

Quantum speed limits for open system dynamics based on a representation-basis-dependent ℓ_w^p -seminorm

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We report a family of quantum speed limits (QSLs) that give evolution time lower bounds between an initial and a final state whose separation is described by a certain representation-basis-dependent norm derived from the weighted ℓ_w^p -seminorm. These QSLs are applicable to open, closed, time-dependent, or time-independent systems in finite-dimensional Hilbert spaces whose density matrices are piecewise time differentiable. They can be extended to systems over separable Hilbert spaces as well. Crucially, these QSLs are valid for arbitrary operators, not just density matrices, provided that a modest technical condition is fulfilled. When compared to the existing QSLs applied to pure state time-independent Hamiltonian evolution, qubit spontaneous emission, high-fidelity gate implementation, coherent state photon loss, and operator coherence or dephasing, ours consistently show improved sharpness in most cases, along with greater universality and still retaining computational efficiency.

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I. INTRODUCTION

Quantum speed limit (QSL) studies the fundamental limits on the minimum evolution time between two states. Originated from the so-called time-energy uncertainty relation, it has developed from a theoretical curiosity inquiry to a useful tool for the understanding and engineering of quantum technologies. The first QSL was found by Mandelstam and Tamm in 1945. This so-called MT bound says that evolution time between two states under the action of a time-independent Hamiltonian is lower bounded by the reciprocal of the standard deviation of the system energy [1]. Interestingly, the MT bound can be generalized to general time-dependent open system dynamics. It reads as [2]

$$\tau \geq \tau_{\text{MT}} \equiv \frac{\hbar \Theta_{\text{Bures}}(\rho_\tau, \rho_0)}{\frac{1}{\tau} \int_0^\tau \Delta E_t dt}, \quad (1)$$

where $\Theta_{\text{Bures}}(\rho_\tau, \rho_0) = \cos^{-1}[\text{Tr}(\sqrt{\sqrt{\rho_0}\rho_\tau\sqrt{\rho_0}})]$ is the Bures angle between the initial state ρ_0 and final state ρ_τ , and ΔE_t is the standard deviation of the state energy at time t . Here the Bures angle is the only parameter used to describe the separation between the initial and final states. From now on, we simply refer to the time-dependent generalization bound τ_{MT} as the MT bound. In 1998, Margolus and Levitin discovered a complementary QSL for time-independent Hamiltonian evolution. This evolution time lower bound is proportional

to the reciprocal of the system's mean energy relative to the ground state [3].

QSL research has been progressing rapidly in the last two decades. Theoretical works have extended QSL from its original formulation in closed, time-independent systems to include more general cases. They include open quantum systems in which interaction with the environment is non-negligible [4–6] as well as time-varying driven systems [7]. Such generalizations are important to study the ultimate evolution time limits in realistic quantum devices used in quantum information processing.

An important research direction is to find more powerful and general QSL bounds. Some of them involve geometric ideas such as the Bures angle [8,9]. Others use information theory concepts like quantum Fisher information [9,10] and generalized Renyi entropy [11]. While many QSLs are expressed in terms of moments of the expectation values of energies of the evolving state over time, these are not the only possibilities. For instance, QSL bounds involving coherence [12], generalized Renyi relative entropy [11], and Kirkwood-Dirac quasiprobabilities [6] have been reported. Additionally, the relation between QSL and quantum correlations has been the focus of many studies, with results showing how entanglement and other nonclassical correlations can affect the speed of quantum evolution [13–15]. In parallel, experimental groups have achieved significant progress in the verification of QSLs in different physical platforms, such as superconducting circuits [16], NMR [17], and cavity electrodynamic systems [18]. These experiments confirm the validity of the fundamental bounds and enable their application for optimizing quantum protocols.

QSL has many important implications. For quantum computing, it gives a fundamental limit for the speed of quantum gates and algorithms [19]. For quantum metrology, it limits the precision of quantum sensors [10,20]. Furthermore, QSL is now considered a key concept in quantum thermodynamics,

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and gives insight into the efficiency and power of quantum engines [21]. As we continue to advance the frontiers of quantum science, a deep understanding of the QSL will be necessary to utilize the full potential of the quantum realm.

In this work, we introduce several QSLs applicable to arbitrary quantum dynamics, including open and time-dependent systems, and extend their applicability beyond density matrices to any operators in separable Hilbert space as long as certain technical conditions are fulfilled. Our main contributions are as follows:

(1) *Unified QSL framework.* We introduce a representation-basis-dependent weighted $\ell_w^{p,\max}$ -norm derived from the weighted ℓ_w^p -seminorm, and use it to obtain three closely related QSLs for finite-dimensional systems that are applicable for both open and closed dynamics as long as the density matrices are piecewise time differentiable. We further extend this formalism to countably infinite-dimensional systems. More importantly, our framework generalizes traditional-state-based QSLs to linear operators. Unlike most QSLs on market, ours are representation basis dependent in general. In fact, our QSLs are neither based on distinguishability measure nor monotone under completely positive trace-preserving (CPTP) maps.

(2) *Efficient and numerically stable computation.* For finite-dimensional systems, two of our QSLs can be computed efficiently in a numerically stable manner. Besides, our remaining QSL, which requires optimization over the representation basis, can be efficiently computed in many cases of practical interest. Nevertheless, just like a generic optimization problem, we do not see a general pattern of the optimal representation basis that gives us the best QSL. All these comments apply also to computing our QSLs for countably infinite-dimensional systems that can be uniformly approximated by discretization or truncation.

(3) *Demonstration of wide applicability.* We illustrate the wide applicability of our QSL-s by showcasing their use in the following five systems:

- (i) closed time-independent Hamiltonian evolution of certain two-dimensional qubit and four-dimensional qudit system that saturate one of our QSL bounds;
- (ii) spontaneous emission in a qubit, benchmarking against the Deffner-Lutz (DL) bound [4];
- (iii) high-fidelity gate operations in nitrogen-vacancy (NV) centers in diamond, comparing robustness against the MT bound [22];
- (iv) photon loss of a coherent state, demonstrating the applicability and superiority of our bound over the DL bound [4] for countably infinite-dimensional quantum systems; and
- (v) dephasing and coherence generation for general Hilbert space operators, illustrating the formalism's extensibility beyond density matrices [23].

Just like most QSLs on market (Ref. [24] is a notable exception), ours are based on a single parameter describing the separation between the initial and final states. We defer the study of QSL in the multiple parameters situation to our subsequent work.

We begin by stating our motivation and intuition behind before introducing our QSLs and their proofs in Sec. II. Then we illustrate the use of these QSLs using

five examples covering time-independent, time-dependent, finite-dimensional, and countably infinite-dimensional system evolution as well as general Hilbert space operator QSLs in Sec. III. Finally, we summarize our findings and point out a few meaningful future research directions in Sec. IV.

II. QUANTUM SPEED LIMITS BASED ON A REPRESENTATION-DEPENDENT-WEIGHTED ℓ_w^p -SEMINORM

A. Main results and proofs

We wanted to find efficiently and accurately computable QSLs that are more powerful than existing ones for a wide range of finite-dimensional systems including time-dependent and open ones. Besides, it would be nice if these QSLs can be generalized to infinite-dimensional systems. To achieve these goals, we need a density matrix distance measure or equivalently a norm that make use of as much information on the density matrices of the system as possible. Based on this demand, we introduce the following norm for finite-dimensional systems. And we are going to discuss its generalization to countably infinite-dimensional systems later in this section. Since a number of symbols and notations are needed to better articulate our formulation and analysis, we summarize them in Table I for easy reference.

Let $\rho_\tau \equiv \rho(\tau)$ be the density matrix of a n -dimensional quantum state as a function of time variable τ . In a fixed orthonormal basis $\mathcal{B}$ representation, its matrix elements can be written as a n^2 -dimensional vector through the following column-stacking vectorization [25]:

$$\text{vec}_{\mathcal{B}}[\rho(\tau)] = (\rho(\tau)_{11}, \rho(\tau)_{12}, \dots, \rho(\tau)_{nn}) \in \mathbb{C}^{n^2}. \quad (2)$$

We further denote time-dependent difference vector by

$$\Delta_{\mathcal{B}}(\tau) = \text{vec}_{\mathcal{B}}(\rho_\tau) - \text{vec}_{\mathcal{B}}(\rho_0). \quad (3)$$

(We shall omit the basis subscript $\mathcal{B}$ in vec and Δ if it is either understood from the context or unimportant.) Lastly, for any fixed $p \geq 1$ and nonzero real vector $w \in \mathbb{R}^{n^2}$ whose elements are non-negative, we equip the set of all n^2 -dimensional vectors (and hence also the set of all density matrices) with the $\ell_w^{p,\max}$ -norm, or more precisely the $\ell_w^{p,\max}([n^2])$ -norm with $[n^2]$ denotes the set $\{0, 1, 2, \dots, n^2 - 1\}$, where

$$\begin{aligned} \mathcal{D}_{p,w,\mathcal{B}}^{\max}(\rho_\tau, \rho_0) &= \|\Delta_{\mathcal{B}}(\tau)\|_{p,w}^{\max} \\ &\equiv \max_{P \in S_{n^2}} \left[\sum_{j=1}^{n^2} w_{P(j)} |\Delta_{\mathcal{B},j}(\tau)|^p \right]^{1/p} \quad (4a) \\ &= \left\{ \sum_{j=1}^{n^2} w_j^\downarrow [|\Delta_{\mathcal{B}}(\tau)|_j^\downarrow]^p \right\}^{1/p}. \quad (4b) \end{aligned}$$

Here S_{n^2} is the symmetric group on n^2 elements and for any real vector v , $v_j^\downarrow$ refers to the j th resultant element by arranging elements of v in descending order. Besides, Corollary II.4.4 in Ref. [26] is used to obtain Eq. (4b) from Eq. (4a). Since $\ell_w^p([n^2])$ is a seminorm for all w , our construction in Eq. (4a) automatically makes $\|\cdot\|_{p,w}^{\max}$ a seminorm. But actually, $\|\cdot\|_{p,w}^{\max}$ is a norm in case some of the elements in w are

TABLE I. List of frequently used notations.

Notation	Meaning
$\rho_t, \rho(t)$	Density matrix at time t
$\mathbb{L}$	Lindblad superoperator
p, q	Hölder conjugate pair
w	Nonzero weight vector
$\bar{w}$	Reordered version of w defined in Eq. (5)
$\mathcal{B}$	Orthonormal basis
$\mathcal{B}_c, \mathcal{B}_d, \mathcal{B}_F, \mathcal{B}_E$	Computational, diagonal, Fock bases, and energy eigenbasis
$v_j^\downarrow$	The j th element of a vector v arranged in descending order
$s_j^\downarrow(A)$	The j th singular value of A arranged in descending order
$\mathbb{1}$	The vector whose elements are all 1
$\mathbb{1}_j$	The vector those first j elements are 1 and the rest are 0
$S_{n^2}, S_{\mathbb{N}}$	Symmetric groups of $[n^2]$ and $\mathbb{N}$
$\text{vec}(\cdot), \text{vec}_{\mathcal{B}}(\cdot)$	Column-stacking vectorization defined in Eq. (2)
$\Delta(\tau), \Delta_{\mathcal{B}}(\tau)$	Time-dependent difference vector defined in Eq. (3)
$\mathcal{D}_{p,w,\mathcal{B}}^{\max}(\cdot, \cdot)$	Metric defined in Eqs. (4) and (9)
$\ \cdot\ _{p,w}^{\max}$	$\ell_w^{p,\max}([n^2])$ - or $\ell_w^{p,\max}(\mathbb{N})$ -norm defined in Eqs. (4) and (9)
$\mathcal{D}_{p,\bar{w},\mathcal{B}}(\cdot, \cdot)$	Pseudometric defined in Eq. (5)
$\ 1\ _{p,\bar{w}}$	Seminorm defined in Eq. (5)
$\tau_{\text{CZ}}, \tau_{\text{DL}}, \tau_{\text{MT}}$	CZ, DL, and MT QSL bounds
$\tau^{\text{int}}, \tau_{p,w^\downarrow,\mathcal{B}}^{\text{int}}$	Basic integral form QSL in Eq. (6a)
$\tau^{\text{sup}}, \tau_{p,w^\downarrow,\mathcal{B}}^{\text{sup}}$	Basic supremum form QSL in Eq. (6b)
$\tau_{\text{opt}}^{\text{int}}$	Optimized form QSL

0. The reason is that if $\|\Delta_{\mathcal{B}}(\tau)\|_{p,w}^{\max} = 0$, Eq. (4b) and the fact that w is a nonzero vector whose elements are non-negative imply that $w_1^\downarrow > 0$ and $|\Delta_{\mathcal{B}}(\tau)|_1^\downarrow = 0$. Thus, $|\Delta_{\mathcal{B}}(\tau)|_1^\downarrow = 0$ for all j . Consequently, $\Delta_{\mathcal{B}}(\tau) = 0$. Last but not least, by continuity, we define $\mathcal{D}_{\infty,w,\mathcal{B}}^{\max}(\rho_t, \rho_0)$ as $\|\Delta_{\mathcal{B}}(\tau)\|_{\infty,w}^{\max} = |\Delta_{\mathcal{B}}(\tau)|_1^\downarrow$. Readers may refer to the two examples in Secs. III A and III B on explicit computation of $\mathcal{D}_{p,w,\mathcal{B}}^{\max}(\cdot, \cdot)$ in two simple cases.

Our intuition is that the norm property of $\mathcal{D}_{p,w,\mathcal{B}}^{\max}$ demands $\mathcal{D}_{p,w,\mathcal{B}}^{\max}(\rho_\tau, \rho_0) \leq \int_0^\tau |\frac{d\mathcal{D}_{p,w,\mathcal{B}}^{\max}}{dt}| dt$. So, an upper bound for the integrand gives a QSL for the system. The smaller the bound, the more powerful the QSL. Nevertheless, this method is tricky in actual numerical calculation because the optimal $P \in S_{n^2}$ that maximizes the above integrand is time dependent. Accurate computation requires precise determination of the time when the optimal P changes. Besides, theoretical analysis of the convergence of this integral is nontrivial. Hence, we consider a variation of the theme. But, before further discussion, we need to introduce the following notation.

Let P be the permutation in S_{n^2} that maximizes Eq. (4a). We denote $\bar{w}$ the reordered version of w given by $\bar{w}_j = w_{P(j)}$ for all j . Using this notation,

$$\begin{aligned} \mathcal{D}_{p,w,\mathcal{B}}^{\max}(\rho_\tau, \rho_0) &= \left[\sum_{j=1}^{n^2} \bar{w}_j |\Delta_{\mathcal{B},j}(\tau)|^p \right]^{1/p} \\ &\equiv \mathcal{D}_{p,\bar{w},\mathcal{B}}(\rho_\tau, \rho_0) \equiv \|\Delta_{\mathcal{B}}(\tau)\|_{p,\bar{w}}. \end{aligned} \quad (5)$$

Surely, this permuted weight vector $\bar{w}$ depends on $p, w, \mathcal{B}$ and may vary with $\Delta_{\mathcal{B}}(\tau)$. Besides, it is not unique in general.

Nonetheless, Eq. (5) holds irrespective of the choice of P (and hence $\bar{w}$).

Using this notation, the seminorm property of $\ell_w^p([n^2])$ plus Eq. (4b), we have $\mathcal{D}_{p,w,\mathcal{B}}^{\max}(\rho_\tau, \rho_0) \leq \int_0^\tau |\frac{d\mathcal{D}_{p,\bar{w},\mathcal{B}}}{dt}| dt$. Now the weight vector $\bar{w}$ used throughout the integral over time is fixed. Surely, the smaller the upper bound for this integrand, the better the resultant QSL. This is the core concept used in our QSLs stated in the Theorem 1 below.

Theorem 1. Let ρ_t be the density matrix of a finite-dimensional quantum system at time t . Suppose that the evolution of ρ_t is governed by the Lindblad superoperator $\mathbb{L}$. Suppose further that ρ_t is differentiable for all $t \in [0, \tau]$ except possibly at finitely many times $t_1 < t_2 < \dots < t_{k-1}$. Then for any $p \in [1, \infty]$, the evolution time $\tau \geq 0$ obeys

$$\tau \geq \tau_{p,w^\downarrow,\mathcal{B}}^{\text{int}} \equiv \frac{\mathcal{D}_{p,w,\mathcal{B}}^{\max}(\rho_\tau, \rho_0)}{\frac{1}{\tau} \int_0^\tau \|\text{vec}_{\mathcal{B}}(\mathbb{L}\rho_t)\|_{p,\bar{w}} dt} \quad (6a)$$

$$\geq \tau_{p,w^\downarrow,\mathcal{B}}^{\text{sup}} \equiv \frac{\mathcal{D}_{p,w,\mathcal{B}}^{\max}(\rho_\tau, \rho_0)}{\max_{j=0}^{k-1} \sup_{t \in (t_j, t_{j+1})} \|\text{vec}_{\mathcal{B}}(\mathbb{L}\rho_t)\|_{p,\bar{w}}}, \quad (6b)$$

where $t_0 = 0$ and $t_k = \tau$ if $\mathcal{D}_{p,w,\mathcal{B}}^{\max}(\rho_\tau, \rho_0) > 0$.

Proof. Note that $\tau > 0$ for $\mathcal{D}_{p,w,\mathcal{B}}^{\max}(\rho_\tau, \rho_0) > 0$. Thus, $\mathcal{D}_{p,w,\mathcal{B}}^{\max}(\rho_t, \rho_0)$ and $\mathcal{D}_{p,\bar{w},\mathcal{B}}(\rho_t, \rho_0)$ can be regarded as functions of $t \in [0, \tau]$. As ρ_t satisfies the Lindblad equation $\dot{\rho}_t = \mathbb{L}\rho_t$ and ρ_t can be regarded as a finite-dimensional linear operator, $\mathcal{D}_{p,w,\mathcal{B}}^{\max}(\rho_t, \rho_0)$ is finite for all $t \in [0, \tau]$. Moreover, it is differentiable at $t \in J \equiv \bigcup_{j=0}^{k-1} (t_j, t_{j+1})$ by the differentiability

assumption of ρ_t . Clearly,

$$\begin{aligned} \frac{d\mathcal{D}_{p,w,B}^{\max}}{dt} &= \frac{d\mathcal{D}_{p,\bar{w},B}}{dt} = \mathcal{D}_{p,\bar{w},B}^{1-p} \sum_{j=1}^{n^2} \bar{w}_j |\Delta_{B,j}|^{p-1} \\ &\quad \times \operatorname{sgn}(\Delta_{B,j}) \frac{d \operatorname{vec}_B(\rho_t)_j}{dt}. \end{aligned} \quad (7)$$

Let $p > 1$ and $1/p + 1/q = 1$. By applying Hölder inequality to $\bar{w}_j^{1/p} \frac{d}{dt} \operatorname{vec}_B(\rho_t)_j$ and $\bar{w}_j^{1/q} |\Delta_{B,j}|^{p-1} \operatorname{sgn}(\Delta_{B,j})$ and followed by multiplying both sides of the resultant inequality by $\mathcal{D}_{p,\bar{w},B}^{1-p}$, we obtain

$$\left| \frac{d\mathcal{D}_{p,w,B}^{\max}}{dt} \right| \leq \left\| \frac{d \operatorname{vec}_B(\rho_t)}{dt} \right\|_{p,\bar{w}} = \|\operatorname{vec}_B(\mathbb{L}\rho_t)\|_{p,\bar{w}} \quad (8)$$

for all $t \in J$. Here, $\|\cdot\|_{p,\bar{w}}$ is the $\ell_{\bar{w}}^p([n^2])$ -seminorm. By taking limits on p , it is clear that this inequality holds also for the case of $p = 1$ or ∞ .

Observe that $\mathbb{L}$ is a bounded superoperator acting on finite-dimensional density matrices. Therefore, $\sup_{t \in J} \|\operatorname{vec}_B(\mathbb{L}\rho_t)\|_{p,\bar{w}}$ is well defined and finite. In addition, Lebesgue dominated convergence theorem guarantees that $\int_0^\tau \|\operatorname{vec}_B(\mathbb{L}\rho_t)\|_{p,\bar{w}} dt$ exists. Furthermore, these integral and supremum must be positive for otherwise $\mathcal{D}_{p,w,B}^{\max}(\rho_\tau, \rho_0) = 0$. Consequently, inequalities (6a) and (6b) are obtained by integrating Eq. (8) over time. This completes our proof. ■

Inequalities (6a) and (6b) are our basic QSLs. We call $\tau_{p,w^\downarrow,B}^{\text{int}}$ or simply τ^{int} the basic integral form QSL. And we call $\tau_{p,w^\downarrow,B}^{\text{sup}}$ or simply τ^{sup} the basic supremum form QSL. Several remarks are in place.

(1) Theorem 1 is still valid for the trivial case of $\mathcal{D}_{p,w,B}^{\max}(\rho_\tau, \rho_0) = 0$ if we interpret $\tau_{p,w^\downarrow,B}^{\text{int}} = \tau_{p,w^\downarrow,B}^{\text{sup}} = 0$ with no regard to the values of the denominators in the right-hand side of inequalities (6a) and (6b).

(2) Our QSLs are derived by bounding the norm $\mathcal{D}_{p,w,B}^{\max}(\rho_\tau, \rho_0)$ through integrating over the rate of change of the auxiliary seminorm $\|\operatorname{vec}_B(\mathbb{L}\rho_t)\|_{p,\bar{w}}$. That is to say, our derivation makes no use of the norm property of $\mathcal{D}_{p,w,B}^{\max}$. So, our QSLs are valid for any $\mathcal{D}_{p,w,B}$, too. However, these extended QSLs are generally less sharp. That is why they are not studied here.

(3) Changing basis in ρ_t corresponds to a change of B in $\mathcal{D}_{p,w,B}^{\max}(\rho_\tau, \rho_0)$. Since both $\mathcal{D}_{p,w,B}^{\max}$ and $\|\operatorname{vec}_B(\mathbb{L}\rho_t)\|_{p,\bar{w}}$ depend on B , our basic QSLs in inequality (6) are not unitarily invariant in general. In fact, this is what we observe in all the applications to be reported in Sec. III. (But, see Sec. II B 1 for an important exception.) Consequently, though having similar forms, our basic QSLs are different from those of the MT bound [2] as well as the ℓ^p - or L^p -norms-based ones reported in Refs. [21,27,28] for theirs are unitarily invariant. More comparison between these QSLs will be given in Sec. II B 1 below.

(4) Since unitary transformation is a special case of a CPTP map, a direct and interesting consequence of the basis-dependent property of the metric $\mathcal{D}_{p,w,B}^{\max}$ is that this metric is not a monotone under the CPTP map. (But again, please refer to Sec. II B 1 for an important exception.) As a consequence, unlike the MT bound, our metric $\mathcal{D}_{p,w,B}^{\max}$ together with our QSLs are in general not based on distin-

guishability measure of quantum states. Here we remark that nondistinguishability-based metric is not uncommon in quantum information science. The Frobenius norm is a famous example.

(5) $\mathcal{D}_{p,w,B}^{\max}(\rho_t, \rho_0)$ and $\|\operatorname{vec}_B(\mathbb{L}\rho_t)\|_{p,\bar{w}}$ are observables for any time t . Determining them may require state tomography or positive operator-valued measure (POVM) (on multiple and identical copies of the state of the system) at all times $t \in [0, \tau]$. As ρ_t traces out a path in the column-stacking vectorization $\operatorname{vec}_B[\rho(t)]$, $\tau_{p,w^\downarrow,B}^{\text{int}}$ and $\tau_{p,w^\downarrow,B}^{\text{sup}}$ originate from the average and supremum rate of change of this path, respectively. Combined with Remark 3 above, our two nonunitary-invariant QSLs offer rooms for optimization that are not present in all unitary invariant QSLs.

(6) Our QSLs are unchanged when w is replaced by λw for any $\lambda > 0$. Thus, without loss of generality, we may fix $w_1^\downarrow = 1$. In addition, if all but one of the components of w are 0 (that is to say, $w_j^\downarrow = 0$ for all $j > 1$), then Eq. (4b) implies that our QSLs do not depend on p .

(7) For given p, w, B , and in the event that the corresponding $\bar{w}$ is not unique, then surely we pick the one that maximizes either inequality (6a) or (6b). Interestingly, for all the cases with nonunique $\bar{w}$ we are going to study in Sec. III, the values of τ^{int} and τ^{sup} do not depend on the choice of $\bar{w}$. It is instructive to find out why in the future.

(8) Although τ^{int} generally gives a better lower bound than τ^{sup} , finding it requires precisely known ρ_t that may not be available (due to uncertainty in the initial state or the evolution model, for instance) as well as integration which can be computationally taxing sometimes. In contrast, calculating τ^{sup} or its lower bound is often simpler, more robust, and computationally less demanding. As we shall illustrate in Secs. III B and III C, τ^{sup} occasionally gives good QSL bounds.

Surely, we may further optimize these basic QSLs over p, w as well as the basis B used to construct ρ_t and ρ_0 . It turns out that optimizing τ^{int} is more useful; and we denote this optimized form QSL by $\tau_{\text{opt}}^{\text{int}}$ and refer to it as the (fully) optimized form. If $\|\operatorname{vec}_B(\mathbb{L}\rho_t)\|_{p,\bar{w}}$ is sufficiently piecewise smooth, then optimization over p can be done efficiently at least for finding local maxima. Likewise, optimization over the basis in constructing $\Delta_B(\tau)$ and $\operatorname{vec}_B(\mathbb{L}\rho_t)$ is a variational optimization problem for smooth function on a compact Lie group that can be done efficiently at least for local solutions [29–31]. Finally, efficient optimization over w can be found in Sec. II B 2. To summarize, local maxima for optimizing $\tau_{p,w,B}^{\text{int}}$ can be found efficiently. In practice, we find that the optimized form QSL $\tau_{\text{opt}}^{\text{int}}$ can be found efficiently almost all the time. In contrast, we do not know of any analytical way to find τ^{int} . In fact, readers are going to see in Sec. III that there is no obvious pattern for the basis B needed in the simpler problem of maximizing $\tau_{p,w^\downarrow,B}^{\text{int}}(\rho_\tau, \rho_0)$ or $\tau_{p,w^\downarrow,B}^{\text{sup}}(\rho_\tau, \rho_0)$ given p and $w^\downarrow$.

Along a different line, we point out that our derivation does not make use of density matrix property. In fact, our QSLs are still valid if ρ_t is substituted by a linear operator A , even a non-Hermitian one as long as the matrix elements of the operator are continuous for all $t \in [0, \tau]$, differentiable for all $t \in J$, and $\|\operatorname{vec}_B(\mathbb{L}A)\|_{p,\bar{w}}$ is bounded or more generally dominated by a Lebesgue integrable function over time interval $[0, \tau]$.

Last but not least, our QSL bounds can be extended to states evolving in countably infinite-dimensional Hilbert spaces. In this case, for a fixed basis $\mathcal{B}$, we may vectorize any linear operator in basically the same way as in Eq. (2) by writing, for instance, the column-stacking vector for the density matrix σ_τ of a countably infinite-dimensional Hilbert space as $\text{vec}_{\mathcal{B}}(\sigma_\tau) = (\text{vec}_{\mathcal{B}}(\sigma_\tau)_{j+1})_{j \in \mathbb{N}}$. [It may be more apparent to use $\mathbb{N}^2$ as the index set for j so as to emphasize the action of column-stacking matrix elements of σ_τ . However, we choose to use $\mathbb{N}$ which has the same cardinality of $\mathbb{N}^2$ instead because there is a natural ordering of $\mathbb{N}$ so that we could talk about $w_{j+1}^\downarrow$ for any $j \in \mathbb{N}$ without ambiguity in the same way as in Eq. (4b). Here we use $j+1$ instead of j as the index so that the notation $w_{j+1}^\downarrow$ coincides with the standard usage in matrix analysis.] Then for any nonzero vector $w = (w_{j+1})_{j \in \mathbb{N}}$ with $w_{j+1} \geq 0$, we may extend the definition in Eq. (4) to

$$\begin{aligned} \mathcal{D}_{p,w,\mathcal{B}}^{\max}(\rho_\tau, \rho_0) &= \|\Delta_{\mathcal{B}}(\tau)\|_{p,w}^{\max} \\ &\equiv \max_{P \in \mathbb{S}_{\mathbb{N}}} \left[\sum_{j \in \mathbb{N}} w_{P(j)+1} |\Delta_{\mathcal{B},j+1}(\tau)|^p \right]^{1/p} \quad (9a) \\ &= \left\{ \sum_{j \in \mathbb{N}} w_{j+1}^\downarrow [|\Delta_{\mathcal{B}}(\tau)|_{j+1}^\downarrow]^p \right\}^{1/p} \quad (9b) \end{aligned}$$

whose validity in notation and as a norm will be justified in Appendix A. Using this extended definition and denoting the element in $\mathbb{S}_{\mathbb{N}}$ that maximizes Eq. (9a) by $\bar{w}$, we state our QSLs for countably infinite-dimensional system below whose proof can be found in Appendix B.

Theorem 2. Theorem 1 holds for countably infinite-dimensional quantum systems provided that the following conditions are satisfied:

- (1) $\mathcal{D}_{p,w,\mathcal{B}}^{\max}(\rho_t, \rho_0)$ is finite for $t \in [0, \tau]$.
- (2) $\sum_{j=0}^{\ell} \frac{df_j(t)}{dt}$ converges uniformly in each of the time intervals (t_m, t_{m+1}) as $\ell \rightarrow \infty$ for $m = 0, 1, \dots, k$, where $f_j(t) = w_{j+1}^\downarrow [|\Delta_{\mathcal{B}}(t)|_{j+1}^\downarrow]^p$.
- (3) For all $t \in J$, $\|[\bar{w}_{j+1}^{1/p} \frac{d}{dt} \text{vec}_{\mathcal{B}}(\rho_t)_{j+1}]_{j \in \mathbb{N}}\|_{p,1}$ and $\|[\bar{w}_{j+1}^{1/q} |\Delta_{\mathcal{B},j+1}(t)|^{p-1} \text{sgn}(\Delta_{\mathcal{B},j+1}(t))]_{j \in \mathbb{N}}\|_{q,1}$ are finite where 1 denotes the vector whose elements are all 1.
- (4) $\|\text{vec}_{\mathcal{B}}(\mathbb{L}\rho_t)\|_{p,\bar{w}} \leq g(t)$ for some $g(t) \in L^1[0, \tau]$.

Here, $\tau_{p,w^\downarrow,\mathcal{B}}^{\sup}$ is regarded as 0 if the denominator in the right-hand side of Eq. (6b) is infinite.

B. Properties of the basic bounds

1. Sufficient condition for the bounds to be representation basis independent

As pointed out in Remark 3, $\tau_{p,w^\downarrow,\mathcal{B}}^{\text{int}}$ and $\tau_{p,w^\downarrow,\mathcal{B}}^{\sup}$ are representation basis dependent in general. Nevertheless, there is a notable exception. Observe that any linear operator A acting on $\mathcal{H}^n$ must satisfy $\sum_{j,k} |A_{jk}|^2 = \sum_{j,k} \langle j|A|k\rangle \langle k|A^\dagger|j\rangle = \text{Tr}AA^\dagger = \sum_j s_j^\downarrow(A)^2$, where $s_j^\downarrow(A)$ denotes that j th singular

value of A arranged in descending order. As a result,

$$\|\text{vec}_{\mathcal{B}}(\mathbb{L}\rho_t)\|_{2,\mathbb{1}_{n^2}} = \left[\sum_j s_j^\downarrow(\mathbb{L}\rho_t)^2 \right]^{1/2}, \quad (10)$$

where $\mathbb{1}_j$ is the vector of length n^2 whose first j components are 1 and the remaining $(n^2 - j)$ components are 0. Consequently, our basic QSLs in inequality (6) for $p = 2$ and $w = \mathbb{1}_{n^2}$ are independent of the basis used to construct ρ_t and ρ_0 . In fact, $\|\cdot\|_{2,\mathbb{1}_{n^2}}$ equals the Frobenius norm $\|\cdot\|_2$. In addition, for any sufficiently small time interval $[t, t + \Delta t]$, $\mathbb{L}\rho_t = -i[\sum_j E_j(t) |E_j(t)\rangle \langle E_j(t)|, \rho_t]/\hbar + O(\Delta t^2)$ where $E_j(t)$ and $|E_j(t)\rangle$ are the eigenenergy and the corresponding eigenstate of the system at time t . Hence,

$$\tau_{2,\mathbb{1}_{n^2}}^{\text{int}} = \frac{\|\rho_t - \rho_0\|_2}{\frac{1}{\tau} \int_0^\tau \|\mathbb{L}\rho_t\|_2 dt} = \frac{\hbar \|\rho_t - \rho_0\|_2}{\frac{\sqrt{2}}{\tau} \int_0^\tau \Delta E_t dt} \quad (11a)$$

and

$$\tau_{2,\mathbb{1}_{n^2}}^{\sup} = \frac{\|\rho_t - \rho_0\|_2}{\sup_{t \in [0,\tau]} \|\mathbb{L}\rho_t\|_2} = \frac{\hbar \|\rho_t - \rho_0\|_2}{\sqrt{2} \sup_{t \in [0,\tau]} \Delta E_t}. \quad (11b)$$

In this regard, our basic integral form QSL and the MT bound can be expressed in almost the same form. The difference is that our numerator depends on the Frobenius distance between the initial and final states while theirs depends on the Bures angle. This conclusion is consistent with our observation in Remark 3. Likewise, we conclude by extension that although our QSLs appear to have the same form as in those in Refs. [27,28], they are of very different nature. This is because in the time-independent case, $\|\text{vec}_{\mathcal{B}}(\mathbb{L}\rho_t)\|_{p,\mathbb{1}_{n^2}}$ cannot be expressed in terms of $\langle |E|^p \rangle$. Last but not least, our basic integral form QSL almost equals that of the operator-norm-based bound by Deffner in Ref. [21] for the case of $p = 2$ and $w = \mathbb{1}_{n^2}$ even though Deffner's result focused on infinite-dimensional systems. Here the difference is that Deffner expressed his QSL via fidelity between the initial and final states while ours do not. In addition, our bounds are completely different from his for other values of p or w .

The Ky Fan's maximum principle (see Exercise II.1.13 in Ref. [26]) implies that $s_1^\downarrow(A)^2 = \max_{|j\rangle} \langle j|AA^\dagger|j\rangle = \|\text{vec}(AA^\dagger)\|_{p,\mathbb{1}_1}^{\max}$, where the maximum is over all normalized state $|j\rangle$. So, maybe $\tau_{p,\mathbb{1}_1,\mathcal{B}}^{\text{int}}$ and $\tau_{p,\mathbb{1}_1,\mathcal{B}}^{\sup}$ were representation basis independent. However, these are not true because the states $|j\rangle$ that maximize $\mathcal{D}_{p,w,\mathcal{B}}^{\max}(\rho_\tau, \rho_0)$ and $\|\text{vec}_{\mathcal{B}}(\mathbb{L}\rho_t)\|_{p,\bar{w}}$ need not be the same. In fact, our numerical study shows that the only representation basis independent $\tau_{p,w}^{\text{int}}$ and $\tau_{p,w}^{\sup}$ are the ones using $p = 2$ and $w = \mathbb{1}_{n^2}$.

2. Efficient optimization over w

Interestingly, for a fixed t and a fixed representation basis in computing ρ_t together with ρ_0 , the maximization of our QSLs over w requires only n^2 evaluations of the right-hand side of inequality (6). We only need to consider the case of $p \in [1, \infty)$. By taking the limit $p \rightarrow \infty$, our analysis also holds for the case of $p = \infty$. Since our basic bounds in inequality (6) do not depend on the ordering of elements in w , without loss of generality, we only need to consider the case when elements of w are non-negative and arranged in

descending order. Let $w = (w_i)$ and $w' = (w'_i)$ be two such sequences. We study the behavior of τ^{int} and τ^{sup} when w is replaced by $\lambda w + \bar{\lambda} w'$, where $\lambda \in [0, 1]$ and $\bar{\lambda} = 1 - \lambda$. The right-hand side of both Eqs. (6a) and (6b) can be written in the form $f(\lambda) = (\frac{\lambda N + \bar{\lambda} N'}{\lambda D + \bar{\lambda} D'})^{1/p}$ with $D, D' > 0$ and $N, N' \geq 0$. By elementary algebra, f always attains its maximum value on the domain boundary of λ , namely, 0 or 1. That is to say, the maxima of Eqs. (6a) and (6b) must be attained by using $w = \mathbb{1}_j$ for $j = 1, 2, \dots, n^2$. This is because only when w is in this form that it cannot be decomposed as a weighted sum of two linearly independent vectors with non-negative elements arranged in descending order. Consequently, to maximize our basic QSLs for a fixed representation basis as well as p , we only need to evaluate the right-hand side of Eqs. (6a) and (6b) with n^2 different weights w . This result is significant because, while the introduction of weightings improves the sharpness of the QSL, it keeps the computational efficiency of our approach.

III. APPLICATIONS

The following five examples show the power and applicability of our QSLs across a range of quantum systems. Every curve in the figures below contains about 200 data points. Using the Manopt package in MATLAB [32], each such set of data points except for those concerning coherent state photon loss in Sec. III D, takes less than 10 min of runtime in a typical laptop. The coherent state photon loss simulation takes about 10 h, most of the time is used to search the global maximum solution. Our MATLAB codes are available through the link in Ref. [33].

A. Time-independent Hamiltonian evolution of pure states

We consider time-independent pure state evolution as a warmup.

1. Qubit case

Consider the evolution of the initial state $|\psi_0\rangle = (|E_0\rangle + |E_1\rangle)/\sqrt{2}$ under the time-independent Hamiltonian

$$H = \sum_j j |E_j\rangle \langle E_j|, \quad (12)$$

where $\{|E_j\rangle\}$ is an orthonormal set of energy eigenstates. Clearly, ΔE of this system is $\frac{1}{2}$. Moreover, it is well known that the MT bound τ_{MT} is tight for this state when the evolution time τ lies in $[0, \pi\hbar]$.

Fix a $\tau > 0$ and consider the basis $\mathcal{B}_d$ that diagonalizes $\Delta(\tau)$. Then, $\rho_\tau = \frac{1}{2}(\begin{smallmatrix} 1 & e^{-i\tau/\hbar} \\ e^{i\tau/\hbar} & 1 \end{smallmatrix})$, $\Delta_{\mathcal{B}_d}(\tau) = \text{vec}_{\mathcal{B}_d}(\rho_\tau - \rho_0) = (0 \quad e^{-i\tau/\hbar} - 1 \quad e^{i\tau/\hbar} - 1 \quad 0)^T/2$. As a result, $\mathcal{D}_{1, \mathbb{1}_1, \mathcal{B}_d}^{\text{max}}(\rho_\tau, \rho_0) = s_1^\downarrow[\Delta_{\mathcal{B}_d}(\tau)] = \max(0, |\frac{e^{-i\tau/\hbar} - 1}{2}|, |\frac{e^{i\tau/\hbar} - 1}{2}|, 0) = |\sin(\tau/2\hbar)|$ and the corresponding $\|\text{vec}_{\mathcal{B}_d}[\dot{\rho}(t)]\|_{1, \mathbb{1}_1} = |\cos[(t - \tau/2)/\hbar]/2\hbar| \leq 1/2\hbar$. Hence, $\int_0^\tau \|\text{vec}_{\mathcal{B}_d}[\dot{\rho}(t)]\|_{1, \mathbb{1}_1} dt = \mathcal{D}_{1, \mathbb{1}_1, \mathcal{B}_d}^{\text{max}}(\rho_\tau, \rho_0)$ whenever $\tau \in [0, \pi\hbar]$. In other words, from inequalities (6a) and (6b), $\tau_{1, \mathbb{1}_1, \mathcal{B}_d}^{\text{sup}} = 2\hbar \sin(\tau/2\hbar)$ and $\tau_{1, \mathbb{1}_1, \mathcal{B}_d}^{\text{int}} = \tau$ whenever $\tau \in [0, \pi\hbar]$. So, our basic integral form QSL is also tight for this system when $\tau \in [0, \pi\hbar]$. In contrast, if we pick $\mathcal{B}$ to be the computational basis $\mathcal{B}_c$, by repeating the

same calculation, we obtain $\mathcal{D}_{1, \mathbb{1}_1, \mathcal{B}_c}^{\text{max}}(\rho_\tau, \rho_0) = |\sin(\tau/2\hbar)|$ and $\|\text{vec}_{\mathcal{B}_c}[\dot{\rho}(t)]\|_{1, \mathbb{1}_1} = 1/2\hbar$ for all t . Therefore, both $\tau_{1, \mathbb{1}_1, \mathcal{B}_c}^{\text{sup}}$ and $\tau_{1, \mathbb{1}_1, \mathcal{B}_c}^{\text{int}}$ equal $2\hbar |\sin(\tau/2\hbar)|$, which are not tight whenever $\tau > 0$. This shows the basis dependence of our QSLs.

Surprisingly, in this simple example, for a randomly picked representation basis $\mathcal{B}$, there is at least a quarter chance that $\tau_{1, \mathbb{1}_1, \mathcal{B}}^{\text{int}}$ is tight. This is because $\mathcal{B}$ is obtained by applying the unitary transformation in the form $U = \begin{pmatrix} e^{i\phi} \cos \theta & e^{i\phi} \sin \theta \\ -e^{-i\phi} \sin \theta & e^{-i\phi} \cos \theta \end{pmatrix}$ to $\mathcal{B}_d$ for some $\theta, \phi, \varphi \in \mathbb{R}$. After some simple calculations, we obtain $\mathcal{D}_{1, \mathbb{1}_1, \mathcal{B}}^{\text{max}}(\rho_\tau, \rho_0) = |\cos 2\theta \sin(\tau/2)|$ if $|\cos 2\theta| \geq |\sin 2\theta|$ and $\|\text{vec}_{\mathcal{B}}[\dot{\rho}(t)]\|_{1, \mathbb{1}_1} = |\cos 2\theta \cos[(t - \tau/2)/\hbar] - \sin 2\theta \sin(\phi - \varphi) \sin[(t - \tau/2)/\hbar]|/2\hbar$. Thus, for randomly chosen U , there is at least a quarter chance that $\int_0^\tau \|\text{vec}_{\mathcal{B}}[\dot{\rho}(t)]\|_{1, \mathbb{1}_1} dt = |\cos 2\theta| \sin(\tau/2)$ if $\tau \in [0, \pi\hbar]$. And in this case, both the $|\cos 2\theta|$ factors in the numerator and denominator of $\tau_{1, \mathbb{1}_1, \mathcal{B}}^{\text{int}}$ cancel each other, giving us the value of τ in the end.

We now increase τ to $4\pi\hbar/3 (> \pi\hbar)$. We find numerically that $\tau > \tau_{\text{opt}}^{\text{int}} = 3.20\hbar > \tau_{\text{MT}} = 2\pi\hbar/3 = \tau_{\text{CZ}}$. Here, τ_{CZ} is the optimized Chau-Zeng (CZ) bound which is the best QSL for closed system under time-independent evolution known to date [34, 35]. Our optimized bound is attained when $w = \mathbb{1}_1$ (and hence for any $p \geq 1$). Remarkably, just like the case of $\tau \in [0, \pi\hbar]$, numerous different representation bases can attain our optimized QSL so that our optimized form QSL can be found simply by Monte Carlo method with ≈ 100 sampling bases. We shall say more about this phenomenon in our upcoming applications.

2. Four-dimensional qudit case

Since all known QSLs for a time-independent Hamiltonian in the literature can only be saturated in qubit or qutrit systems [34], we further show the power of our optimized form QSL when applied to the following randomly chosen four-dimensional qudit system with no obvious symmetry or structure. Let us evolve the initial state $|\psi_0\rangle = \sqrt{0.2}|E_0\rangle + \sqrt{0.4}|E_1\rangle + \sqrt{0.3}|E_\pi\rangle + \sqrt{0.1}|E_4\rangle$ under the H in Eq. (12). At time τ , the fidelity between the initial and final states equals $|0.2 + 0.4e^{-i\tau/\hbar} + 0.3e^{-i\pi\tau/\hbar} + 0.1e^{-4i\tau/\hbar}|$. Using this fidelity at $\tau = \tau_c = 3.43\hbar$ and by means of the publicly available program in Ref. [35], we find that the optimized CZ bound [34] for this system is $0.87\hbar$. And by setting $\mathcal{B}$ to be the energy eigenbasis, $\|\text{vec}(\mathbb{L}\rho_t)\|_{p, \bar{w}}$ is time independent for all p and $\bar{w}$. Moreover, our calculation shows that the best basic integral form QSL $\tau^{\text{int}} = \tau^{\text{sup}} = 1.98\hbar$ occurs when $w = \mathbb{1}_1$ or $(\mathbb{1}_2$ and $p \geq 1)$. Obviously, this result is not impressive.

What surprises us is that $\tau_{\text{opt}}^{\text{int}}$ agrees with τ up to at least seven significant figures for any $\tau \in [0, \tau_c]$. This strongly suggests that our QSL $\tau_{\text{opt}}^{\text{int}}$ is saturated for this system up to the evolution time τ_c . Since the CZ bound is the most powerful bound for time-independent Hamiltonian evolution known up to this work [34], our pure state qubit and qudit time-independent Hamilton evolution examples demonstrate the power of our fully optimized form QSL. We find that generally the optimal sets of parameters appear to be degenerate and quite common in a lot of cases, especially when τ is small. In fact, the Monte Carlo method alone is sometimes sufficient to find $\tau_{\text{opt}}^{\text{int}}$ or its approximation, lowering the actual computa-

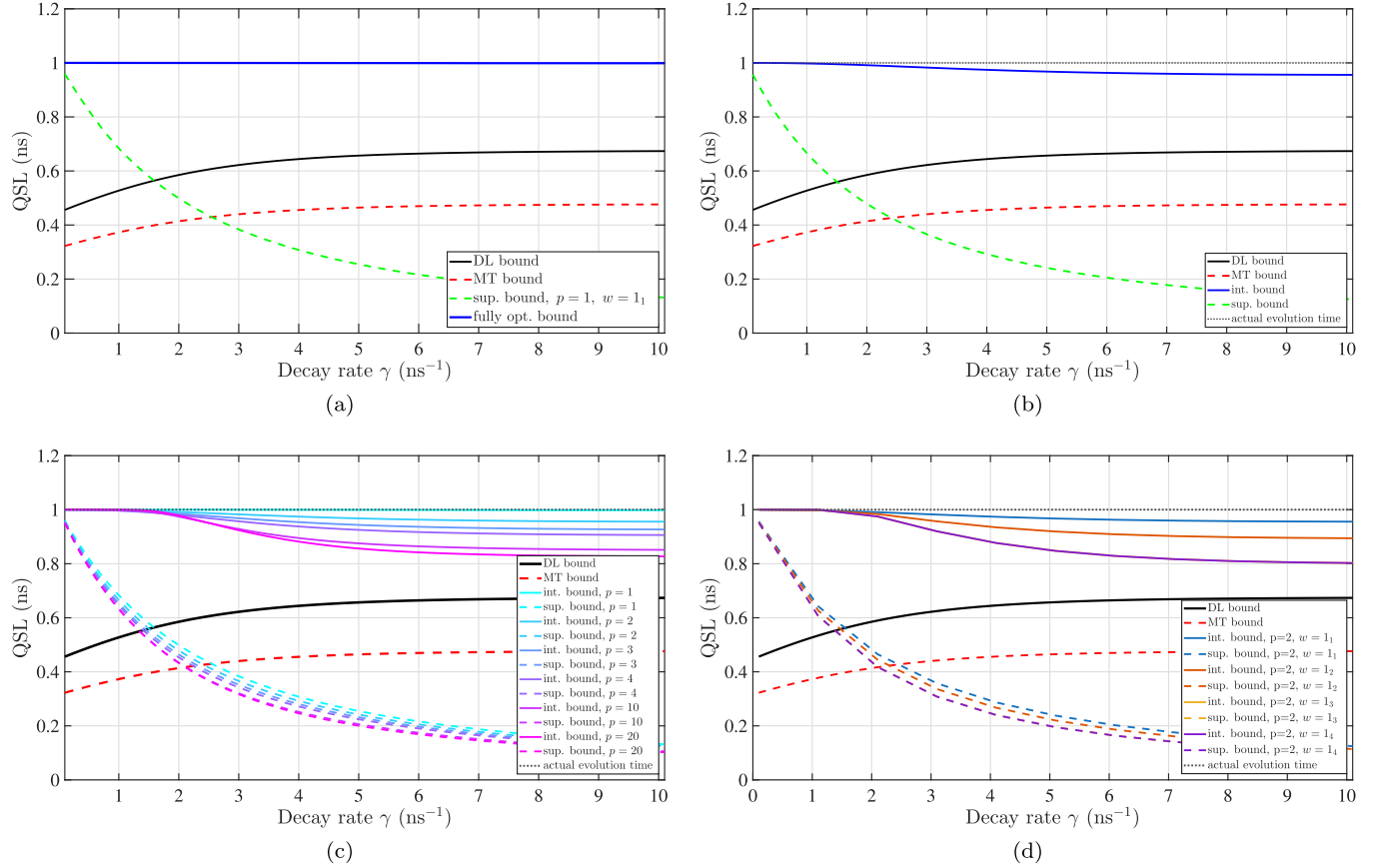


FIG. 1. Various QSLs for a qubit undergoing spontaneous emission against the decay rate γ when $\tau = 1$. (a) Compares $\tau_{1,1_1,B_E}^{\sup}$, $\tau_{1,1_1,B_E}^{\text{int}}$ with the MT and DL bounds. Note that $\tau_{1,1_1,B_E}^{\text{int}}$ equals the actual evolution time τ . (b) Compares $\tau_{2,1_4,B_E}^{\sup}$ and $\tau_{2,1_4,B_E}^{\text{int}}$ with the MT and DL bounds. (c), (d) Vary p and w while keeping the other parameters fixed to those used in (b), respectively.

tional complexity cost. These observations are generic as they are found in all open and closed systems we have studied. We further remark that for the qudit system we have studied here, it is possible to find $\tau > \tau_c$ such that $\tau_{\text{opt}}^{\text{int}} = \tau$. Moreover, the set of τ 's for this to occur appears to be a fractal. This deserves further investigations in the future.

B. A qubit undergoing spontaneous emission

Consider a qubit initialized to the state $(|0\rangle + |1\rangle)/\sqrt{2}$. We study its spontaneous emission to the ground state $|0\rangle$ under the dynamics governed by the Lindblad equation in the interaction picture [4]

$$\frac{d\rho_t}{dt} = \mathbb{L}(\rho_t) = \gamma \left(|0\rangle \langle 1| \rho_t |1\rangle \langle 0| - \frac{1}{2} \{ |1\rangle \langle 1|, \rho_t \} \right), \quad (13)$$

where γ is the emission rate. Surely, the density matrix of the qubit at time t written in the energy eigenbasis is

$$\rho_t = \frac{1}{2} \begin{pmatrix} 2 - e^{-\gamma t} & e^{-\gamma t/2} \\ e^{-\gamma t/2} & e^{-\gamma t} \end{pmatrix} \quad (14)$$

and the corresponding vectorized state equals $\text{vec}(\rho_t) = (2 - e^{-\gamma t} \quad e^{-\gamma t/2} \quad e^{-\gamma t/2} \quad e^{-\gamma t})^T / 2$.

We compare the performance of our QSLs with the one reported by Deffner and Lutz [4], namely,

$$\tau_{\text{DL}} = \max \left(\frac{1}{\Lambda_{\tau}^{\text{op}}}, \frac{1}{\Lambda_{\tau}^{\text{tr}}}, \frac{1}{\Lambda_{\tau}^{\text{hs}}} \right) \sin^2 \Theta_{\text{Bures}}(\rho_{\tau}, \rho_0), \quad (15)$$

where $\Lambda_{\tau}^{\text{op/tr/hs}} = \frac{1}{\tau} \int_0^{\tau} \|\mathbb{L}(\rho_t)\|_{\text{op/tr/hs}} dt$ with op, tr, and hr denoting the operator, trace, and Hilbert-Schmidt norms, respectively. As $\|\mathbb{L}(\rho_t)\|_{\text{tr}} = 2\|\mathbb{L}(\rho_t)\|_{\text{op}}$, Eq. (15) equals $\tau_{\text{DL}} = \max(1/\Lambda_{\tau}^{\text{op}}, 1/\Lambda_{\tau}^{\text{hs}}) \sin^2 \Theta_{\text{Bures}}(\rho_{\tau}, \rho_0)$. It turns out that the operator norm gives the tightest bound for this system [4]; and we refer to it as the DL bound. Moreover, the bound derived from the Hilbert-Schmidt norm serves as a generalization of the MT bound for open systems. Here we simply call it the MT bound.

As for our QSLs, we first consider the case of $p = 1$, $w = 1_1$, and $\mathcal{B} = \mathcal{B}_E \equiv \{|0\rangle, |1\rangle\}$. For any $\tau \geq 0$, it is clear from Eq. (14) that $\mathcal{D}_{1,1_1,B_E}^{\max}(\rho_{\tau}, \rho_0) = (1 - e^{-\gamma\tau})/2$ and the corresponding $\|\text{vec}_{\mathcal{B}_E}(\dot{\rho}_t)\|_{1,1_1} = \gamma e^{-\gamma t}/2 \leq \gamma/2$. Hence, $\tau_{1,1_1,B_E}^{\sup} = (1 - e^{-\gamma\tau})/\gamma$ and $\tau_{1,1_1,B_E}^{\text{int}} = \tau$. In other words, our basic integral form QSL is tight for all $\tau \geq 0$. Actually, similar calculations give $\tau_{2,1_1,B_E}^{\sup} = \tau_{1,1_1,B_E}^{\sup}$ and $\tau_{2,1_1,B_E}^{\text{int}} = \tau_{1,1_1,B_E}^{\text{int}}$ so that $\tau_{\text{opt}}^{\text{int}}$ may sometimes be attained using more than one set of parameters.

To study the performance of our basic supremum form QSL, we follow Ref. [4] by fixing the evolution time $\tau = 1$.

As shown in Fig. 1(a), $\tau_{1,\mathbb{1},\mathcal{B}_E}^{\text{sup}}$ already beats both the MT and DL bounds when $\gamma < 1.6 \text{ ns}^{-1}$ and $\gamma < 2.5 \text{ ns}^{-1}$, respectively. This shows the effectiveness of our supremum form QSL as pointed out in Remark 8.

To further investigate the dependence of our QSLs on p , w , and $\mathcal{B}$, we consider the case of $\tau = 1$, $p = 2$, $w = \mathbb{1}_4$, and $\mathcal{B} = \mathcal{B}_E$. As shown in Fig. 1(b), the basic integral form QSL $\tau_{2,\mathbb{1}_4,\mathcal{B}_E}^{\text{int}}$ always wins whereas the basic supremum form QSL $\tau_{2,\mathbb{1}_4,\mathcal{B}_E}^{\text{sup}}$ beats the DL and MT bounds when $\gamma < 1.5 \text{ ns}^{-1}$ and $\gamma < 2.4 \text{ ns}^{-1}$, respectively. Figure 1(c) shows the effect of p by fixing $p = 2$ and $\mathcal{B} = \mathcal{B}_E$ and Fig. 1(d) depicts the effect of varying w while fixing the other two parameters. Both figures highlight the effectiveness of our bounds over wide parameter ranges.

To summarize, a significant advantage of our proposed bounds is that, even in the absence of full parameter optimization, the baseline formulation provides a remarkably tight constraint on the quantum evolution time. This is particularly valuable in scenarios where accurate system evolution is impossible and/or an exhaustive optimization search is either computationally intractable or experimentally unfeasible. Our results demonstrate that even for suboptimal parameter choices, the bound remains close to the saturation (actual evolution time), affirming its practical utility.

C. High-fidelity gates using an NV center In diamond

Next, we consider the dynamics of spin 1 of a NV center in diamond, a more complicated system that serves as a promising platform for quantum information processing. We analyze the time required to perform a high-fidelity quantum gate and compare our QSLs with the MT bound obtained in Ref. [22].

The spin operators $\hat{S}_x$, $\hat{S}_y$, and $\hat{S}_z$ in the basis $\{|m_s = +1\rangle, |m_s = 0\rangle, |m_s = -1\rangle\}$ are

$$\begin{aligned} \hat{S}_x &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \text{and} \\ \hat{S}_z &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \end{aligned} \quad (16)$$

And the time-dependent Hamiltonian governing the spin dynamics is given by $\hat{H}(t) = \hat{H}_0 + \hat{H}_c(t)$, where $\hat{H}_0$ is the static Hamiltonian

$$\hat{H}_0 = \hbar(D\hat{S}_z^2 + \gamma_e B_0 \hat{S}_z) \quad (17)$$

with D being the zero-field splitting parameter and γ_e is the electron gyromagnetic ratio. Besides, the control Hamiltonian $\hat{H}_c(t)$ representing the interaction with a time-dependent magnetic field component B_1 applied perpendicular to the NV axis is given by

$$\hat{H}_c(t) = \hbar\gamma_e B_1 [f_x(t)\hat{S}_x + f_y(t)\hat{S}_y], \quad (18)$$

with the control field direction switching at half the total evolution time τ :

$$(f_x(t), f_y(t)) = \begin{cases} (1, 0) & \text{if } t < \tau/2, \\ (0, 1) & \text{if } t \geq \tau/2. \end{cases} \quad (19)$$

The system is initialized in the state $|\psi_0\rangle = |m_s = +1\rangle$. Here we set $D = 2\pi \times 2.87 \text{ GHz}$, $\gamma_e = 2\pi \times 28.0345 \text{ GHz/T}$, and $B_0 = 0.05 \text{ T}$ in our computation.

Figure 2(a) shows that $\tau_{\text{opt}}^{\text{int}}$ wins the MT bound for all values of B_0/B_1 , often by a wide margin. Moreover, the fully optimized p , w , and $\mathcal{B}$ depend on B_0/B_1 in a complicated way without obvious patterns or trends. Figures 2(b)–(2d) show the dependence of p , w , and $\mathcal{B}$ by fixing the other parameters to $p = 2$, $w = \mathbb{1}_9$, and $\mathcal{B}$ equals the computational basis $\mathcal{B}_c$, respectively. They tell us that optimizing either p , w , and $\mathcal{B}$ alone is already quite effective in getting a good bound. Further note that all bounds exhibit rather huge fluctuations over the magnetic field ratio B_0/B_1 . Nevertheless, our QSLs, particularly for larger p , tend to show smaller variance. A small-variance QSL is experimentally advantageous, as it provides a more robust estimate of the minimal time required for high-fidelity gate operations, especially in the presence of minor parameter drifts [22]. This analysis demonstrates that by optimizing over both p and w , our QSLs can offer a sharpness and less fluctuating bound than the existing method for complex, time-dependent quantum systems. Finally, we remark that while our supremum form bounds are generally less effective than the MT bound, they are close and very occasionally beat it.

D. Photon loss in an infinite-dimensional bosonic mode

Here we demonstrate the extensibility of our framework to countably infinite-dimensional systems by considering a single bosonic mode undergoing photon loss. This is a paradigmatic open quantum system model describing dissipation in optical or microwave cavities. The dynamics is governed by the Lindblad master equation [36]

$$\frac{d\rho_t}{dt} = \mathbb{L}\rho_t = \kappa \left(a\rho_t a^\dagger - \frac{1}{2} \{a^\dagger a, \rho_t\} \right), \quad (20)$$

where a is the annihilation operator and κ is the photon decay rate. The Hilbert space is spanned by the Fock states $\mathcal{B}_F \equiv \{|n\rangle\}_{n=0}^\infty$. We initialize the system to the coherent state $\rho_0 = |\alpha_0\rangle\langle\alpha_0|$ for some fixed $\alpha_0 \in \mathbb{C}$. Under the dynamics of Eq. (20), the state remains pure and coherent, evolving as $\rho_t = |\alpha(t)\rangle\langle\alpha(t)|$ with amplitude $\alpha(t) = \alpha_0 e^{-\kappa t/2}$.

Interestingly, our basic QSLs for $p = 2$ and $w = \mathbb{1}$, namely, the uniform weight with all elements in the vector $\mathbb{1}$ being 1, can be calculated analytically. Note that the analysis in Sec. II B 1 is valid for infinite-dimensional systems so that $\tau_{2,\mathbb{1},\mathcal{B}}^{\text{int}}$ and $\tau_{2,\mathbb{1},\mathcal{B}}^{\text{sup}}$ are basis independent. In fact, analogous to Eq. (10), we know that

$$\begin{aligned} \|\text{vec}(\rho_\tau - \rho_0)\|_{2,\mathbb{1}} &= \sqrt{\sum_j s_j^\dagger(\rho_\tau - \rho_0)} = \sqrt{\text{Tr}[(\rho_\tau - \rho_0)(\rho_\tau - \rho_0)^\dagger]} \\ &= \sqrt{2 - 2|\langle\alpha(0)|\alpha(\tau)\rangle|^2} \\ &= \sqrt{2 - 2\exp[-|\alpha_0|^2(1 - e^{-\kappa\tau/2})^2]}. \end{aligned} \quad (21)$$

To calculate $\|\text{vec}(\mathbb{L}\rho_t)\|_{2,\mathbb{1}}$, we observe that $|\dot{\alpha}(t)\rangle = \kappa[|\alpha(t)|^2 - \alpha(t)a^\dagger|\alpha(t)\rangle]/2$ and hence $\langle\dot{\alpha}(t)|\alpha(t)\rangle = 0$.

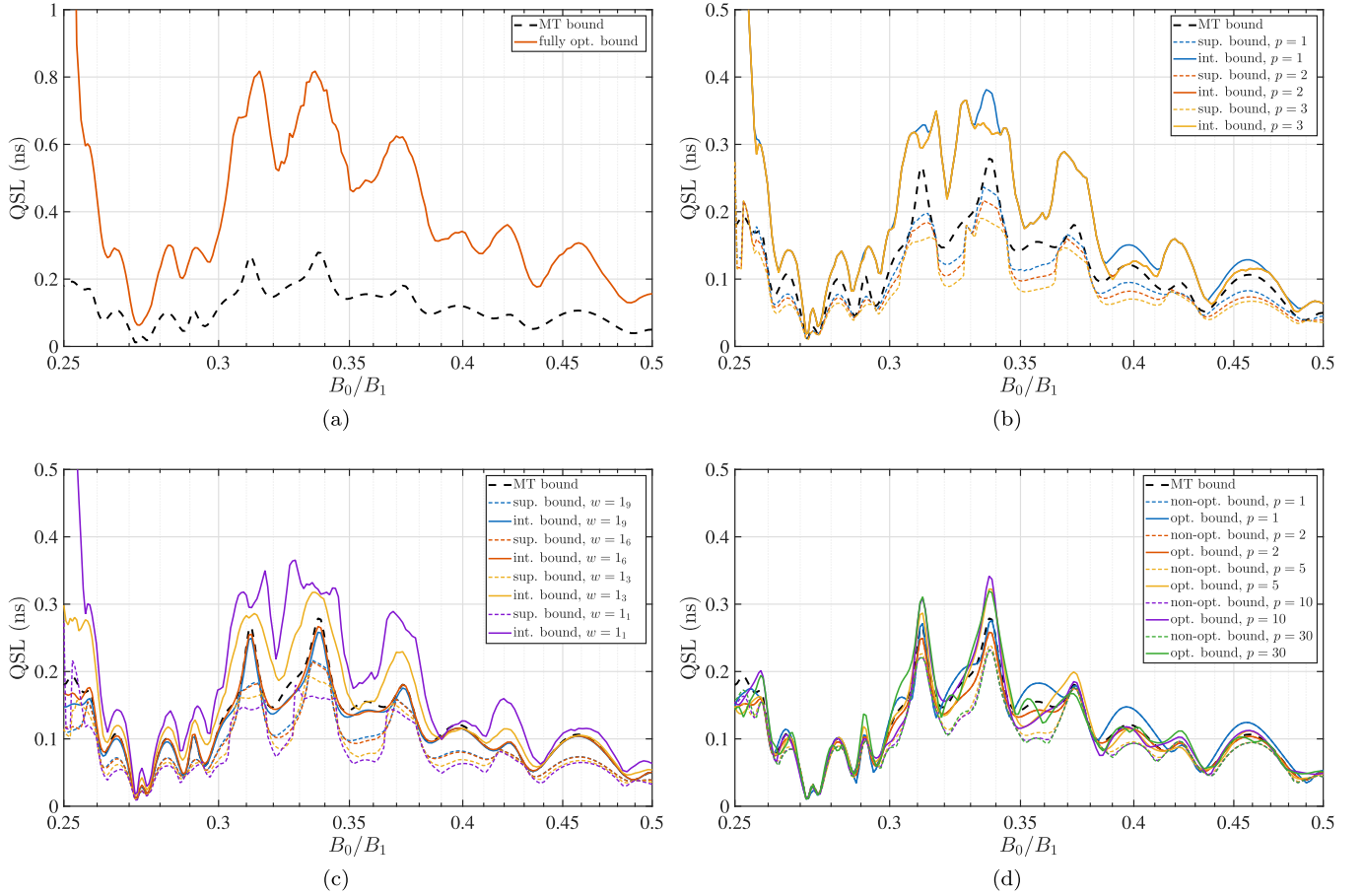


FIG. 2. Various QSLs for the gate operation defined in Sec. III C on an NV-center spin against the magnetic field ratio B_0/B_1 . For simplicity, we set $\hbar = 1$ in all plots. (a) Shows the fully optimized $\tau_{\text{opt}}^{\text{int}}$ bound. It uses a wider y-axis scale from the rest for better illustration. For (b), (c), and (d), we vary the parameters p , w , and $\mathcal{B}$ by fixing the remaining parameters to $p = 2$, $w = \mathbb{1}_9$, and $\mathcal{B}$ as the computational basis $\mathcal{B}_c$, respectively.

Therefore,

$$\begin{aligned} \|\text{vec}(\mathbb{L}\rho_t)\|_{2,1} &= \sqrt{\text{Tr}(\dot{\rho}_t \dot{\rho}_t^\dagger)} = \sqrt{2 \langle \dot{\alpha}(t) | \dot{\alpha}(t) \rangle} \\ &= \frac{\kappa |\alpha(t)|}{\sqrt{2}} = \frac{\kappa |\alpha_0| e^{-\kappa t/2}}{\sqrt{2}} \leq \frac{\kappa |\alpha_0|}{\sqrt{2}} \end{aligned} \quad (22)$$

for all $t \geq 0$.

We now proceed via Theorem 2. By means of the facts that $\langle m | \mathbb{L}\rho_t | n \rangle$ is smooth and decays exponentially for large m, n , it is straightforward to check that conditions in Theorem 2 hold for any $p \geq 1$, $w = \mathbb{1}$ and any basis $\mathcal{B}$. This result allows us to compute our QSLs with ease. In particular, we get our two analytical QSLs

$$\tau_{2,1}^{\text{int}} \equiv \tau_{2,1,\mathcal{B}}^{\text{int}} = \frac{\tau \sqrt{1 - \exp[-|\alpha_0|^2(1 - e^{-\kappa\tau/2})^2]}}{|\alpha_0|(1 - e^{-\kappa\tau/2})} \quad (23)$$

and

$$\tau_{2,1}^{\text{sup}} \equiv \tau_{2,1,\mathcal{B}}^{\text{sup}} = \frac{2\sqrt{1 - \exp[-|\alpha_0|^2(1 - e^{-\kappa\tau/2})^2]}}{\kappa |\alpha_0|} \quad (24)$$

for any basis $\mathcal{B}$. Whereas for a general p and $\mathcal{B}$, our two QSLs for $w = \mathbb{1}$ can be numerically computed.

Figure 3 compares our bounds with the DL bound [4] for different decay rates κ by fixing $\tau = 1$ and $\alpha_0 = 2$. The thin solid lines in Fig. 3(a) are the analytical QSLs stated in Eqs. (23) and (24), while the circles and squares denote the numerical values obtained by computing the corresponding QSLs in the truncated Fock orthonormal set $\{|n\rangle\}_{n=0}^{19}$. We expect our numerical computational results to be a good approximation of the analytical result because the eigenenergies of this system are discrete and the states concerned are well approximated in this truncated Hilbert space [37]. Indeed, the excellent agreement between these two verifies that our discrete approximation is numerically stable and converges rapidly to the true infinite-dimensional system. Because of this, all the numerical QSLs plotted in Fig. 3 are computed in this truncated Fock orthonormal set. Consistent with Eq. (6), the integral forms $\tau_{2,1}^{\text{int}}$ and $\tau_{3,1,\mathcal{B}_F}^{\text{int}}$ are always better than the corresponding supremum forms $\tau_{2,1}^{\text{sup}}$ and $\tau_{3,1,\mathcal{B}_F}^{\text{sup}}$. Besides, $\tau_{3,1,\mathcal{B}_F}^{\text{sup}} \geq \tau_{2,1}^{\text{sup}}$ and $\tau_{3,1,\mathcal{B}_F}^{\text{int}} \geq \tau_{2,1}^{\text{int}}$. More importantly, these two basic integral form QSL bounds consistently beat the DL bound over the entire range of decay rates investigated. This highlights the power of our framework. Furthermore, the thick solid lines in Fig. 3(b) are the QSL obtained by optimizing over the basis $\mathcal{B}$ after fixing $p = 3$ and $w = \mathbb{1}$. They show

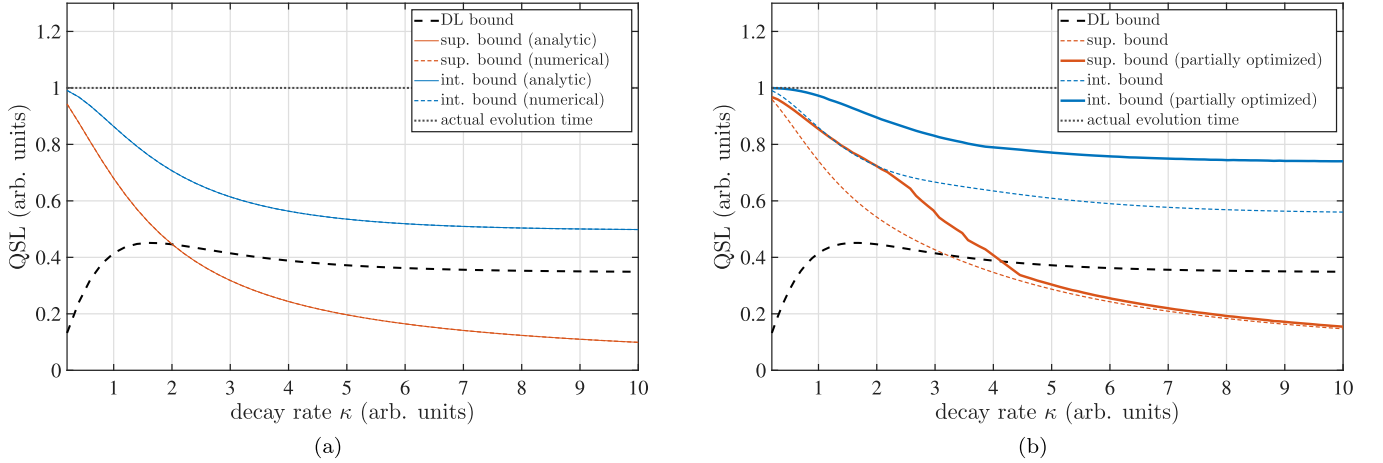


FIG. 3. Various QSLs for a bosonic mode undergoing photon loss plotted against the decay rate κ , with fixed evolution time $\tau = 1$ and initial amplitude $\alpha_0 = 2$. In (a), which uses $p = 2$ and $w = 1$, thin solid lines are the analytical results while markers are numerical ones computed using the truncated Fock orthonormal set $\{|n\rangle\}_{n=0}^{19}$. In (b), which uses $p = 3$ and $w = 1$, all results are numerically computed in the same truncated Fock orthonormal set. The thin dashed and thick solid lines are for the basic bounds and the partially optimized ones defined in the text, respectively.

that optimizing over the basis alone yields significantly tighter bounds at the expense of a much longer runtime due to the huge number of local maxima encountered in the optimization process. This confirms that our QSLs extend naturally to countably infinite-dimensional open systems.

E. Quantumness measures

We now apply our QSLs to study the noncommutativity of operators. Noncommutativity, quantified by the off-diagonal elements of one observable within the eigenbasis of another, is a measure of quantumness and a fundamental driver of state evolution in quantum dynamics [23,38]. Here we use dephasing and coherence generation processes to demonstrate the versatility of our framework. Specifically, we adopt the models from Ref. [23] which derived separate QSLs for these two processes. In contrast, our unified QSL is applicable to both.

1. Dephasing process

First, we consider the dephasing process where the observable is initialized to $\mathcal{A}_0 = \sigma_y$ and evolved under the Hamiltonian $H = \sigma_x$. The time-evolved observable is given by

$$\mathcal{A}_t = e^{-\gamma t/2} \begin{bmatrix} -\sin 2t & i(\frac{\gamma}{4} \sin 2t - \cos 2t) \\ -i(\frac{\gamma}{4} \sin 2t - \cos 2t) & \sin 2t \end{bmatrix}, \quad (25)$$

and the corresponding value of “quantumness” between the observables $\mathcal{A}_t$ and $\mathcal{A}_0$ equals [23]

$$Q(\mathcal{A}_0, \mathcal{A}_t) \equiv 2\|\mathcal{A}_0, \mathcal{A}_t\|^2 = 16e^{-\gamma t} \sin^2 2t. \quad (26)$$

For this specific process, the specialized QSL

$$T \geq T_Q \equiv \frac{\sqrt{Q(\mathcal{A}_0, \mathcal{A}_T)}}{\sqrt{2\langle\|\mathcal{A}_0, \mathcal{L}^\dagger(\mathcal{A}_T)\|\rangle_{\text{hs}}}_T} \quad (27)$$

has been derived in Ref. [23], where $\langle \cdot \rangle_T$ denotes the average of its argument over the time interval $[0, T]$.

2. Coherence generation process

Second, we analyze coherence generation in a system initialized in the state $\rho_0 = |0\rangle\langle 0|$, which evolves via the Hamiltonian $H = \sigma_x$. The dynamics of the density matrix follows:

$$\frac{d\rho_t}{dt} = -i[H, \rho_t] + \frac{\gamma}{2}(\sigma_z \rho_t \sigma_z - \rho_t). \quad (28)$$

At time t , the state is

$$\rho_t = \frac{1}{2} + e^{-\gamma t/2} \times \begin{bmatrix} \frac{1}{2} \cos 2t + \frac{\gamma}{8} \sin 2t & \frac{i}{2} \sin 2t \\ -\frac{i}{2} \sin 2t & -(\frac{1}{2} \cos 2t + \frac{\gamma}{8} \sin 2t) \end{bmatrix}. \quad (29)$$

The coherence, measured with respect to the observable $\mathcal{A} = \sigma_z$, is found to be

$$C(\rho_t, \mathcal{A}) \equiv -\frac{1}{2} \sum_k \text{Tr}\{\sqrt{\rho_t}, |k\rangle\langle k|\}^2 = e^{-\gamma t/2} |\sin 2t|, \quad (30)$$

where the sum is over any orthonormal basis $\{|k\rangle\}$ of the system. The specialized bound

$$T \geq T_C \equiv \frac{\sqrt{2}|\sqrt{C(\rho_0, \mathcal{A})} - \sqrt{C(\rho_T, \mathcal{A})}|}{\langle \sqrt{\sum_k \|\partial_t \sqrt{\rho_t} |k\rangle\langle k|\|^2} \rangle_T} \quad (31)$$

has been found in Ref. [23].

We find that in both the dephasing and coherence generation processes, $\tau_{\text{opt}}^{\text{int}}$ is attained at $p = 2$ and $w = 1$. (Therefore, they are representation basis independent.) As shown in Figs. 4(a) and 5, our bounds are tight when $t < 0.35$ in both dephasing and coherence generation. Moreover, they are better than the specialized bounds of Ref. [23] for longer

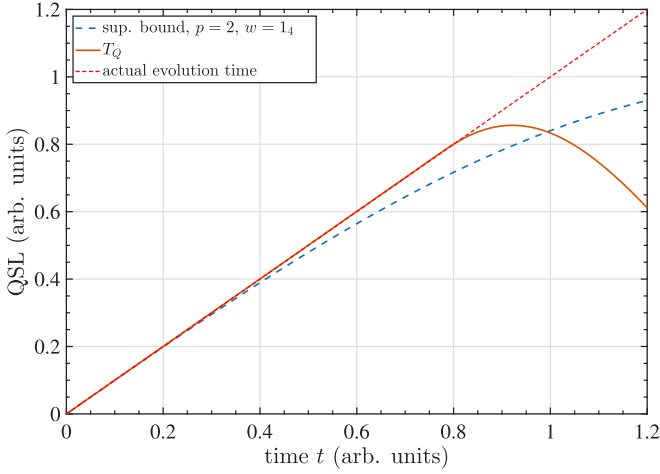


FIG. 4. $\tau_{\text{opt}}^{\text{sup}}$ and T_Q for the qubit coherence generation.

evolution times ($t > 1.0$ for dephasing and $t > 1.1$ for coherence generation). Although the specialized bounds are tighter for very short times, our QSLs are more general, perform better at large t , do not decrease at large t , and, more significantly, highlight the robustness of our framework in generalizing to operator dynamics beyond density matrices, a feature critical for addressing diverse quantum phenomena.

IV. DISCUSSIONS AND OUTLOOK

In summary, we have introduced a family of easy-to-use nonunitary-invariant $\ell_w^{p,\max}$ -norm-based QSLs that offers the benefits of universality and computational simplicity. Unlike most of the QSLs on market, ours are basis representation dependent and are not monotone under the action of CPTP maps. Furthermore, just like a typical optimization problem, we do not find any obvious pattern on the choice of representation basis as well as the values of p and w in the optimized bound $\tau_{\text{opt}}^{\text{int}}$. Demonstrating wide applicability, the family encompasses both closed and open quantum systems, under time-dependent and time-independent Hamiltonians. We have also extended the scope of the QSL from density matrices to

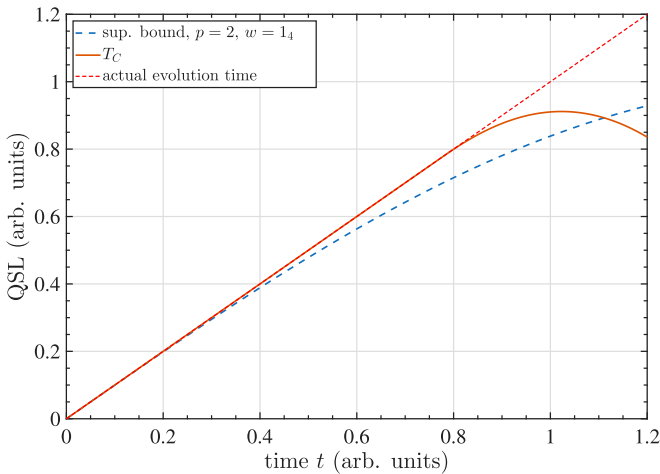


FIG. 5. $\tau_{\text{opt}}^{\text{sup}}$ and T_C for the qubit coherence generation.

general quantum operators. As illustrated by the five provided examples, our family of QSLs is powerful enough to give the best bounds known to date in many cases, particularly so when the evolution time τ is relatively short. In fact, the optimized form $\tau_{\text{opt}}^{\text{int}}$ is often tight when τ is small. Besides, even the less model-dependent and computationally less intensive basic supremum form τ^{sup} can be quite powerful in a lot of cases. We believe that the use of the representation-basis-dependent $\|\text{vec}_{\mathcal{B}}(\mathbb{L}\rho_t)\|_{p,\bar{w}}$ and its time integral are our keys of success. In fact, using representation-basis-dependent norm is a unique feature among all the QSLs that we aware of in the literature.

From our five illustrative examples, we do not see any pattern on the optimized p , w , and $\mathcal{B}$ used. Even for the case when p and w are fixed, we have no general analytical method to find the optimized $\mathcal{B}$. Nevertheless, these numerical examples also show that representation basis optimization can be done in practice very quickly when τ is small because optimized bases are common. Although we have studied the underlying reason analytically for the pure qubit state evolution under a time-independent Hamiltonian in Sec. III A 1, a sound explanation for a general system is still lacking. It is instructive to research on it in the future.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [33].

APPENDIX A: JUSTIFICATION OF THE VALIDITY OF EQ. (9)

We justify the validity of Eq. (9) as follows. First, let us consider the case when the right-hand side of both Eqs. (9a) and (9b) are finite. In this case, the series involved in both equations are absolutely convergent. By rearranging the order of the sum, we may assume without loss of generality that $|\Delta_{\mathcal{B},j+1}(\tau)|^p = [|\Delta_{\mathcal{B}}(\tau)|_{j+1}^\downarrow]^p$ for all $j \in \mathbb{N}$. In this way, we see that there is a $P \in S_{\mathbb{N}}$ such that the sum in the right-hand side of Eq. (9a) equals that of the right-hand side of Eq. (9b). We now claim that if $P \in S_{\mathbb{N}}$ with $w_{P(j)+1} \neq w_{j+1}^\downarrow$ for some $j \in \mathbb{N}$, then $\sum_{j \in \mathbb{N}} w_{P(j)+1} |\Delta_{\mathcal{B},j+1}(\tau)|^p < \sum_{j \in \mathbb{N}} w_{j+1}^\downarrow [|\Delta_{\mathcal{B}}(\tau)|_{j+1}^\downarrow]^p$. Suppose P is one such element in $S_{\mathbb{N}}$. Let ℓ be the smallest element in $\mathbb{N}$ obeying $w_{P(\ell)+1} \neq w_{\ell+1}^\downarrow$ whose existence is guaranteed in our setting. More importantly, $w_{P(\ell)+1} < w_{\ell+1}^\downarrow$. Let $P' \in S_{\mathbb{N}}$ be the one sending ℓ to $P(\ell)$, $P(\ell)$ to ℓ , and keeping all other elements fixed. Obviously, $\sum_{j \in \mathbb{N}} w_{P(j)+1} |\Delta_{\mathcal{B},j+1}(\tau)|^p < \sum_{j \in \mathbb{N}} w_{P'(j)+1} |\Delta_{\mathcal{B},j+1}(\tau)|^p$. This proves our claim as $P' \circ P \in S_{\mathbb{N}}$. A direct consequence of this claim is that the maximum of Eq. (9a) is attained when $w_{P(j)+1} = w_{j+1}^\downarrow$ for all $j \in \mathbb{N}$. This proves the validity of Eq. (9) if $\mathcal{D}_{p,w,\mathcal{B}}^{\max}(\rho_\tau, \rho_0)$ is finite. Using the above argument, it is easy to see that the right-hand side of Eq. (9a) is infinite if and only if the right-hand side of Eq. (9b) is also infinite. Therefore, the use

of maximum in Eq. (9a) is justified and Eq. (9b) is valid in all cases.

Finally, $\mathcal{D}_{p,w,\mathcal{B}}^{\max}$ is clearly a seminorm when ρ_τ and ρ_0 are countably infinite-dimensional density matrices. It is a norm for whenever $\rho_\tau \neq \rho_0$, $|\Delta_{\mathcal{B}}(\tau)|_1^\downarrow > 0$, and hence $\mathcal{D}_{p,2,\mathcal{B}}^{\max}(\rho_\tau, \rho_0) > 0$.

APPENDIX B: PROOF OF THEOREM 2

Here we prove Theorem 2 by modifying the proof of Theorem 1. By Condition 1, $\mathcal{D}_{p,w,\mathcal{B}}^{\max}(\rho_\tau, \rho_0)$ is finite. Hence, $\tau > 0$ and $\mathcal{D}_{p,\tilde{w},\mathcal{B}}^{\max}(\rho_t, \rho_0)$ and $\mathcal{D}_{p,\tilde{w},\mathcal{B}}(\rho_t, \rho_0)$ can be regarded as functions of $t \in [0, \tau]$. Next, using the notations in that proof, we need to show that for every $t \in J$, $\mathcal{D}_{p,\tilde{w},\mathcal{B}}(\rho_t, \rho_0)$ is differentiable and

$$\begin{aligned} \frac{d\mathcal{D}_{p,w,\mathcal{B}}^{\max}}{dt} &= \frac{d\mathcal{D}_{p,\tilde{w},\mathcal{B}}}{dt} = \mathcal{D}_{p,\tilde{w},\mathcal{B}}^{1-p} \sum_{j \in \mathbb{N}} \tilde{w}_{j+1} |\Delta_{\mathcal{B},j+1}|^{p-1} \\ &\quad \times \operatorname{sgn}(\Delta_{\mathcal{B},j+1}) \frac{d\operatorname{vec}_{\mathcal{B}}(\rho_t)_{j+1}}{dt}. \end{aligned} \quad (\text{B1})$$

To do so, it suffices to prove that

$$\frac{d}{dt} \sum_{j \in \mathbb{N}} f_j(t) = \sum_{j \in \mathbb{N}} \frac{df_j(t)}{dt} \quad (\text{B2})$$

whenever $t \in J$. For any such j , clearly $df_j(t)/dt$ exists. Moreover, Condition 1 implies that $\sum_{j \in \mathbb{N}} f_j(t)$ converges for all $t \in [0, \tau]$. Together with Condition 2 as well as Theorem 9.14 in Ref. [39], we conclude that Eq. (B2) and hence Eq. (B1) hold.

Let $p > 1$ and $1/p + 1/q = 1$. Applying Hölder inequality to the two vectors in Condition 3 and followed by multiplying both sides of the resultant inequality by $\mathcal{D}_{p,\tilde{w},\mathcal{B}}^{1-p}$, we arrive at

$$\left| \frac{d\mathcal{D}_{p,w,\mathcal{B}}^{\max}}{dt} \right| \leq \left\| \frac{d\operatorname{vec}_{\mathcal{B}}(\rho_t)}{dt} \right\|_{p,\tilde{w}} = \|\operatorname{vec}_{\mathcal{B}}(\mathbb{L}\rho_t)\|_{p,\tilde{w}} \quad (\text{B3})$$

for all $t \in J$, where $\|\cdot\|_{p,\tilde{w}}$ is the $\ell_{\tilde{w}}^p(\mathbb{N})$ -seminorm. Surely, inequality (B3) holds for $p = 1$ or ∞ by taking the corresponding limits on p .

Finally, Condition 4 and Lebesgue dominated convergence theorem imply the existence of $\int_0^\tau \|\operatorname{vec}_{\mathcal{B}}(\mathbb{L}\rho_t)\|_{p,\tilde{w}} dt$. This completes our proof of Theorem 2.

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