

Condensation of vanishing photon emission rates in random atomic clouds

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In the collective photon emission from atomic clouds both the atomic transition frequency and the decay rate are modified compared to a single isolated atom, leading to the effects of superradiance and subradiance. In this article, we analyze the properties of the Euclidean random matrix associated with the radiative dynamics of a cold atomic cloud, previously investigated in the contexts of photon localization and Dicke super- and subradiance. We present evidence of a phase transition, surprisingly controlled by the cooperativeness parameter, rather than the spatial density or the diagonal disorder. The numerical results corroborate the occurrence of such a phase transition at a critical value of the cooperativeness parameter, above which the lower edge of the spectrum vanishes, exhibiting a macroscopic accumulation of eigenvalues. Independent evaluations based on the two phenomena provide the same value for the critical cooperativeness parameter.

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I. INTRODUCTION

Many-body physics and emerging collective effects present many surprising features with theoretical, numerical, and experimental challenges. A fruitful approach to discover novel features often exploits interdisciplinary research and concepts, not directly focusing on detailed implementation and applications. Motivated by the recent development of open quantum many-body systems, in this article we investigate novel features based on a detailed analysis of the eigenvalues of a random matrix involved in the description of collective decay processes. Following Dicke's pioneering article [1], the investigation of cooperative effects in atom-photon interactions has attracted increasing attention from both theoretical and experimental points of view. The interaction of photons with atomic ensembles produces complex collective phenomena [2], which profoundly differ from the behavior of individual components of the system. For instance, in the collective photon emission from atomic clouds [3] both the atomic transition frequency [4,5] and the decay rate are modified compared to a single isolated atom. This rate can be larger or smaller than the decay rate Γ of an isolated atom, yielding *superradiance* [6–9] and *subradiance* [10–13], respectively. The possibility to con-

trol the emergence of collective phenomena paves the way for a wide range of applications, e.g., for quantum information processing and quantum communication [14–17]. Motivated by this ongoing effort on cooperative effects and light localization in two-level systems, implemented, for instance, in cold atomic clouds, we focus on the spectral properties of a Euclidean random matrix, relevant for photon localization and Dicke super- and subradiance [18–20]. The same random matrix was the subject of previous analyses on photon localization [21] and cooperativeness [22] (see [23] for a vector-model generalization). We note that phase transitions in different Euclidean random matrices have been identified with a phase transition controlled either by the spatial density of the scatterers [24,25] or by additional diagonal disorder [26]. In this article we show that a phase transition has been found in the Euclidean random matrix as studied in [21–23], surprisingly controlled by the so-called cooperativeness parameter, which is different from the spatial density and the diagonal disorder parameter. Even though the approach we follow is based on eigenvalue statistics rather than observables of a driven system or the evolution of an interacting many-body system, understanding an eigenvalue problem allows for complementary insight into the underlying emerging properties of complex systems.

II. THE PHYSICAL SYSTEM AND THE MATRIX S

The system under investigation is based on a random distribution of resonant scatterers, such as in a cold cloud of N

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identical atoms with fixed random positions $\mathbf{r}_j$, $j = 1, \dots, N$, independently sampled from a three-dimensional Gaussian distribution with zero mean and variance $\sigma^2 > 0$ [6,11,22,27]. We treat the atoms as two-level systems with transition frequency $\omega_a = ck_a$. For the atom-photon interaction we use a dipole model in the rotating wave and scalar approximations [28,29] (the vectorial case was addressed, e.g., in [23]). We do not consider near-field dipole-dipole coupling and thus resort to the so-called scalar model. *A posteriori*, this will be justified by the low-density nature of the limit [22]. Furthermore, we work in the weak-excitation limit, restricting ourselves to the single-excitation sector, with at most one emitted photon or one atom in the excited state at a time.

In this setting, let us denote by $\beta_j(t)$ the probability amplitude that the j th atom is excited. The total excitation probability of the atomic cloud is $P(t) = \boldsymbol{\beta}^\dagger(t)\boldsymbol{\beta}(t)$, where $\boldsymbol{\beta}(t)$ is a column vector with components $\beta_j(t)$. After the laser is switched off, the evolution equation of the excitation probability reads [6,27,30]

$$\dot{P}(t) = -\Gamma \boldsymbol{\beta}^\dagger(t) S \boldsymbol{\beta}(t), \quad (1)$$

where Γ is the linewidth of the atomic transition. The emission-rate matrix S is a Euclidean random matrix [31–34], whose elements are given by

$$S_{ij} = \text{sinc} \left(\sqrt{\frac{N}{b}} \|\mathbf{x}_i - \mathbf{x}_j\| \right), \quad i, j = 1, \dots, N, \quad (2)$$

with $\mathbf{x}_i = \mathbf{r}_i/\sigma$ being the rescaled dimensionless atomic positions. These are independently sampled from a three-dimensional standard Gaussian probability density function

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} e^{-\|\mathbf{x}\|^2/2}, \quad \mathbf{x} \in \mathbb{R}^3, \quad (3)$$

for which the *cooperativeness parameter* b is defined as

$$b = \frac{N}{(k_a \sigma)^2}. \quad (4)$$

Note that this parameter is proportional to the optical depth used in other works but differs by a numerical factor. While photon-mediated interactions also induce coherent dynamics [22,35], diagonalizing the emission-rate matrix (2) provides direct information about independent decay channels since its elements represent, in the Weisskopf-Wigner approximation, the overlap between the asymptotic states $\int d\mathbf{k} e^{-i\mathbf{k} \cdot \mathbf{r}_j} (ck - ck_a + i\Gamma/2)^{-1} |\mathbf{k}\rangle$ of photons emitted by atoms in $\mathbf{r}_j$ [29].

It was shown that the cooperativeness parameter b is the relevant scaling parameter from both physical and mathematical points of view to describe the cooperative nature of the cloud emission (see Refs. [6,10,11,22,36]). Henceforth, we turn to studying the asymptotic behavior of S in Eq. (2) when both the numerator and the denominator in (4) tend to infinity, with their ratio kept fixed [22]. More specifically, we investigate the spectrum of S in the limit $N \rightarrow \infty$ for various finite values of b . Note that this corresponds to a vanishing atomic density and finite optical depth limit. Indeed, the dimensionless peak density, i.e., the density at the center of the Gaussian cloud multiplied by the cube of the wavelength $\lambda_a = 2\pi/k_a$, is given by $\rho_0 \lambda_a^3 = (2\pi b)^{3/2} N^{-1/2}$ and hence vanishes in the limit $N \rightarrow \infty$ with b fixed. Although a vector model of

photon-mediated interaction would generally provide a more detailed description, we show in Appendix A that the scalar and vector models provide equivalent information in the limit of $N \rightarrow \infty$ and a fixed cooperativeness parameter, up to a rescaling of the latter [23]. Therefore, the results obtained for the investigated phase transition can be generalized to the vector model, while the differences between models become crucial for Anderson localization [25], emerging at finite density.

The matrix S is positive semidefinite and satisfies the trace condition $\text{Tr} S = N$. Therefore, the eigenvalues λ_i , $i = 1, \dots, N$, of S are non-negative, satisfy $\sum_i \lambda_i = N$, and have unit mean. See [22] for more details. The eigenvalue $\lambda = 1$ corresponds to the decay rate Γ of a single isolated atom; $0 \leq \lambda < 1$ corresponds to the subradiant modes of the matrix S in (2), with a longer lifetime, and $1 < \lambda \leq N$ is associated with superradiant modes, with a decay rate larger than Γ . Figure 1 displays the eigenvalue distribution $p(\lambda)$ for several values of the cooperativeness parameter b , with a distribution that is very different for each value considered. For small values of b the spectrum is symmetric around the isolated-atom emission peak at $\lambda = 1$, while for increasing b the effects of super- and subradiance become more pronounced. For large values of b [Fig. 1(d)] the spectrum is characterized by an accumulation of eigenvalues in the subradiant sector, compensated by the presence of very few large eigenvalues.

In this article we focus on the very subradiant part of the spectrum when $\lambda \ll 1$. It is clear in Fig. 1 that the spectrum displays a significant accumulation of small eigenvalues as b is increased. These eigenvalues signal the presence of an important subradiant component in the dissipative dynamics of the atomic gas. We want to investigate this feature and, in particular, scrutinize the phenomenon of the accumulation of vanishing eigenvalues. We shall see that this corresponds to a phase transition which is controlled by the cooperativeness parameter b .

Our strategy is as follows. We first analyze the behavior of the minimum eigenvalue $\lambda_{\min}$ and prove that it vanishes above a threshold value b_c of the cooperativeness and in the $N \rightarrow \infty$ limit. Then we compute the limit of the fraction of vanishing eigenvalues N_0/N . We anticipate that this corresponds to a finite fraction of eigenvalues vanishing in the $N \rightarrow \infty$ limit, which appears at the value $b = b_c$ at which the density of eigenvalues touches the value $\lambda = 0$. All these features point to the presence of a phase transition.

III. SMALL-EIGENVALUE DISTRIBUTION

Let us first analyze the cumulative distribution

$$\mathbb{P}(\lambda) = \int_0^\lambda p(y) dy \quad (5)$$

of the emission rates for small λ 's ($\lambda < 10^{-2}$) for different values of the cooperativeness b . The cumulative distribution is displayed in Fig. 2 for different values of b in the range [3,7]. The numerical precision of our simulations is 10^{-13} , which means that eigenvalues below 10^{-13} are, in practice, indistinguishable from zero. We note the absence of eigenvalues in the interval between $\lambda = 0$ and $\lambda_{\min} > 0$ in Fig. 2(a) for $b \leq 4.7$, which becomes smaller as b tends to a critical

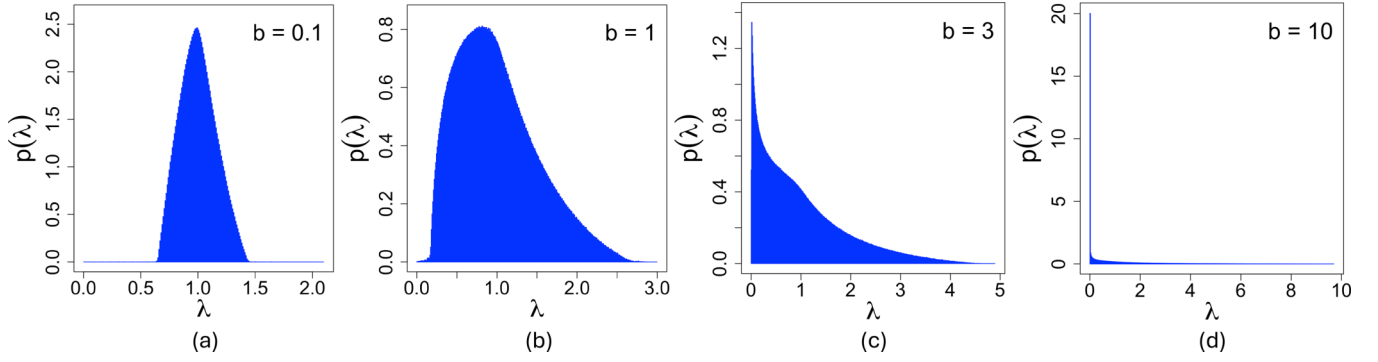


FIG. 1. Histograms of the normalized eigenvalue distribution $p(\lambda)$ obtained from 10 realizations of the matrix S in Eq. (2) with $N = 20000$ and different values of the cooperativeness parameter b .

value $b_c \approx 4.7$. In Fig. 2(b) the cumulative distributions in the range $10^{-13} \leq \lambda \leq 10^{-5}$ for $4.3 \leq b \leq 5$ are monitored, and we observe that the interval shrinks to zero at

$$b_c = 4.73(5). \quad (6)$$

At this point, a single nondecaying mode is born, with an infinite lifetime (in the $N \rightarrow \infty$ limit). As the cooperativeness b is increased above b_c , we also observe the appearance of the fast increase of $\mathbb{P}(0)$. As we can observe in Fig. 2(c), for $4.8 \leq b \leq 7$, this behavior is compatible with a *finite fraction* of vanishing eigenvalues and therefore nondecaying modes, which increases as b is increased beyond b_c . The asymptotic extent of this fraction will be scrutinized in what follows.

A. Study of the minimum eigenvalue

Let us now focus on the minimum eigenvalue of the spectrum of S , corresponding to the most subradiant state. We have taken into account several values of b and computed the mean minimum eigenvalue $\lambda_{\min}$, averaged over several realizations of the matrix S , for different values of the atom number N . Evidence of the transition at b_c emerges in Fig. 3, where $\lambda_{\min}$ is plotted as a function of b . Note that, here, $\lambda_{\min}$ is the minimum eigenvalue obtained for the corresponding b and for the largest achievable matrix. For all the values of $b > 4.735$, $\lambda_{\min}$ falls

below our numerical precision, thus being compatible with zero. Also, $\lambda_{\min}$ displays a scaling which is fully compatible with zero in the $N \rightarrow \infty$ limit (see Appendix B).

The presence of the transition at $b_c = 4.735$ can also be deduced from Fig. 3 from the first derivative of the $\lambda_{\min}$ vs b curve: In correspondence of the critical point, it has a tangent compatible with the vertical (note the log scale on the vertical axis).

We conclude that the behavior of the minimum eigenvalue represents quantitative evidence of a phase transition at b_c . Indeed, phase transitions related to the eigenvalue distribution touching the origin have been observed in other physical systems, e.g., in the entanglement spectrum of a bipartite system [37].

The occurrence of a phase transition associated with the sudden divergence of $p(0)$, driven by increasing b , is made evident by the study of the convergence of the locator expansion perturbation theory [38–42]. This argument, presented in Appendix C, captures the origin of the transition as well as the correct scaling with N , predicting a critical value that overestimates the numerical result by a factor ~ 2 . However, what that argument cannot explain is the fact that the minimum eigenvalue $\lambda_{\min}$ touches zero exactly at the same time at which a finite eigenvalue density forms at $\lambda = 0$, as shown below. Controlling $\lambda_{\min}$ requires a degree of control over the large

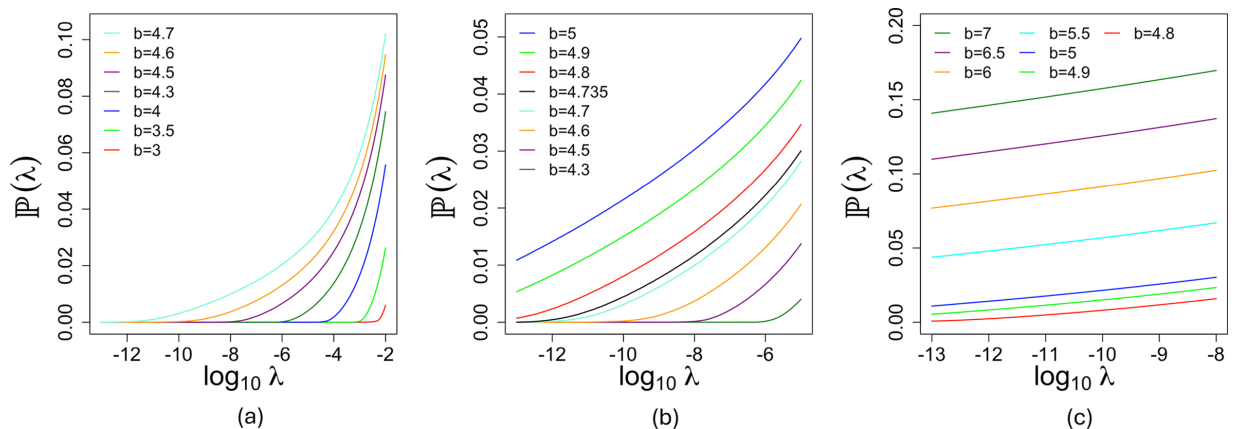


FIG. 2. Cumulative distribution $\mathbb{P}(\lambda) = \int_0^\lambda p(y)dy$ of small eigenvalues for (a) $b < b_c$, (b) b close to b_c , and (c) $b > b_c$. The distributions are obtained with $N = 20000$.

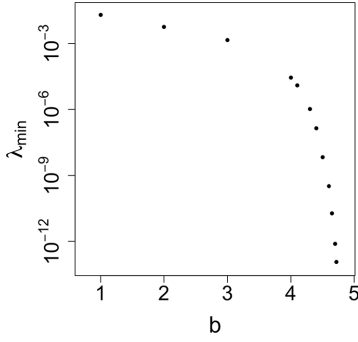


FIG. 3. Plot of the minimum emission rate $\lambda_{\min}$ as a function of b below the critical value $b_c = 4.735$. Note the log scale on the vertical axis. Here, $\lambda_{\min}$ is the minimum eigenvalue obtained for the corresponding b and for $N = 30\,000$, which is the largest achievable matrix size in numerical simulations.

deviations of the density $p(\lambda)$, which typically is not captured by perturbative calculations. We are therefore left to rely on the numerical results, which, however, seem quite robust.

B. Fraction of eigenvalues in the condensate

For $b \gtrsim b_c$ the spectrum of S starts to exhibit an accumulation of eigenvalues close to zero, which becomes larger for increasing values of b . We count the fraction of these eigenvalues for different values of b and N . Considering the finite numerical precision, we count as part of this *condensate* all the eigenvalues of S smaller than 10^{-13} .

At fixed b , we count the fraction N_0/N of vanishing eigenvalues and, as explained in Appendix D, extrapolate for $N \rightarrow \infty$ the fraction of the condensate at zero. The evolution of this fraction as a function of b is presented in Fig. 4. In Fig. 4(a), the presence of a transition at a critical value close to 4.7 can be observed. Below this threshold there is no vanishing eigenvalue (even for large matrix sizes), while above

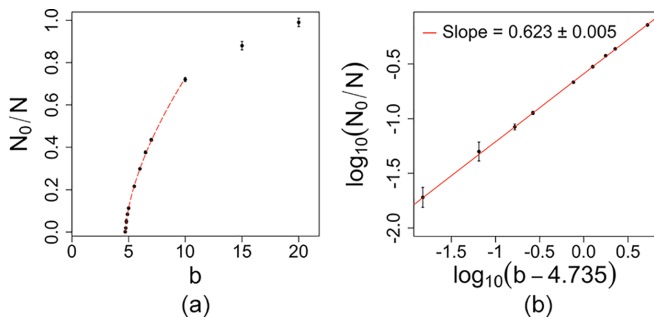


FIG. 4. Asymptotic fraction of vanishing eigenvalues as a function of b . (a) shows that for $b < 4.7$ there are no eigenvalues below 10^{-13} ; for $b = 4.9$ the condensate fraction is less than 0.1, and it increases as b grows, approaching unity for $b = 20$. (b) shows the linear relationship between $\log_{10}(N_0/N)$ and $\log_{10}(b - b'_c)$, where $b'_c = 4.735$ is the critical value above which the spectrum exhibits an accumulation of vanishing eigenvalues. The slope of the linear fit yields the critical exponent $\beta \simeq 0.623$ of the power law (7). This power law was used to fit the transition region in (a) (red dashed curve). Note that the critical value b'_c coincides with the value of b_c determined by studying the minimum eigenvalue.

the threshold the asymptotic fraction of vanishing eigenvalues grows. It approaches unity for $b = 20$, indicating that almost all eigenvalues vanish in this regime.

It is interesting to estimate the critical value b'_c above which the spectrum exhibits the transition toward an accumulation of vanishing eigenvalues and to assess whether near this critical point the fraction of subradiant vanishing eigenvalues grows according to a power law:

$$\frac{N_0}{N} \propto (b - b'_c)^\beta, \quad (7)$$

where β is a critical exponent. To this end, we focus on the data in Fig. 4(a) corresponding to values of b from 4.75 to 10. These data are used to plot $\log_{10}(N_0/N)$ as a function of $\log_{10}(b - b'_c)$ for several values of b'_c close to 4.7, and for each of them we perform a linear fit, thus checking whether Eq. (7) is satisfied. From this protocol, we estimate the critical value b'_c as the one yielding the best linear-regression fit.

The result of this fitting procedure is shown in Fig. 4(b) for the optimal value of b'_c , indicating that indeed, the fraction of vanishing eigenvalues obeys the power law (7) near the transition, with a critical exponent $\beta \simeq 0.623$. Note that the critical value b'_c is compatible with the value of b_c in Eq. (6), found for the birth of the nondecaying eigenmode, thus demonstrating the consistence between the different approaches in characterizing the transition. The power law (7) is used to fit the transition region in Fig. 4(a) (red dashed curve). We note that the occurrence of weak phase transitions with the resulting formation of a nonzero singular component at the edge of the eigenvalue spectrum is a universal phenomenon which has been reported in many physics problems related to random matrices [43,44] and in random networks [45,46].

We finally remark that the Markovian assumption of neglecting photon propagation delays is well motivated for optical atomic transitions (see [11]), where non-Markovian effects cannot emerge at distances smaller than tens of meters.

IV. CONCLUSIONS

The results obtained point to the existence of a phase transition, occurring at a critical value b_c of the cooperativeness parameter b . As b increases beyond b_c , the smallest eigenvalue $\lambda_{\min}$ of the matrix S approaches the extremely subradiant boundary $\lambda = 0$, and the spectrum starts exhibiting an accumulation of vanishing eigenvalues in the subradiant sector. In other words, there is a single transition point at which both phenomena occur, which we found to be at $b_c = 4.73(5)$: For $b \geq b_c$ the smallest eigenvalue in the asymptotic limit $N \rightarrow +\infty$ reaches zero, and the spectrum begins to display an accumulation of vanishing eigenvalues. We identified a mechanism for which the presence of eigenvalues at zero can occur by analyzing the perturbation theory for small b for the density of eigenvalues $p(0)$, showing that it diverges above a certain b_c^0 . Our perturbative value of $b_c^0 = 11.2$ overestimates the numerically obtained $b_c = 4.73(5)$, in line with our understanding of the validity of this kind of derivation.

We note that while the Anderson localization transition has been predicted for a fixed atomic density [24,25], the transi-

tion reported here occurs, in the large- N limit, at a vanishing atomic density, but at a finite optical depth. In this sense, it is directly related to the cooperativeness of the system (4), which is the control parameter for super- and subradiance [2]. In the dilute limit, vanishing two-body interactions are compensated by both their long-range nature and the growing number of coupled atoms, providing another signature of the cooperative nature of the transition. In this context, it would be particularly interesting to understand how disorder of the energy coupling, either from the dipole-dipole excitation-exchange (Hamiltonian) terms or from energy broadening (diagonal disorder), could affect the emission-rate condensation transition. Finally, we note that in an experimental setup, the vanishing emission-rate transition may compete with decoherence mechanisms other than spontaneous emission, such as finite-temperature effects [47–49], which may set a lower bound on the emission rates [50].

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APPENDIX A: CONNECTION WITH THE VECTOR MODEL

Consider the imaginary part of the vector Green’s matrix that describes the pairwise interaction between dipoles in positions $\{\mathbf{r}_i\}_{i \in \{1, \dots, N\}}$ [23]:

$$\Lambda_{i\alpha, j\beta} = S_1(k_a r_{ij}) P_{\perp, \alpha\beta}(\hat{\mathbf{r}}_{ij}) + S_2(k_a r_{ij}) P_{\parallel, \alpha\beta}(\hat{\mathbf{r}}_{ij}), \quad (\text{A1})$$

with $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, $r_{ij} = |\mathbf{r}_{ij}|$, and $\hat{\mathbf{r}}_{ij} = \mathbf{r}_{ij}/r_{ij}$, where the distance-dependent functions are defined as

$$S_1(x) = \frac{3}{2} \left(\frac{\sin(x)}{x} + \frac{x \cos(x) - \sin(x)}{x^3} \right), \quad (\text{A2})$$

$$S_2(x) = -3 \frac{x \cos(x) - \sin(x)}{x^3} \quad (\text{A3})$$

and

$$P_{\perp, \alpha\beta}(\hat{\mathbf{r}}) = \delta_{\alpha\beta} - \hat{r}_\alpha \hat{r}_\beta, \quad P_{\parallel, \alpha\beta}(\hat{\mathbf{r}}) = \hat{r}_\alpha \hat{r}_\beta \quad (\text{A4})$$

are the transverse and longitudinal projectors, respectively. The terms $\Lambda_{i\alpha, j\beta} = \delta_{\alpha\beta}$ are consistent with the $x \rightarrow 0$ limits of S_1 and S_2 . This immediately provides the result

$$\text{Tr}(\Lambda) = 3N. \quad (\text{A5})$$

The computation of

$$\begin{aligned} \text{Tr}(\Lambda^2) &= \sum_{i,j=1}^N \sum_{\alpha, \beta=x,y,z} (\Lambda_{i\alpha, j\beta})^2 \\ &= 3N + \sum_{i \neq j} \sum_{\alpha, \beta=x,y,z} (\Lambda_{i\alpha, j\beta})^2 \end{aligned} \quad (\text{A6})$$

is simplified by considering that

$$\sum_{\alpha\beta} [P_{\perp, \alpha\beta}(\hat{\mathbf{r}})]^2 = \sum_{\alpha} P_{\perp, \alpha\alpha}(\hat{\mathbf{r}}) = 2, \quad (\text{A7})$$

$$\sum_{\alpha\beta} [P_{\parallel, \alpha\beta}(\hat{\mathbf{r}})]^2 = \sum_{\alpha} P_{\parallel, \alpha\alpha}(\hat{\mathbf{r}}) = 1, \quad (\text{A8})$$

$$\sum_{\beta} P_{\perp, \alpha\beta}(\hat{\mathbf{r}}) P_{\parallel, \alpha\beta}(\hat{\mathbf{r}}) = 0. \quad (\text{A9})$$

Therefore,

$$\text{Tr}(\Lambda^2) = 3N + \sum_{i \neq j} [2S_1^2(k_a r_{ij}) + S_2^2(k_a r_{ij})]. \quad (\text{A10})$$

If the dipole positions are random, with independent and identical distributions, the expectation value of $\text{Tr}(\Lambda^2)$ can be used to estimate the second moment of the distribution of the eigenvalues λ of Λ :

$$\langle \lambda^2 \rangle = \frac{\langle \text{Tr}(\Lambda^2) \rangle}{3N} = 1 + \frac{N-1}{3} [2\langle S_1^2(k_a r) \rangle + \langle S_2^2(k_a r) \rangle]. \quad (\text{A11})$$

In the case of an isotropic Gaussian distribution with the standard deviation σ along the three directions, the dimensionless distances $R = r/\sigma$ are distributed according to the probability density [22]

$$p(R) = \frac{1}{\sqrt{4\pi}} R^2 \exp\left(-\frac{R^2}{4}\right). \quad (\text{A12})$$

Defining the dimensionless parameter $M = (k_a \sigma)^2$, we obtain

$$\begin{aligned} \langle S_1^2(k_a r) \rangle &= \int_0^\infty dR S_1^2(\sqrt{M}R) p(R) \\ &= \frac{3}{64M^3} [12M^2 + 6M + 1 - 8\sqrt{\pi} M^{3/2} \text{erf}(2\sqrt{M}) \\ &\quad - e^{-4M} (12M^2 + 10M + 1)], \end{aligned} \quad (\text{A13})$$

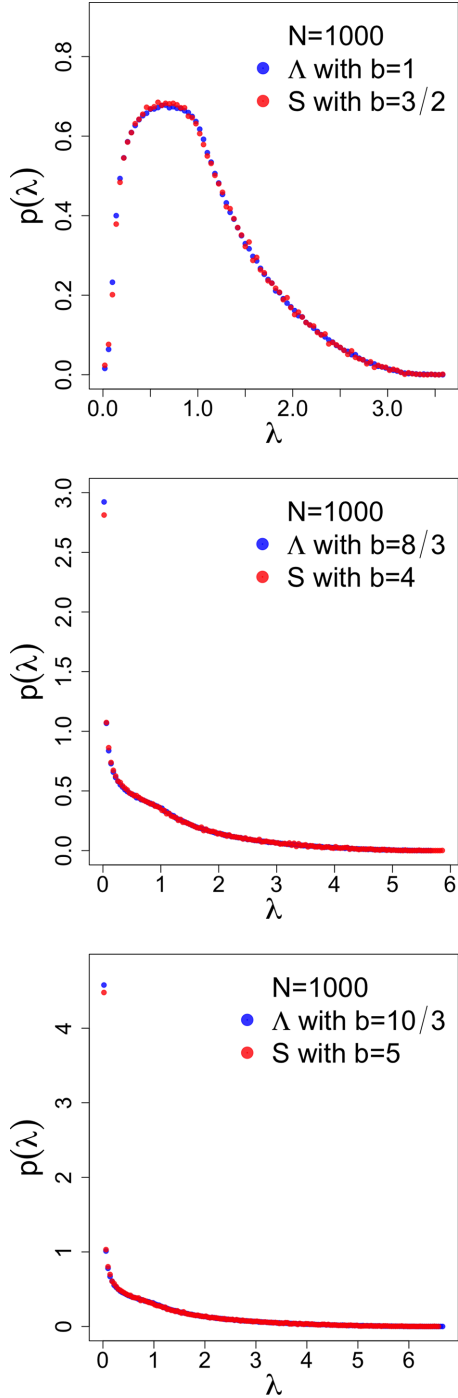


FIG. 5. Comparison between the eigenvalue distributions of the scalar-model S matrix and the vector-model Λ matrix defined in Eq. (A1). The S matrix is diagonalized with the values $b = 3/2, 4, 5$, while the value of b for Λ is scaled by a factor of $2/3$. The distributions, represented as histograms with a bin size of 0.04 , are obtained by accumulating the spectra of 20 matrices each, referred to $N = 1000$ atoms. Note that the discrepancies observed in the first bin of the middle panel (0.11) and bottom panel (0.10) are smaller than the corresponding standard errors for both the scalar-model distribution (0.19 for $b = 4$, 0.26 for $b = 5$) and the vector-model distribution (0.20 for $b = 8/3$, 0.20 for $b = 10/3$).

$$\begin{aligned} \langle S_2^2(k_a r) \rangle &= \int_0^\infty dR S_2^2(\sqrt{M}R) p(R) \\ &= \frac{3}{16M^3} [1 - 6M + 4\sqrt{\pi}M^{3/2}\text{erf}(2\sqrt{M}) \\ &\quad + e^{-4M}(2M - 1)]. \end{aligned} \quad (\text{A14})$$

As in the scalar case, the variance $(\Delta\lambda)^2 = \langle \lambda^2 \rangle - \langle \lambda \rangle^2 = \langle \lambda^2 \rangle - 1$ converges to a finite value in the $N \rightarrow \infty$ limit only if M scales linearly with N . In this case, defining

$$b = \frac{N}{M} = \frac{N}{(k_a \sigma)^2}, \quad (\text{A15})$$

we obtain

$$\langle \lambda^2 \rangle \rightarrow 1 + \frac{3}{8}b, \quad (\text{A16})$$

with the only nontrivial contribution coming from $\langle S_1^2(k_a r) \rangle$.

The above result shows that the first and second moments of the eigenvalue distributions of the scalar model matrix

$$S_{ij} = \frac{\sin(k_a r_{ij})}{k_a r_{ij}} = \text{sinc}\left(\sqrt{\frac{N}{b}} x_{ij}\right) \quad (\text{A17})$$

coincide in the $N \rightarrow \infty$ case, up to a scaling of the parameter b , which must be smaller by a factor of $2/3$ in the vector-model matrix (see Eq. (22) in Ref. [22] for the scalar model result; see Ref. [23] for considerations on the scaling factor between the two models). Moreover, in Fig. 5 we numerically show that the analogy goes well beyond the second moment: In the reported cases, with $b = 3/2, 4, 5$, the two eigenvalue distributions practically overlap already at $N = 1000$.

APPENDIX B: EVALUATION OF THE MINIMUM EIGENVALUE

Here, we further detail the protocol to obtain the minimum eigenvalue $\lambda_{\min}$ presented in Fig. 3 in the main text and its scaling. At a fixed value of b we compute the minimum eigenvalue obtained from the diagonalization of S , and average it over several realizations of the particle positions. The behavior of $\lambda_{\min}$ with the atom number N is shown in Fig. 6. We emphasize that all eigenvalues must be non-negative.

For $b \geq 4.735$ we find an accurate linear dependence between $1/\log_{10}(\lambda_{\min})$ and $1/\log_{10} N$. It is worth noting that such scaling has been predicted in infinite-dimensional localization [40–42]. Remarkably, the linear fit holds for small values of N as well. This fit is valid up to the working precision (10^{-13}) of the numerical diagonalization algorithm (yellow area in Fig. 6).

For $b < 4.735$ the dependence is no longer linear [see Fig. 6(b)], but rather bends towards a nonvanishing asymptotic value (that is, $\lambda_{\min}^\infty > 0$), indicating the presence of a gap. One may try to estimate such an asymptotic value, but that would depend on some (arbitrary) nonlinear fitting function. Figure 3 displays the smallest (average) $\lambda_{\min}$ that we are able to simulate, corresponding to the case with $N = 30\,000$. Since all curves tend to decrease monotonically with $1/\log_{10} N$, this yields a conservative estimate for $\lambda_{\min}^\infty$. Nonetheless, Fig. 3

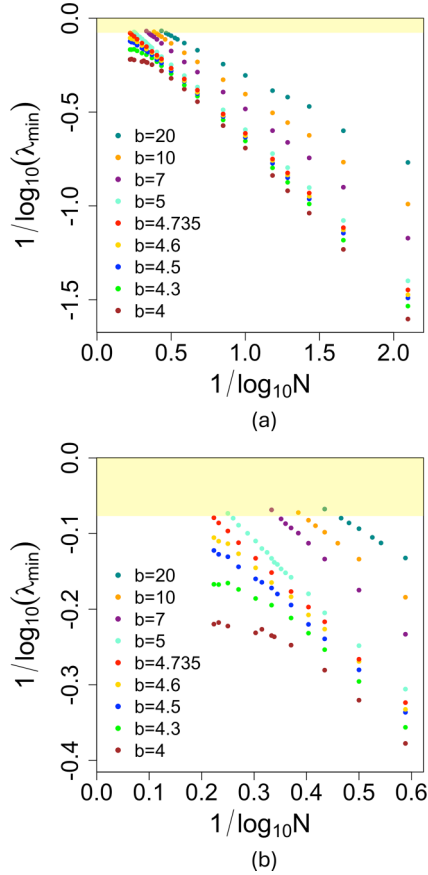


FIG. 6. Plots of $1/\log_{10}(\lambda_{\min})$ vs $1/\log_{10} N$ obtained for different values of b . The horizontal yellow region corresponds to the working precision (10^{-13}) of the diagonalization algorithm. The eigenvalues falling in this area are considered to be zero within the numerical precision.

displays a vertical tangent corresponding to the critical point b_c .

The values of $\lambda_{\min}$ obtained for $b > b_c = 4.735$, numerically obtained for smaller values of N , cannot be distinguished from zero within our numerical precision. However, above the threshold, as can be observed in Fig. 6(a), it is expected to go to zero in the $N \rightarrow \infty$ limit.

Finally, we note that, as emphasized in the main text, it is by no means obvious that the fraction of the condensate and the gap of the minimum eigenvalue point to the same phase transition at $b = b_c$. The simultaneous occurrence rather suggests that they are complementary manifestations of the phase transition.

APPENDIX C: A PERTURBATIVE ESTIMATE OF THE TRANSITION

One can obtain a rough understanding of the nature of the transition with the following reasoning, which is perturbative for $b \ll 1$. Note that for $b = 0$ the matrix is diagonal with a degenerate spectrum $\lambda_i = 1$. For small b , the off-diagonal matrix elements are small, with typical values

$$S_{ij} \simeq \frac{(-1)^{\phi_{ij}}}{\sqrt{N/b} \ell_{ij}}, \quad (\text{C1})$$

where $\ell_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$ is the distance between points i and j . We approximate $\sin(\sqrt{N/b} \ell_{ij})$ with a random sign. The transition occurs when the edge of the band of eigenvalues reaches $\lambda = 0$. Note that if the matrix were not positive definite, the band would go from localized around $\lambda = 1$ to being a semicircle of size $\sqrt{N}$ centered around 1.

The eigenvalue density is expected to be

$$p(\lambda) = \frac{1}{\pi N} \sum_i \text{Im} \langle i | \frac{1}{S - \lambda - i0^+} | i \rangle. \quad (\text{C2})$$

The value $p(0)$ will be nonzero only if there are delocalized states at $\lambda = 0$, while all the unperturbed localized states are at energy $\lambda_i^{(0)} = 1$. In the localization perturbation theory [38,39], the condition for this to happen is that the amplitude to get from any localized state i to any other j becomes of $O(1)$. In perturbation theory this is proportional to

$$A_{i \rightarrow j} = \sum_{p \in \text{Path}(i \rightarrow j)} \prod_{(kl) \in p} \frac{S_{kl}}{\lambda - \lambda_l^{(0)}}. \quad (\text{C3})$$

Using approximation (C1) and setting $\lambda = 0$ and $\lambda_l^{(0)} = 1$, we obtain

$$A_{i \rightarrow j} = \sum_p (-1)^{\phi_p} \left(\frac{b^{1/2}}{\sqrt{N}} \right)^N \prod_{(kl) \in p} \frac{1}{\ell_{kl}}, \quad (\text{C4})$$

where ϕ_p is the random sign assigned to the amplitude of the path p .

The presence of $\sqrt{N}$ in the denominator means that any single path will contribute a vanishing fraction to this sum. The largest contribution will come from the nonrepeating paths that are largest in number. In this case it is easy to see that choosing any sequence $i \rightarrow i_1 \rightarrow \dots \rightarrow i_{N-2} \rightarrow j$ gives $(N-2)! \sim N!$ paths. Now we must assume that ℓ_{kl} are independent and identically distributed numbers, which leads to

$$A_{i \rightarrow j} \simeq \sum_{p=1}^{N!} (-1)^{\phi_p} \left(\frac{b^{1/2}}{\sqrt{N}} \right)^N \frac{1}{\bar{\ell}^N}, \quad (\text{C5})$$

where $\bar{\ell}$ is the geometric average of ℓ_{kl} , which can be computed (under the assumption of statistical independence) using the distribution of points in Eq. (3), which leads to $\bar{\ell} = \exp(1 - \gamma/2) = 2.037$. Hence, we find the typical value of the random amplitude is

$$A_{i \rightarrow j} \propto \left(\frac{b^{1/2}}{\bar{\ell}} \right)^N \frac{1}{N^{N/2}} \sum_{p=1}^{N!} (-1)^{\phi_p} \sim \left(\frac{b^{1/2}}{\bar{\ell}} \right)^N \frac{\sqrt{N!}}{N^{N/2}} \sim \left(\frac{b}{b_c^0} \right)^{N/2}, \quad (\text{C6})$$

with

$$b_c^0 = \bar{\ell}^2 e = 11.277. \quad (\text{C7})$$

Thus, under all the above approximations, a transition can be found at a finite value of b which overestimates by approximately a factor of 2.4 the one found numerically. However, reexamining the approximations, one can identify several factors (the dressing of the denominators, the role of large fluctuations of the amplitudes, the correlations of the variables

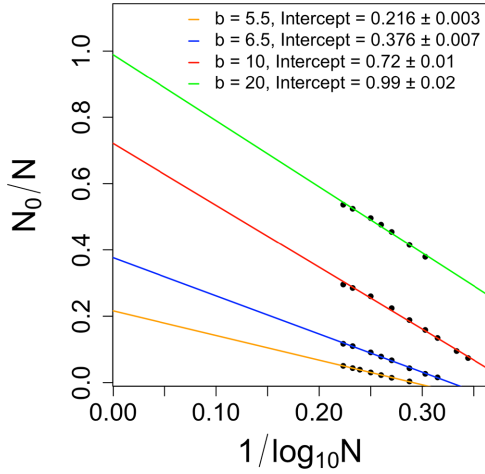


FIG. 7. Fraction N_0/N of eigenvalues below 10^{-13} as a function of $1/\log_{10} N$ for $b = 5.5, 6.5, 10, 20$. The intercept of the linear fits with the vertical axis corresponds to the asymptotic fraction of eigenvalues in the condensate as $N \rightarrow +\infty$. As can be seen, the intercept increases with b , approaching unity for $b = 20$.

ℓ_{kl} , etc.) which can lead to a decrease in the b_c estimate. In any case, this qualitative calculation captures the origin of the transition (based on a large delocalization of the eigenstates hypothesis) and the correct scaling with N (the N factors of $\sqrt{N}$ which are compensated by $\sqrt{N!}$): These properties are shared with the exact solution.

APPENDIX D: FINITE-SIZE CORRECTIONS FOR THE CONDENSATE FRACTION

We explain here how to deal with finite-size corrections when deriving the fraction of subradiant vanishing eigenvalues for $b > b_c$ (Fig. 4 in the main text). At fixed values of b and N , we count the number N_0 of eigenvalues below 10^{-13} , our machine numerical precision, and compare it to the total number of eigenvalues N . This procedure is repeated for at least 10 realizations of the matrix S . We then vary N . A few examples are shown in Fig. 7, where we plot N_0/N vs $(\log_{10} N)^{-1}$, obtaining a linear fit. This allows us to extrapolate the fitting line back to the vertical axis ($N \rightarrow \infty$). This is the value plotted in Fig. 4.

- [1] R. H. Dicke, Coherence in spontaneous radiation processes, *Phys. Rev.* **93**, 99 (1954).
- [2] W. Guerin, M. T. Rouabah, and R. Kaiser, Light interacting with atomic ensembles: Collective, cooperative and mesoscopic effects, *J. Mod. Opt.* **64**, 895 (2017).
- [3] D. Petrosyan and K. Mølmer, Collective emission of photons from dense, dipole-dipole interacting atomic ensembles, *Phys. Rev. A* **103**, 023703 (2021).
- [4] R. Röhlberger, K. Schlage, B. Sahoo, S. Couet, and R. Ruffer, Collective Lamb shift in single-photon superradiance, *Science* **328**, 1248 (2010).
- [5] J. Keaveney, A. Sargsyan, U. Krohn, I. G. Hughes, D. Sarkisyan, and C. S. Adams, The cooperative Lamb shift in an atomic nanolayer, *Phys. Rev. Lett.* **108**, 173601 (2012).
- [6] M. O. Araújo, I. Krešić, R. Kaiser, and W. Guerin, Superradiance in a large and dilute cloud of cold atoms in the linear-optics regime, *Phys. Rev. Lett.* **117**, 073002 (2016).
- [7] M. Gross and S. Haroche, Superradiance: An essay on the theory of collective spontaneous emission, *Phys. Rep.* **93**, 301 (1982).
- [8] M. O. Scully and A. A. Svidzinsky, The super of superradiance, *Science* **325**, 1510 (2009).
- [9] N. E. Rehler and J. H. Eberly, Superradiance, *Phys. Rev. A* **3**, 1735 (1971).
- [10] T. Bienaimé, N. Piovella, and R. Kaiser, Controlled Dicke subradiance from a large cloud of two-level systems, *Phys. Rev. Lett.* **108**, 123602 (2012).
- [11] W. Guerin, M. O. Araújo, and R. Kaiser, Subradiance in a large cloud of cold atoms, *Phys. Rev. Lett.* **116**, 083601 (2016).
- [12] A. Cipris, N. A. Moreira, T. S. do Espírito Santo, P. Weiss, C. J. Villas-Boas, R. Kaiser, W. Guerin, and R. Bachelard, Subradiance with saturated atoms: Population enhancement of the long-lived states, *Phys. Rev. Lett.* **126**, 103604 (2021).
- [13] Y. A. Fofanov, I. M. Sokolov, R. Kaiser, and W. Guerin, Subradiance in dilute atomic ensembles: Role of pairs and multiple scattering, *Phys. Rev. A* **104**, 023705 (2021).
- [14] K. Hammerer, A. S. Sørensen, and E. S. Polzik, Quantum interface between light and atomic ensembles, *Rev. Mod. Phys.* **82**, 1041 (2010).
- [15] H. J. Kimble, The quantum internet, *Nature (London)* **453**, 1023 (2008).
- [16] L.-M. Duan, M. D. Lukin, J. I. Cirac, and P. Zoller, Long-distance quantum communication with atomic ensembles and linear optics, *Nature (London)* **414**, 413 (2001).
- [17] M. O. Scully, Single photon subradiance: Quantum control of spontaneous emission and ultrafast readout, *Phys. Rev. Lett.* **115**, 243602 (2015).
- [18] A. Svidzinsky and J.-T. Chang, Cooperative spontaneous emission as a many-body eigenvalue problem, *Phys. Rev. A* **77**, 043833 (2008).
- [19] A. A. Svidzinsky, J.-T. Chang, and M. O. Scully, Cooperative spontaneous emission of N atoms: Many-body eigenstates, the effect of virtual Lamb shift processes, and analogy with radiation of N classical oscillators, *Phys. Rev. A* **81**, 053821 (2010).
- [20] S. E. Skipetrov and A. Goetschy, Eigenvalue distributions of large Euclidean random matrices for waves in random media, *J. Phys. A* **44**, 065102 (2011).
- [21] E. Akkermans, A. Gero, and R. Kaiser, Photon localization and Dicke superradiance in atomic gases, *Phys. Rev. Lett.* **101**, 103602 (2008).
- [22] V. Viggiano, R. Bachelard, F. D. Cunden, P. Facchi, R. Kaiser, S. Pascazio, and F. V. Pepe, Cooperative photon emission rates in random atomic clouds, *Phys. Rev. A* **108**, 063701 (2023).
- [23] L. Bellando, A. Gero, E. Akkermans, and R. Kaiser, Roles of cooperative effects and disorder in photon localization: The case of a vector radiation field, *Eur. Phys. J. B* **94**, 49 (2021).

- [24] L. Bellando, A. Gero, E. Akkermans, and R. Kaiser, Cooperative effects and disorder: A scaling analysis of the spectrum of the effective atomic Hamiltonian, *Phys. Rev. A* **90**, 063822 (2014).
- [25] S. E. Skipetrov and I. M. Sokolov, Absence of Anderson localization of light in a random ensemble of point scatterers, *Phys. Rev. Lett.* **112**, 023905 (2014).
- [26] G. L. Celardo, M. Angeli, F. Mattiotti, and R. Kaiser, Localization of light in three dimensions: A mobility edge in the imaginary axis in non-Hermitian Hamiltonians, *Europhys. Lett.* **145**, 35002 (2024).
- [27] T. Bienaimé, M. Petruzzo, D. Bigerni, N. Piovella, and R. Kaiser, Atom and photon measurement in cooperative scattering by cold atoms, *J. Mod. Opt.* **58**, 1942 (2011).
- [28] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Photons and Atoms: Introduction to Quantum Electrodynamics* (Wiley-Interscience, New York, 1989).
- [29] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions: Basic Processes and Applications* (Wiley, New York, 1992).
- [30] Ph. W. Courteille, S. Bux, E. Lucioni, K. Lauber, T. Bienaimé, R. Kaiser, and N. Piovella, Modification of radiation pressure due to cooperative scattering of light, *Eur. Phys. J. D* **58**, 69 (2010).
- [31] M. Mézard, G. Parisi, and A. Zee, Spectra of Euclidean random matrices, *Nucl. Phys. B* **559**, 689 (1999).
- [32] E. Bogomolny, O. Bohigas, and C. Schmit, Spectral properties of distance matrices, *J. Phys. A* **36**, 3595 (2003).
- [33] A. Goetschy and S. E. Skipetrov, Euclidean random matrices and their applications in physics, [arXiv:1303.2880](https://arxiv.org/abs/1303.2880).
- [34] C. Bordenave, On Euclidean random matrices in high dimension, *Electron. Commun. Probab.* **18**, 1 (2013).
- [35] A. Asenjo-Garcia, M. Moreno-Cardoner, A. Albrecht, H. J. Kimble, and D. E. Chang, Exponential improvement in photon storage fidelities using subradiance and “selective radiance” in atomic arrays, *Phys. Rev. X* **7**, 031024 (2017).
- [36] M. O. Araújo, W. Guerin, and R. Kaiser, Decay dynamics in the coupled-dipole model, *J. Mod. Opt.* **65**, 1345 (2018).
- [37] P. Facchi, U. Marzolino, G. Parisi, S. Pascazio, and A. Scardicchio, Phase transitions of bipartite entanglement, *Phys. Rev. Lett.* **101**, 050502 (2008).
- [38] F. Pietracaprina, V. Ros, and A. Scardicchio, Forward approximation as a mean-field approximation for the Anderson and many-body localization transitions, *Phys. Rev. B* **93**, 054201 (2016).
- [39] C. R. Laumann, A. Pal, and A. Scardicchio, Many-body mobility edge in a mean-field quantum spin glass, *Phys. Rev. Lett.* **113**, 200405 (2014).
- [40] B. L. Altshuler, V. E. Kravtsov, A. Scardicchio, P. Sierant, C. Vanoni, Renormalization group for Anderson localization on high-dimensional lattices, [arXiv:2403.01974](https://arxiv.org/abs/2403.01974).
- [41] P. Sierant, M. Lewenstein, and A. Scardicchio, Universality in Anderson localization on random graphs with varying connectivity, *SciPost Phys.* **15**, 045 (2023).
- [42] C. Vanoni, B. L. Altshuler, V. E. Kravtsov, and A. Scardicchio, Renormalization group analysis of the Anderson model on random regular graphs, *Proc. Natl. Acad. Sci. USA* **121**, e2401955121 (2024).
- [43] F. D. Cunden, P. Facchi, M. Ligabò, and P. Vivo, Universality of the weak pushed-to-pulled transition in systems with repulsive interactions, *J. Phys. A* **51**, 35LT01 (2018).
- [44] F. D. Cunden, P. Facchi, M. Ligabò, and P. Vivo, Third-order phase transition: Random matrices and screened Coulomb gas with hard walls, *J. Stat. Phys.* **175**, 1262 (2019).
- [45] F. L. Metz and J. D. Silva, Spectral density of dense random networks and the breakdown of the Wigner semicircle law, *Phys. Rev. Res.* **2**, 043116 (2020).
- [46] J. D. Silva and F. L. Metz, Analytic solution of the resolvent equations for heterogeneous random graphs: Spectral and localization properties, *J. Phys. Complexity* **3**, 045012 (2022).
- [47] P. Weiss, A. Cipris, M. O. Araújo, R. Kaiser, and W. Guerin, Robustness of Dicke subradiance against thermal decoherence, *Phys. Rev. A* **100**, 033833 (2019).
- [48] A. S. Kuraptsev and I. M. Sokolov, Influence of atomic motion on the collective effects in dense and cold atomic ensembles, *Phys. Rev. A* **101**, 033602 (2020).
- [49] A. S. Kuraptsev and I. M. Sokolov, Motional effects in dynamics of fluorescence of cold atomic ensembles excited by resonance pulse radiation, *Phys. Rev. A* **108**, 032814 (2023).
- [50] P. Lassègues, M. A. Fernandes Biscassi, M. Morisse, A. Cidrim, P. G. Santos Dias, H. Eneriz, R. C. Teixeira, R. Kaiser, R. Bachelard, and M. Hugbart, Transition from classical to quantum loss of light coherence, *Phys. Rev. A* **108**, 042214 (2023).