

Frequency shifts heralding ground-state squeezing and entanglement of two coupled harmonic oscillators

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It is often argued that two linearly coupled quantum harmonic oscillators, even when cooled to their ground state, display no inherently quantum features beyond quantized energy levels. Here, we challenge this view by showing that their classical observables encode genuinely quantum features. In particular, we demonstrate that the characteristic frequency shifts of the normal modes observed in coupled oscillators signal nonclassical correlations and ground-state entanglement at zero temperature, corresponding to two-mode squeezing between the uncoupled modes. From a complementary perspective, these two effects, frequency shifts and squeezing, represent the same underlying phenomenon but expressed in different mode bases. What appears as a spectral renormalization in one description manifests itself as entanglement in the other. Frequency shifts therefore constitute an entanglement witness accessible via standard spectroscopy. When the underlying squeezing is not directly measurable, it can be still exploited to enhance the signal-to-noise ratio in precision frequency measurements of individual oscillators without requiring squeezed quantum noise. This uncovers an alternative route to quantum-enhanced sensing within systems traditionally regarded as classical, offering insight into how signatures of quantumness persists across the quantum-to-classical boundary.

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I. INTRODUCTION

Coupled harmonic oscillators are among the most fundamental and commonly used systems in physical modeling [1–3]. They serve as minimal but solvable models for a wide range of physical phenomena [4], from vibrational modes in molecules [5–8], phonons in solids [9], through biophysics [10–12] to cavity modes in quantum optics [13–16] and circuit quantum electrodynamics [17]. Due to their analytic tractability and ubiquity [18], they played a central role in developing both classical and quantum theories of motion, decoherence [19–21], and measurement [22–24]. When two quantum harmonic oscillators are coupled linearly, the resulting system remains Gaussian and exactly solvable [25–30], leading to the widespread conjecture that this dynamics is, in essence, classical [31–33]. Indeed, many aspects of the coupled dynamics (such as normal mode formation, energy exchange, and resonance shifts) can be derived entirely from the framework of classical physics [34]. As a result, coupled oscillator systems are commonly viewed as belonging to the classical domain unless explicitly driven into non-Gaussian regimes, effectively simulated through driving [35], or subjected to strong measurements that reveal nonlocal correlations. This perception

has contributed to the prevailing notion that linearly coupled quantum oscillators are quantum in name only, lacking the genuinely nonclassical features as entanglement or squeezing. Hence they are frequently used as benchmarks or pedagogical examples where quantum and classical predictions coincide [36,37] and the quantum nature of the system is irrelevant.

In this work, we challenge this prevailing view by showing that coupled quantum harmonic oscillators, even when appearing classical in their dynamical behavior, can harbor deeply quantum correlations. Specifically, we reveal that the frequency shifts (changes in the normal-mode resonance frequencies) arising from mode hybridization can serve as indirect signatures of two-mode squeezing and entanglement between the original uncoupled modes. These correlations, though not directly measurable in standard normal mode observables, leave a classical imprint on the system's response, allowing the frequency shifts to be used as entanglement witnesses. Furthermore, we show that the inherent latent squeezing can be harnessed to enhance the precision of frequency estimation for the individual oscillators. In this way, the system enables quantum-enhanced sensing in a platform typically considered classical. Our results contribute to a more nuanced understanding of the quantum-to-classical transition and point to alternative opportunities for exploiting quantum resources in minimally engineered or weakly measured systems.

II. TWO COUPLED HARMONIC OSCILLATORS

Interacting systems are often coupled to their environments via their normal modes [38–42]. In this mode-diagonal

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scenario, standard dissipation and readout effectively address the normal modes, which makes entanglement between the original subsystems not directly accessible to such channels. The essential requirement is that the normal modes are spectrally resolved, such that dissipation occurs in the eigenmode basis rather than in the basis of the uncoupled subsystems. Experimentally, this corresponds to the strong-coupling regime, where the coherent coupling exceeds the relevant dissipation and decoherence rates, leading to a resolvable normal-mode splitting [43]. Although this description is particularly important in ultrastrong light-matter coupling, the ultrastrong-coupling regime itself is not required for the validity of our results. While loss and noise can degrade correlations, revealing or erasing entanglement operationally requires access to joint observables or engineered, nonlocal dissipation. To illustrate this, consider two coupled harmonic oscillators described by the Hamiltonian

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} + \Omega \hat{b}^\dagger \hat{b} + \frac{g}{2} (\hat{a} + \hat{a}^\dagger)(\hat{b} + \hat{b}^\dagger), \quad (1)$$

where $\hat{a}$ ($\hat{b}$) annihilates an excitation with energy ω (Ω), and g is the coupling strength between the oscillators. The ground state of this system can be written as [44–46] (see also Appendix A)

$$|\text{GS}\rangle = \exp \left[\theta \frac{(\hat{a} - \hat{a}^\dagger)(\hat{b} + \hat{b}^\dagger)\omega + (\hat{a} + \hat{a}^\dagger)(\hat{b} - \hat{b}^\dagger)\Omega}{2\sqrt{\omega\Omega}} \right] \times \exp \left[\frac{\xi_+}{2} (\hat{b}^2 - \hat{b}^{\dagger 2}) \right] \exp \left[\frac{\xi_-}{2} (\hat{a}^2 - \hat{a}^{\dagger 2}) \right] |0\rangle_a |0\rangle_b, \quad (2)$$

where the subscripts a and b refer to the energy eigenbases of the uncoupled harmonic oscillators with frequencies ω and Ω , respectively. We also introduce a mixing angle (note that we keep $\Omega > \omega$ as a convention)

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{2g\sqrt{\omega\Omega}}{\Omega^2 - \omega^2} \right), \quad (3)$$

and two single-mode squeezing parameters

$$\xi_- = -\frac{1}{2} \log \left(\frac{\sqrt{\omega^2 + \Omega^2 - \sqrt{(\omega^2 - \Omega^2)^2 + 4g^2\omega\Omega}}}{\sqrt{2}\omega} \right) > 0, \quad (4)$$

$$\xi_+ = -\frac{1}{2} \log \left(\frac{\sqrt{\omega^2 + \Omega^2 + \sqrt{(\omega^2 - \Omega^2)^2 + 4g^2\omega\Omega}}}{\sqrt{2}\Omega} \right) < 0, \quad (5)$$

which are real. Note that for $g = \sqrt{\omega\Omega} \equiv g_c$, the squeezing parameter ξ_- becomes infinite which marks the instability threshold. In general, the squeezing parameters $\xi_{\pm}$ are related to the redistribution of quantum fluctuations between modes $\hat{a}$ and $\hat{b}$.

The above ground state (2) is clearly entangled [47,48]. However, when transformed into the energy eigenbasis of the full interacting system, the Hamiltonian takes the form of two

decoupled harmonic oscillators

$$\hat{H} = \omega e^{-2\xi_-} \hat{c}^\dagger \hat{c} + \Omega e^{-2\xi_+} \hat{d}^\dagger \hat{d}, \quad (6)$$

with the ground state simply given by

$$|\text{GS}\rangle = |0\rangle_c |0\rangle_d, \quad (7)$$

where the subscripts c and d refer to the energy eigenbasis of the coupled harmonic oscillators with normal-mode frequencies $\omega e^{-2\xi_-}$ and $\Omega e^{-2\xi_+}$.

At first sight, it may seem paradoxical that the ground state is a product state in the normal-mode basis, while the original bare modes can exhibit strong correlations and entanglement. This does not mean that entanglement is created or removed by a simple change of basis. Once a tensor-product structure of the Hilbert space is fixed, entanglement is invariant under local basis transformations. The key point is that the bare modes and the normal modes correspond to different decompositions of the total system into subsystems, i.e., to different tensor-product structures of the Hilbert space. A state that is separable with respect to the normal-mode partition may therefore appear entangled with respect to the original mode partition, without any change in the physical state itself. This distinction is also relevant in relativistic quantum information, where different observers may naturally adopt different mode decompositions of a quantum field. For example, inertial and uniformly accelerated observers define different mode partitions, leading to different entanglement structures and to phenomena such as the Unruh effect [49–51]. More generally, relative motion or spacetime curvature can modify the physically relevant subsystem decomposition and hence the observed entanglement properties [52,53].

Measurement in an arbitrary basis is generally nontrivial. Most quantum systems are not measured directly but rather through coupling to an external environment, often via the electromagnetic field [24,54]. For systems with a linear energy structure (such as linearly interacting harmonic oscillators), if driven by a coherent state of light, they will emit a coherent state in return, regardless of the complexity of their internal quantum states [23,55,56]. In the weak system-bath coupling regime, where the Born-Markov and secular approximations apply, the reduced dynamics is naturally described in the eigenbasis of the interacting Hamiltonian (1). As a result, the relevant dissipative channels are most conveniently expressed in terms of the hybrid normal modes described by $\hat{c}$ and $\hat{d}$, rather than the bare modes $\hat{a}$ and $\hat{b}$. In the language of the master equation, the dominant dissipative jump operators are then associated with the normal modes $\hat{c}$ and $\hat{d}$. From this perspective, the environment directly probes the normal-mode degrees of freedom, for which the ground state is simply $|0\rangle_c |0\rangle_d$. As a result, it may appear that no information about the entanglement between the original modes $\hat{a}$ and $\hat{b}$ is imprinted onto the environment, since the dissipation channels are defined in terms of the separable eigenmodes rather than the bare-mode partition in which the entanglement is present.

However, this is not entirely true. The quantumness of the system is indirectly encoded in the normal mode frequencies of the interacting oscillators. Measuring shifts in these frequencies can, therefore, serve as an entanglement witness. To be specific, the squeezing parameters characterizing the

ground state at $T = 0$ can be obtained by measuring the normal mode frequency since

$$\xi_- = -\frac{1}{2} \log \frac{\omega_-}{\omega}, \quad (8)$$

$$\xi_+ = -\frac{1}{2} \log \frac{\omega_+}{\Omega}, \quad (9)$$

where $\omega_- = \omega e^{-2\xi_-}$ and $\omega_+ = \Omega e^{-2\xi_+}$ are the normal mode frequencies of the interacting harmonic oscillators (1). In other words, although no correlations between mode a and b can be detected directly, the squeezing of the ground state can be indirectly inferred from the frequencies of the modes c and d , provided that there are no thermal fluctuations, the Hamiltonian is known, and the parameters are well controlled. This interpretation further assumes that dissipation remains weak enough that the system-bath coupling does not significantly modify the normal-mode structure. In this regime, the dissipative dynamics is governed by transitions between the eigenstates of the interacting Hamiltonian, and the relevant decay channels can be approximately associated with the normal modes c and d .

III. INSEPARABILITY CRITERION

Let us have a look now at entanglement. In continuous variable quantum systems, such as optical modes described by canonical quadrature operators, entanglement can be detected via the inseparability criterion proposed by Duan *et al.* [47] and independently by Simon [48]. Consider quadrature operators of two bosonic modes, for mode a

$$\hat{x}_a = \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \quad \text{and} \quad \hat{p}_a = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger), \quad (10)$$

and analogous for mode b . Define the Einstein-Podolsky-Rosen (EPR)-like operators

$$\hat{u} = \hat{x}_a + \hat{x}_b, \quad \hat{v} = \hat{p}_a - \hat{p}_b. \quad (11)$$

Then, for all separable states, the following inequality holds:

$$\Delta^2 \hat{u} + \Delta^2 \hat{v} \geq 1. \quad (12)$$

Therefore, violation of this bound

$$\Delta^2 \hat{u} + \Delta^2 \hat{v} < 1, \quad (13)$$

is a sufficient condition for entanglement between modes a and b . This criterion is especially useful for Gaussian states and is experimentally accessible via homodyne detection provided that one can measure it in the basis of modes a and b .

This is not the case for interacting systems, if the environment is not coupled through the original bare modes a and b , but rather through the hybrid eigenmodes c and d . Their corresponding quadrature operators are defined analogously as

$$\hat{x}_c = \frac{1}{2}(\hat{c} + \hat{c}^\dagger), \quad \hat{p}_c = \frac{1}{2i}(\hat{c} - \hat{c}^\dagger), \quad (14)$$

$$\hat{x}_d = \frac{1}{2}(\hat{d} + \hat{d}^\dagger), \quad \hat{p}_d = \frac{1}{2i}(\hat{d} - \hat{d}^\dagger). \quad (15)$$

In other words, the normal modes define the natural dissipation channels and appear as the jump operators in the

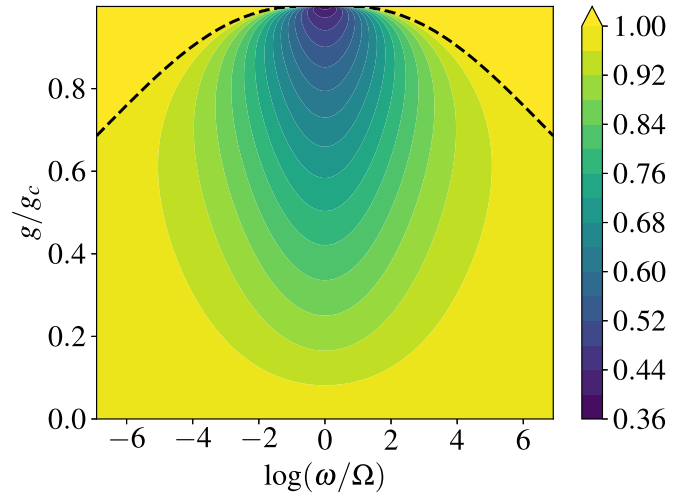


FIG. 1. Inseparability criterion at $T = 0$ ($\beta = \infty$) versus ω/Ω and g/g_c (explicit form given in the Appendix B). The scale is truncated at 1 to highlight the entangled regime (enclosed with a dashed line), with maximal entanglement at $\omega = \Omega$.

corresponding master equation. Nevertheless, the EPR operators $\hat{u}$ and $\hat{v}$ can be rewritten in terms of the modes c and d . For the resonant condition $\omega = \Omega$ (used here for clarity), and evaluated in the ground state, the inseparability criterion simplifies to (see Appendix B)

$$\frac{e^{-2\xi_-}}{2} + \frac{e^{2\xi_+}}{2} = \frac{\omega_-}{2\omega} + \frac{\omega}{2\omega_+} < 1, \quad (16)$$

which shows that measuring the frequency shifts (or equivalently the normal-mode frequencies themselves, for instance spectroscopically) provides direct access to the inseparability criterion and thus to the entanglement between the original modes a and b .

However, in the limit $\Omega \gg \omega$, one expects no entanglement because of the large separation of energy scales between the two oscillators. In this regime, the condition for inseparability simplifies to

$$\frac{e^{-2\xi_-}}{2} + \frac{e^{2\xi_+}}{2} = \frac{\omega_-}{2\omega} + \frac{\omega}{2\omega_-} < 1, \quad (17)$$

which cannot be satisfied, confirming that the system remains separable in this limit. The general condition (see Appendix B) is presented in Fig. 1.

As expected, the maximal amount of entanglement appears close to the instability point $g = g_c$ in the resonant case $\omega = \Omega$ (see Appendix C for analogous results obtained using logarithmic negativity).

IV. FINITE TEMPERATURE

All the discussion above assumes perfect knowledge of the system's temperature. In particular, it assumes that the system is in its ground state. Under this assumption, the frequency measurement can be interpreted as a conditional witness of entanglement, valid only at zero temperature. However, in realistic scenarios, the system is typically in a mixed state due to finite thermal excitations, and the question naturally arises:

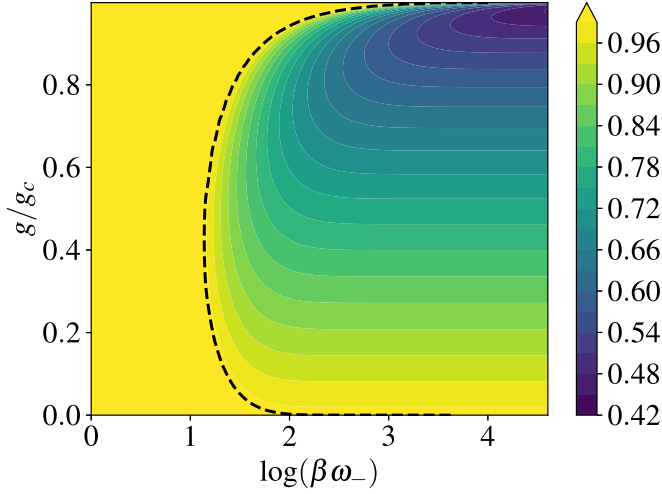


FIG. 2. Inseparability criterion as a function of g/g_c and the dimensionless temperature $\beta\omega_- \equiv \omega_-/T$ for the resonant case $\omega = \Omega$. Increasing temperature reduces the entangled region, with entanglement vanishing beyond a specific temperature dependent on g/g_c . The scale is truncated at 1 to highlight the entangled regime (enclosed with a dashed line). Note that ω_- is a function of g/g_c .

how does thermal noise affect the underlying entanglement and its relation to frequency shifts? To address this, we now consider the case of finite temperature in the resonant case $\omega = \Omega$. The system is then described by a thermal (Gibbs) state as

$$\hat{\rho} = \sum_i p_i |e_i\rangle \langle e_i|, \quad (18)$$

where $|e_i\rangle$ are the eigenstates of the Hamiltonian (6) and the Boltzmann weights are given by

$$p_i = \frac{e^{-E_i\beta}}{Z}, \quad \text{with} \quad Z = \sum_i e^{-E_i\beta}, \quad (19)$$

where E_i is the eigenvalue corresponding to $|e_i\rangle$, $\beta = 1/T$ is the inverse temperature (in units where $k_B = 1$), and Z is the partition function ensuring normalization.

To do it we rewrite the inseparability criterion in the normal mode basis (see the Appendix B). After some algebra, it is straightforward to show that for a thermal state, we obtain

$$e^{-2\xi_-} \left(\langle \hat{c}^\dagger \hat{c} \rangle + \frac{1}{2} \right) + e^{2\xi_+} \left(\langle \hat{d}^\dagger \hat{d} \rangle + \frac{1}{2} \right) < 1, \quad (20)$$

where $\langle \hat{c}^\dagger \hat{c} \rangle$ and $\langle \hat{d}^\dagger \hat{d} \rangle$ are the number of thermal excitations in modes c and d , respectively,

$$e^{-2\xi_-} \left(\frac{1}{e^{\beta\omega_-} - 1} + \frac{1}{2} \right) + e^{2\xi_+} \left(\frac{1}{e^{\beta\omega_+} - 1} + \frac{1}{2} \right) < 1, \quad (21)$$

which confirms that knowing the temperature and the frequencies of noninteracting and interacting harmonic oscillators is sufficient to certify entanglement between modes a and b . The inseparability criterion from Eq. (21) as a function of the interaction strength and temperature is presented in Fig. 2.

The analysis above shows that frequency shifts do not merely signal entanglement through the inseparability criterion, but remain quantitatively connected to the degree of two-mode squeezing even at finite temperature. The same

microscopic correlations responsible for inseparability also determine the macroscopic spectral response of the system. This establishes a direct link between entanglement and frequency renormalization: the observed spectral shifts are not simply static features of the coupled system, but signatures of the underlying quantum correlations that can be exploited for metrological enhancement. As a result, the same frequency shifts that witness entanglement also encode the potential metrological advantage. In the following, we show how these hidden correlations, embedded in the structure of the system rather than in its noise statistics, translate into improved parameter estimation, thereby providing a direct connection between correlations, sensitivity, and observable frequency shifts.

V. QUANTUM ENHANCED SENSITIVITY THROUGH SQUEEZED STATES

Let us first revisit the conventional route to quantum-enhanced sensitivity based on squeezing of quantum noise. For this purpose we consider a single quantum harmonic oscillator governed by the Hamiltonian

$$\hat{H} = \omega \hat{a}^\dagger \hat{a}. \quad (22)$$

In frequency metrology, one typically prepares a coherent state by displacing the vacuum [31,33,55] and monitors the system's dynamics to estimate the oscillator frequency [57–59]. A common choice of measurement is the position quadrature implemented via homodyne detection [23,60]

$$\hat{x}_a = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \quad (23)$$

whose expectation value evolves in time as

$$\langle \hat{x}_a \rangle = |\alpha| \cos(\omega t), \quad (24)$$

where $|\alpha|^2$ is the mean excitation number and t is the evolution time. For a coherent state, the quantum uncertainty in $\hat{x}$ remains constant

$$\Delta^2 \hat{x}_a = \langle \hat{x}_a^2 \rangle - \langle \hat{x}_a \rangle^2 = \frac{1}{4}. \quad (25)$$

The quality of the measurement can be quantified by the signal-to-noise ratio

$$S(\omega) = \frac{\mathcal{M}(\partial_\omega \langle \hat{x}_a \rangle)^2}{\Delta^2 \hat{x}_a}, \quad (26)$$

where $\mathcal{M}$ (in the following set to 1 for simplicity) is the number of measurements performed to construct the expectation value of $\hat{x}$. To improve precision one reduces the quantum noise by using a squeezed (ξ assumed to be real) coherent state of the form

$$\hat{D}(\alpha) \hat{S}(\xi) |0\rangle, \quad (27)$$

where $\hat{D}(\alpha)$ and $\hat{S}(\xi)$ are the displacement and squeezing operators, respectively. For such a state, the noise in the $\hat{x}$ quadrature at the initial time becomes

$$\Delta^2 \hat{x}_a = \frac{e^{-2\xi}}{4}, \quad (28)$$

leading to an enhanced signal-to-noise ratio

$$S(\omega) = \frac{(\partial_\omega \langle \hat{x}_a \rangle)^2}{\Delta^2 \hat{x}_a} \leq \frac{4|\alpha|^2 t^2}{e^{-2\xi}}. \quad (29)$$

Here, ξ is the squeezing parameter. For $\xi = 0$, we recover the standard quantum limit, while for $\xi > 0$, the signal-to-noise ratio is enhanced due to reduced quantum noise of the $\hat{x}$ observable.

This is precisely why squeezed states are so valuable in quantum metrology. They allow one to surpass the standard quantum limit by enlarging the signal-to-noise ratio through quantum noise reduction and thus improve the measurement precision.

VI. QUANTUM ENHANCED SENSITIVITY WITH SQUEEZING THE OSCILLATOR'S FREQUENCY

Now consider a frequency measurement on an interacting harmonic oscillator from Eq. (6) (here we concentrate on one of the modes only)

$$\hat{H} = \omega e^{-2\xi_-} \hat{c}^\dagger \hat{c}, \quad (30)$$

still with the aim to precisely estimate the bare frequency of this oscillator without interaction ω [61]. We assume that the system is probed in the regime where the coherent interaction dominates over dissipation and the normal modes of the interacting Hamiltonian are spectroscopically well resolved. In this weak-dissipation, Born-Markov regime, the reduced dynamics is naturally described in the eigenbasis of the interacting system, and external driving at the corresponding resonance frequency selectively addresses the normal modes $\hat{c}$ and $\hat{d}$. In this sense, the preparation of the coherent state $|\alpha\rangle_c$ should be understood as coherent excitation of the dressed mode $\hat{c}$, rather than an independent preparation of the bare modes $\hat{a}$ and $\hat{b}$. For an initial coherent state $|\alpha\rangle_c$, where the subscript c indicates the new basis, it is straightforward to calculate the signal-to-noise ratio. The noise is of course

$$\langle \hat{x}_c^2 \rangle - \langle \hat{x}_c \rangle^2 = \frac{1}{4}, \quad (31)$$

as it should be for a coherent state, but an interesting thing happens with the signal

$$\partial_\omega \langle \hat{x}_c \rangle = -|\alpha|t \sin \omega_- t \times \partial_\omega \omega_-. \quad (32)$$

We find that the signal is multiplied by a derivative of the effective frequency with respect to the frequency of the non-interacting harmonic oscillator that we want to measure. This derivative can be explicitly evaluated. For a sufficiently far off-resonant case ($\omega \ll \Omega$ taken for simplicity in the following) it turns out to be

$$\begin{aligned} \partial_\omega \omega_- &= \partial_\omega [\omega \exp(-2\xi_-)] = \partial_\omega [\omega \sqrt{1 - g^2/\Omega\omega}] \\ &= \exp(2\xi_-) \left[1 - \frac{g^2}{2\omega\Omega} \right] > 1, \end{aligned} \quad (33)$$

which for large squeezing ($g \sim \sqrt{\omega\Omega}$) simplifies to

$$\partial_\omega \omega_- = \frac{\exp(2\xi_-)}{2}. \quad (34)$$

As a result, the signal-to-noise ratio becomes

$$S(\omega) = \frac{(\partial_\omega \langle \hat{x}_c \rangle)^2}{\Delta^2 \hat{x}_c} \leq \frac{|\alpha|^2 t^2}{\exp(-4\xi_-)}. \quad (35)$$

The upper bound is obtained under the assumption of ideal coherent evolution over a known evolution time t , with no additional decoherence, dissipation, or technical noise affecting the dynamics. It can be saturated when the system evolves purely under the effective Hamiltonian and the measurement is optimized for the estimation of ω , such that the quadrature $\hat{x}_c$ is destructively measured $\mathcal{M} \gg 1$ times at the operating point where the response is maximal (e.g., when $|\sin(\omega_- t)| \sim 1$). In practice, losses and imperfect readout reduce the attainable signal-to-noise ratio below this ideal limit. We further emphasize that estimating the bare frequency ω from the measured normal-mode frequency ω_- requires prior knowledge of the remaining system parameters entering the transformation, in particular, the coupling strength g and the second bare frequency Ω . These quantities are assumed to be independently calibrated using standard spectroscopic methods, for example, through normal-mode splitting, time-domain oscillations, or separate measurements in regimes where the coupling is known or negligible. The enhanced sensitivity discussed here should therefore be understood as conditional on accurate characterization of these auxiliary parameters, as is common in precision metrology where the estimation of a target parameter relies on a well-calibrated underlying model.

Although the system appears entirely classical and no direct squeezing of quantum noise is involved, the signal-to-noise ratio is nonetheless enhanced. The improvement is governed by the squeezing parameter associated with the inaccessible modes a and b . In other words, it is possible to obtain a “quantum” enhanced measurement within a seemingly classical system. This highlights that the classical observables of coupled harmonic oscillators are fundamentally shaped by their underlying quantum correlations. Even in the absence of overt quantum features, quantum resources might remain operative and exploitable. Additionally, one could also create squeezing in the new mode $\hat{c}$ and combining the squeezing effects at two different levels [62,63].

VII. CONCLUSION AND DISCUSSION

We demonstrated that linearly coupled quantum harmonic oscillators, systems, often regarded as essentially classical due to their Gaussian states and analytical solvability, can harbor hidden quantum correlations. These manifest themselves through measurable, seemingly classical observables. In particular, we showed that coupling-induced frequency shifts encode two-mode squeezing between the original oscillator modes. From two complementary viewpoints, the frequency shifts and the squeezing describe the same physical effect of coupling, merely expressed in different mode bases. In the normal-mode picture it appears as a frequency renormalization, while in the original basis it manifests as two-mode squeezing. The measurement basis can be switched by quenching the intermode coupling, so spectroscopy probes the dressed modes while, after the quench, quadrature readout addresses the bare modes, which is already feasible with tun-

TABLE I. Summary of the main relations established in this work. The table indicates which experimentally accessible quantities allow one to infer the corresponding physical properties together with the assumptions under which the relations are valid.

Experimental measurement	Physical quantity inferred	Reference	Assumptions
Normal-mode frequencies $\omega_{\pm}$	Ground-state squeezing $\xi_{\pm}$	Eqs. (8) and (9)	Known Hamiltonian; resonant case ($\omega = \Omega$); weak dissipation; resolved normal modes; ground state.
Normal-mode frequencies $\omega_{\pm}$	Ground-state entanglement (Duan criterion)	Eq. (16)	Resonant case ($\omega = \Omega$); ground state. General expression given in Appendix B.
Normal-mode frequencies $\omega_{\pm}$ and temperature T	Finite-temperature entanglement	Eq. (21)	Resonant case ($\omega = \Omega$); thermal equilibrium; known temperature.
Normal-mode frequency ω_{-}	Enhanced frequency sensitivity	Eq. (35)	Far off-resonant regime ($\Omega \gg \omega$); coherent excitation of the normal mode; calibrated system parameters.

able couplers. Although the results were derived in different physical regimes throughout the article, they all establish a common connection between experimentally accessible frequency measurements and the underlying quantum properties of the coupled oscillators. For convenience, these relations are summarized in Table I, which serves as a quick reference to the main results and the assumptions under which they hold.

Although the underlying squeezing might not be directly observable, it leaves a clear imprint on the system's response, which can be harnessed for quantum-enhanced frequency estimation. Strikingly, this enhancement arises without the need to manipulate quantum noise directly. Instead of relying on traditional noise squeezing to reduce measurement uncertainty, the quantum advantage stems from the implicit squeezing embedded in the eigenstates of the interacting system. This mechanism leads to a signal-to-noise improvement that scales as $\exp(4\xi)$, an enhancement over the standard $\exp(2\xi)$ scaling achieved through explicit squeezing [61]. Hence in suitable cases, quantum resources might be more efficiently exploited when encoded into the system's structure rather than externally imposed.

Our findings challenge the commonly held belief that squeezing has no classical counterpart. We propose that the metrological advantage typically attributed to quantum squeezing can, in certain systems, appear as classical frequency shifts. This suggests a subtle reinterpretation of squeezing. While the underlying correlations are quantum in origin, their effects may be accessed through classical measurements of the system response [64,65]. Our results blur the conventional boundary between classical and quantum regimes, showing that systems traditionally considered classical can allow for equal measurement advantages as a quantum model when appropriately interpreted.

From a wider perspective, this also sheds some light on the classical-to-quantum transition, suggesting that the emergence of quantum-enhanced behavior need not coincide with abrupt changes in nonclassicality indicators; it may instead unfold gradually encoded in how a system responds to parameter variations [66,67]. In this view, classical observables, such as frequencies, can serve as sensitive witnesses of latent quantum correlations, even in regimes where traditional signatures of quantumness are absent [68,69]. This opens a promising direction for quantum metrology in platforms

where direct quantum state control is limited. Weakly interacting or passive systems, which require no state preparation beyond coherent displacements, may still provide access to quantum resources, hidden in plain sight as classical observables [59]. Ultimately, this motivates a deeper investigation into how quantum correlations can be embedded, witnessed, and utilized across a broad range of physical systems, reframing our understanding of the quantum-classical divide in both conceptual and practical terms.

Beyond the specific system considered here, the present framework is not restricted to the ultrastrong-coupling regime. Rather, it applies whenever the interacting system exchanges excitations with its environment through spectrally resolved normal modes, i.e., in the strong-coupling regime where the coherent interaction exceeds the relevant dissipation and decoherence rates. Under these conditions, dissipation is naturally described in the normal-mode basis, which constitutes the central assumption of our analysis. Furthermore, the results can be naturally extended to interacting spin systems that admit a mapping to a harmonic oscillator, such as the Lipkin-Meshkov-Glick model [70–73]. This offers a direct route to certify entanglement (in the form of spin squeezing) by simply monitoring how the energy gap is modified by spin-spin interactions. A more intriguing challenge, however, is to uncover how the energy gap relates to spin squeezing in less symmetric systems that cannot be reduced to a harmonic oscillator description. We anticipate that this connection will hinge on the interplay between the gap and the symmetry properties of the eigenstates [74], potentially opening additional pathways for generating scalable spin squeezing in many-body spin systems. Establishing such a relation would not only broaden the range of experimentally accessible platforms, but could also provide a unifying framework linking spectral properties and metrological performance in complex quantum matter.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: TRANSFORMING THE OPERATORS

The Hamiltonian can be diagonalized by the unitary transformation [45,46]

$$\hat{U} = \exp \left[\theta \frac{(\hat{a} - \hat{a}^\dagger)(\hat{b} + \hat{b}^\dagger)\omega + (\hat{a} + \hat{a}^\dagger)(\hat{b} - \hat{b}^\dagger)\Omega}{2\sqrt{\omega\Omega}} \right] \times \exp \left[\frac{\xi_+}{2}(\hat{b}^2 - \hat{b}^{\dagger 2}) \right] \exp \left[\frac{\xi_-}{2}(\hat{a}^2 - \hat{a}^{\dagger 2}) \right], \quad (\text{A1})$$

with parameters

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{2g\sqrt{\omega\Omega}}{\Omega^2 - \omega^2} \right), \quad (\text{A2})$$

$$\xi_- = -\frac{1}{2} \log \left[\frac{\sqrt{\omega^2 + \Omega^2 - \sqrt{(\omega^2 - \Omega^2)^2 + 4g^2\omega\Omega}}}{\sqrt{2}\omega} \right], \quad (\text{A3})$$

$$\xi_+ = -\frac{1}{2} \log \left[\frac{\sqrt{\omega^2 + \Omega^2 + \sqrt{(\omega^2 - \Omega^2)^2 + 4g^2\omega\Omega}}}{\sqrt{2}\Omega} \right]. \quad (\text{A4})$$

Applying $\hat{U}$ to the bare annihilation operators gives the Bogoliubov transformations

$$\begin{aligned} \hat{c} &= \hat{U}\hat{a}\hat{U}^\dagger = \frac{\cos\theta}{2\sqrt{\omega\omega_-}}[(\omega + \omega_-)\hat{a} + (\omega - \omega_-)\hat{a}^\dagger] \\ &\quad + \frac{\sin\theta}{2\sqrt{\omega\omega_+}}[(\omega + \omega_+)\hat{b} + (\omega - \omega_+)\hat{b}^\dagger], \\ \hat{d} &= \hat{U}\hat{b}\hat{U}^\dagger = \frac{\cos\theta}{2\sqrt{\Omega\omega_+}}[(\Omega + \omega_+)\hat{b} + (\Omega - \omega_+)\hat{b}^\dagger] \\ &\quad - \frac{\sin\theta}{2\sqrt{\Omega\omega_-}}[(\Omega + \omega_-)\hat{a} + (\Omega - \omega_-)\hat{a}^\dagger], \quad (\text{A5}) \end{aligned}$$

with analogous expressions for the Hermitian conjugates.

Each bare mode is thus mapped to a superposition of creation and annihilation operators of both oscillators. The transformation combines a rotation with single-mode squeezing, thereby providing the metric that consistently relates observables in the bare and normal-mode representations.

APPENDIX B: GENERAL INSEPARABILITY CRITERION FOR THE GROUND STATE

In the main text we considered the resonant case $\omega = \Omega$ for which the inseparability criterion acquires a particularly simple form. Here we derive the corresponding expression for arbitrary frequencies. We begin with the Duan inseparability criterion

$$\Delta^2\hat{u} + \Delta^2\hat{v} < 1, \quad (\text{B1})$$

where

$$\hat{u} = \hat{x}_a + \hat{x}_b, \quad \hat{v} = \hat{p}_a - \hat{p}_b. \quad (\text{B2})$$

The quadrature operators of the bare modes are related to those of the normal modes through the Bogoliubov transformation (A5). Introducing the quadratures

$$\hat{x}_c = \frac{\hat{c} + \hat{c}^\dagger}{2}, \quad \hat{p}_c = \frac{\hat{c} - \hat{c}^\dagger}{2i}, \quad (\text{B3})$$

and analogously for mode d , one finds after rewriting Eqs. (A5) that the EPR operators become linear combinations of the normal-mode quadratures

$$\hat{u} = \alpha_c\hat{x}_c + \alpha_d\hat{x}_d, \quad \hat{v} = \beta_c\hat{p}_c + \beta_d\hat{p}_d, \quad (\text{B4})$$

where the coefficients $\alpha_{c,d}$ and $\beta_{c,d}$ are fully determined by the Bogoliubov transformation (A5).

Since the ground state of the interacting Hamiltonian is simply

$$|GS\rangle = |0\rangle_c|0\rangle_d, \quad (\text{B5})$$

the normal modes are independent and all cross correlations vanish

$$\langle\hat{x}_c\hat{x}_d\rangle = \langle\hat{p}_c\hat{p}_d\rangle = 0. \quad (\text{B6})$$

Consequently,

$$\Delta^2\hat{u} = \alpha_c^2\langle\hat{x}_c^2\rangle + \alpha_d^2\langle\hat{x}_d^2\rangle, \quad (\text{B7})$$

$$\Delta^2\hat{v} = \beta_c^2\langle\hat{p}_c^2\rangle + \beta_d^2\langle\hat{p}_d^2\rangle. \quad (\text{B8})$$

Using the vacuum fluctuations of the two normal modes

$$\langle\hat{x}_c^2\rangle = \frac{e^{-2\xi_-}}{4}, \quad \langle\hat{p}_c^2\rangle = \frac{e^{2\xi_-}}{4}, \quad (\text{B9})$$

$$\langle\hat{x}_d^2\rangle = \frac{e^{-2\xi_+}}{4}, \quad \langle\hat{p}_d^2\rangle = \frac{e^{2\xi_+}}{4}, \quad (\text{B10})$$

and substituting the explicit coefficients obtained from Eqs. (A5), straightforward algebra yields

$$\begin{aligned} \Delta^2\hat{u} + \Delta^2\hat{v} &= \frac{(\omega\Omega + \omega_- \omega_+)}{8\omega\omega_- \Omega \omega_+} [(\omega + \Omega)(\omega_- + \omega_+) \\ &\quad - (\omega_- - \omega_+)((\omega - \Omega)\cos(2\theta) - 2\sqrt{\omega\Omega}\sin(2\theta))]. \quad (\text{B11}) \end{aligned}$$

Therefore, the inseparability condition becomes

$$\begin{aligned} &\frac{(\omega\Omega + \omega_- \omega_+)}{8\omega\omega_- \Omega \omega_+} [(\omega + \Omega)(\omega_- + \omega_+) - (\omega_- - \omega_+) \\ &\quad \times ((\omega - \Omega)\cos(2\theta) - 2\sqrt{\omega\Omega}\sin(2\theta))] < 1. \quad (\text{B12}) \end{aligned}$$

Using the definition of the mixing angle

$$\tan(2\theta) = \frac{2g\sqrt{\omega\Omega}}{\Omega^2 - \omega^2}, \quad (\text{B13})$$

one obtains

$$\cos(2\theta) = \frac{\Omega^2 - \omega^2}{\sqrt{(\Omega^2 - \omega^2)^2 + 4g^2\omega\Omega}}, \quad (\text{B14})$$

$$\sin(2\theta) = \frac{2g\sqrt{\omega\Omega}}{\sqrt{(\Omega^2 - \omega^2)^2 + 4g^2\omega\Omega}}, \quad (\text{B15})$$

which immediately leads to

$$\frac{(\omega\Omega + \omega_-\omega_+)\left[(\omega + \Omega)(\omega_- + \omega_+) + \frac{(\omega_- - \omega_+)(-4\omega\Omega g - (\omega - \Omega)^2(\omega + \Omega))}{\sqrt{(\omega^2 - \Omega^2)^2 + 4\omega\Omega g^2}}\right]}{8\omega\omega_- \Omega\omega_+} < 1. \quad (\text{B16})$$

Finally, expressing the interaction strength through the normal-mode frequencies

$$g = \sqrt{\frac{(\omega_-^2 - \omega_+^2)^2 - (\omega^2 - \Omega^2)^2}{4\omega\Omega}}, \quad (\text{B17})$$

we can rewrite the inseparability criterion just with four frequencies

$$\frac{(\omega\Omega + \omega_-\omega_+)\left[(\omega + \Omega)(\omega_- + \omega_+) - \frac{2\sqrt{\omega\Omega}\sqrt{(\omega_-^2 - \omega_+^2)^2 - (\omega^2 - \Omega^2)^2} + (\omega + \Omega)(\omega - \Omega)^2}{\omega_- + \omega_+}\right]}{8\omega\omega_- \Omega\omega_+} < 1. \quad (\text{B18})$$

For a resonant condition, $\omega = \Omega$, and $\omega_+ > \omega_-$ the above expression simplifies to Eq. (16) from the main text.

APPENDIX C: LOGARITHMIC NEGATIVITY FOR TWO COUPLED HARMONIC OSCILLATORS

Since the ground state of the system is completely determined by its normal-mode and noninteracting frequencies, any measure of quantum correlations, including entanglement, can be expressed directly in terms of these frequencies. To quantify the bipartite entanglement between two bosonic modes, we use now the *logarithmic negativity* [75,76], which is one of the most commonly employed entanglement measures for Gaussian states due to its computational simplicity and clear operational meaning.

For a two-mode Gaussian state described by the covariance matrix σ , the logarithmic negativity is defined as

$$E_N = \max[0, -\log_2(2\tilde{\nu}_-)], \quad (\text{C1})$$

where $\tilde{\nu}_-$ denotes the smallest *symplectic eigenvalue* of the *partially transposed* covariance matrix. This quantity captures the degree to which the partial transposition operation (analogous to time reversal of one mode) violates the physicality condition of the covariance matrix. The threshold for separability is given by $2\tilde{\nu}_- = 1$.

(1) If $2\tilde{\nu}_- \geq 1$, the state is separable (no entanglement).

(2) If $2\tilde{\nu}_- < 1$, the state is entangled and the smaller the value of $\tilde{\nu}_-$, the stronger the entanglement.

1. Model and canonical quadratures

We consider the bilinear Hamiltonian used in the main text

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} + \Omega \hat{b}^\dagger \hat{b} + \frac{g}{2}(\hat{a} + \hat{a}^\dagger)(\hat{b} + \hat{b}^\dagger), \quad (\text{C2})$$

describing two harmonic oscillators of frequencies ω and Ω coupled via a position-position interaction of strength g . Introducing canonical quadratures (setting $\hbar = 1$)

$$\hat{X}_a = \frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger), \quad \hat{P}_a = i\sqrt{\frac{\omega}{2}}(\hat{a}^\dagger - \hat{a}), \quad (\text{C3})$$

$$\hat{X}_b = \frac{1}{\sqrt{2\Omega}}(\hat{b} + \hat{b}^\dagger), \quad \hat{P}_b = i\sqrt{\frac{\Omega}{2}}(\hat{b}^\dagger - \hat{b}), \quad (\text{C4})$$

we recover the canonical commutation relations $[\hat{X}_j, \hat{P}_k] = i\delta_{jk}$. In these coordinates, the Hamiltonian becomes

$$\hat{H} = \frac{1}{2}(\hat{P}_a^2 + \hat{P}_b^2) + \frac{1}{2}(\omega^2 \hat{X}_a^2 + \Omega^2 \hat{X}_b^2) + g\sqrt{\omega\Omega} \hat{X}_a \hat{X}_b. \quad (\text{C5})$$

It is convenient to write the position part as $\frac{1}{2}\mathbf{X}^T V \mathbf{X}$, where $\mathbf{X} = (\hat{X}_a, \hat{X}_b)^T$ and

$$V = \begin{pmatrix} \omega^2 & g\sqrt{\omega\Omega} \\ g\sqrt{\omega\Omega} & \Omega^2 \end{pmatrix},$$

while the momentum part remains diagonal, $\frac{1}{2}\mathbf{P}^T \mathbf{P}$.

2. Normal modes and diagonalization

To find the normal modes, we diagonalize the symmetric matrix V . Its eigenvalues correspond to the squared normal-mode frequencies

$$\omega_\pm^2 = \frac{\omega^2 + \Omega^2}{2} \pm \sqrt{\left(\frac{\omega^2 - \Omega^2}{2}\right)^2 + g^2 \omega \Omega}. \quad (\text{C6})$$

The associated orthogonal transformation $R(\theta)$ corresponds to a rotation by an angle θ , determined by

$$\tan(2\theta) = \frac{2g\sqrt{\omega\Omega}}{\Omega^2 - \omega^2} = \sqrt{\frac{(\omega_-^2 - \omega_+^2)^2}{(\omega^2 - \Omega^2)^2} - 1}. \quad (\text{C7})$$

Introducing the rotated quadratures

$$\begin{pmatrix} \hat{X}_+ \\ \hat{X}_- \end{pmatrix} = R(\theta)^T \begin{pmatrix} \hat{X}_a \\ \hat{X}_b \end{pmatrix}, \quad \begin{pmatrix} \hat{P}_+ \\ \hat{P}_- \end{pmatrix} = R(\theta)^T \begin{pmatrix} \hat{P}_a \\ \hat{P}_b \end{pmatrix}, \quad (\text{C8})$$

the Hamiltonian takes the simple form

$$\hat{H} = \frac{1}{2} \sum_{s=\pm} (\hat{P}_s^2 + \omega_s^2 \hat{X}_s^2), \quad (\text{C9})$$

i.e., two independent harmonic oscillators with frequencies $\omega_\pm$.

3. Ground-state covariances

The ground state of each normal mode is a vacuum with covariances

$$\langle \hat{X}_s^2 \rangle = \frac{1}{2\omega_s}, \quad \langle \hat{P}_s^2 \rangle = \frac{\omega_s}{2}, \quad s = \pm. \quad (\text{C10})$$

There are no cross correlations between position and momentum in this basis. Transforming back to the local quadratures yields the covariance blocks (with $c = \cos \theta$, $s = \sin \theta$)

$$\begin{aligned} X_{11} &= c^2 \frac{1}{2\omega_+} + s^2 \frac{1}{2\omega_-}, & X_{22} &= s^2 \frac{1}{2\omega_+} + c^2 \frac{1}{2\omega_-}, \\ X_{12} &= cs \left(\frac{1}{2\omega_+} - \frac{1}{2\omega_-} \right), \end{aligned} \quad (\text{C11})$$

$$\begin{aligned} P_{11} &= c^2 \frac{\omega_+}{2} + s^2 \frac{\omega_-}{2}, & P_{22} &= s^2 \frac{\omega_+}{2} + c^2 \frac{\omega_-}{2}, \\ P_{12} &= cs \left(\frac{\omega_+}{2} - \frac{\omega_-}{2} \right). \end{aligned} \quad (\text{C12})$$

Thus, the full covariance matrix in the ordering $(\hat{X}_a, \hat{X}_b, \hat{P}_a, \hat{P}_b)$ is block-diagonal, $\sigma = \text{diag}(X, P)$.

4. Partial transpose and reduction to a 2×2 problem

Partial transposition with respect to mode b acts in phase space as $\hat{P}_b \mapsto -\hat{P}_b$. This operation leaves the position block unchanged but flips the sign of P_{12} . Denoting the modified momentum block by $\tilde{P}$, the partially transposed covariance matrix is $\tilde{\sigma} = X \oplus \tilde{P}$. For block-diagonal covariances of this type, the squares of the symplectic eigenvalues of $\tilde{\sigma}$ coincide with the eigenvalues of the 2×2 matrix

$$M = X \tilde{P}. \quad (\text{C13})$$

Let $\mu_{\pm}$ be the eigenvalues of M ($\mu_+ \geq \mu_-$). Then the partially transposed symplectic eigenvalues are $\tilde{\nu}_{\pm} = \sqrt{\mu_{\pm}}$, and the relevant one for entanglement is the smaller, $\tilde{\nu}_- = \sqrt{\mu_-}$.

5. Closed form for $\text{tr } M$ and μ_-

Using Eqs. (C11) and (C12), and recalling that the element P_{12} changes sign under partial transposition, one obtains

$$\text{tr } M = X_{11}P_{11} + X_{22}P_{22} - 2X_{12}P_{12}. \quad (\text{C14})$$

It is worth noting that the specific choice of dimensionless or rescaled quadratures (as defined in the main text) does not affect this expression nor the subsequent results. After some straightforward algebra, the above relation can be simplified to the compact form

$$\text{tr } M = \frac{1}{4\omega_+\omega_-} [\omega_+^2 + \omega_-^2 - \cos^2(2\theta)(\omega_+ - \omega_-)^2]. \quad (\text{C15})$$

For the pure ground state, the determinant factorizes as

$$\det M = (\det X)(\det \tilde{P}) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}.$$

Since M is a 2×2 matrix, its eigenvalues are

$$\mu_{\pm} = \frac{1}{2} \left[\text{tr } M \pm \sqrt{(\text{tr } M)^2 - \frac{1}{4}} \right]. \quad (\text{C16})$$

Hence, the smaller eigenvalue is

$$\mu_- = \frac{1}{2} \left[\text{tr } M - \sqrt{(\text{tr } M)^2 - \frac{1}{4}} \right], \quad (\text{C17})$$

from which the smallest symplectic eigenvalue follows as

$$\tilde{\nu}_- = \sqrt{\mu_-}. \quad (\text{C18})$$

6. Resonant case: Simple analytical expressions

In the resonant situation, $\omega = \Omega$, the diagonalization simplifies considerably. The mixing angle becomes $\theta = \pi/4$, for which $\cos(2\theta) = 0$. The normal-mode frequencies reduce to

$$\omega_{\pm}^2 = \omega^2 \pm g\omega, \quad \text{i.e.,} \quad \omega_{\pm} = \omega \sqrt{1 \pm \frac{g}{\omega}}. \quad (\text{C19})$$

Substituting these relations into the above expressions gives

$$\text{tr } M = \frac{1}{4} \left(\frac{\omega_+}{\omega_-} + \frac{\omega_-}{\omega_+} \right), \quad (\text{C20})$$

and with $\det M = 1/16$, one obtains the particularly transparent result

$$\tilde{\nu}_- = \frac{1}{2} \sqrt{\frac{\omega_-}{\omega_+}}. \quad (\text{C21})$$

The logarithmic negativity then follows as

$$E_N = \frac{1}{2} \log_2 \left(\frac{\omega_+}{\omega_-} \right) = \frac{1}{2} \log_2 \left(\sqrt{\frac{1 + g/\omega}{1 - g/\omega}} \right). \quad (\text{C22})$$

These expressions show that the entanglement grows with increasing coupling g , and diverges as $g \rightarrow \omega$, where the lower normal-mode frequency softens to zero, signaling the approach to an instability.

7. Finite temperature

If the two normal modes are thermally populated with mean occupations $n_{\pm} = (e^{\beta\omega_{\pm}} - 1)^{-1}$, the variances $\langle \hat{X}_s^2 \rangle$ and $\langle \hat{P}_s^2 \rangle$ acquire the multiplicative factors $(2n_{\pm} + 1)$. Repeating the rotation and partial transpose steps yields, in the resonant case

$$\tilde{\nu}_-(T) = \frac{1}{2} \sqrt{\frac{2n_- + 1}{2n_+ + 1}} \times \frac{\omega_-}{\omega_+}, \quad (\text{C23})$$

and the thermal logarithmic negativity is then

$$E_N(T) = \max\{0, -\log_2[2\tilde{\nu}_-(T)]\}. \quad (\text{C24})$$

As expected, thermal excitations degrade the entanglement, which vanishes when $2\tilde{\nu}_-(T) \geq 1$.

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