive Fibonacci unit cells (top row) and the growth rates within instability regions (shown for $n = 5$ in the bottom row). (a) and (b) use the high-contrast parameter values $\varepsilon_A = 0.3$ and $\varepsilon_B = 0.2$, whereas (c) and (d) use the low-contrast parameter values $\varepsilon_A = 0.95$ and $\varepsilon_B = 0.9$ (in both cases, the background is $\varepsilon_0 = 1$).

DATA AVAILABILITY

The software that supports the findings of this article is openly available [55].

APPENDIX: DEPENDENCE ON PARAMETER VALUES

In this study, we have used nondimensional parameter values to demonstrate the methods for estimating blow-up. It is worth highlighting that the observed phenomena are not particular to the values chosen. For the above examples, we used $\varepsilon_A = 0.8$ and $\varepsilon_B = 0.6$, along with a normalized background value of $\varepsilon_0 = 1$. In this nondimensional model, the most critical property is the contrast between the values of $\Omega(t)$.

If we increase the contrast between the values of $\Omega(t)$, by reducing the values of ε_A and ε_B , then we expect the instabil-

ity gaps to broaden and the blow-up rates to increase. This is consistent with the behavior of spatial photonic crystals, where band gaps get wider and deeper for higher contrasts. This is seen in Fig. 5(a), where we use the values $\varepsilon_A = 0.3$ and $\varepsilon_B = 0.2$ and show the k gaps for successive Fibonacci unit cells, indexed by n . We see that the k gaps are much broader and fill most of the range of k shown. In Fig. 5(b), we show the growth rate r_5 corresponding to the $\mathcal{F}_5$ unit cell. Notice the scale on the vertical axis: The growth rate is several orders of magnitude larger than it was in Fig. 3(b).

For smaller contrasts, we expect the k gaps to be smaller. In Fig. 5(c), we show the k gaps for the low-contrast parameters $\varepsilon_A = 0.95$ and $\varepsilon_B = 0.9$. Here, the k gaps are much smaller. Figure 5(d) shows the corresponding growth rate, which is trivially equal to 1 for many of the k values shown and only grows significantly within the main k gaps.

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