

Achieving optimal multiparameter quantum estimation in optomechanics

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Accurate simultaneous estimation of temperature and effective detuning is crucial for understanding, controlling, and optimizing optomechanical systems. Temperature characterizes the thermal state and mechanical properties, while detuning governs the light-motion interaction, influencing stability and enabling diverse quantum applications. Precise estimates of these parameters are therefore essential to exploit the full potential of such systems in high-precision metrology and the exploration of macroscopic quantum phenomena. In this work we investigate the simultaneous estimation of the effective detuning Δ and the mirror temperature T in a Fabry-Pérot cavity system, with a particular focus on the performance of heterodyne detection. Using analytical expressions for the quantum Fisher information matrices (QFIM) derived from the symmetric logarithmic derivative (SLD) and the right logarithmic derivative (RLD), we demonstrate that the quantum Cramér-Rao bound associated with the RLD is consistently lower than that based on the SLD, highlighting superior estimation precision. This gap between the SLD- and RLD-based bounds becomes especially pronounced in certain physical regimes, emphasizing the relevance of choosing the appropriate formalism in noncommutative estimation scenarios. From a physical perspective, the improved estimation performance stems from the fact that temperature directly influences thermal fluctuations in the mechanical mode, while the effective detuning modulates the strength of the optomechanical interaction, shaping the information exchange between optical and mechanical subsystems. We showed that using higher optical power, greater cavity decay rate, increased photon and phonon numbers, and reduced coupling strength improves the precision of both parameter estimates. These physical mechanisms affect the available quantum resources in the system—such as squeezing, entanglement, and state purity—which in turn determine the ultimate sensitivity achievable. Furthermore, our analysis shows that heterodyne detection can, under certain conditions, approach the ultimate precision set by the QFIM. Thus, a measurement strategy based on heterodyne detection could enable the efficient estimation of temperature, achieving performance close to that of ideal quantum-limited strategies.

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I. INTRODUCTION

The primary goal of quantum metrology, first introduced by Holevo [1] and Helstrom [2], is to measure the parameters defining a given quantum system with higher precision. Quantum metrology thus aims to develop methods that leverage quantum resources to determine the ultimate precision limits under the rules of quantum estimation theory. This approach enables the design of novel quantum sensors, addressing the growing demand for both highly efficient and cost-effective detectors. It establishes the fundamental limit of a physical system, determining the precision with which its parameters can be estimated using a quantum probe. This limit, known as the quantum Cramér-Rao bound (QCRB) [3–6], represents the lowest achievable variance in parameter estimation and

can typically be reached in the context of single-parameter estimation. However, when multiple parameters are estimated simultaneously, saturating this bound becomes more challenging due to the incompatibility of optimal measurements associated with different estimated parameters [7–12]. This issue has led to the development of multiparameter quantum metrology, which aims to generalize certain conditions that allow for the saturation of the QCRB, thereby achieving maximal precision.

In general, the determination of the QCRB relies on the calculation of the Quantum Fisher information matrix (QFIM) [13–15]. This matrix plays a crucial role in multiparameter quantum metrology, as the inverse of this matrix defines the fundamental precision limits in the simultaneous estimation of multiple parameters. Therefore, increasing the QFIM is a key challenge in improving multiparameter estimation protocols. The QFIM is involved in various fields, including phase estimation [16–24], thermometry [25–27], time [28,29], and frequency measurement [30,31], as well as the estimation of squeezing parameters [32–36]. It is also applied in optical interferometry for gravitational wave detection [37] and magnetic field measurement [38–41]. Moreover, it is used in the estimation of the Unruh-Hawking effect [42,43] and in

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the quantification of entanglement and quantum coherence [44–46].

The diverse potential applications of quantum metrology have driven the development of theoretical techniques for calculating the elements of the QFIM. Within this framework, an analytical approach to derive the QFIM for continuous-variable (CV) bosonic systems described by Gaussian states has been proposed. Recently, these states have attracted significant interest in the field of quantum information [47–49], primarily for two key reasons. First, they provide particularly convenient theoretical tools while being easily generated and manipulated experimentally. Second, they have numerous applications in quantum optics [50], teleportation channels [51,52], and cavity optomechanics [53–55]. The latter explores various quantum phenomena arising from radiation-pressure interactions between optical and mechanical degrees of freedom [56], such as the cooling of mechanical oscillators [57–60], optomechanically induced transparency [61–65], and photon and phonon blockade effects [66–68]. Moreover, the relevance of the Gaussian representation is further supported by remarkable experimental observations in Bose-Einstein condensates [69–71]. In addition, recent studies have explored the emergence of quantum collective phenomena in multimode cavity optomechanics platforms, demonstrating how the collective motion of multiple mechanical oscillators can enhance optomechanical coupling strength and enable ground-state cooling of collective modes [72]. Furthermore, robust entanglement generation schemes have been proposed based on breaking the dark-mode effect via synthetic magnetism, which enables strong light-vibration entanglement even at nonzero temperatures, thereby significantly improving noise tolerance in optomechanical networks [73]. These advances highlight new pathways for realizing complex quantum states and enhancing the precision of metrological protocols in hybrid quantum systems.

In parallel, various experimental milestones have confirmed the feasibility of implementing such theoretical optomechanical models. For instance, strong radiation-pressure interactions between optical and mechanical modes have been observed [74], and ground-state cooling of mechanical oscillators has been demonstrated using sideband cooling techniques [75,76]. In addition, quantum signatures such as radiation-pressure shot noise [77] and mechanical zero-point motion [78] have been observed experimentally. Heterodyne detection—central to our estimation scheme—is also well established in cavity optomechanics, enabling simultaneous access to both quadratures of the optical field and providing robust information extraction from the output signal [56]. These achievements confirm that the components of our model are compatible with state-of-the-art experimental platforms.

The evaluation of the QFIM for Gaussian states presents a major challenge due to the complexity of the density matrix formalism. To overcome this difficulty, many studies have adopted the phase-space formalism, which allows describing a Gaussian state solely through its first and second moments. Given the central role of Gaussian states and the significance of multiparameter quantum estimation theory in optimizing measurement precision, it is essential to unify these two approaches within a coherent analytical framework.

In this work we consider an experimental scheme for the simultaneous estimation of two distinct parameters in an optomechanical system. Our study focuses on an optical cavity where a resonant laser beam interacts with a moving (micro) mirror. This setup is based on realistic experimental parameters aligned with current advancements in optics and microfabrication. More specifically, we consider a Fabry-Pérot optical cavity in which one of the mirrors, being significantly lighter than the other, undergoes displacement due to radiation pressure. The displacement of the second mirror involves the excitation of multiple degrees of freedom, each associated with a specific resonance frequency. However, by incorporating a bandpass filter in the detection scheme [79], it is possible to isolate a single mode. Consequently, we focus on a single mechanical mode of the mirror, modeled as a harmonic oscillator, which interacts with the optical field through radiation pressure. In such a system, temperature governs the level of thermal noise in the mechanical resonator, directly affecting decoherence and limiting the observability of quantum effects. On the other hand, the effective detuning Δ sets the strength and nature of the optomechanical interaction by tuning the system into or out of resonance. It impacts crucial phenomena such as sideband cooling, optomechanical entanglement, and squeezing [80–82]. These two parameters influence the steady-state photon and phonon populations, the strength of quantum correlations, and the amount of information available in the output field. Understanding their roles is key to optimizing control strategies and achieving high-precision measurements in quantum optomechanical platforms. To assess the ultimate precision of parameter estimation, we analyze the QFIM using both the symmetric logarithmic derivative (SLD) [83] and the right logarithmic derivative (RLD) [84,85]. The phase-space formalism proves particularly advantageous in this context, as it significantly simplifies the computation of key quantities in multiparameter quantum estimation and provides analytically tractable results for continuous-variable systems.

This paper is structured as follows: Section II presents the enhancement model under investigation, a Fabry-Pérot optical cavity system, along with its associated dynamical equations. Section III introduces the fundamental tools for multiparameter estimation of Gaussian states, deriving the expressions and formulations for the QFIM based on the SLD and RLD operators. It also computes the expression for the classical Fisher information (CFI) using the heterodyne detection method. Section IV applies the results from Sec. III to compute both the QFIM and the CFI, illustrating the practical implications of our findings. Finally, Sec. VI provides a brief conclusion.

II. FABRY-PÉROT CAVITY OPTICAL SYSTEM

A. The model

The system we have chosen in this work is a Fabry-Pérot optical cavity containing two mirrors considered to have different masses. The first mirror (M1) is fixed, while the second (M2) is mobile and moves under the influence of the force exerted by the radiation pressure due to the light intensity. As shown in Fig. 1, the cavity is driven by a coherent laser field with power $\mathcal{P}$ and frequency ω_L . Nevertheless, a

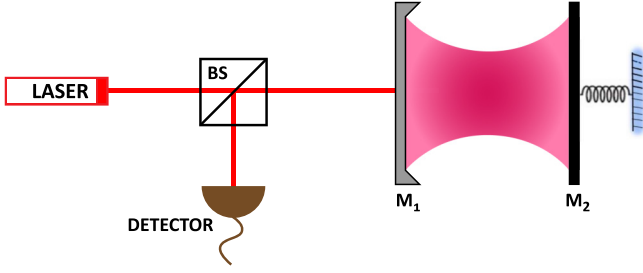


FIG. 1. Schematic representation of an optomechanical single-sided Fabry-Pérot cavity driven by a coherent laser field with frequency ω_L and power $\mathcal{P}$. The output field of the cavity is redirected toward a heterodyne detection setup by using a beam splitter (BS).

single-frequency approach can be adopted if a bandpass filter is applied in the detection scheme [79] and if the interaction between modes is insignificant (negligible). Consequently, we will only consider a single mode of the movable mirror, represented as a quantum-mechanical harmonic oscillator with frequency ω_m . The dynamics of our system are governed by the following Hamiltonian [80,86]:

$$H = \hbar\omega_c b^\dagger b + \frac{\hbar\omega_m}{2}(p^2 + q^2) - \hbar g b^\dagger b q + i\hbar\mathcal{E} (e^{-i\omega_L t} b^\dagger - e^{i\omega_L t} b), \quad (1)$$

where $b^\dagger$ (b) are the creation (annihilation) operators of the mode corresponding to the cavity. The operators p and q represent the momentum and position of the mode associated with the mirror, respectively. g is the coupling coefficient, written as a function of the cavity frequency ω_c , and it broadens when the intensity of the interactive field is zero (in the absence of a field), as given by $g = (\omega_c/L)\sqrt{\hbar/m\omega_m}$. Here, ω_m is the frequency of the mechanical mode, and the effective mass associated with it is denoted by m [79]. In Eq. (1) the last expression is the Hamiltonian associated to the laser field with frequency ω_L and amplitude $|\mathcal{E}| = \sqrt{2\kappa\mathcal{P}/\hbar\omega_L}$ related to the decay rate κ and the power $\mathcal{P}$.

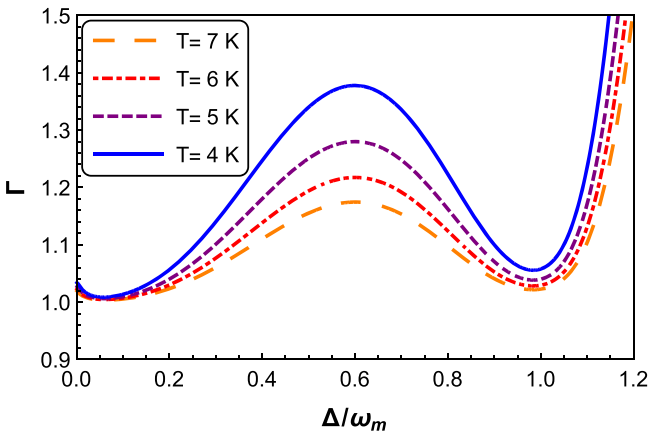


FIG. 2. The ratio between the SLD and RLD QCRB (Γ) as a function of the normalized detuning Δ for different values of the mirror temperature T .

B. Langevin equations

Our study focuses on a noisy system and must take into account the photon losses inside the cavity. Thus, the dynamics in the Heisenberg representation can be described, in the rotating frame, by the set of the following nonlinear Langevin equations:

$$\partial_t q = \omega_m p, \quad (2a)$$

$$\partial_t p = -\omega_m q - \gamma_m p + g b^\dagger b + \zeta, \quad (2b)$$

$$\partial_t b = -(\kappa + i\Delta_0)b + i g b q + \mathcal{E} + \sqrt{2\kappa} b^{\text{in}}, \quad (2c)$$

where $\Delta_0 = \omega_c - \omega_L$ [56] represents the laser detuning, γ_m is the damping rate associated with the mechanical mode, and the input noise for the cavity is denoted by b^{in} . The latter is expressed through the following nonzero correlation function [87,88]:

$$\langle b^{\text{in}\dagger}(t) b^{\text{in}}(t') \rangle = \delta(t - t'). \quad (3)$$

The correlation function is also used to define the Hermitian Brownian noise operator ζ , which can be expressed as a function of the mirror temperature and the Boltzmann constant K , as follows [87]:

$$\langle \zeta(t) \zeta(t') \rangle = \frac{\gamma_m}{\omega_m} \int \frac{d\omega_0}{2\pi} e^{-i\omega_0(t-t')} \omega_0 \left(\coth \left[\frac{\hbar\omega_0}{2KT} \right] + 1 \right). \quad (4)$$

C. Linearization of the Langevin equations

It is always possible to reformulate a Heisenberg operator as a stable value, represented by a c number, combined with an additional fluctuation operator having a zero mean. We then have $q = q_s + \delta q$, $b = \beta_s + \delta b$, and $p = p_s + \delta p$. By substituting these expressions into the equalities defined in Eqs. (2), the equations separate into two distinct sets: one consisting of quantum Langevin equations corresponding to the fluctuation operators, and the other comprising the nonlinear algebraic equations for the steady-state values [89]. The values in the steady-state regime are expressed as $\beta_s = \mathcal{E}/(\kappa + i\Delta)$, $q_s = g|\beta_s|^2/\omega_m$, and $p_s = 0$, with the first equation being nonlinear and defining the amplitude of the interacting field β_s , where $\Delta = \Delta_0 - g^2|\beta_s|^2/\omega_m$ [56] represents the effective detuning of the cavity. Our objective becomes simplified when we assume a laser with a very high input power $\mathcal{P}$, that is, $\beta_s \gg 1$. As a result, the nonlinear terms $\delta b^\dagger \delta b$ and $\delta b \delta q$ can be neglected, leading to the following linear quantum Langevin equations:

$$\partial_t \delta q = \omega_m \delta p, \quad (5a)$$

$$\partial_t \delta p = -\omega_m \delta q - \gamma_m \delta p + G \delta x + \zeta, \quad (5b)$$

$$\partial_t \delta \mathcal{X} = -\kappa \delta \mathcal{X} + \Delta \delta \mathcal{Y} + \sqrt{2\kappa} \mathcal{X}^{\text{in}}, \quad (5c)$$

$$\partial_t \delta \mathcal{Y} = G \delta q - \Delta \delta \mathcal{X} - \kappa \delta \mathcal{Y} + \sqrt{2\kappa} \mathcal{Y}^{\text{in}}. \quad (5d)$$

It is necessary to note that the four equations in Eqs. (5) are derived by choosing an appropriate phase convention for the cavity field, ensuring that b_s becomes real.

In this context we define the noise operators $\mathcal{X}^{\text{in}} = (b^{\text{in}} + b^{\text{in}\dagger})/\sqrt{2}$, and $\mathcal{Y}^{\text{in}} = (b^{\text{in}} - b^{\text{in}\dagger})/i\sqrt{2}$, which correspond to

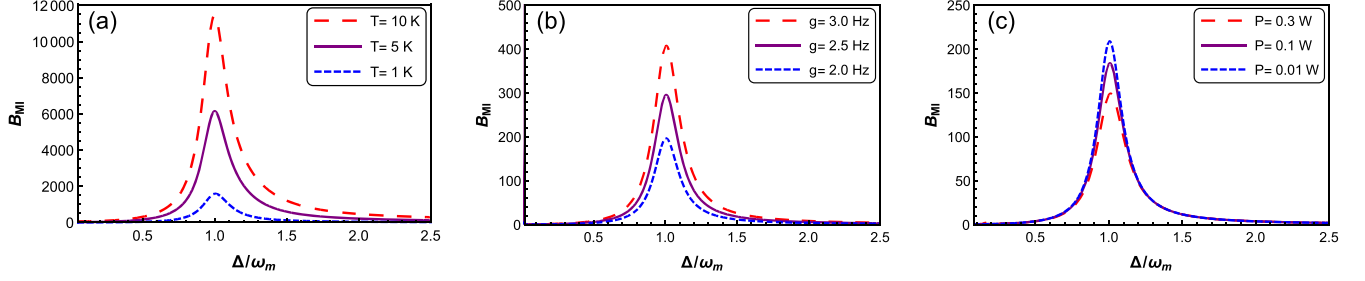


FIG. 3. Most informative QCRB ($\mathcal{B}_{\text{ML}}$) as a function of the normalized detuning Δ : (a) for different mirror temperatures T , (b) for different optomechanical couplings g , and (c) for different driving powers $\mathcal{P}$. The parameters used are identical to those in Fig. 2.

the quadratures of the fields $\delta\mathcal{X} = (\delta b + \delta b^\dagger)/\sqrt{2}$ and $\delta\mathcal{Y} = (\delta b - \delta b^\dagger)/i\sqrt{2}$. The key point is that the effective optomechanical coupling $G = g\beta_s\sqrt{2}$ represents the correlation between the oscillator and the quantum fluctuations of the field. This coupling is significantly enhanced, allowing for the generation of substantial optomechanical entanglement.

D. The covariance matrix

The linearized Langevin equations given by Eq. (5) are written in a general form as $\partial_t \mathbf{u}(t) = \mathbf{A}\mathbf{u}(t) + \mathbf{n}(t)$, where the quadrature vector $\mathbf{u}(t)$ is defined as $\mathbf{u}^T(t) = [\delta q(t), \delta p(t), \delta\mathcal{X}(t), \delta\mathcal{Y}(t)]$, $\mathbf{n}^T(t) = [0, \zeta(t), \sqrt{2\kappa}\mathcal{X}^{\text{in}}(t), \sqrt{2\kappa}\mathcal{Y}^{\text{in}}(t)]$ is the noise vector, and the derivative matrix $\mathbf{A}$ is given as follows:

$$\mathbf{A} = \begin{pmatrix} 0 & \omega_m & 0 & 0 \\ -\omega_m & -\gamma_m & G & 0 \\ 0 & 0 & -\kappa & \Delta \\ G & 0 & -\Delta & -\kappa \end{pmatrix}. \quad (6)$$

The system is said to be stable if and only if the eigenvalues of the matrix $\mathbf{A}$ have negative real parts. This stability can be further analyzed using the Routh-Hurwitz criterion [90], which provides two key constraints on the system parameters that must be satisfied to ensure stability [80],

$$2\kappa\gamma_m[\Delta^4 + (-2\omega_m^2 + \gamma_m^2 + 2\kappa^2 + 2\gamma_m\kappa)\Delta^2 + (\omega_m^2 + \kappa^2 + \gamma_m\kappa)^2] + G^2\Delta\omega_m(2\kappa + \gamma_m)^2 > 0, \quad (7a)$$

$$\omega_m^2(\Delta^2 + \kappa^2) - \omega_m G^2 \Delta > 0. \quad (7b)$$

Since the dynamics are linearized and the noises b^{in} and ζ are quantum Gaussian noises with zero mean, the quantum steady state of the fluctuations corresponds to a bipartite Gaussian state with zero mean. Consequently, the correlation matrix describing this Gaussian state is a covariance matrix of dimension 4×4 , whose elements are generally calculated as $V_{ij} = \langle u_i(t)u_j(t) + u_j(t)u_i(t) \rangle / 2$ [80,91]. To determine the covariance matrix (CM) in the steady state under stability conditions, we solve the following linear algebraic Lyapunov equation [92]:

$$\mathbf{A}\mathbf{V} + \mathbf{V}\mathbf{A}^T = -\mathbf{D}, \quad (8)$$

where $\mathbf{D}$ is the diffusion matrix, with its elements defined as follows: $\langle n_k(t)n_l(t') + n_l(t')n_k(t) \rangle / 2 = \Phi_{kl}(t-t') = \mathcal{D}_{kl}\delta(t-t')$. It can then be noted that $\mathbf{D} = \text{diag}[0, \gamma_m(2n+1), \kappa, \kappa]$. The matrix $\mathbf{V}$ can be directly

calculated from the solution of Eq. (8) and is written in block form as follows:

$$\mathbf{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}, \quad (9)$$

where V_{11} , V_{22} , and V_{12} are 2×2 matrices respectively defined by $V_{11} = \text{diag}[a_{11}/b_{11}, a_{22}/b_{22}]$,

$$V_{22} = \begin{pmatrix} a_{33}/b_{11} & -a_{34}/b_{11} \\ -a_{34}/b_{11} & a_{44}/b_{11} \end{pmatrix},$$

$$V_{12} = \begin{pmatrix} a_{13}/b_{11} & -a_{14}/b_{11} \\ a_{23}/b_{22} & a_{24}/b_{22} \end{pmatrix}.$$

The expressions for the elements of the last three matrices are defined by

$$a_{11} = -\left(\omega_m \left(G^2 \kappa \mu + \gamma_m (2\kappa(\kappa^2 + \Delta^2)^2 (\Delta^2 + (\gamma_m + \kappa)^2) + G^2 \Delta (4(\Delta - \kappa)\kappa(\kappa + \Delta) + \gamma_m(\Delta^2 - 3\kappa^2))\omega_m + 4\kappa(\Delta^2 + \kappa^2)(-\Delta^2 + \kappa(\gamma_m + \kappa))\omega_m^2 - 2G^2 \Delta \kappa \omega_m^3 + 2\kappa(\Delta^2 + \kappa^2)\omega_m^4 \coth \left[\frac{\omega_m \hbar}{2KT} \right] \right) \right), \quad (10)$$

$$a_{13} = G\omega_m \left(\kappa (-2\gamma_m(\Delta^2 + \kappa^2)\omega_m(-\Delta^2 + \kappa(\gamma_m + \kappa) + \omega_m^2) + G^2(-2\Delta(\gamma_m + \kappa)(\Delta^2 + \kappa^2) + \gamma_m\Delta\omega_m^2)) - \gamma_m\Delta(2\kappa(\Delta^2 + \kappa^2)(\Delta^2 + (\gamma_m + \kappa)^2) + G^2\Delta(\gamma_m + 2\kappa)\omega_m - 2\kappa(\Delta^2 + \kappa^2)\omega_m^2) \coth \left[\frac{\omega_m \hbar}{2KT} \right] \right), \quad (11)$$

$$a_{14} = \left(G\omega_m \kappa \left(-2\gamma_m\Delta(\gamma_m + 2\kappa)(\Delta^2 + \kappa^2)\omega_m + G^2(2(\gamma_m + \kappa)^2(\Delta^2 + \kappa^2) + \gamma_m\kappa\omega_m^2) + \gamma_m\mu \coth \left[\frac{\omega_m \hbar}{2KT} \right] \right) \right), \quad (12)$$

$$a_{22} = 2G^2\kappa((\kappa + \gamma_m)(\Delta^2 + \kappa^2) + \kappa\omega_m^2) + \gamma_m(2\kappa(\Delta^2 + \kappa^2)(\Delta^2 + (\kappa + \gamma_m)^2) + G^2\Delta(\gamma_m + 2\kappa)\omega_m + 4\kappa(-\Delta^2 + \kappa(\gamma_m + \kappa))\omega_m^2 + 2\kappa\omega_m^4) \coth \left[\frac{\omega_m \hbar}{2KT} \right], \quad (13)$$

$$a_{23} = 2G\gamma_m \kappa \left((\gamma_m + \kappa)(\Delta^2 + \kappa^2) + \kappa \omega_m^2 - \Delta(\gamma_m + 2\kappa)\omega_m \coth \left[\frac{\omega_m \hbar}{2KT} \right] \right), \quad (14)$$

$$a_{24} = G\kappa \left(-2\gamma_m \Delta(\Delta^2 + \kappa^2) - G^2(\gamma_m + 2\kappa)\omega_m + 2\gamma_m \Delta \omega_m^2 - 2\gamma_m \omega_m (-\Delta^2 + \kappa(\gamma_m + \kappa) + \omega_m^2) \coth \left[\frac{\omega_m \hbar}{2KT} \right] \right), \quad (15)$$

$$a_{33} = \left(\kappa (G^4 \Delta^2 (2\kappa + \gamma_m)\omega_m + \mathbf{v} + G^2 \Delta (2\gamma_m(\kappa^2 + \Delta^2)(\Delta^2 + (\kappa + \gamma_m)^2) - (-2\gamma_m^2 \kappa + 4\kappa(\kappa^2 + \Delta^2) + \gamma_m(5\Delta^2 + \kappa^2))\omega_m^2 + \gamma_m \omega_m^4)) - G^2 \Delta \omega_m \lambda \right), \quad (16)$$

$$a_{34} = G^2 \kappa \omega_m (G^2 \Delta(\gamma_m + \kappa)(\gamma_m + 2\kappa) + \gamma_m \omega_m (-\gamma_m \Delta^2 - 3\Delta^2 \kappa + \gamma_m \kappa^2 + \kappa^3 + \kappa \omega_m^2) + \lambda), \quad (17)$$

$$a_{44} = \left(\kappa (\mathbf{v} + G^2 \Delta (2\gamma_m(\Delta^2 + \kappa^2)(\Delta^2 + (\gamma_m + \kappa)^2) + (6\gamma_m^2 \kappa - 4\kappa(\Delta^2 + \kappa^2) + \gamma_m(-7\Delta^2 + 5\kappa^2))\omega_m^2 + 3\gamma_m \omega_m^4) - G^4 \omega_m (2\gamma_m^2 \kappa - 4\Delta^2 \kappa + 2\kappa^3 + \gamma_m (-3\Delta^2 + 4\kappa^2 + \omega_m^2))) + G^2 \gamma_m \omega_m (-2\kappa^3(\Delta^2 + (\gamma_m + \kappa)^2) + G^2 \Delta(\gamma_m + 2\kappa)\omega_m - (2\kappa(\Delta^2 + \kappa^2) + \gamma_m(\Delta^2 + 2\kappa^2))\omega_m^2) \coth \left[\frac{\omega_m \hbar}{2KT} \right] \right), \quad (18)$$

$$b_{11} = (2(G^2 \Delta - (\kappa^2 + \Delta^2)\omega_m)1/2 b_{22}), \quad (19)$$

$$b_{22} = 4\gamma_m \kappa (\Delta^2 + \kappa^2)(\Delta^2 + (\kappa + \gamma_m)^2) + 2G^2 \Delta (2\kappa + \gamma_m)^2 \omega_m + 8\gamma_m \kappa (-\Delta^2 + \kappa(\kappa + \gamma_m)\omega_m^2 + 4\gamma_m \kappa \omega_m^4), \quad (20)$$

$$\mu = 2\kappa(\kappa^2 + \Delta^2)(\Delta^2 + (\kappa + \gamma_m)^2) - G^2 \Delta (2\kappa + \gamma_m)\omega_m + 2(\kappa + \gamma_m)(\Delta^2 + \kappa^2)\omega_m^2, \quad (21)$$

$$\mathbf{v} = 2\gamma_m(\Delta^2 + \kappa^2)\omega_m (-(\Delta^2 + \kappa^2)(\Delta^2 + (\kappa + \gamma_m)^2) - 2(-\Delta^2 + \kappa(\kappa + \gamma_m))\omega_m^2 - \omega_m^4), \quad (22)$$

$$\lambda = \gamma_m \Delta (2\kappa(\Delta^2 + (\gamma_m + \kappa)^2) + \gamma_m \omega_m^2) \coth \left[\frac{\omega_m \hbar}{2KT} \right]. \quad (23)$$

III. MULTIPARAMETER ESTIMATION THEORY IN GAUSSIAN QUANTUM STATES

A quantum or classical system is generally defined by a Hamiltonian, associated with a density matrix ρ_φ , where $\varphi = \{\varphi_1, \varphi_2, \dots, \varphi_n\}$ denotes a set of unknown parameters to be estimated. These parameters, assumed to lie within an open subset of $\mathbb{R}^n$, are typically encoded into the systems dynamics through an evolution map θ_φ . To estimate them, one first

prepares a probe state ρ which evolves under the transformation $\rho \xrightarrow{\theta_\varphi} \rho_\varphi$. Then, by performing a generalized measurement (POVM) $\{\Pi_x\}$ on the output state ρ_φ , a measurement outcome x is obtained with probability $P(x/\varphi) = \text{Tr}[\rho_\varphi \Pi_x]$. An estimator $\hat{\varphi} = \{\hat{\varphi}_1, \hat{\varphi}_2, \dots, \hat{\varphi}_n\}$ is then constructed based on the measurement data to infer the values of the unknown parameters. Consequently, the quantum Cramér-Rao bound provides an optimal limit achievable in quantum estimation theory, a framework widely applied in metrology protocols to achieve maximum precision in parameter estimation. To compute the QCRB, the QFIM must be determined, commonly calculated using two methods based on the SLD and the RLD. These derivatives are derived from specific differential equations [93–95],

$$\partial_{\varphi_i} \hat{\rho} = \frac{1}{2} \{ \hat{\rho} \hat{L}_{\varphi_i}^S \} \quad (\text{SLD}), \quad (24)$$

$$\partial_{\varphi_i} \hat{\rho} = \hat{\rho} \hat{L}_{\varphi_i}^R \quad (\text{RLD}), \quad (25)$$

where ∂_{φ_i} represents the partial derivative, and $\{A, B\}$ denotes the anticommutation between A and B . Thus, the QFIMs associated with the SLD and RLD are respectively given by

$$\mathcal{H}_{\varphi_i \varphi_j} = \frac{1}{2} \text{Tr}[\hat{\rho} \{ \hat{L}_{\varphi_i}^S, \hat{L}_{\varphi_j}^S \}], \quad (26)$$

$$\mathcal{F}_{\varphi_i \varphi_j} = \text{Tr}[\hat{\rho} \hat{L}_{\varphi_i}^R \hat{L}_{\varphi_j}^{R\dagger}]. \quad (27)$$

The QCRBs associated are expressed respectively by

$$\text{cov}(\hat{\varphi}) \geq \frac{1}{\nu} \mathcal{H}^{-1}, \quad (28)$$

$$\text{cov}(\hat{\varphi}) \geq \frac{1}{\nu} \text{Re}(\mathcal{F}^{-1}) + |\text{Im}(\mathcal{F}^{-1})|, \quad (29)$$

where $\mathcal{F}^{-1} = \text{Re}(\mathcal{F}^{-1}) + i|\text{Im}(\mathcal{F}^{-1})|$. The symbol $|\bullet|$ represents the absolute value of the quantity $\bullet$, and ν symbolizes the number of measurements taken. The covariance matrix $\text{cov}(\hat{\varphi})$ is defined as [96]

$$\begin{aligned} \text{cov}(\varphi_i, \varphi_j) &= E[(\varphi_i - E(\varphi_i))(\varphi_j - E(\varphi_j))] \\ &= E(\varphi_i \varphi_j) - E(\varphi_i)E(\varphi_j). \end{aligned} \quad (30)$$

In particular, when $i \neq j$, the estimation method at an individual level is equivalent to $\mathcal{H}_{\varphi_i \varphi_j} = \mathcal{F}_{\varphi_i \varphi_j} = 0$. To do this, theorists have quantified the optimal measure of a parameter by the variance. Thus, Eqs. (28) and (29) take the following form:

$$\text{Var}(\varphi_i) \geq \frac{1}{\nu} \mathcal{H}_{\varphi_i \varphi_i}^{-1}, \quad (31)$$

$$\text{Var}(\varphi_i) \geq \frac{1}{\nu} (\mathcal{F}_{\varphi_i \varphi_i, \text{Re}}^{-1} + |\mathcal{F}_{\varphi_i \varphi_i, \text{Im}}^{-1}|). \quad (32)$$

Applying the trace operator to the two saturated Eqs. (31) and (32) corresponds to the sum over the variances of the estimated parameters. The saturation of Eqs. (31) and (32) represents an optimal evolution of the estimated parameter. They then become as follows:

$$\sum_i^N \text{Var}(\varphi_i) \geq B_S = \frac{1}{\nu} \text{Tr}[\mathcal{H}^{-1}], \quad (33)$$

$$\sum_i^N \text{Var}(\varphi_i) \geq B_R = \frac{1}{\nu} (\text{Tr}[\mathcal{F}_{\text{Re}}^{-1}] + \text{Tr}[|\mathcal{F}_{\text{Im}}^{-1}|]). \quad (34)$$

Generally, the challenge of simultaneously estimating multiple parameters arises from the incompatibility of their optimal measurements, a consequence of the fundamental noncommutativity of quantum mechanics. This noncommutativity prevents the simultaneous attainment of the optimal limits associated with the RLD and SLD, as optimizing the measurement for one parameter typically degrades the accuracy of others. Furthermore, the optimal estimator for the RLD bound might not correspond to a physically realizable Positive Operator-Valued Measure. However, despite the potential differences in optimal measurements for individual parameters, a single, simultaneous measurement can still be designed to achieve both of these limits. Conversely, it is well established that the quantum Fisher information (QFI) based on the SLD remains lower than that based on RLD in the estimation of a single parameter. However, the QFI based on the SLD provides a strictly precise bound for the parameters [2]. In the context of quantum estimation theory for multiple parameters, several studies [7–10,12,54] have relied on the SLD during the calculation steps. These studies have generally shown that the saturation of the QCRB given by Eqs. (28) and (31) is valid only if

$$\text{Tr}[\hat{\rho}[\hat{L}_{\varphi_i}^S, \hat{L}_{\varphi_j}^S]] = 0. \quad (35)$$

According to Ref. [97], it is obvious that condition (35) can be reformulated in another equivalent form as follows:

$$\text{Im}(\text{Tr}[\hat{\rho} \hat{L}_{\varphi_i}^S \hat{L}_{\varphi_j}^S]) = 0. \quad (36)$$

The failure to satisfy the above condition implies that the RLD bound becomes more significant than the SLD bound.

First, it is important to note that the two bounds, B_S associated with SLD and B_R associated with RLD, differ, especially in their mathematical derivations, as each is determined by a distinct method. To identify the most relevant bound, the authors of Refs. [93,94] introduced a quantity that corresponds to the maximum between the two QCRBs, based on B_S and B_R . This quantity is called the most informative Cramér-Rao quantum bound (B_{MI}) and is defined as follows:

$$B_{\text{MI}} = \text{Min}\{B_S, B_R\}, \quad (37)$$

with

$$B_S = \frac{\mathcal{H}_{TT} + \mathcal{H}_{\Delta\Delta}}{\nu(\mathcal{H}_{\Delta\Delta} \cdot \mathcal{H}_{TT} - \mathcal{H}_{T\Delta} \cdot \mathcal{H}_{\Delta T})},$$

$$B_R = \frac{\text{Re}(\mathcal{F}_{TT}) + \text{Re}(\mathcal{F}_{\Delta\Delta})}{\nu \text{Re}(\det(\mathcal{F}))} + \frac{|\text{Im}(\mathcal{F}_{TT})| + |\text{Im}(\mathcal{F}_{\Delta\Delta})|}{\nu \text{Im}(\det(\mathcal{F}))}. \quad (38)$$

To determine the minimum value between B_S and B_R , we compared the two bounds by calculating their ratio, given by

$$\Gamma = \frac{B_S}{B_R}. \quad (39)$$

Generally, the ratio Γ takes the following three different cases:

$$\begin{aligned} \text{if } \Gamma > 1 &\iff B_{\text{MI}} = B_R, \\ \text{if } \Gamma < 1 &\iff B_{\text{MI}} = B_S, \\ \text{if } \Gamma = 1 &\iff B_{\text{MI}} = B_S = B_R. \end{aligned} \quad (40)$$

From Eq. (37), the optimal measurements of multiparameter protocols depend solely on the informative QCRB, which is

expressed by the following inequality:

$$\sum_i^N \text{Var}(\varphi_i) \geq \frac{1}{\nu} B_{\text{MI}}. \quad (41)$$

In the framework of quantum metrology, the calculation of the QFI provides a higher estimation precision for a parameter compared to the CFI. In this section we explicitly determine the SLD and RLD operators for quantum Gaussian states. Based on these expressions, we derive the QFIMs associated with the SLD and RLD, respectively. Finally, we obtain the analytical expression of the CFI based on the SLD operator for the one-parameter estimation.

A. Quantum context based on SLD and RLD

Generally, the elements of the QFIM directly depend on the SLD and the RLD, given respectively by Eqs. (24) and (25). To determine the expressions for the QFIM elements for Gaussian states, we consider that the SLD and RLD can also be written in a quadratic form in terms of the canonical operators as follows:

$$\hat{L}_{\varphi_i}^S = \hat{L}^{S_0} + \hat{L}_j^{S_1} \hat{\mathbf{r}}_j + \hat{L}_{kl}^{S_2} \hat{\mathbf{r}}_k \hat{\mathbf{r}}_l, \quad (\text{SLD}) \quad (42)$$

where the vector $\hat{\mathbf{r}} = [\hat{q}_1, \hat{p}_1, \dots, \hat{q}_n, \hat{p}_n]^T$ and the three terms $\hat{L}^{S_0}$, $\hat{L}^{S_1}$, and $\hat{L}^{S_2}$ are given by (detailed calculation provided in Refs. [83,94])

$$\hat{L}_{\varphi_i}^{S_0} = -\frac{1}{2} \text{Tr}[V \hat{L}_{\varphi_i}^{S_2}] - d_r^T \hat{L}_{\varphi_i}^{S_1} - d_r^T \hat{L}_{\varphi_i}^{S_2} d_r, \quad (\in \mathbb{C}), \quad (43a)$$

$$\hat{L}_{\varphi_i}^{S_1} = 2V^{-1} \partial_{\varphi_i} d_r - 2\hat{L}_{\varphi_i}^{S_2} d_r, \quad (\text{vector in } \mathbb{C}_{2N}), \quad (43b)$$

$$\text{vec}[\hat{L}_{\varphi_i}^{S_2}] = (V^\dagger \otimes V + \Omega \otimes \Omega)^+ \text{vec}[\partial_{\varphi_i} V],$$

(matrix in $\mathbb{C}_{2N \times 2N}$), (43c)

where $d_r = \langle \mathbf{r} \rangle$ is a displacement vector, and $\Omega = \bigoplus_{k=1}^n i \sigma_y$, with σ_y as the second Pauli matrix. Similarly, the expression for the RLD is given as follows:

$$\hat{L}_{\varphi_i}^R = \hat{L}^{R_0} + \hat{L}_j^{R_1} \hat{\mathbf{r}}_j + \hat{L}_{kl}^{R_2} \hat{\mathbf{r}}_k \hat{\mathbf{r}}_l, \quad (\text{RLD}) \quad (44)$$

where the calculation of the quantities below is provided in Ref [94]:

$$\hat{L}_{\varphi_i}^{R_0} = -\frac{1}{2} \text{Tr}[\mathcal{J}_+ \hat{L}_{\varphi_i}^{R_2}] - d_r^T \hat{L}_{\varphi_i}^{R_1} - d_r^T \hat{L}_{\varphi_i}^{R_2} d_r, \quad (\in \mathbb{R}), \quad (45a)$$

$$\hat{L}_{\varphi_i}^{R_1} = 2\mathcal{J}_+^{-1} \partial_{\varphi_i} d_r - 2\hat{L}_{\varphi_i}^{R_2} d_r, \quad (\text{vector in } \mathbb{R}_{2N}), \quad (45b)$$

$$\text{vec}[\hat{L}_{\varphi_i}^{R_2}] = (\mathcal{J}^\dagger \otimes \mathcal{J})^+ \text{vec}[\partial_{\varphi_i} V],$$

(matrix in $\mathbb{R}_{2N \times 2N}$). (45c)

$\mathcal{J}$ is a matrix of dimension $2N \times 2N$ that can be expressed in terms of the covariance matrix $\mathcal{V}$ Eqs. (9) and the matrix Ω as $\mathcal{J} = 2\mathcal{V} + i\Omega$, and $\text{vec}[m]$ represents the vectorization of the matrix m , which constructs a column vector by rearranging the columns of the matrix m one after the other, in the form

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mapsto \quad \text{vec}[m] = (a, c, b, d)^T. \quad (46)$$

By inserting the expressions of the SLD and RLD into Eqs. (26) and (27), respectively, the QFIM-SLD and QFIM-RLD are given in the following form (see Refs. [83,94]):

$$\mathcal{H}_{\varphi_i\varphi_j} = 2 \text{vec}[\partial_{\varphi_i}\mathcal{V}]^\dagger \mathfrak{M}^{-1} \text{vec}[\partial_{\varphi_j}\mathcal{V}] + \partial_{\varphi_i}d_r^T \mathcal{V}^{-1} \partial_{\varphi_j}d_r, \quad (47)$$

$$\mathcal{F}_{\varphi_i\varphi_j} = 2 \text{vec}[\partial_{\varphi_i}\mathcal{V}]^\dagger \Sigma^{-1} \text{vec}[\partial_{\varphi_j}\mathcal{V}] + 2\partial_{\varphi_i}d_r^T \mathcal{J}^{-1} \partial_{\varphi_j}d_r, \quad (48)$$

with $\mathfrak{M} = (4\mathcal{V}^\dagger \otimes \mathcal{V} + \Omega \otimes \Omega)$, and $\Sigma = (\mathcal{J}^\dagger \otimes \mathcal{J})$. Similarly, when $\mathfrak{M}$ and Σ are two noninvertible matrices (the inverse of these matrices does not exist), we move to a more general case. In this case, we replace the invertible matrices $\mathfrak{M}^{-1}$ and Σ^{-1} with $\mathfrak{M}^+$ and Σ^+ , where the “+” index, known as the Moore-Penrose pseudoinverse, is considered a generalization of the matrix inverse [98,99]. Consequently, the QFIMs SLD and RLD take the form of the following expressions:

$$\mathcal{H}_{\varphi_i\varphi_j} = 2 \text{vec}[\partial_{\varphi_i}\mathcal{V}]^\dagger \mathfrak{M}^+ \text{vec}[\partial_{\varphi_j}\mathcal{V}] + \partial_{\varphi_i}d_r^T \mathcal{V}^{-1} \partial_{\varphi_j}d_r, \quad (49)$$

$$\mathcal{F}_{\varphi_i\varphi_j} = 2 \text{vec}[\partial_{\varphi_i}\mathcal{V}]^\dagger \Sigma^+ \text{vec}[\partial_{\varphi_j}\mathcal{V}] + 2\partial_{\varphi_i}d_r^T \mathcal{J}^{-1} \partial_{\varphi_j}d_r. \quad (50)$$

B. Classical context

We consider a very powerful measurement strategy using positive operator-valued measures POVM $\hat{\Pi}_x \in (\text{all POVMs})$, which minimizes the mean-squared error (MSE) of parameter estimation and provides better saturation of the QCRB for an unbiased estimator. The POVM Π_x satisfies the completeness condition $\sum_x \hat{\Pi}_x^\dagger \hat{\Pi}_x = \mathbb{1}$ [100–103]. Quantum systems, during their measurements, produce a conditional probability distribution that is responsible for providing the maximal Fisher information. The expression for this probability is given as follows:

$$P(x/\varphi) = \text{Tr}[\rho_\varphi \Pi_x], \quad (51)$$

where $P(x/\varphi)$ represents the conditional probability of estimating the parameter x at the end of the measurement, when φ defines the exact value of the parameter. Consequently, to measure this parameter, the authors of Refs. [100–103] have defined the classical Fisher information (CFI) as follows:

$$H_\varphi^c = \int \frac{1}{P(x/\varphi)} \left[\frac{\partial P(x/\varphi)}{\partial \varphi} \right]^2 dx. \quad (52)$$

In general, $\mathcal{F}(x) \geq H^c$, where $\mathcal{F}(x)$ represents the QFI for a single parameter x . To achieve the exceptional case, i.e., $\mathcal{F}(x) = H^c$, it is necessary to select an optimal measurement strategy. This strategy, based on the SLD, is defined by $\partial_\varphi \rho = (\hat{\rho} \hat{L} + \hat{L} \hat{\rho})/2$ for the estimation of a single parameter. However, this optimal configuration can be challenging to implement experimentally and may even be inaccessible in

certain cases. This suggests that the maximum estimation precision defined by the QFI might not be practically achievable.

In this work we present the CFI obtained through heterodyne detection. We also demonstrate that this CFI can closely approach the improved QFI generated by our system. Heterodyne detection serves as a key strategy for Gaussian measurement, where the field to be measured is combined with a reference field having a different frequency. This technique allows simultaneous access to both optical quadratures (position and momentum), which is particularly advantageous for multiparameter estimation. Compared to homodyne detection, which measures only a single quadrature, heterodyne detection provides more complete information about the optical state. It is also less sensitive to phase fluctuations of the local oscillator and is widely used in experimental setups [104,105]. Formally, heterodyne detection relies on a projection operator based on coherent states expressed as $[|\alpha\rangle\langle\alpha|/\pi]$ [56]. This allows for determining the CFI associated with this measurement method, as described in Refs. [54,106]:

$$H_\varphi = \frac{1}{2} \text{Tr}[(V^{-1} \partial_\varphi V)^2] + \partial_\varphi d_r^T V^{-1} \partial_\varphi d_r. \quad (53)$$

The new covariance matrix is expressed as a function of the covariance matrix that describes our system [Eq. (9)] and is given by $V = \mathcal{V} + \mathbb{1}_4$, in which $\mathbb{1}_4$ is the identity matrix of dimension 4×4 .

IV. APPLICATION TO THE MODEL

In this section we examine how various physical parameters affect the precision of estimating the effective detuning Δ and the mirror temperature T . The temperature governs the thermal state of the mechanical resonator, influencing the average phonon number, the decoherence rate, and the accessibility of quantum regimes. A precise estimation of T is therefore crucial for minimizing thermal noise and enabling efficient cooling or nonclassical state generation [81,91]. Similarly, the effective detuning Δ , defined as the frequency mismatch between the driving laser and the optical cavity, directly modulates the optomechanical coupling strength and affects the populations of intracavity photons and mechanical phonons. This parameter is fundamental to phenomena such as sideband cooling, quantum squeezing, and entanglement [81,82]. Its precise real-time estimation is required to ensure optimal control of the system in the presence of laser drifts or environmental perturbations [80]. In what follows we analyze the role of the Fisher information in enhancing the accuracy of estimating both parameters, with a particular emphasis on the mirror temperature. A comparison between the classical and quantum Fisher information is performed to highlight their differences in estimation performance. Additionally, we identify the most informative quantum Cramér-Rao bound ($\mathcal{B}_{\text{MI}}$), which governs the precision in estimating both the effective detuning Δ and the mirror temperature T . In this context, based on Eqs. (47), (48), and (53), we compute the QFIMs and CFI, respectively, as follows:

$$\mathcal{H}_{\Delta T} = 2 \begin{bmatrix} \text{vec}[\partial_\Delta \mathcal{V}]^\dagger \mathfrak{M}^{-1} \text{vec}[\partial_\Delta \mathcal{V}] + \frac{1}{2} \partial_\Delta d_r^T \mathcal{V}^{-1} \partial_\Delta d_r & \text{vec}[\partial_\Delta \mathcal{V}]^\dagger \mathfrak{M}^{-1} \text{vec}[\partial_T \mathcal{V}] + \frac{1}{2} \partial_\Delta d_r^T \mathcal{V}^{-1} \partial_T d_r \\ \text{vec}[\partial_T \mathcal{V}]^\dagger \mathfrak{M}^{-1} \text{vec}[\partial_\Delta \mathcal{V}] + \frac{1}{2} \partial_T d_r^T \mathcal{V}^{-1} \partial_\Delta d_r & \text{vec}[\partial_T \mathcal{V}]^\dagger \mathfrak{M}^{-1} \text{vec}[\partial_T \mathcal{V}] + \frac{1}{2} \partial_T d_r^T \mathcal{V}^{-1} \partial_T d_r \end{bmatrix}, \quad (54)$$

$$\mathcal{F}_{\Delta T} = 2 \begin{bmatrix} \text{vec}[\partial_{\Delta} \mathcal{V}]^{\dagger} \Sigma^{-1} \text{vec}[\partial_{\Delta} \mathcal{V}] + \partial_{\Delta} d_r^{\text{T}} \mathcal{J}^{-1} \partial_{\Delta} d_r & \text{vec}[\partial_{\Delta} \mathcal{V}]^{\dagger} \Sigma^{-1} \text{vec}[\partial_T \mathcal{V}] + \partial_{\Delta} d_r^{\text{T}} \mathcal{J}^{-1} \partial_T d_r \\ \text{vec}[\partial_T \mathcal{V}]^{\dagger} \Sigma^{-1} \text{vec}[\partial_{\Delta} \mathcal{V}] + \partial_T d_r^{\text{T}} \mathcal{J}^{-1} \partial_{\Delta} d_r & \text{vec}[\partial_T \mathcal{V}]^{\dagger} \Sigma^{-1} \text{vec}[\partial_T \mathcal{V}] + \partial_T d_r^{\text{T}} \mathcal{J}^{-1} \partial_T d_r \end{bmatrix}, \quad (55)$$

$$H_T = \frac{1}{2} (\text{Tr}[(V^{-1} \partial_T V)^2] + 2 \partial_T d_r^{\text{T}} V \partial_T d_r), \quad (56)$$

where

$$d_r = \left[\frac{2 g \kappa \mathcal{P}}{\hbar \omega_L (\kappa^2 + \Delta^2)}, 0, \frac{2 \sqrt{\kappa \mathcal{P}}}{\sqrt{\hbar \omega_L} \sqrt{\kappa^2 + \Delta^2}}, 0 \right]^{\text{T}} \quad (57)$$

is a displacement vector.

The QFIMs defined in the last two equations are highly complex to compute analytically due to the presence of the inverse matrices $\mathfrak{M}$ and Σ , both of which are 16×16 matrices. This makes the calculation particularly challenging in our case. Therefore, the evaluation is performed entirely numerically. The parameters used in the numerical simulations are normalized as Δ/ω_m and T/T_0 . Consequently, the corresponding Fisher information elements are also dimensionless; $\tilde{F}_{\Delta\Delta} = \omega_m^2 F_{\Delta\Delta}$, $\tilde{F}_{TT} = T_0^2 F_{TT}$, and $\tilde{F}_{\Delta T} = \omega_m T_0 F_{\Delta T}$. Similarly, the associated dimensionless QCRBs are expressed as $\tilde{\text{Var}}(\Delta/\omega_m) = (F^{-1})_{\Delta\Delta}/\omega_m^2$ and $\tilde{\text{Var}}(T/T_0) = (F^{-1})_{TT}/T_0^2$. This normalization ensures a consistent dimensional analysis and enables direct comparison across different parameter regimes. Therefore, the Fisher information and QCRBs shown in the subsequent figures are presented in normalized units, illustrating the behavior of the estimation precision for both Δ and T . To determine the most effective method for achieving better estimation results, we evaluate the ratio Γ , as given by Eq. (39). Unless otherwise specified, the numerical parameters used in this study are as follows:

$$\begin{aligned} \kappa/2\pi &= 10^5 \text{ Hz}, \quad \omega_m/2\pi = 10^6 \text{ Hz}, \quad \omega_L/2\pi = 2.8 \times 10^{14} \text{ Hz}, \\ \gamma_m/2\pi &= 10^2 \text{ Hz}, \quad \mathcal{P} = 50 \text{ mW}, \quad T = 100 \text{ mK}, \quad g = 2 \text{ Hz}. \end{aligned}$$

These parameters are chosen to be consistent with values reported in state-of-the-art optomechanical experiments and theoretical proposals [56,74,75,77]. For example, cavity decay rates on the order of $\kappa/2\pi \approx 10^5$ Hz and mechanical resonance frequencies around $\omega_m/2\pi \approx 10^6$ Hz are typical for Fabry-Pérot cavities with micromechanical oscillators [56,74]. Mechanical damping rates of $\gamma_m/2\pi \approx 10^2$ Hz are common in high- Q mechanical resonators [56], and cryogenic operating temperatures near $T \approx 100$ mK can be routinely achieved with dilution refrigerators [76,78,107,108]. The laser frequency $\omega_L/2\pi \approx 2.8 \times 10^{14}$ Hz corresponds to near-infrared light (1064 nm), a standard wavelength for optical cavity experiments [56]. The input power $P = 50$ mW is well within the range used for driving optical cavities, and the single-photon optomechanical coupling rate $g \approx 2$ Hz lies in the weak-coupling regime commonly observed in cavity optomechanics [56]. These considerations confirm that the chosen parameter set is experimentally feasible with current technology. Figure 2 illustrates the evolution of the ratio Γ between the two QCRBs based on the SLD and RLD as a function of the effective cavity detuning Δ for different values of the mirror temperature T . From the figure we observe

that the ratio Γ increases consistently with temperature. It reaches its minimum values for low Δ , particularly around $\Delta = \omega_m$, and attains its maximum at around $\Delta/\omega_m = 0.6$ and for $\Delta > \omega_m$. Beyond this point, Γ increases rapidly with both Δ and T . These results show that Γ always remains greater than or equal to 1 for any value of the effective detuning Δ and mirror temperature T within the parameter regimes considered here; this is not a universal property, and Γ may be less than 1 in other physical systems or parameter regimes. In detail, the three bounds $\mathcal{B}_{\text{MI}}$, $\mathcal{B}_R$, and $\mathcal{B}_S$ coincide when $\Delta = 0$ and when $\Delta = \omega_m$, and $\mathcal{B}_M = \mathcal{B}_R$ in all other cases. Overall, the QCRB based on the RLD provides better and more efficient estimation results compared to the one based on the SLD. Physically, the parameter Γ reflects the relative efficiency of the two quantum bounds (SLD and RLD) in estimating parameters within the optomechanical system. Its introduction enables a direct comparison between these two fundamental approaches, helping to identify which provides the best estimation performance under varying system conditions. Such analysis is essential for guiding methodological choices and optimizing metrological performance in practical experimental settings.

Figure 3 illustrates the most informative quantum Cramér-Rao bound ($\mathcal{B}_{\text{MI}}$), determined from the QFIM based on RLD formalism. This bound is plotted as a function of the effective cavity detuning Δ in three different configurations: Fig. 3(a) for various values of the mirror temperature T , Fig. 3(b) for various values of the optomechanical coupling strength g , and Fig. 3(c) for different optical powers $\mathcal{P}$. It is observed that when the three parameters T , g , and $\mathcal{P}$ are fixed, the QCRB reaches minimal values at both low and high values of Δ , indicating a large amount of accessible information in the system for the estimation of the two parameters. Conversely, when Δ approaches the mechanical frequency ω_m , the QCRB becomes very large, reflecting a significant loss of quantum information available for accurately estimating Δ and T . In this intermediate region, the interaction between the optical field and the mechanical mode is enhanced but thermal and optical effects become strongly intertwined, making it difficult to distinguish the individual contributions of T and Δ . As a result, multiparameter estimation becomes less efficient in this region, which acts as a critical point of precision loss. On the other hand, when the ratio Δ/ω_m is very small or significantly greater than 1.5, the QCRB becomes minimal or nearly vanishes, indicating improved precision in parameter estimation. In the regime $\Delta \ll \omega_m$, the system exhibits relatively weak optomechanical interaction. Although thermal effects dominate, they are more easily identifiable, which facilitates the estimation of the temperature T . In contrast, when $\Delta/\omega_m > 1.5$, the system operates in a more linear and stable dynamical regime, where Δ and T influence the system more independently. In these regimes, thermal and optomechanical effects are better separated, allowing for a clearer extraction of quantum information related to each parameter. Thus, the

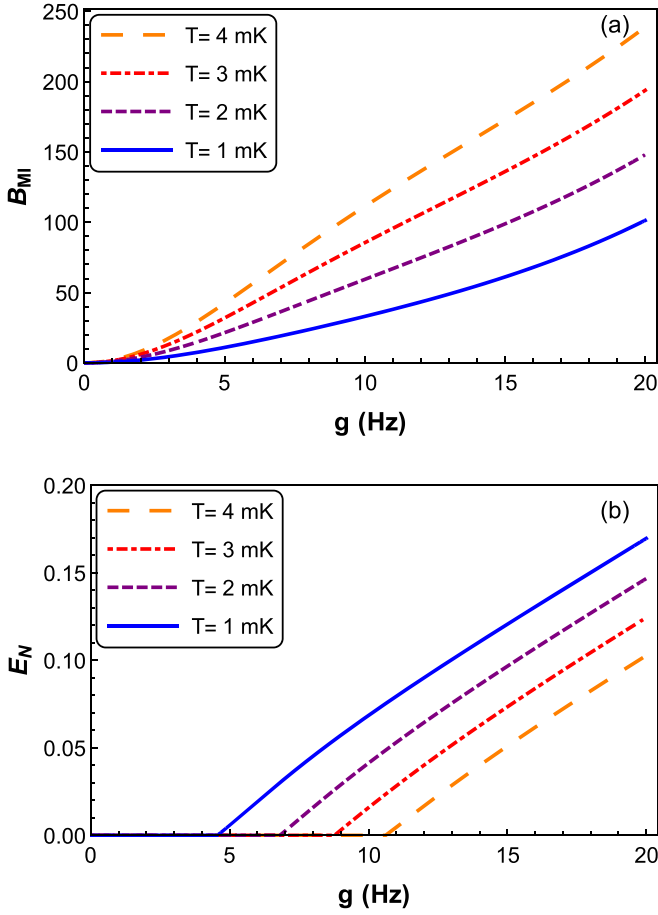


FIG. 4. Plot of the most informative Cramér-Rao bound $\mathcal{B}_{\text{MI}}$ (a) and the logarithmic negativity E_N (b) as a function of the optomechanical coupling g for different mirror temperatures T . All the parameters used are identical to those in Fig. 2.

shape of the QCRB curve highlights a fundamental tradeoff between the richness of physical interactions and the ability to accurately estimate parameters. Furthermore, it is observed that the QCRB increases with the temperature T , with the coupling strength g , and decreases with increasing optical power $\mathcal{P}$. This behavior is explained by the impact each of these parameters has on the quantum information accessible in the system. A high temperature amplifies the thermal noise of the mechanical mode, which disrupts quantum coherence and introduces additional uncertainty, thereby reducing estimation precision. Similarly, an excessively strong optomechanical coupling can induce nonlinearities and complex correlations between parameters, complicating their discrimination and negatively affecting multiparameter estimation. Lastly, a low optical power reduces the useful signal injected into the cavity, limiting the system's sensitivity to variations in physical parameters. These combined effects lead to an overall increase in the QCRB, revealing a degradation in estimation precision across the considered regimes.

As shown in Fig. 4, a parallel evolution of the QCRB Fig. 4(a) and the logarithmic negativity E_N Fig. 4(b), an indicator of quantum entanglement, is observed as a function of the optomechanical coupling g and for different mirror temperatures T . Both quantities increase with g , which can be

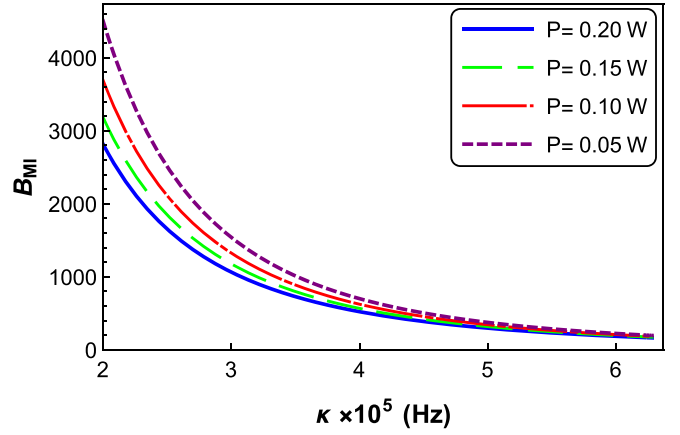


FIG. 5. The most informative QCRB ($\mathcal{B}_{\text{MI}}$) as functions of the decay rate κ for different driving powers $\mathcal{P}$. All the parameters used are identical to those in Fig. 2.

explained by the fact that stronger coupling enhances the interactions between the optical field and the mechanical mode. This strengthens the entanglement E_N but also makes parameter estimation more complex, especially in the multiparameter regime, leading to an increase in the QCRB. In other words, although entanglement is an asset in quantum metrology, excessively strong coupling can introduce cross-correlations and additional quantum noise, making the precise extraction of information more difficult. Regarding the effect of temperature, the QCRB increases with T because higher temperatures amplify the thermal noise of the mechanical resonator, reducing measurement fidelity and harming estimation precision. In contrast, the negativity E_N decreases with T , as entanglement is highly sensitive to decoherence induced by the thermal environment. At high temperatures, thermal noise destroys quantum correlations, thereby reducing E_N . These two behaviors reflect the typical duality of quantum systems: although both increase with the optomechanical coupling g , they evolve in opposite directions with respect to the temperature T . This illustrates the subtle tradeoff between quantum correlations and metrological robustness under thermal effects.

To further investigate the estimation accuracy of the parameters T and Δ , we have plotted the QCRB as a function of the decay rate κ for different values of the optical power $\mathcal{P}$ in Fig. 5. The results show that the QCRB is maximal for low values of κ and decreases progressively with the increase in both κ and $\mathcal{P}$. This trend can be explained by the fact that at low dissipation, the information remains largely confined within the system, making it more difficult to extract. However, a moderate increase in κ facilitates access to the information encoded in the system's state, thus improving the estimation accuracy, as indicated by the observed decrease in QCRB. Furthermore, the simultaneous increase in both optical power $\mathcal{P}$ and decay rate κ leads to an increase in the average number of photons and phonons, strengthening the optomechanical interaction and enhancing the amount of information available for estimation. Thus, the joint optimization of $\mathcal{P}$ and κ plays a crucial role in improving the measurement accuracy in the studied hybrid optomechanical system, enabling a better estimation of the parameters T and

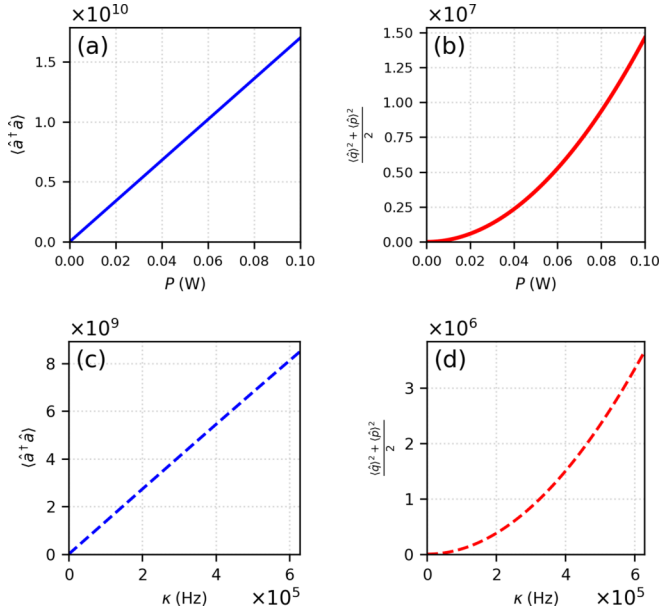


FIG. 6. Plot of the average number of photons [phonons] as a function of (a) [(b)] the driving power $\mathcal{P}$ and (c) [(d)] the decay rate κ . The parameters used are identical to those in Fig. 2.

Δ , particularly for high values of laser power $\mathcal{P}$ and decay rate κ .

To elucidate the physical origin of the enhanced estimation precision previously observed, Figs. 6(a) and 6(b) show that both optical power $\mathcal{P}$ and decay rate κ influence the estimation precision in the optomechanical system. Increasing the optical power $\mathcal{P}$ leads to an increase in the average number of photons and phonons in the system, which enhances the optomechanical interaction and, consequently, the estimation precision. This is reflected in a decrease in the QCRB and a corresponding increase in the QFI. Indeed, a higher number of quanta allows for better state discrimination and more efficient extraction of quantum information, thereby enhancing the system's ability to estimate its parameters. Similarly, increasing the decay rate κ also improves the estimation precision, though in a different manner [Figs. 6(c) and 6(d)]. While dissipation may seem counterintuitive in a quantum framework, here it makes the states more accessible for estimation by reducing information loss. This reduction in QCRB with the increase in κ demonstrates that controlled dissipation can facilitate the extraction of quantum information in such systems. Thus, this figure supports the overall objective of our study—optimizing the precision in estimating the temperature T and the effective detuning Δ —by identifying how specific physical parameters (such as power and cavity decay) influence the information carriers that govern estimation efficiency. This positive correlation between the estimation precision and the parameters $\mathcal{P}$ and κ is further supported by the results presented in Figs. 3(c) and 5. To provide a clear comparison between the fully quantum and heterodyne detection scenarios, we focus here exclusively on estimating a single parameter, either temperature or effective detuning. Figure 7 presents three curves illustrating different physical quantities as functions of the mirror temperature T . The first curve, shown in dash-dotted red, represents the QFI based

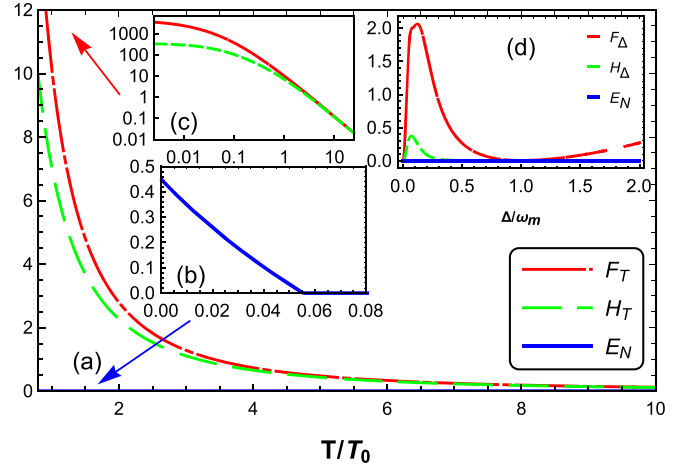


FIG. 7. Variation of the QFI $\mathcal{F}$, the CFI H , and the logarithmic negativity E_N as functions of the temperature T [panels (a)–(c)] and the normalized detuning Δ [panel (d)]. In panels (a)–(c), the estimated parameter is T with $g = 45$ Hz, while in panel (d), the estimated parameter is Δ . We choose a reference temperature of $T_0 = 400$ mK, and the other parameters are the same as in Fig. 2.

on the RLD formalism. The second, in dashed green, corresponds to the CFI obtained via heterodyne detection, which is based on the SLD. Finally, the third curve, in blue solid lines, shows the logarithmic negativity E_N , used as a measure of entanglement in the system. In these graphs we have chosen a coupling constant $g = 45$ Hz, since the logarithmic negativity becomes zero for lower values of g . In contrast, the Fisher information—both quantum and classical—remains well defined and computable for different values of g . It is observed that all three quantities reach their maximum at low mirror temperature T and rapidly decrease as T increases. From a physical standpoint, this decline is explained by the effect of thermal fluctuations, which tend to degrade the quantum correlations within the system, thus making precise estimation of the temperature T more difficult. Moreover, we note that the maxima and minima of the Fisher information and the logarithmic negativity coincide. This suggests that a high degree of quantum correlations is associated with greater information sensitivity, leading to better estimation of the parameter T in the considered regime. In other words, optimal estimation is achieved when quantum correlations are maximal. In the second part of the analysis, we compare the maximum precision offered by the QFI with that provided by the CFI resulting from heterodyne detection H_T in order to assess their respective efficiencies. The results show that the inequality $\mathcal{F}_T \geq H_T$ is always satisfied. More precisely, within the parameter range studied, H_T is equal to or very close to $\mathcal{F}_T$, indicating that the precision achieved via heterodyne detection is nearly equivalent to that offered by the QFI. This means that heterodyne detection, which measures the quadratures associated with position and momentum in an optomechanical system, effectively optimizes the extraction of information about the mirror temperature T . This result is based on a fundamental physical principle: heterodyne detection exploits the frequency shift between two optical fields to achieve high measurement sensitivity. In the Gaussian state framework,

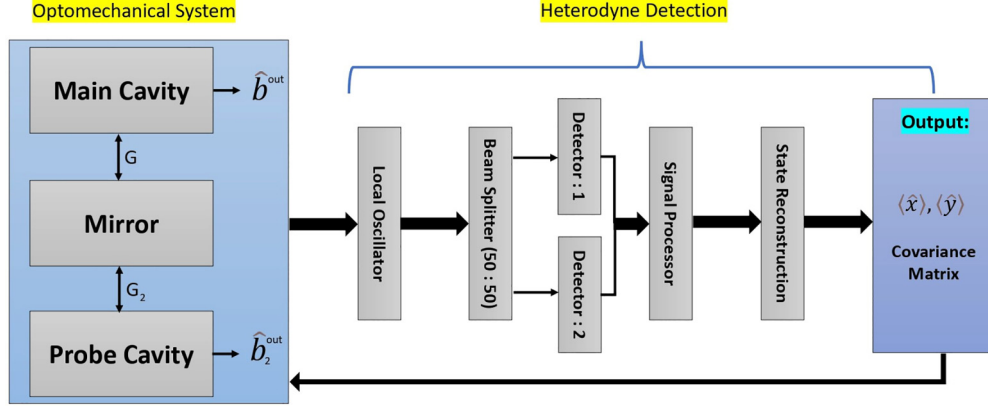


FIG. 8. Schematic Representation of the Reconstruction of the First-Order Moment and Covariance Matrix of a Gaussian State via Heterodyne Detection.

all relevant information about the system is encoded in the first-order moments and in the covariance matrix, which is composed of quadrature observables. By analyzing the interference signal generated by heterodyne detection, it becomes possible to accurately access the mode quadratures, making it possible to reconstruct a probability distribution sensitive to the mirror temperature T . From this distribution, and using Eq. (52), a CFI can be extracted that closely approximates the QFI. In theory, the SLD associated with the temperature T can be determined from the moments and the covariance matrix in order to estimate this parameter. However, the experimental implementation of such an optimal measurement remains, in practice, difficult to realize.

The analysis presented in Figs. 3–7 provides a comprehensive understanding of how key system parameters affect the simultaneous estimation of the effective detuning Δ and the mirror temperature T . We observe that the QCRB, which sets the ultimate limit for estimation precision, increases—meaning estimation becomes less precise—with increasing mirror temperature T and optomechanical coupling strength g . This behavior is due to the enhanced thermal noise and nonlinear interactions that degrade the information content in the output field. Conversely, higher cavity decay rate κ and stronger optical power $\mathcal{P}$ improve estimation precision by increasing the rate of information extraction and enhancing the intracavity photon population, respectively. These trends highlight the importance of optimizing the balance between coupling strength, decoherence sources, and the systems interaction with its environment, which governs both information leakage and extraction. Moreover, our results show that under well-chosen conditions, heterodyne detection can provide estimation performances approaching the quantum limit, thereby offering a practically accessible strategy for high-precision thermometry and detuning sensing in optomechanical platforms.

V. EXPERIMENTAL FEASIBILITY OF THE PROPOSED SCHEME

From a practical standpoint, it is essential to discuss the experimental feasibility of our scheme. First, we recall that optomechanical systems exhibit excellent performance and

that hybrid optomechanical platforms have already been experimentally implemented for the investigation of various quantum effects [56]. In the context of our study, which focuses on the estimation of the parameters Δ and T , it is necessary to perform heterodyne detection of the quadratures of the optical modes [109]. This step allows one to obtain the elements of the optical covariance matrix and the displacement vector, which are required for the experimental reconstruction of the Fisher information and the QCRB. This approach is justified by the fact that the system state is Gaussian and can therefore be fully characterized by the first and second moments of the quadratures.

Figure 8 illustrates a schematic setup where heterodyne detection is employed to simultaneously access the first-order moments and reconstruct the covariance matrix of the Gaussian state, thus enabling a direct experimental validation of our theoretical predictions. However, the measurement of the mechanical mode remains less straightforward. To make it feasible, one may consider adding a second Fabry-Pérot cavity adjacent to the first, formed by the movable mirror and a third fixed mirror. The parameters of this secondary cavity can be adjusted so as to enable the measurement of the position and momentum of the mirror from the output signal via heterodyne detection. Assuming that the movable mirror has unit reflectivity on both sides, thereby eliminating direct optical coupling between the two cavities, the annihilation operator of the secondary cavity then satisfies a linearized version of Eq. (2c):

$$\partial_t \delta b = -(\kappa + i\Delta)\delta b + iG\delta q + \sqrt{2\kappa} \delta b^{\text{in}}. \quad (58)$$

Thus, the linearized form of the annihilation operator of the additional cavity b_2 is written as

$$\partial_t \delta b_2 = -(\kappa_2 + i\Delta_2)\delta b_2 + iG_2\delta q + \sqrt{2\kappa_2} \delta b_2^{\text{in}}, \quad (59)$$

where Δ_2 , κ_2 , G_2 , and δb_2^{in} denote the effective detuning, decay rate, effective optomechanical coupling strength, and input noise of the second optical mode, respectively. Furthermore, $G_2 = (\omega_{c2}/L_2) \times \sqrt{\hbar/m\omega_m}$, with L_2 and ω_{c2} representing the length and frequency of the additional cavity. By choosing parameters such that $\Delta_2 = \omega_m \gg \kappa_2$, G_2 , the rotating-wave approximation (RWA) can be applied [110]. In

this case, Eq. (59) can be expressed in the rotating frame with respect to $\Delta_2 = \omega_m$ for the slow variables $\delta\tilde{O}(t) = \delta O(t) \exp\{-i\omega_m t\}$, neglecting terms oscillating at $2\omega_m$, as follows:

$$\partial_t \delta\tilde{b}_2 = -\kappa_2 \delta\tilde{b}_2 + i\frac{G_2}{\sqrt{2}}\delta\tilde{m} + \sqrt{2\kappa_2} \delta\tilde{b}_2^{\text{in}}, \quad (60)$$

where $\delta m = (\delta q + i\delta p)/\sqrt{2}$. We now consider the bad-cavity regime, $\kappa_2 \gg G_2$, in which the additional optical mode adiabatically follows the dynamics of the movable mirror, yielding

$$\delta\tilde{b}_2 = i\frac{G_2}{\kappa_2\sqrt{2}}\delta\tilde{m} + \sqrt{2/\kappa_2} \tilde{b}_2^{\text{in}}. \quad (61)$$

By applying the input–output relation [88] given by $\tilde{b}_2^{\text{out}} = \tilde{b}_2^{\text{in}} - \sqrt{2\kappa_2} \delta\tilde{b}_2$, we finally obtain

$$\tilde{b}_2^{\text{out}} = -\tilde{b}_2^{\text{in}} - i\frac{G_2}{\sqrt{\kappa_2}}\delta\tilde{m}. \quad (62)$$

For an appropriate choice of parameters ($\Delta_2 = \omega_2$, $\kappa_2 \gg G_2$, ...), the dynamics of the mirror ($\delta\tilde{m}$) can be inferred from the homodyne or heterodyne measurement of the additional optical mode b_2 .

VI. CONCLUSION

In quantum metrology and quantum information theory, the precise estimation of temperature T and effective detuning Δ play a fundamental role in the understanding, control, and optimization of optomechanical systems. Temperature determines the thermal state of mechanical oscillators, influencing key properties such as the mechanical damping rate and thermal noise. Effective detuning, on the other hand, governs the strength and nature of the light-mechanical interaction, and its precise estimation enables optimal tuning for various applications, including cooling, amplification, and entanglement generation. An inaccurate estimation may lead to unstable regimes or self-sustained oscillations, compromising the system's stability. In this work we demonstrated the ad-

vantages of the quantum Cramér-Rao bound (QCRB) for multiparameter estimation in an optomechanical system by explicitly computing the quantum Fisher information matrices (QFIMs) associated with the symmetric logarithmic derivative (SLD) and right logarithmic derivative (RLD) operators within the framework of multimode Gaussian states. We also formulated the saturation condition of the QCRB for the RLD operator and applied this formalism to a Fabry-Pérot cavity system, thereby illustrating the theoretical scope of the method. Additionally, we computed the classical Fisher information (CFI) obtained via heterodyne detection and compared it with the QFI, suggesting that this approach is a viable strategy for parameter estimation.

Our analysis revealed that the estimation of effective detuning becomes less precise as it approaches the mechanical frequency, simultaneously affecting the precision of temperature estimation. In contrast, increasing the optical power $\mathcal{P}$, the cavity decay rate κ , and reducing the temperature T and optomechanical coupling strength g leads to a decrease in the QCRB, indicating a significant enhancement in the amount of information accessible about both estimated parameters. We also showed that increasing the mean number of photons and phonons in the system improves the estimation by further lowering the QCRB. Finally, the evolution of the logarithmic negativity, consistent with both the QFI and CFI, demonstrates a reduction in entanglement with increasing temperature T , confirming the coherence between the systems correlation structure and the information accessible through estimation. Therefore, our theoretical results indicate that the heterodyne detection method is an effective strategy for joint parameter estimation in optomechanical systems.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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