

Simple analytical model describing the collective nonlinear response of an ensemble of two-level emitters weakly coupled to a waveguide

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We model and investigate the collective nonlinear optical response of an ensemble of two-level emitters that are weakly coupled to a waveguide featuring a single spatial mode. Our approach generalizes the insight that photon-photon correlations in the light scattered by a single two-level emitter result from two-photon interference to the case of many emitters. Using our model, developed for coherent driving in the weak-saturation regime, we study different configurations for probing the nonlinear response of the ensemble, e.g., through the waveguide or via external illumination, and derive analytical expressions for the second-order quantum coherence function, $g^{(2)}(\tau)$, as well as for the squeezing spectrum of the output light in the waveguide, $S_\theta(\omega)$. For the transmission of resonant guided light, we recover the same predictions as previously made with more involved theoretical models when analyzing experimental results regarding $g^{(2)}(\tau)$ [Prasad *et al.*, *Nat. Photon.* **14**, 719 (2020)] and $S_\theta(\omega)$ [Hinney *et al.*, *Phys. Rev. Lett.* **127**, 123602 (2021)]. We also study the transmission of light that is detuned from the transition of the two-level emitter, a situation that we recently studied experimentally [Cordier *et al.*, *Phys. Rev. Lett.* **131**, 183601 (2023)]. Our model predictions show how the collectively enhanced nonlinear response of weakly coupled emitters can be harnessed to generate nonclassical states of light using ensembles ranging from a few to many emitters.

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I. INTRODUCTION

Modeling many-body systems is a major aim and challenge in quantum physics. In this context, the description of the scattering of light by an ensemble of quantum emitters is a ubiquitous yet inherently complex problem. Over the past decades, a considerable amount of work has been devoted to the study of collective radiative phenomena. Progress has been made both theoretically [1–10] and experimentally [11–24]. A large part of this research has focused on the single excitation regime or on situations where all emitters are located in a small region of space and couple to the same optical mode, as initially studied by Dicke [1]. Accounting at the same time for extended ensembles of spatially arranged emitters and many excitations is analytically and even numerically challenging. However, such spatially extended ensembles occur in numerous experimental platforms [13,14,16,17,23,25].

Here, we treat the problem of many two-level emitters that are weakly coupled to a propagating spatially single light mode. Our approach is based on the insight that photon-photon correlations in the light scattered by a single two-level

emitter result from two-photon interference [26–31]. Starting from a description similar to Refs. [27,28], we generalize this concept to many waveguide-coupled two-level emitters and introduce an intuitively accessible model, which allows us to calculate the quantum nonlinear response of the ensemble of emitters and to derive analytical expressions for the second-order quantum coherence function and the squeezing spectrum of the output light. In order to check the validity of our model, we compare its predictions to those obtained in Ref. [32], where it has been shown theoretically that continuous-wave input laser light can be transformed into perfectly antibunched or strongly bunched light, depending on the number of coupled emitters, and find perfect agreement. Compared to previous theoretical descriptions, the simplicity of our approach results in a significant computational advantage and provides a much more intuitive picture. Furthermore, it can account for different probing configurations, e.g., through the waveguide or via external illumination, and different spatial arrangements of the emitters. Using our model predictions, we exemplify how to exploit the collective nonlinear response of an ensemble of emitters for, e.g., generating quantum light with tailored photon statistics or for realizing a narrowband source of squeezed light. Finally, while the transformation of laser light into antibunched light with a single quantum emitter is well established [33] and was recently also shown to be possible with large ensembles of weakly coupled emitters [34], here, we anticipate that this is also possible with a small number (for example, $N = 3$) of suitably arranged emitters by illuminating them simultaneously from the outside and through the waveguide. This makes our model broadly applicable, from solid-state platforms with

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few spectrally identical emitters [35] to cold-atom nanophotonics, where their number can reach the thousands.

We seek to model the transformation of a weak classical continuous-wave monochromatic light field upon interaction with an ensemble of N two-level emitters. We assume the latter to be weakly coupled to the same propagating optical mode, which is defined, e.g., by a single-mode optical waveguide [36]. This situation can be experimentally realized with various emitters, such as laser-cooled atoms, quantum dots, color centers, molecules in organic host material, or excitons, coupled to nanofibers [25,37,38], or to photonic waveguides [35,39–49]. In the microwave regime, this can also be implemented with microwave qubits coupled to a transmission line [50,51]. While realizations with laser-cooled atoms allow one to couple large numbers of weakly coupled identical emitters [15,21,22,52], to date, solid-state emitters allow only the coupling of a few identical emitters, with, however, much larger coupling strength [53–58].

This article is organized as follows: In Sec. II we explain the significance of the coherent and incoherent component with the pedagogical example of resonance fluorescence from a single emitter where the resulting antibunched light can be understood as destructive interference of both components [26,29–31]. Furthermore, we discuss the different illumination configurations for a single emitter (Secs. II A–II C). In Sec. III we introduce our model, which extends this interference concept from a single emitter to an ensemble of emitters coupled to the same waveguide mode. We then apply our model to case studies under different excitation geometries. First, we recover with our model known results for emitters resonantly excited through the waveguide [32] (Sec. III A), and then extend it to new geometries such as illumination under the Bragg (Sec. III B) and anti-Bragg angle (Sec. III C), as well as a combined illumination from outside the waveguide and through the waveguide (Sec. III D).

II. SINGLE EMITTER

A. Single emitter, external illumination

Let us first consider the case of a single emitter coupled to a spatial single-mode waveguide as sketched in Fig. 1. The emitter is considered a two-level system with a population decay rate of the excited state Γ . The emitter is driven by an external coherent single-mode field at frequency ω_L with a resonant Rabi frequency Ω and a detuning $\Delta = \omega_L - \omega_0$ from the atomic transition frequency ω_0 . In the following, we assume $\Omega = |\Omega|e^{i\phi}$ to be complex, with the phase ϕ being defined by the phase of the driving field in the absence of emitters. In our model, we focus on the weak drive regime $|\Omega| \ll \Gamma$, where the driving field can be approximated by a truncated coherent state with at most two photons, see Appendix A.

Moreover, our study focuses on weakly coupled emitters, i.e., $\beta \ll 1$, where $\beta = \Gamma_{\text{wg}}/\Gamma$ denotes the ratio between the scattering rate in the forward direction of the waveguide Γ_{wg} and the total scattering rate Γ . The case of weak coupling, $\beta \ll 1$, thus implies that for a single excited atom the emission rate into the waveguide is much smaller than the total emission rate, $\Gamma_{\text{wg}} \ll \Gamma$. Throughout the article, we assume

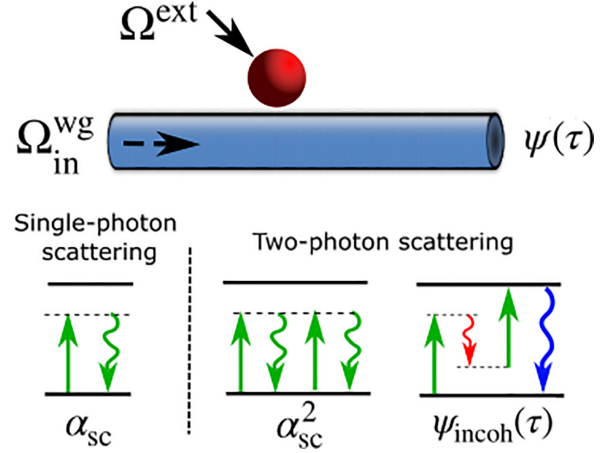


FIG. 1. Top: A single emitter coupled to a waveguide mode. The emitter can be driven externally (Ω^{ext}) and/or through the waveguide ($\Omega_{\text{in}}^{\text{wg}}$). Bottom: Possible one- and two-photon scattering processes with the coherent single-photon scattering amplitude α_{sc} , the coherent two-photon scattering amplitude α_{sc}^2 , and the incoherent two-photon scattering amplitude $\psi_{\text{incoh}}(\tau)$, which is given in Eq. (4).

for all simulations and figures the numerical value $\beta = 0.01$ if not stated otherwise. The light-matter interaction can be decomposed into different scattering processes, which in the weak-saturation regime are the single- and two-photon coherent scattering and two-photon incoherent scattering [26], as shown in Fig. 1.

While the rate of coherently scattered single-photons scales like $R_{\text{coh}} \propto |\Omega \cdot g_{\Delta}|^2$, the amplitude of coherently scattering a single-photon into the forward direction of the guided mode is given by

$$\alpha_{\text{sc}} = -\frac{\Omega}{2\sqrt{\beta}\Gamma} g_{\Delta}, \quad (1)$$

where we introduce a power-independent photon generation coefficient

$$g_{\Delta} = \frac{2\beta}{1 - 2i\Delta/\Gamma}. \quad (2)$$

The minus sign in Eq. (1) stems from the π phase shift between the scattered light field and the driving field for on-resonant excitation [59].

The interference mechanisms leading to (anti)bunched light originate from the structure of the two-photon wave function ψ . This wave function can be decomposed into a coherent and an incoherent two-photon scattering contribution. In the time domain, these describe the amplitudes for scattering two photons into the waveguide with a temporal separation τ . The amplitude of the coherent component is given by

$$\psi_{\text{coh}}(\tau) \equiv \psi_{\text{coh}} = \alpha_{\text{sc}}^2, \quad (3)$$

while the amplitude of incoherently scattering two photons with a time delay τ is given by [32,60]

$$\psi_{\text{incoh}}(\tau) = -\alpha_{\text{sc}}^2 e^{-\Gamma|\tau|/2} e^{i\Delta|\tau|}. \quad (4)$$

For $\tau = 0$, the amplitude of this nonlinear scattering process is equal in magnitude to the coherent amplitude of scattering two photons into the waveguide ψ_{coh} . For finite

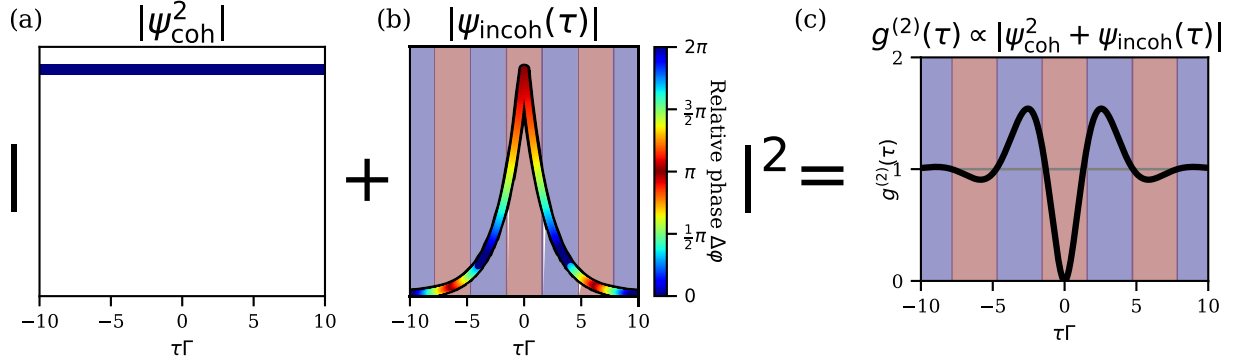


FIG. 2. Scattered two-photon wave functions from a single emitter at detuning $\Delta/\Gamma = 1$ under external illumination: (a) The coherent component ψ_{coh} is time independent and has a constant phase. (b) The incoherent component $\psi_{\text{incoh}}(\tau)$ is a localized function, and the colors correspond to the phase difference to the coherent component $\Delta\varphi = \text{Arg}\{\psi_{\text{incoh}}(\tau)\} - \text{Arg}\{\psi_{\text{coh}}\}$. At $\tau = 0$, the phase difference is π such that in (c) antibunching occurs. The second-order correlation function $g^{(2)}(\tau)$ is obtained from (a) and (b) by taking into account the complex phase. The constructive or destructive nature of the interference is indicated by the background shading in (b) and (c) such that at $\tau = 0$ a fully destructive interference occurs. The interference pattern as a function of τ then leads to signatures of Rabi oscillation in $g^{(2)}(\tau)$.

$\tau = t_2 - t_1$, it exponentially decays with $|\tau|$. The photon pairs created from an incoherent scattering process are thus correlated in time with a typical separation $\tau_{\text{exc}} = 1/\Gamma$ —the lifetime of the excited state. This is in contrast to ψ_{coh} that describes two temporally uncorrelated photons.

Combining Eqs. (3) and (4), the total temporal two-photon wave function is then

$$\psi(\tau) = \psi_{\text{coh}} + \psi_{\text{incoh}}(\tau) \quad (5)$$

$$= \alpha_{\text{sc}}^2 (1 - e^{-\Gamma|\tau|/2} e^{i\Delta|\tau|}). \quad (6)$$

In leading order of the driving field (Ω^2), one can directly relate the two-photon wave function to the normalized second-order correlation function [32,61], measured, e.g., with a Hanbury Brown and Twiss setup:

$$g^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger(0) \hat{a}^\dagger(\tau) \hat{a}(\tau) \hat{a}(0) \rangle}{|\langle \hat{a}^\dagger \hat{a} \rangle|^2} \quad (7)$$

$$= \frac{|\psi(\tau)|^2}{P^2}, \quad (8)$$

where $\hat{a}^\dagger(t)$ is the standard bosonic creation operator and $P = |\langle \hat{a}^\dagger \hat{a} \rangle|$ the photon rate measured at the waveguide output. In the low-saturation regime it is dominated by the coherent scattering and given by $P = |\alpha_{\text{sc}}|^2 + \mathcal{O}(\Omega^4)$.

Note that, at zero time delay, both components interfere fully destructively such that $\psi(\tau = 0) = 0$, and consequently, $g^{(2)}(\tau = 0) = 0$ as shown in Fig. 2. This highlights the fact that, in the low-saturation regime, the antibunching of a single two-level emitter results from the destructive interference between coherently and incoherently scattered photons [26].

In the frequency domain, the coherently scattered photons inherit the spectrum of the laser, herein assumed to be a delta function at ω_L as we consider the laser drive to be monochromatic. In contrast, the incoherent scattering can be understood as a degenerate four-wave mixing process where two laser photons are converted into a pair of red- and blue-detuned photons of frequency $\omega_L \pm \omega$, where ω is the detuning from the laser frequency ω_L . The biphoton spectrum $\psi_{\text{incoh}}(\omega) =$

$\int_{-\infty}^{\infty} \psi_{\text{incoh}}(\tau) e^{i\omega\tau} d\tau$ is given by [60]

$$\psi_{\text{incoh}}(\omega) = -\frac{\Omega^2}{2\beta^2\Gamma^2} g_{\Delta+\omega} g_{\Delta-\omega}. \quad (9)$$

For a resonant laser excitation, $\Delta = 0$, this spectrum has a Lorentzian shape of FWHM Γ as shown in Fig. 3. As a consequence of energy conservation, the generated red- and blue-detuned photons are time-frequency entangled with one photon at $\omega_L + \omega$ and one at $\omega_L - \omega$ when expressed in the laboratory frame [60,62,79]. For off-resonant excitation, $\Delta \gtrsim \Gamma$, the spectrum consists of two Lorentzians of width Γ with one being centered on the atomic resonance at ω_0 and

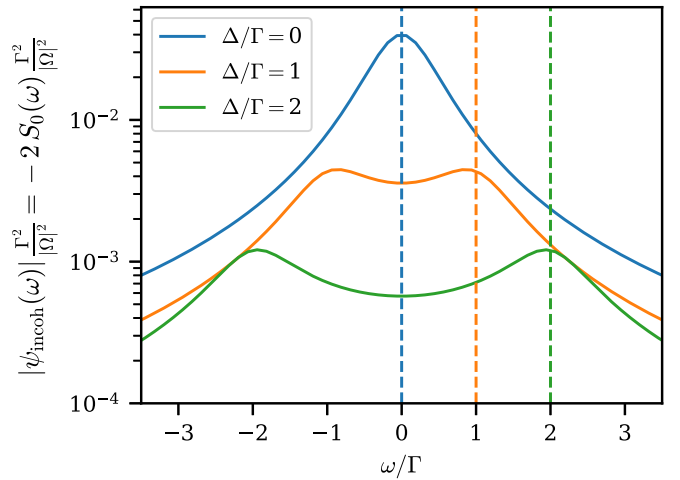


FIG. 3. Frequency spectrum: Absolute value of the incoherent two-photon amplitude $\psi_{\text{incoh}}(\omega)$ for a single emitter coupled with $\beta = 0.01$ to a waveguide and for three different detunings Δ . The spectra are plotted in the excitation laser frame, such that $\omega = 0$ corresponds to the laser frequency, as it would be observed in homodyne detection when measuring the squeezing spectrum $S_0(\omega)$. The dashed lines indicate the respective detunings and thus coincide with the resonance frequency of the atom. For sufficiently large detunings, the spectrum of the entangled photon pairs is peaked around $\pm\Delta$.

one detuned by 2Δ from the atomic resonance, i.e., at $\pm\Delta$ with respect to the driving frequency, see Fig. 3.

Due to their distinct spectrum, the relative balance of coherent and incoherent light can be modified, e.g., via spectral filtering of either the coherent or incoherent component [31,63]. This was recently studied in several experiments [29,30,61,64], demonstrating the intricate relation between photon statistics and the two-photon interference between the coherent and incoherent components.

It is also noteworthy that within the low-saturation regime, the spectrum of the incoherent two-photon amplitude $\psi_{\text{incoh}}(\omega)$ is directly linked to the squeezing spectrum $S_\theta(\omega)$ [65,79] observable via homodyne detection. It is defined as

$$S_\theta(\omega) = \int_{-\infty}^{\infty} \langle : \Delta \hat{X}_\theta(\tau) \Delta \hat{X}_\theta(0) : \rangle e^{i\omega\tau} d\tau \quad (10)$$

$$= -\frac{1}{2} |\psi_{\text{incoh}}(\omega)| \cos[2\theta + \varphi(\omega)], \quad (11)$$

where we introduced the generalized quadrature operator $\hat{X}_\theta(t) = \frac{1}{2} [\hat{a}(t)e^{i\theta} + \hat{a}^\dagger(t)e^{-i\theta}]$ with $\Delta \hat{X}_\theta(t) = \hat{X}_\theta(t) - \langle \hat{X}_\theta(t) \rangle$, $: \dots :$ denoting normal ordering and $\varphi(\omega) = \text{Arg}\{\psi_{\text{incoh}}(\omega)\}$. Here, θ is a phase that, in the case of homodyne detection, is set by the local oscillator. In the case of resonant excitation, squeezing, e.g., quantum noise reduction, is then maximized for $\theta = 0$, while for antisqueezing $\theta = \pi/2$ is required. An interesting quantity in this context is $2\pi S_0(\tau = 0) = \int_{-\infty}^{\infty} d\omega S_0(\omega)$, which corresponds to the total quantum noise reduction obtained by integrating the squeezing spectrum over all frequencies.

B. Single emitter, illumination through the waveguide

Now let us consider the case where the emitter is driven through the waveguide with Rabi frequency $\Omega_{\text{in}}^{\text{wg}}$, see Fig. 1, with the corresponding amplitude of the input coherent state $\alpha_{\text{in}} = \Omega_{\text{in}}^{\text{wg}} / (2\sqrt{\beta}\Gamma)$. The coherent component at the waveguide output α_{out} now consists of the coherently scattered light α_{sc} that interferes with the coherent field that is already present inside the waveguide α_{in} :

$$\alpha_{\text{out}} = \alpha_{\text{in}} + \alpha_{\text{sc}} = t_\Delta \alpha_{\text{in}}, \quad (12)$$

where we define $t_\Delta = 1 - g_\Delta$ as the complex-valued single-atom transmission coefficient. It follows that the coherent component of the two-photon wave function is

$$\psi_{\text{coh}} = t_\Delta^2 \alpha_{\text{in}}^2. \quad (13)$$

Adding the coherent and incoherent component of the two-photon wave function, we obtain the total two-photon wave function:

$$\begin{aligned} \psi(\tau) &= t_\Delta^2 \alpha_{\text{in}}^2 - \alpha_{\text{sc}}^2 e^{-\Gamma|\tau|/2} e^{i\Delta|\tau|} \\ &= \alpha_{\text{in}}^2 \left(t_\Delta^2 - \frac{4\beta^2}{(1 - 2i\Delta/\Gamma)^2} e^{-\Gamma|\tau|/2} e^{i\Delta|\tau|} \right). \end{aligned} \quad (14)$$

In the limit of $\beta \ll 1$ [66], the scattered field of a single atom is much smaller than the input field, $\alpha_{\text{sc}} \ll \alpha_{\text{in}}$, such that the two-photon wave function at the output is nearly identical to the one of the input, $\psi(\tau) \approx \alpha_{\text{in}}^2$, and $g^{(2)}(\tau) \approx 1$. Thus, a single weakly coupled emitter has a negligible effect on the photon statistics of the light transmitted in the waveguide. In

contrast, as discussed later in Sec. III A, the interaction of N weakly coupled emitters can strongly suppress or enhance the two-photon wave function at zero time delay $\psi(\tau = 0)$.

C. Single emitter, combined illumination

Let us now examine the case where the emitter is driven by both an external field and a guided field with Rabi frequencies Ω^{ext} and $\Omega_{\text{in}}^{\text{wg}}$, respectively. Here and in the following, when considering combined illumination, for simplicity we assume the two fields to have a fixed phase relation and identical polarizations and detunings, such that the Rabi frequencies sum up to the total Rabi frequency seen by the atom $\Omega = \Omega^{\text{ext}} + \Omega_{\text{in}}^{\text{wg}}$. The combined scattering amplitude is then

$$\bar{\alpha}_{\text{in}}^{\text{sc}} = -(\Omega_{\text{in}}^{\text{wg}} + \Omega^{\text{ext}}) \frac{(\beta\Gamma)^{1/2}}{\Gamma - 2i\Delta} \quad (15)$$

$$= -\Omega_{\text{in}}^{\text{wg}} \frac{(\beta'\Gamma)^{1/2}}{\Gamma - 2i\Delta}, \quad (16)$$

where we introduced the effective coupling constant

$$\beta' = \beta(1 + \Omega^{\text{ext}}/\Omega_{\text{in}}^{\text{wg}}), \quad (17)$$

which depends on the ratio of the driving field $\Omega_{\text{in}}^{\text{wg}}$ and the field resulting from the external beam Ω^{ext} [67]. In analogy to the transmission and generation coefficient t_Δ and g_Δ introduced in Eqs. (2) and (12), one can introduce an effective transmission t'_Δ and normalized scattering coefficient g'_Δ ,

$$t'_\Delta = 1 - \frac{2\beta'}{1 - 2i\Delta/\Gamma} = 1 - g'_\Delta, \quad (18)$$

and accordingly, the coherent component of the two-photon wave function is given by

$$\begin{aligned} \psi_{\text{coh}} &= \alpha_{\text{out}}^2 \\ &= (\alpha_{\text{in}} + \alpha_{\text{sc}})^2 = \alpha_{\text{in}}^2 t'^2_\Delta. \end{aligned} \quad (19)$$

The incoherent component of two-photon wave function ψ_{incoh} is given by the same expression as in Eq. (4), where α_{sc} is given by Eq. (16).

Finally, adding the expressions in Eqs. (19) and (4), we obtain the two-photon wave function in the case of combined illumination of a single emitter:

$$\psi(\tau) = \alpha_{\text{in}}^2 \left(t'^2_\Delta - \frac{4\beta'^2}{(1 - 2i\Delta/\Gamma)^2} e^{-\Gamma|\tau|/2} e^{i\Delta|\tau|} \right). \quad (20)$$

Note that in general β' is a complex number where $\text{Arg}\{\beta'\}$ is given by the relative phase between Ω^{ext} and $\Omega_{\text{in}}^{\text{wg}}$. In the following, we restrict our discussion to the case where Ω^{ext} and $\Omega_{\text{in}}^{\text{wg}}$ are perfectly in phase, i.e., $\text{Arg}\{\Omega^{\text{ext}}\} = \text{Arg}\{\Omega_{\text{in}}^{\text{wg}}\}$, such that β' is real and $\beta' \leq 1$.

In summary, under combined illumination, a single emitter effectively behaves as if illuminated through the waveguide with an increased coupling constant β' when the fields are in phase, as discussed in detail in Ref. [67].

III. MANY EMITTERS

Let us now consider N emitters weakly coupled to a waveguide. When the distance between the emitters a is large

enough to neglect direct dipole-dipole coupling via the near-field $a \gg \lambda/2$ [7], the interactions between the emitters are dominated by the infinite range interaction mediated via the field in the waveguide. In particular, we want to investigate how the waveguide-mediated interaction between the emitters modifies quantities like the second-order correlation function $g^{(2)}(\tau)$ and the squeezing spectrum $S_\theta(\omega)$ of the transmitted light. Here, we consider the case of unidirectional coupling, i.e., we assume the emission rate into the backward direction of the waveguide to be zero. This reduces the complexity of the problem and can be realized to a good approximation with, e.g., cold-trapped atoms coupled to nanofibers [68]. Moreover, in the weak-coupling regime, many results obtained in this paper for unidirectional coupling are also valid for bidirectional coupling, see Appendix B for a more detailed discussion.

Our approach to modeling an ensemble of N emitters follows a two-step procedure that is valid in the regime of weak driving and weak emitter coupling. Since the contribution from the weak incoherent driving can be neglected, we compute in the first step the coherent fields at the position of each emitter using linear response theory. In a second, more involved step, we calculate the incoherent fields generated by the coherent component and propagate them through the ensemble, as detailed in the following.

The coherent component at the waveguide output now consists of the sum of the input guided field and the coherent scattering amplitudes of all emitters, α_n^{sc} , where the index $n = 1, 2, 3, \dots$ numbers the atoms:

$$\alpha_{\text{out}} = \alpha_{\text{in}} + \sum_{n=1}^N \alpha_n^{\text{sc}}. \quad (21)$$

Here, the coherent scattering amplitudes α_n^{sc} are computed according to Eq. (1) using the local Rabi frequencies Ω_n seen by the n th atoms, which depend on the illumination configuration, see below. As previously, the coherent two-photon component is then given by $\psi_{\text{coh}} = \alpha_{\text{out}}^2$.

For deriving the incoherent two-photon component at the waveguide output, we resort to the frequency domain and find

$$\psi_{\text{incoh}}(\omega) = \sum_{n=1}^N \psi_{\text{incoh}}^{(n)}(\omega) (t_{\Delta+\omega} t_{\Delta-\omega})^{N-n}, \quad (22)$$

where $\psi_{\text{incoh}}^{(n)}(\omega)$ is the frequency-domain representation of the incoherent component of the two-photon amplitude generated by the n th atom in the chain. We calculate it according to Eq. (9) using the local Rabi frequency Ω_n , thereby assuming that the incoherent scattering process is only driven by the coherent component of the incident field. The individually scattered incoherent amplitudes are then summed up, leading to a coherent buildup that arises in the waveguide, a collective emission by effectively interacting two-level emitters. The two-photon component is reduced due to absorption by the remaining emitters in the waveguide. This effect appears in the model [Eq. (22)] as $t_{\Delta+\omega} t_{\Delta-\omega}$, which represents the amplitude transmission coefficient for an incoherently scattered photon pair—one- and one red-detuned—propagating past a single atom. Note that, here, we neglect higher-order nonlinear interaction where incoherently scattered photon

pairs experience a second nonlinear scattering. Neglecting these higher-order incoherent scattering processes is a good approximation in the regime of weak coupling ($\beta \ll 1$), see Sec. III A where we explicitly compare our result with Ref. [32].

The output total two-photon amplitude in the time domain is thus given by $\psi(\tau) = \psi_{\text{coh}} + \psi_{\text{incoh}}(\tau)$. In the following, we apply this formalism to derive quantitative predictions for four experimentally interesting situations.

A. N emitters, illumination through the waveguide

First, we discuss the case where N atoms are solely excited through the waveguide mode. This setting was theoretically studied in Refs. [32] and [65] based on a scattering matrix formalism and higher-order cumulant expansion, respectively, with a discussion of the resonant excitation case ($\Delta = 0$). Here, we extend this discussion to off-resonant excitation, which has been experimentally studied and theoretically analyzed [69] using the model presented here.

The coherent amplitude output is given by Eq. (21). To evaluate it, we observe that the n th atom is driven by the coherent amplitude $\alpha_n = \alpha_{n-1} + \alpha_{n-1}^{\text{sc}} = t_{\Delta} \alpha_{n-1} = t_{\Delta}^{n-1} \alpha_{\text{in}}$. One finds that the coherent component of the two-photon wave function decays exponentially according to Beer-Lambert's law:

$$\psi_{\text{coh}} = \alpha_{\text{out}}^2 = t_{\Delta}^{2N} \alpha_{\text{in}}^2. \quad (23)$$

In order to compute the incoherent component of the output according to Eq. (22) in this configuration, we now have to calculate the local Rabi frequencies, which are given by

$$\Omega_n^{\text{wg}} = 2\sqrt{\beta\Gamma}\alpha_n = 2\sqrt{\beta\Gamma}t_{\Delta}^{n-1}\alpha_{\text{in}} = t_{\Delta}^{n-1}\Omega_{\text{in}}^{\text{wg}}. \quad (24)$$

Inserting this expression into the incoherent two-photon component [see Eq. (9)] emitted by the n th emitter into the waveguide, we obtain

$$\psi_{\text{incoh}}^{(n)}(\omega) = -\frac{2(t_{\Delta}^{n-1}\Omega_{\text{in}}^{\text{wg}})^2}{\beta\Gamma} g_{\Delta} g_{\Delta+\omega} g_{\Delta-\omega}. \quad (25)$$

Summing up the incoherent contribution of all emitters at the waveguide output [see Eq. (22)], we obtain the incoherent component of the output two-photon wave function:

$$\begin{aligned} \psi_{\text{incoh}}(\omega) &= (\Omega_{\text{in}}^{\text{wg}})^2 \sum_{n=1}^N (t_{\Delta}^2)^{n-1} \tilde{\psi}_{\text{incoh}}(\omega) (t_{\Delta+\omega} t_{\Delta-\omega})^{N-n} \\ &= (\Omega_{\text{in}}^{\text{wg}})^2 \tilde{\psi}_{\text{incoh}}(\omega) \times \begin{cases} \frac{t_{\Delta}^{2N} - (t_{\Delta+\omega} t_{\Delta-\omega})^N}{t_{\Delta}^2 - t_{\Delta+\omega} t_{\Delta-\omega}} & \omega \neq 0, \\ N(t_{\Delta}^2)^{N-1} & \omega = 0, \end{cases} \end{aligned} \quad (26)$$

where in the last step we computed the geometric sum explicitly and we introduce the normalized incoherent two-photon amplitude

$$\tilde{\psi}_{\text{incoh}}(\omega) = \frac{\psi_{\text{incoh}}^{(n)}(\omega)}{(\Omega_n^{\text{wg}})^2} = \frac{\psi_{\text{incoh}}^{(n)}(\omega)}{(t_{\Delta}^{n-1}\Omega_{\text{in}}^{\text{wg}})^2}. \quad (27)$$

As evident from Eq. (26), the incoherent component of the output two-photon wave function $\psi_{\text{incoh}}(\omega)$ results from an interplay of (1) attenuation of coherent light along the ensemble,

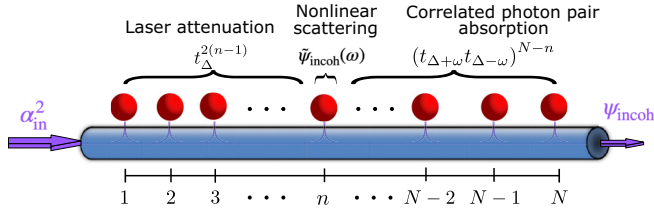


FIG. 4. Contribution of the n th atom to the incoherent two-photon component of the transmitted light. The first $n-1$ emitters attenuate the input light, which is then incoherently scattered by the n th emitter. Then, the following $N-n$ emitters act as a dispersive and absorptive medium for this incoherent two-photon component.

(2) incoherent scattering at each atom, and (3) absorption of the incoherent photons by the following atoms, as sketched in Fig. 4. From the total two-photon wave function at the output, $\psi(\tau)$, we obtain the second-order correlation function $g^{(2)}(\tau)$:

$$g^{(2)}(\tau) = \frac{|\alpha_{\text{in}}^2 t_{\Delta}^{2N} + \psi_{\text{incoh}}(\tau)|^2}{|\alpha_{\text{in}} t_{\Delta}^N|^4}. \quad (28)$$

Here, as $\psi_{\text{incoh}} \propto \alpha_{\text{in}}^2$, $g^{(2)}(\tau)$ is independent of the input power. This is a consequence of the truncation to the two-photon number state used here [70].

It is noteworthy that, due to the infinite-range interaction, these results are independent of the emitter's distances as long as near-field effects can be neglected. Furthermore, Eq. (26) is valid for arbitrary atom-laser detunings and thus goes beyond what has been discussed in Ref. [32]. In the following, we will see that off-resonant excitation leads to richer interference phenomena compared to the resonant case. This stems from the fact that, in the detuned case, ψ_{coh} and $\psi_{\text{incoh}}(\tau)$ are complex quantities, while they are real in the resonant case.

In this context, it is useful to define the phase difference between the incoherent and coherent component at zero time delay, $\Delta\varphi = \text{Arg}\{\psi_{\text{incoh}}(\tau=0)\} - \text{Arg}\{\psi_{\text{coh}}\}$. Depending on $\Delta\varphi$, the incoherent two-photon component will interfere with the coherent component either destructively [when $\Delta\varphi = (2m+1)\pi$, $m \in \mathbb{Z}$] or constructively (when $\Delta\varphi = 2m\pi$, $m \in \mathbb{Z}$). This is illustrated in Figs. 5(b) and 5(c), where we show the evolution of both the coherent and incoherent two-photon amplitude in the complex plane as a function of the number of emitters for two different detunings. The coherent component still decays exponentially with the number of emitters, as in the case for resonant excitation.

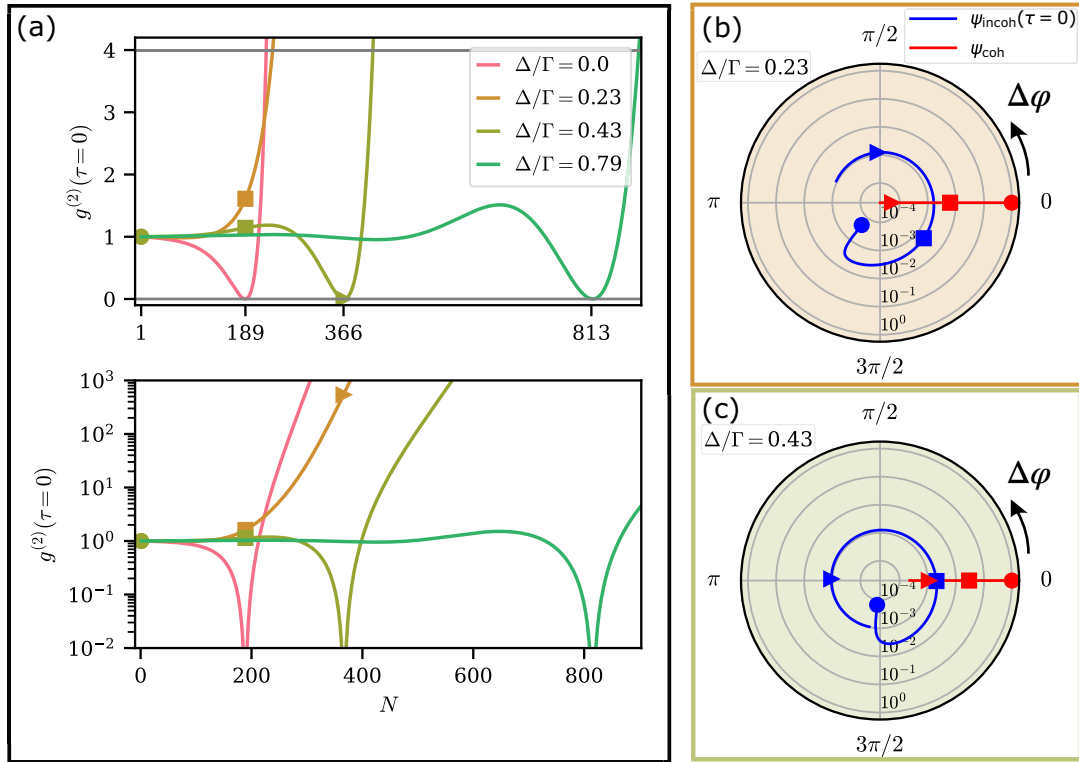


FIG. 5. Bunching and antibunching for the case of illumination through the waveguide as a function of atom number N for different detunings Δ . (a) Second-order correlation function at zero time delay, $g^{(2)}(\tau=0)$, in linear- and log scale as a function of emitter number N and for $\beta = 0.007$ as in Ref. [69]. (b) Complex polar plot in logarithmic representation of the coherent amplitude ψ_{coh} and incoherent photon amplitude $\psi_{\text{incoh}}(\tau=0)$ at zero time delay for $\Delta/\Gamma = 0.23$ with the amplitudes normalized to α_{in} . For this detuning, both constructive and destructive interference occur; however, in the regime where destructive interference occurs the two components have very dissimilar magnitudes, and we obtain only negligible antibunching [for $\psi_{\text{incoh}}(\tau=0) \ll \psi_{\text{coh}}$] or strong bunching [for $\psi_{\text{incoh}}(\tau=0) \gg \psi_{\text{coh}}$]. (c) Same representation for $\Delta/\Gamma = 0.43$, for which a perfect destructive interference occurs at $N = 366$. The markers in (a)–(c) correspond to the same atom numbers: the circle, square, and triangle mark $N = 1$, $N = 189$, and $N = 366$, respectively.

By comparison, the incoherent two-photon amplitude at zero time delay $\psi_{\text{incoh}}(\tau = 0)$ shows a more complex dynamics. Its magnitude first linearly increases due to the collective enhancement in forward scattering [71]. For a larger atom number, the magnitude of the two-photon component decreases subexponentially with N [32].

We note that, for larger detunings, even richer dynamics takes place. Due to the dispersive action of the following atoms, the amplitude of the incoherent two-photon component emitted by atoms at the beginning of the chain can destructively interfere with that emitted by atoms at the end of the chain. This can lead to a vanishing incoherent component $\psi_{\text{incoh}}(\tau = 0)$ for a certain combination of detuning and atom number (see Appendix C and Fig. 13 therein).

Turning now to Fig. 5(a), it is apparent that for off-resonant excitation the combined dynamics of ψ_{coh} and $\psi_{\text{incoh}}(\tau = 0)$ leads to an oscillatory behavior of $g^{(2)}(\tau = 0)$ as a function of N , which stems from the N dependence of $\Delta\varphi$, see Fig. 5(b), leading to alternating constructive and destructive interference of the two components. In general, this prevents the observation of perfect antibunching [$g^{(2)}(\tau = 0) = 0$]. Surprisingly, when scanning N and Δ , we find that several combinations lead to perfect destructive interference and, hence, to perfect photon antibunching [see pink and green lines in Fig. 5(c)]. Specifically, this occurs when the moduli of both components are equal, $|\psi_{\text{coh}}| = |\psi_{\text{incoh}}(\tau = 0)|$, and their phase difference is $\Delta\varphi = (2m + 1)\pi$, $m \in \mathbb{Z}$. While the perfect antibunching point on resonance ($\Delta = 0$) has been predicted and observed in Refs. [32,34], perfect antibunching for off-resonant excitations has only recently been observed [69] based on the predictions of this model.

We now compare our results to Mahmoodian *et al.* [32], where the two-photon wave function has been calculated using a scattering formalism together with the Bethe-Ansatz technique [62]. In contrast to our model, this approach yields exact solutions for any value of β . In Fig. 6 we compare the incoherent component $|\psi_{\text{incoh}}(\tau = 0)|$ calculated with the two approaches. Both predictions agree very well for $\beta \ll 1$. For larger β deviations are expected, as our model does not take into account higher-order scattering processes. Note that, in contrast to Ref. [32], our model provides a closed expression for $\psi_{\text{incoh}}(\tau = 0)$, which is given by the frequency integral of the analytical expression given in Eq. (26). Thus, its numerical complexity does not depend on N , contrary to the numerical calculation based on Ref. [32]. Consequently, predictions made with our model are numerically much less expensive for large N .

B. External illumination: Bragg angle

The next setting under study is an array of atoms externally illuminated by a resonant plane wave as sketched in Fig. 7. Let us first consider an illumination under the Bragg angle Θ such that the partial amplitudes scattered by all N emitters interfere constructively in the waveguide. This Bragg condition is fulfilled when

$$\cos \Theta = \frac{m\lambda}{a} - \frac{\lambda}{\lambda_f}, \quad (29)$$

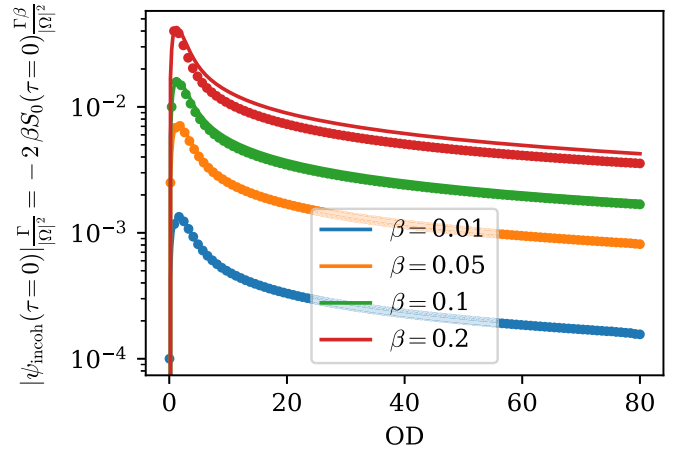


FIG. 6. Comparison of the predictions by Mahmoodian *et al.* [32] (dots) and from the model of this paper (solid lines) of the incoherent two-photon amplitude $|\psi_{\text{incoh}}(\tau = 0)|$ [or equivalently, the quantum noise reduction integrated over the full squeezing spectrum $S_0(\tau = 0)$] as a function of optical depth $OD = 4\beta N$ and for different values of β . For small β a very good agreement is achieved, even for very large ensembles. For larger β deviations occur due to higher-order scattering processes.

where $\lambda_f = \lambda/n_{\text{eff}}$ is the wavelength inside the waveguide, n_{eff} is the effective refractive index of the guided mode, a is the lattice spacing, and m is the Bragg order. This situation resembles the situation studied in Refs. [9,72]. We neglect again free-space interaction between the atoms such that we only consider Bragg conditions fulfilling $a \gtrsim \lambda$. We note that possible voids in the atomic array would not affect our predictions.

For a plane wave, each emitter is subject to an identical external driving field with a Rabi frequency of magnitude $|\Omega^{\text{ext}}|$. The first emitter scatters a coherent amplitude α_1^{sc} into the waveguide which constructively interferes with the scattered amplitude from the second emitter according to the Bragg condition of Eq. (29). After N emitters, and taking into account absorption, the output two-photon coherent amplitude inside the waveguide is

$$\alpha_{\text{out}} = \sum_{n=1}^N t_0^{N-n} \alpha_1^{\text{sc}} = \frac{1 - t_0^N}{1 - t_0} \alpha_1^{\text{sc}} = \frac{1 - t_0^N}{2\beta} \alpha_1^{\text{sc}}, \quad (30)$$

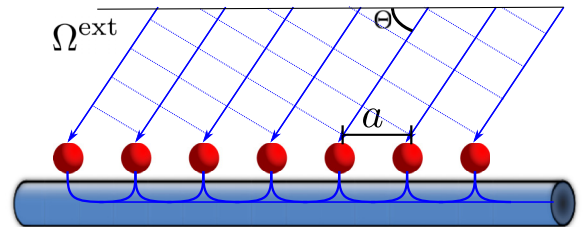


FIG. 7. Sketch of the external illumination by a resonant plane wave, where a is the distance between the atoms. Every atom is driven by a field with a Rabi frequency of magnitude $|\Omega^{\text{ext}}|$ under a Bragg angle Θ .

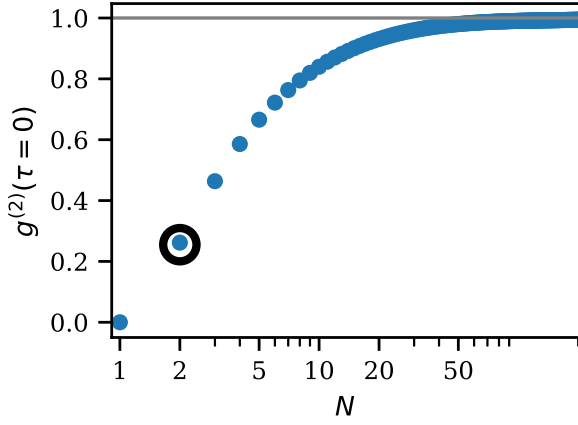


FIG. 8. Second-order correlation function at zero time delay $g^{(2)}(0)$ under the Bragg condition as a function of N for $\beta = 0.01$. As in the excitation scheme with a single emitter from the side (Sec. II A), the point of perfect antibunching appears at $N = 1$. For larger N , $g^{(2)}(0)$ approaches 1. Interestingly, for $N = 2$ an antibunching of $g^{(2)}(0)$ of $1/4$ is predicted (black circle).

where α_1^{sc} is the coherent scattering amplitude of the first emitter. The atoms are now driven by the field that results from the interference between the external drive and the fiber-guided field. At resonance and under Bragg condition, these two fields are in phase opposition at each atom. Specifically, the Rabi frequency seen by the n th emitter is given by $\Omega_n = t_0^{n-1} \Omega^{\text{ext}}$, which interestingly decreases with the atom number. This exponential decrease is consistent with findings in Ref. [73] on Bragg scattering from optically dense ensembles. From the driving amplitude Ω_n we obtain the incoherent two-photon amplitude

$$\psi_{\text{incoh}}(\omega) = (\Omega^{\text{ext}})^2 \sum_{n=1}^N (t_0^*)^{n-1} \tilde{\psi}_{\text{incoh}}(\omega) (|t_\omega|^2)^{N-n}, \quad (31)$$

using $t_{-\omega} = t_\omega^*$. Interestingly, the expression of $\psi_{\text{incoh}}(\omega)$ in Eq. (31) is the same as for an illumination through the waveguide in Eq. (26) for $\Delta = 0$, i.e., both incoherent two-photon amplitudes are equal if the external Bragg field drives the first atom with the same strength as in the case where we probe the system through the waveguide. The coherent two-photon amplitude, however, differs strongly, and for external illumination we obtain

$$\psi_{\text{coh}} = \frac{(1 - t_0^N)^2}{4\beta^2} (\alpha_1^{\text{sc}})^2. \quad (32)$$

This leads to the normalized second-order correlation function

$$g^{(2)}(\tau) = \frac{[(\alpha_1^{\text{sc}})^2 (1 - t_0^N)^2 + 4\beta^2 \psi_{\text{incoh}}(\tau)]^2}{|\alpha_1^{\text{sc}} (1 - t_0^N)|^4}. \quad (33)$$

As expected, for $N = 1$ one gets perfect antibunching $g^{(2)}(0) = 0$ (fluorescence from a single emitter), and for large N the output light approaches $g^{(2)}(0) = 1$ as shown in Fig. 8.

It is interesting to note that for $N = 2$ we obtain $g^{(2)}(0) \simeq 1/4$. This is in contrast to the criteria of $g^{(2)}(0) \leq 1/2$ that is commonly used to verify the presence of a single quantum emitter. A $g^{(2)}(0)$ smaller than $1/2$, in our case, is due

to the constructive interference of the incoherent photons under Bragg illumination from different emitters. In contrast, if we consider N emitters with shot-to-shot fluctuations of their relative distances, one recovers the common criteria $g^{(2)}(0) = 1 - 1/N$ (see Appendix D). We note that $g^{(2)}(0) < 1/2$ from two coherently driven emitters has been predicted in the literature [74,75].

C. External illumination: Anti-Bragg

Now we look at the anti-Bragg configuration [76], i.e., the case where emitted coherent fields interfere destructively inside the waveguide, in contrast to the Bragg case. Again, we restrict our discussion to the resonant case and the anti-Bragg angle ζ is given by

$$\cos \zeta = \left(m + \frac{1}{2}\right) \frac{\lambda}{a} - \frac{\lambda}{\lambda_f}. \quad (34)$$

In this anti-Bragg configuration, the coherent amplitude inside the waveguide depends on whether the atom number is odd or even:

$$\psi_{\text{coh}} = \begin{cases} \alpha_{\text{sc}}^2 & \text{for } N \text{ odd,} \\ 0 & \text{for } N \text{ even.} \end{cases} \quad (35)$$

Thus, in both cases, the scattering amplitude is close to zero for small coupling strength β . In the following, we will neglect the driving of the atoms via this small field α_{sc} [77] and assume that all emitters are solely driven by the external drive Ω^{ext} .

The incoherent two-photon field thus becomes

$$\psi_{\text{incoh}}(\omega) = (\Omega^{\text{ext}})^2 \sum_{n=1}^N \tilde{\psi}_{\text{incoh}}(\omega) (|t_\omega|^2)^{N-n}, \quad (36)$$

where in contrast to the previous two configurations, Eqs. (26) and (31), there is no term for the driving laser absorption $(t_0^2)^{n-1}$ appearing. This geometric sum can then be simplified to

$$\psi_{\text{incoh}}(\omega) = (\Omega^{\text{ext}})^2 \tilde{\psi}_{\text{incoh}}(\omega) \frac{1 - |t_\omega|^{2N}}{1 - |t_\omega|^2}. \quad (37)$$

In contrast to the previous case studies, the driving Rabi frequency remains constant along the chain, which leads to a continuous buildup of the incoherent component with N and different spectral properties of the incoherent component. In Fig. 9 we show this enhanced collective effect on $|\psi_{\text{incoh}}(\omega)|$. It is apparent that, for large enough N , such that $|t_{\omega=0}|^N \simeq 0$, $\psi_{\text{incoh}}(\omega)$ exhibits a plateau around $\omega = 0$. This plateau asymptotically reaches a value of $-\alpha_{\text{sc}}^2 \Gamma / [\beta(1 - \beta)]$, as indicated by the black dashed line in Fig. 9.

The strong incoherent component $\psi_{\text{incoh}}(\omega)$ can be directly linked to squeezed light via Eq. (11), and we obtain a maximal squeezing of

$$S_0(\omega) = -\frac{(\Omega^{\text{ext}})^2}{\Gamma^2(2 - 2\beta)} \quad (38)$$

within the plateau, and we note that Eq. (38) is only valid for $\Omega^{\text{ext}} \ll \Gamma$ [78].

To quantify the strength of the atom's nonlinearity, we integrate $\psi_{\text{incoh}}(\omega)$ over all frequencies, $\psi_{\text{incoh}}(\tau = 0) =$

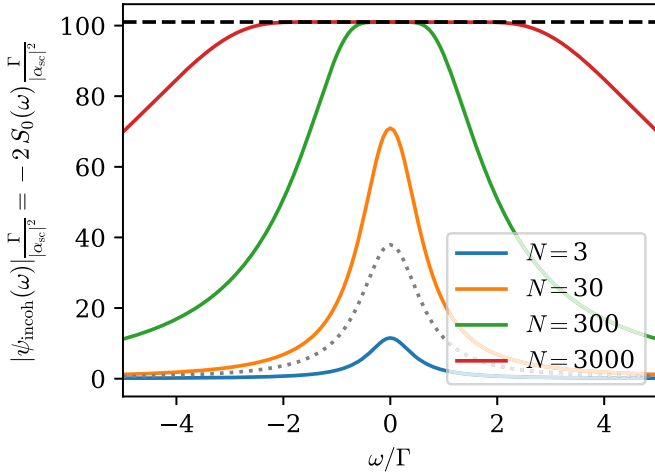


FIG. 9. Spectrum of the incoherent component $|\psi_{\text{incoh}}(\omega)|\Gamma/|\alpha_{\text{sc}}|^2$ for atoms illuminated under the anti-Bragg angle condition and $\Delta = 0$. The curves are obtained for $\beta = 0.01$ and different atom numbers. For small N a coherent buildup takes place, while for large N the spectrum saturates to $\alpha_{\text{sc}}^2/[\beta(1-\beta)]$ as shown by the black dashed line. For comparison, the dotted gray line shows $|\psi_{\text{incoh}}(\omega)|\Gamma/|\alpha_{\text{sc}}|^2$ for illumination through a waveguide [79] (or equivalently in Bragg configuration) for the same β and the atom number that maximizes $|\psi_{\text{incoh}}(\omega)|$ at $\omega = 0$. This large incoherent amplitude $|\psi_{\text{incoh}}|$ relates directly to a relatively strong amplitude squeezed light $S_0(\omega)$.

$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \psi_{\text{incoh}}(\omega)$. This quantity can be experimentally inferred from the noise reduction in the amplitude quadrature [79].

To compare different configurations, we normalize $\psi_{\text{incoh}}(\tau)$ to the scattering rate of the first emitter $(\alpha_{\text{sc}}^1)^2$. With this normalization, $\psi_{\text{incoh}}(\tau)$ becomes independent of the input power, and the three different configurations are compared in Fig. 10. From Fig. 10(a) it is apparent that for the

same number of atoms $|\psi_{\text{incoh}}(\tau = 0)|$ reaches much larger values in the anti-Bragg configuration than under illumination through the waveguide (or, equivalently, for the Bragg configuration). This larger amount of incoherent photon pairs is due to the fact that the driving strength along the chain does not decrease, in contrast to the exponentially damped driving in both Bragg illumination and illumination through the waveguide.

Let us finally discuss the normalized second-order correlation function $g^{(2)}(\tau)$ in the anti-Bragg configuration, see Fig. 10(b). In our approximation of Eq. (8), it diverges for every even number of atoms, since the coherent power $P_{\text{coh}} \propto |\alpha_{\text{out}}|^2$ is zero [80]. For odd atom numbers, one obtains

$$g^{(2)}(\tau) = \left| 1 - \frac{\psi_{\text{incoh}}(\tau)}{\alpha_{\text{sc}}^2} \right|^2. \quad (39)$$

While the coherent two-photon component is independent of N with a value of α_{sc}^2 , the incoherent two-photon component $\psi_{\text{incoh}}(\tau = 0)$ grows much faster than in the two other configurations. This leads to strong bunching already for much smaller atom numbers than when illuminating through the waveguide, see Fig. 10(b). Thus, the anti-Bragg configuration is promising for solid-state emitters, where obtaining large ensembles of identical emitters is challenging. For example, already for two emitters, perfect bunching [i.e., diverging $g^{(2)}(\tau = 0)$] is predicted [81]. Note that the underlying destructive interference could even be obtained for randomly spaced emitters with inhomogeneous transition strengths and frequencies. This only requires individual addressability, including the ability to accordingly adapt the local phases and amplitudes of the external drive, e.g., with a spatial light modulator (SLM). The increased yield of correlated photon pairs could then be of practical use for narrowband photon pair sources or heralded single-photon sources.

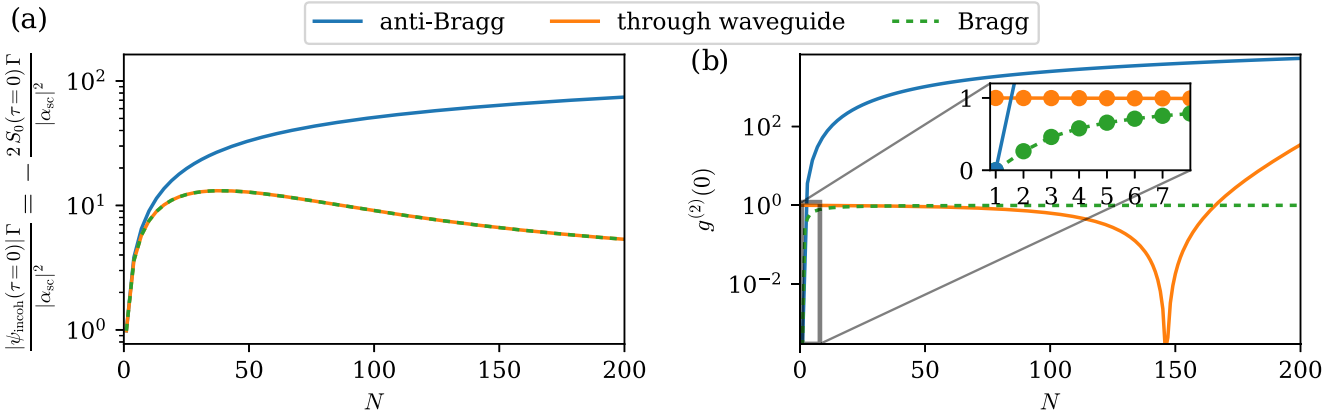


FIG. 10. (a) Incoherent component of the two-photon wave function $|\psi_{\text{incoh}}(\tau = 0)|\Gamma/|\alpha_{\text{sc}}|^2$ as a function of the number of emitters N , and which directly relates to the quantum noise reduction (squeezing) integrated over the full spectrum $S_0(\tau = 0)$. In contrast to the Bragg or transmission illumination, $|\psi_{\text{incoh}}(\tau = 0)|$ grows much faster in the anti-Bragg configuration. For comparison, the two-photon field is normalized to the scattered coherent field α_{sc}^2 of the first emitter in both cases. (b) The second-order correlation function $g^{(2)}(0)$ as a function of emitters N in solid blue for the anti-Bragg configuration and for odd N . For comparison, $g^{(2)}(\tau = 0)$ for the transmission configuration is shown in orange and for the Bragg configuration in green, where $g^{(2)}(0)$ quickly reaches 1 after an initial perfect antibunching for $N = 1$ (shown in the inset zoom). Both figures are calculated for $\beta = 0.01$.

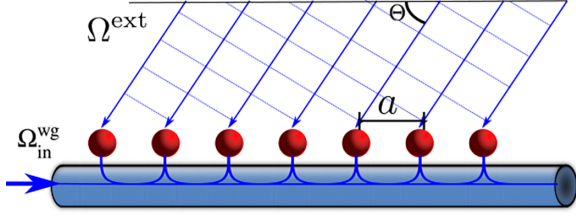


FIG. 11. Combined illumination through the waveguide and externally under the Bragg angle Θ . This configuration effectively increases the coupling strength β' , leading to strong nonlinear effects with fewer emitters.

D. Combined illumination through waveguide and external field

In this last section, we study the case where many atoms are driven simultaneously via the waveguide and an external field as sketched in Fig. 11 with both fields in phase at the atoms. As discussed in Sec. II C for a single atom and $\beta \ll 1$, the light in the fiber effectively behaves as if the atoms were solely illuminated through the waveguide with a rescaled coupling constant β' .

Then, this concept of an effectively increased coupling constant greatly simplifies the problem. Specifically, the output coherent photon component for N emitters and resonant drive ($\Delta = 0$) is given by

$$\alpha_{\text{out}} = \alpha_{\text{in}} \prod_{n=1}^N t'(n), \quad (40)$$

where the individual amplitude transmission coefficients are given by $t'(n) = 1 - 2\beta'(n)$. Here, the coupling strength of the n th emitter is given by

$$\beta'(n)/\beta = 1 + \Omega^{\text{ext}}/(\Omega_n^{\text{wg}} t_0^{n-1}). \quad (41)$$

For the incoherent component, the creation process behaves as if each coupling constant was $\beta'(n)$. However, the transmission of the incoherent photon pairs $|t_\omega|^{2(N-n)}$ continues to scale with the bare value of β . This leads to

$$\frac{\psi_{\text{incoh}}(\omega)}{\alpha_{\text{in}}^2} = - \sum_{n=1}^N \left(\prod_{i=1}^{n-1} t'(i) \right) \frac{2g'(n) |g'_\omega(n)|^2}{\beta'(n)\Gamma} (|t_\omega|^2)^{N-n}, \quad (42)$$

where $|\alpha_{\text{in}}|^2$ is the power sent through the waveguide and $g'_\omega(n) = \frac{2\beta'(n)}{1-2i\omega/\Gamma}$ the position-dependent photon generation coefficient. In analogy to $t'(n)$, we omit the subscript for $g'(n)$, as here we assumed $\Delta = 0$.

Let us discuss some concrete examples. For a typical value of $\beta = 0.01$, as obtained in cold-atom nanofiber setups, introducing an external Bragg field with an amplitude twice that of the nanofiber-guided light ($\Omega^{\text{ext}}/\Omega_n^{\text{wg}} = 2$) enhances the effective coupling strength of the first atom to $\beta' = 3\beta$, and to even larger values for the following atoms (see Appendix E). This drastically changes the behavior of the ensemble. Particularly, the number of emitters required for perfect antibunching in transmission [34] reduces from 146 to 29 as shown in Fig. 12. For moderately larger $\beta = 0.03$, as readily obtained in solid-state systems, and $\Omega^{\text{ext}}/\Omega_n^{\text{wg}} = 10$,

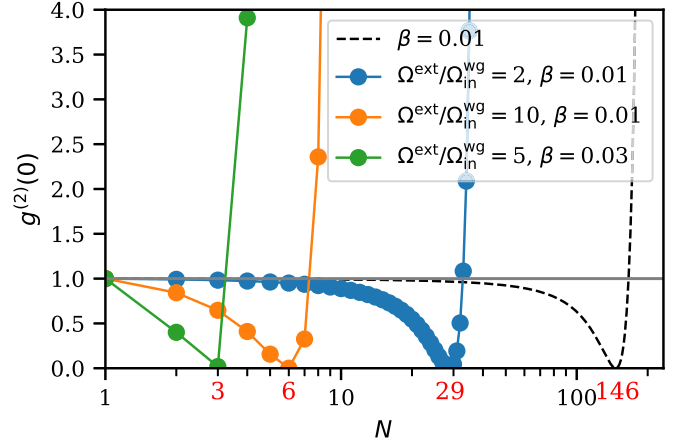


FIG. 12. The second-order correlation function $g^{(2)}(0)$ as a function of the number of emitters in combined illumination under the Bragg angle and through the waveguide for $\Delta = 0$ for three exemplary parameter sets. For comparison, the black dashed line shows the case of pure waveguide illumination with a bare coupling constant $\beta = 0.01$. By adding the external Bragg beam with a field strength $\Omega^{\text{ext}}/\Omega_n^{\text{wg}} = 2$ at the first emitter, the collective nonlinear interaction of the ensemble is enhanced such that for $N = 29$ a point of perfect antibunching is reached (orange line). For $\Omega^{\text{ext}}/\Omega_n^{\text{wg}} = 10$ this point is already reached with six atoms. When increasing the coupling strength, this effect becomes more pronounced, and with $\beta = 0.03$ perfect antibunching can be reached with three emitters and $\Omega^{\text{ext}}/\Omega_n^{\text{wg}} = 5$. The solid lines are guides to the eye.

the number of emitters required for perfect antibunching even reduces to only three.

These last two examples illustrate how an external drive can help reduce the number of identical quantum emitters required to obtain (anti)bunched light from an ensemble. Such a reduction is especially interesting for implementations using solid-state systems where obtaining large number of identical emitters is challenging. Moreover, as discussed in Sec. III C, the required phase relationship between the waveguide light and the external drive could even be met for randomly placed emitters by adapting the local phases, e.g., with a SLM. Moreover, a variation of the bare coupling strength between the emitters can also be compensated with a SLM, or could be readily taken into account in our model. Thus, based on this approach, strong antibunching or bunching can be reached in many experimental situations and physical platforms.

IV. CONCLUSION AND SUMMARY

In summary, we have presented a theoretical model to efficiently calculate the nonlinear response of an ensemble of N quantum emitters weakly coupled ($\beta \ll 1$) to a propagating optical field that is spatially single mode in the low-saturation regime ($|\Omega| \ll \Gamma$). Using this model, we derive simple analytic expressions for the incoherent amplitude of the two-photon wave function, $\psi_{\text{incoh}}(\omega)$, for numbers of emitters for which alternative state-of-the-art methods [32,65] are numerically costly. This allows us to infer both the squeezing spectrum and the second-order correlation function. We show how large ensembles as well as countable numbers of identical weakly coupled emitters can produce quantum states of light

with sub- or super-Poissonian statistics and squeezing. While this is technologically achievable with cold atoms coupled to nanofibers as demonstrated in Refs. [34,69,79], obtaining large ensembles of identical emitters is challenging with solid-state systems. To circumvent this limitation, we present two techniques that allow one to arbitrarily reduce the number of emitters required to observe these strong quantum effects.

It is an open question how the discussed effects extend to larger saturation. Recently, on-resonant illumination through the waveguide was studied also for larger saturation parameters [65] but remains unexplored in the off-resonant case, which is also covered by our model. Furthermore, our model could be extended such that it also yields reliable predictions for larger values of β . In this regime, interesting effects such as photon number sorting [82], topologically protected edge states [83,84], and many-body-localized phases [85] have been predicted. Here, we expect our model to yield physical insights in the case of large ensembles where other approaches exclusively offer numerical predictions.

ACKNOWLEDGMENTS

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DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding authors upon reasonable request.

APPENDIX A: INPUT AND OUTPUT STATES

In our model, we focus on the weak-drive regime $|\Omega| \ll \Gamma$, where the driving field can be approximated by a truncated coherent state with at most two photons,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \left(|0\rangle + \alpha |1\rangle + \frac{\alpha^2}{\sqrt{2}} |2\rangle \right), \quad (\text{A1})$$

where $\alpha = \alpha_{\text{in}} \sqrt{T}$. Here, T defines a quantization time chosen such that $\Gamma^{-1} \ll T \ll |\alpha_{\text{in}}|^{-2}$, which is always possible in the weak excitation regime. Using this truncation allows us to describe the output state after the interaction with the ensemble of emitters as [32]

$$|\text{out}\rangle = |\text{out}\rangle_{1 \rightarrow 1} + |\text{out}\rangle_{2 \rightarrow 2} + |\text{out}\rangle_{2 \rightarrow 1}, \quad (\text{A2})$$

where $|\text{out}\rangle_{X \rightarrow Y}$ describes a state of the combined system of guided and unguided modes with Y of the X input photons scattered into the forward direction of the waveguide. Cor-

respondingly, for the component $|\text{out}\rangle_{2 \rightarrow 1}$, one of the two input photons has not been scattered into the waveguide mode but into unguided modes [86]. Note that we omit states with $Y = 0$ as they do not contribute to the observables of interest [second-order correlation function $g^{(2)}$ and squeezing spectrum $S_\theta(\omega)$] [32,79].

In the weak drive regime, where single-photon scattering dominates, the total output power is predominantly governed by the power of the coherent component of the output state, which scales linearly with the power of the driving field, $|\alpha_{\text{out}}|^2 \propto |\Omega|^2/\Gamma$. By contrast, the power of the incoherent component of the output state scales quadratically with the power of the driving field, $P_{\text{incoh}} \propto |\Omega|^4/\Gamma^3$. However, while the incoherent component only contributes to a negligible fraction of the total output power in the weak-drive regime, it plays a crucial role in observables such as the normalized second-order correlation function, $g^{(2)}(\tau)$, or the squeezing spectrum, $S_\theta(\omega)$. These two observables can both be inferred from the *temporal two-photon wave function*, $\psi(t_1, t_2)$, that is associated with $|\text{out}\rangle_{2 \rightarrow 2}$ in the following way [32]:

$$|\text{out}\rangle_{2 \rightarrow 2} = A \iint \psi(\tau = t_2 - t_1) \hat{a}^\dagger(t_1) \hat{a}^\dagger(t_2) dt_1 dt_2 |0\rangle. \quad (\text{A3})$$

Here, the normalization A is chosen such that the coincidence count rate is $G^{(2)}(\tau) = \langle \hat{a}^\dagger(0) \hat{a}^\dagger(\tau) \hat{a}(\tau) \hat{a}(0) \rangle = |\psi(\tau)|^2$.

APPENDIX B: UNIDIRECTIONAL COUPLING

In this article, we assume a perfectly chiral system where all emitters scatter only in the forward direction of the waveguide. This assumption is motivated by several factors: First, such chiral scattering exists naturally in setups based on cold atoms coupled to nanofibers [87]. Second, unidirectional coupling drastically reduces the complexity of the calculation, and third, for *illumination through the waveguide* we expect only small differences between chiral and symmetric coupling [32]. The same is expected to hold for the *Bragg configuration* and *anti-Bragg configuration*.

APPENDIX C: ILLUMINATION THROUGH THE WAVEGUIDE: APPROXIMATE FORMULAS FOR $\psi_{\text{incoh}}(\tau)$ AND $g^{(2)}(\tau)$ FOR RESONANT EXCITATION $\Delta = 0$

In the configuration where the laser illuminates the emitters through the waveguide (Sec. III A), the magnitude of the coherent component ψ_{coh} decays exponentially with N for any Δ , while the incoherent component ψ_{incoh} exhibits a more complex dynamics, as discussed in the following.

1. Off-resonant

As shown in Fig. 13(a), for a detuned excitation field $\psi_{\text{incoh}}(\tau = 0)$ shows a complicated dynamics as a function of N and Δ . Since the phase of the generated incoherent component changes along the waveguide, even destructive interference between the newly generated incoherent component and the preexisting one can occur, as shown by the orange line in Fig. 13(a).

Still, a simple picture can be provided by introducing the optical depth $\text{OD} = -2N \log(|t_\Delta|)$ to distinguish the small-

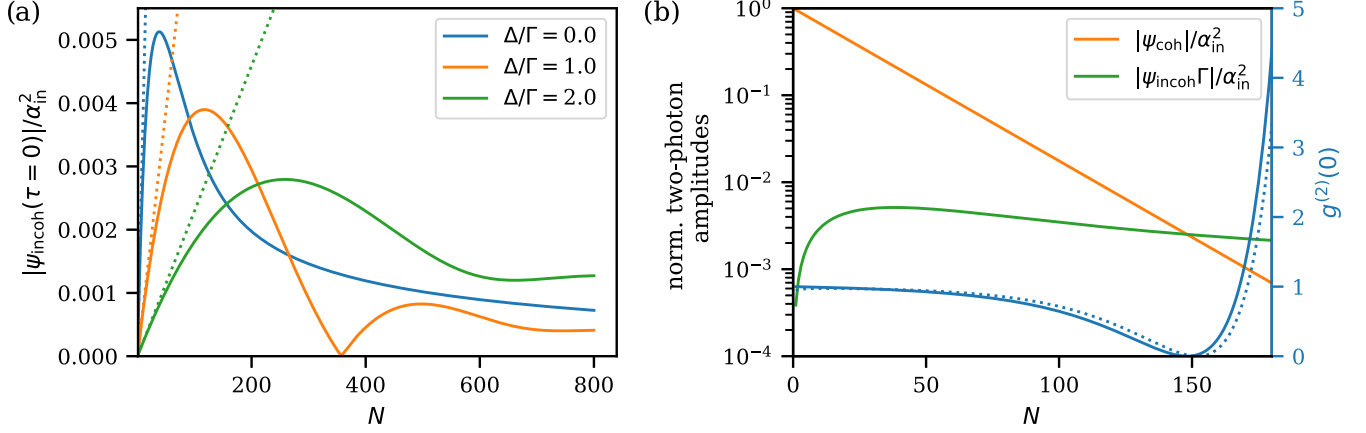


FIG. 13. (a) The incoherent component $\psi_{\text{incoh}}(\tau = 0)$ for $\beta = 0.01$ and different detuning Δ as a function of the emitter N . In the off-resonant case $\psi_{\text{incoh}}(\tau = 0)$ shows a complicated dynamic. For a small number of emitters (low-OD regime) it first increases linearly with N due to the coherent enhancement. For larger N a more complicated picture emerges due to interferences between the light scattered by different emitters that can even lead to a cancellation of $\psi_{\text{incoh}}(\tau = 0)$. The dotted lines show the low-OD approximation of coherent buildup in forward scattering. (b) Evolution of the incoherent and coherent component at zero delay, as well as the corresponding $g^{(2)}(\tau = 0)$ as a function of N for the same β and $\Delta = 0$. At $N = 150$ the point of destructive interference occurs, and for larger N strong bunching can be observed [32,34]. The dotted line is the further approximation of Eq. (C4) with good agreement for small β .

and large-OD regime. For small OD the absorption can be neglected, i.e., $t_{\omega}^N \simeq 1$ for all ω . This implies that both the laser attenuation $t_{\Delta}^{2(n-1)}$ and the absorption of the incoherent component $(t_{\Delta+\omega}t_{\Delta-\omega})^{N-n}$ can be neglected, and Eq. (26) simplifies to

$$\psi_{\text{incoh}}(\omega) \underset{\text{OD} \ll 1}{=} N\psi_{\text{incoh}}^{(1)}(\omega), \quad (\text{C1})$$

where $\psi_{\text{incoh}}^{(1)}(\omega)$ is the incoherent component scattered by the first emitter. Equation (C1) reflects the coherent buildup of the incoherent component in forward direction due to collective enhancement, which is shown in Fig. 13(a) by the dotted lines.

2. Resonant excitation

It is instructive to consider the case of resonant excitation as in Ref. [34], which is sketched in Fig. 13(b) for $\tau = 0$. The incoherent component (green) first increases linearly with N in the low-OD regime. This is the regime where collective effects dominate. For larger OD, absorption cannot be neglected anymore, and $\psi_{\text{incoh}}(\tau = 0)$ decreases subexponentially. Due to this subexponential decay the coherent component reaches at some point the magnitude of the exponentially decaying coherent component. At the point where both amplitudes are equal, a destructive interference in the two-photon wave function $\psi(\tau = 0)$ takes places, which leads to antibunching with $g^{(2)}(0) = 0$.

Mathematically, one can obtain insight by the following approximations: For small β , the discretization of the atoms can be neglected and $t_{\omega}^N \simeq e^{-\frac{\text{OD}}{2-4i\omega/\Gamma}}$. For a resonant drive ($\Delta = 0$) and in first order in β , Eq. (26) simplifies to

$$\psi_{\text{incoh}}(\omega) \underset{\text{OD} \gg 1}{=} \frac{\alpha_{\text{in}}^2 \beta \Gamma}{\omega^2} [e^{-\text{OD} \frac{1}{1+4\omega^2/\Gamma^2}} - e^{-\text{OD}}] \forall \omega \neq 0. \quad (\text{C2})$$

For a larger number of emitters, Eq. (C2) predicts sidebands which develop due to the absorption of the incoherent photon pairs from the ensemble of emitters and which have been

observed in Ref. [79]. Note that Eq. (C2) has a divergence for $\omega = 0$. To simplify Eq. (C2), one can introduce the approximation $1 + 4\omega^2/\Gamma^2 \simeq \omega^2/\Gamma^2$ in the exponent. This is a good approximation in the large-OD regime, where most of the incoherent photons are in the sidebands and around $\omega/\Gamma \simeq 1$ the weight of $\psi_{\text{incoh}}(\omega)$ is small. From this we can calculate

$$\psi_{\text{incoh}}(\tau = 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \psi_{\text{incoh}}(\omega) = -\alpha_{\text{in}}^2 \sqrt{\frac{\beta}{4\pi N}}, \quad (\text{C3})$$

and it follows

$$g^{(2)}(\tau = 0) \approx \left[1 - e^{4\beta N} \sqrt{\frac{\beta}{4\pi N}} \right]^2. \quad (\text{C4})$$

Note that Eq. (C4) represents a very good approximation for $g^{(2)}(0)$ even for a small OD, as shown in Fig. 13. This is due to the fact that in the low-OD limit, $g^{(2)}(0)$ is exponentially dominated by the coherent component $\alpha_{\text{in}} t_{\Delta}^{2N}$ and the error on $\psi_{\text{incoh}}(\tau = 0)$ has a negligible influence. A similar expression was found in Ref. [10] based on a Green's function approach.

APPENDIX D: EXTERNAL EXCITATION WITH N EMITTERS AT RANDOM DISTANCES

In this section we investigate the situation of N emitters which are located at random distances and calculate the second-order correlation function $g^{(2)}(\tau)$. The light which arrives at atom i is characterized by a phase $e^{i\varphi_i}$. Furthermore, for simplicity we restrict our analysis to resonant excitation ($\Delta = 0$) [88] and the low-OD regime. In this regime, two effects can be neglected: absorption of the coherent component inside the waveguide and absorption of the incoherent component. Then, the power at the waveguide output is given

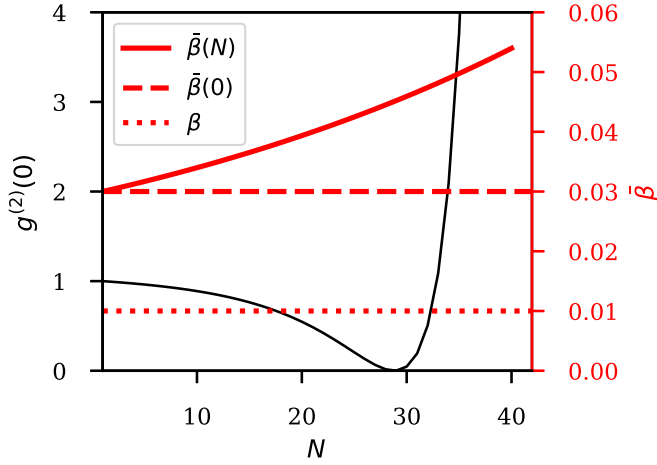


FIG. 14. The left y axis shows $g^{(2)}(0)$ for a combined Bragg and waveguide illumination for the same parameters as the blue line in Fig. 11. The right y axis shows the increase of the effective coupling constant β' along the waveguide.

by

$$P_{\text{wg}} = E \left\{ \left| \sum_{i=1}^N \alpha_{\text{sc}} e^{i\varphi_i} \right|^2 \right\} = N |\alpha_{\text{sc}}|^2, \quad (\text{D1})$$

where $E\{\dots\}$ denotes averaging over the atomic position and/or phases φ_i . Similarly, the second-order correlation

function $G^2(\tau)$ averaged over the atomic distances is

$$G^2(\tau) = E \left\{ \left| \left(\sum_{i=1}^N \alpha_{\text{sc}} e^{i\varphi_i} \right)^2 + \sum_{i=1}^N e^{i2\varphi_i} \alpha_{\text{sc}}^2 e^{-\Gamma|\tau|} \right|^2 \right\}. \quad (\text{D2})$$

For simplicity, let us focus on $\tau = 0$ and $\Delta = 0$, and we obtain

$$g^2(\tau = 0) = \frac{N^2 - N}{N^2} = 1 - \frac{1}{N}. \quad (\text{D3})$$

This corresponds to the standard results for $g^{(2)}(0)$ of a Fock state with photon number n and $n = N$. This is in contrast to our result in the *Bragg configuration*, where we obtain approximately $g^{(2)}(0) \simeq |1 - \frac{1}{N}|^2$ (neglecting absorption).

APPENDIX E: COMBINED ILLUMINATION THROUGH WAVEGUIDE AND EXTERNAL FIELD: DISCUSSION OF $\beta'(n)$.

In principle, as in Ref. [67], the effective coupling constant $\beta'(n)$ is not bound by 1 and $\beta'(n)$ diverges with increasing N . However, we can numerically show that for reasonable parameters $\beta'(n)$ does not exceed 1 around the interesting point of antibunching, as shown in Fig. 14, and our model stays valid with $\beta'(n) \ll 1$.

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