

Non-Markovian noise limits for sustaining entanglement in multiparty quantum states

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The principal resource in several quantum information-processing tasks is entanglement. Building practically useful versions of such tasks, one often needs to consider multipartite systems. And in such scenarios, it is impossible to eliminate the effects of the environment while distributing the resources. We show how information backflow from the environment due to non-Markovianity in the system dynamics affects the resources in multipartite systems, specifically multiparty entanglement. In particular, we find that the entanglement of multiqubit Greenberger-Horne-Zeilinger cat states saturates with time to a nonzero value for a non-Markovian dephasing noise. The multiqubit W states saturate to a higher value. Such nonzero time-saturated values are washed out if the non-Markovianity in the channel is removed. The multiqubit W states provide a richer set of features, including a nonzero value for an asymptotic number of qubits and an even-odd dichotomy in the entanglement with the number of qubits. Moving to a non-Markovian depolarizing channel, we find that both cat and W states exhibit a revival of entanglement after the collapse.

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I. INTRODUCTION

The concept of entanglement remains, perhaps, the most important area of research in quantum information [1,2] due to its crucial significance in applications in quantum technologies as well as for fundamental aspects of quantum theory. Implementation of different information-processing tasks or quantum computation in a distributive scenario [3–8] involves the presence of multipartite entanglement in the physical substrate. Despite its overwhelming importance, multipartite entanglement has been explored to a much lesser extent than bipartite entanglement.

Whenever we consider the physical realization of any information-processing task, a key factor to keep in mind is the interaction of the system with the environment. In the literature, there are different theoretical ways to handle or filter out certain types of errors that occur due to the unavoidable presence of noise [9–19], but they are not always feasible in implementation. In general, the dynamics of such a system is governed by a nonunitary evolution and, with certain restrictions, can be described by the Gorini-Kossakowski-Sudarshan-Lindblad master equation [20,21], which provides a completely positive and trace-preserving evolution of the system with time. The dynamics is often assumed to be “Markovian,” and the typical assumptions involved include an uncorrelated system-environment initial state and an

unaltered environment state whenever it interacts with the system [22]. The typical dynamics of an open quantum system is often non-Markovian, which can be theoretically realized by removing the initial assumptions that are made in the Markovian case [23–29]. There are different ways to detect and quantify non-Markovianity, and two popular ones are the Breuer-Laine-Piilo (BLP) [30] and Rivas-Huelga-Plenio (RHP) measures [31]. The BLP measure is based on the distinguishability of the evolved states, which basically arises from the concept of information backflow from the environment to the system [30]. On the other hand, the RHP measure is conceptualized by checking for nonmonotonicity in the dynamics of system-auxiliary entanglement [31].

In this paper, we consider non-Markovian versions of physically motivated noisy quantum environments [32–35] and find their effects, when applied locally, on multipartite entanglement. We consider the effects on two paradigmatic sets of multiqubit quantum states, viz., Greenberger-Horne-Zeilinger (GHZ) “cat” states [36,37] and W states [38–40]. These states are locally unitarily equivalent for two parties, but already for three parties, they are not interconvertible via stochastic local quantum operations and classical communication (SLOCC) [39]. These states have appeared in numerous applications in quantum information protocols, and there have been a number of studies dealing with the entanglement content of these states in the presence of noise [41–49]. In particular, the robustness of GHZ states in the asymptotic limit was studied in the presence of depolarizing noise in [41]. The effects of noise on multipartite entanglement have largely been considered hitherto within the Markovian domain.

We show that the presence of non-Markovianity in the system dynamics is potentially helpful for sustaining entanglement of the state of the system. We note that the effects of a particular noise on the two different sets of states are significantly different. We begin with the dephasing channel, with a

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non-Markovian element, and show that the multiqubit W state is more robust than the multiqubit GHZ state in sustaining its entanglement contents. We find a dichotomy between even and odd numbers of qubits in the case of multipartite W states when a particular bipartition is considered. Next, we consider depolarizing noise and show that it is possible to retrieve the entanglement of the output state by introducing a memory effect in the dynamics. This is true for both W and GHZ states.

This paper is organized in the following way. In Sec. II, we introduce the initial systems and the noise models we will be working with. We also portray the exact structure of the protocol. Next, in Sec. III, we discuss in some detail the properties of the final-state entanglement in the presence of non-Markovian versions of the considered channels. We present our conclusions in Sec. IV.

II. THE INITIAL SYSTEM AND THE DECOHERENCE MODELS

In this paper, we consider some noise models that may arise in realistic situations when implementing a quantum circuit in practice. We characterize their effects on multipartite systems, where each party is assumed to hold one qubit. The initial state of the system is considered to be either a GHZ or W state. We also assume that the system and the environment are completely uncorrelated at the beginning of the dynamics. Mathematically, the joint initial state of the system and environment is hence considered to be in a product form. For any N -qubit system, the GHZ state has the following form:

$$|\psi\rangle_{\text{GHZ}}^N = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}). \quad (1)$$

On the other hand, the N -qubit W state can be written as

$$|\psi\rangle_W^N = \frac{1}{\sqrt{N}}(|0\rangle^{\otimes(N-1)}|1\rangle + |0\rangle^{\otimes(N-2)}|1\rangle|0\rangle + \cdots + |0\rangle|1\rangle|0\rangle^{\otimes(N-2)} + |1\rangle|0\rangle^{\otimes(N-1)}). \quad (2)$$

The state in Eq. (2) is a single-mode excited W state. We also consider multimode excited W states. Note that the GHZ and single-mode excited W states are SLOCC inequivalent. In [41], the authors dealt with the robustness of GHZ states in the presence of depolarizing noise while the number of parties increases asymptotically. They considered the form of depolarizing noise in which the initial state is modified by the presence of local white noise, and the channel acting on every party transforms the input of that party into a probabilistic mixture in the following way: $\rho \rightarrow (1-p)\rho + p\mathcal{I}/d$, where $\mathcal{I}$ is the corresponding identity matrix, p is the probability of the noise admixed, and d is the corresponding local Hilbert-space dimension. While this is an interesting way to describe noise, it is a somewhat restrictive description of the interaction (of the system with its environments), as it does not explicitly describe the evolution of an initial state with time. The dynamics of the state of the system with time under such noise can be evaluated only by solving the master equation governing the evolution of the state under such noise, with the exact form of the master equation being derived from the interaction Hamiltonian of the system with the environment.

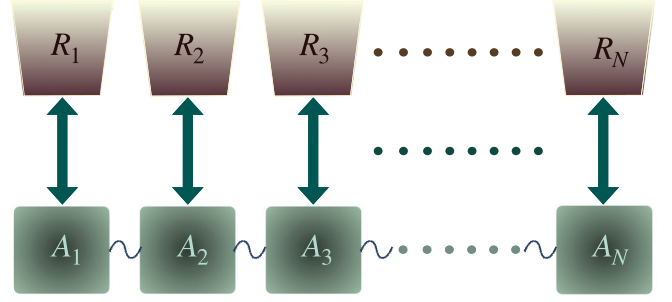


FIG. 1. Schematic diagram of system-environment interaction. Here, N parties, A_1 to A_N , each holding a single qubit, share an entangled state, and each qubit of the system interacts with N identical noninteracting reservoirs, R_1 to R_N .

In our prescription, we have an N -qubit system initialized in either of the states given in Eqs. (1) and (2). These N qubits are considered to interact with N reservoirs individually. We consider that all the reservoirs describing the environment are identical, and the interactions between any qubit and its environment are also considered to be similar. We illustrate this scenario in Fig. 1. Hence, our protocol corresponds to a model in which the system goes through local interaction with a noisy environment. In the literature, one can find many examples of such noisy interactions, as mentioned earlier. For near-term quantum hardware, e.g., involving superconducting qubits or trapped ions, the implementation primarily indicates the existence of local noise interacting with the concerned system [50–53]. The system-environment cluster evolves for a time $\tilde{t}$, and we are interested in the entanglement characterization of the final state. For a single qubit, the evolution of the state with time is governed by the time-local master equation,

$$\frac{d\rho}{d(\omega_0\tilde{t})} = \dot{\rho}(t) = -\frac{i}{\hbar\omega_0}[H(t), \rho(t)] + \frac{1}{\omega_0} \sum_k \gamma_k(t) \times \left(L_k(t)\rho(t)L_k^\dagger(t) - \frac{1}{2}\{L_k^\dagger(t)L_k(t), \rho(t)\} \right), \quad (3)$$

with ω_0 being the frequency of the individual qubit system. We set $\omega_0\tilde{t} = t$. Here, $H(t)$ is the Hamiltonian of the system under consideration, $L_k(t)$ are the Lindblad operators corresponding to the k th bath, and $\gamma_k(t)$ are the corresponding rates of dissipation. The first part of the above equation represents the unitary part of the dynamics, while the second part governs the dissipative scenario (interaction picture). The time dependence of the decay rates introduces memory effects in the system-bath interaction, which, in turn, bring in non-Markovian features in the dynamics of the N -qubit system.

Before moving on to the explicit description of the noise models, here, we discuss some particular features of the initial states considered. Note that the N -qubit states considered are symmetric across any $(N-m)$ vs m bipartition, where any m parties among N parties (holding an m -qubit system) stand together and the remaining $(N-m)$ parties also stand together. This is how the bipartition is created. For an even number of qubits, $1 \leq m \leq N/2$, and for an odd number of qubits, $1 \leq m \leq (N-1)/2$. In the following we provide an example for better understanding. Let us consider a five-qubit system, i.e., $N = 5$, distributed among five parties $A_1, A_2, A_3,$

A_4 , and A_5 . In this case, there can be two unique values for m , and they are $m = 1$ and $m = 2$.

Proposition 1. For a five-qubit system, the GHZ and W states are symmetric across all possible unique bipartitions.

Proof. Let us first consider $m = 1$. In this case, the bipartitions possible are A_1 vs $A_2A_3A_4A_5$, A_2 vs $A_1A_3A_4A_5$, A_3 vs $A_1A_2A_4A_5$, A_4 vs $A_1A_2A_3A_5$, and A_5 vs $A_1A_2A_3A_4$. In all these bipartitions the GHZ state can be written as $(|0\rangle|04\rangle + |1\rangle|154\rangle)/\sqrt{2}$. On the other hand, the W state can be written as $(|0\rangle|14\rangle + |0\rangle|24\rangle + |0\rangle|44\rangle + |0\rangle|84\rangle + |1\rangle|04\rangle)/\sqrt{5}$. Here, the basis for the four-qubit system is considered to be $|i_4\rangle_{i=0}^{15}$. Similarly, for $m = 2$, the bipartitions are A_1A_2 vs $A_3A_4A_5$ and all possible such permutations with two parties standing together and the remaining three parties also standing together. In this situation, the GHZ state can be written as $(|02\rangle|03\rangle + |32\rangle|73\rangle)/\sqrt{2}$, and the W state can be rewritten as $(|02\rangle|13\rangle + |02\rangle|23\rangle + |02\rangle|43\rangle + |12\rangle|03\rangle + |22\rangle|03\rangle)/\sqrt{5}$. Here, the bases for the two-qubit and three-qubit systems are $|j_2\rangle_{j=0}^3$ and $|k_3\rangle_{j=0}^7$, respectively. Hence, both states are symmetric under the particular bipartition considered. ■

The above claim can be extended to any N . The noise models are considered in such a way that they affect each qubit in the system in the same way. Hence, the interaction with the environment does not change the initial symmetry of the considered states. Hence, for the entanglement, we calculate the logarithmic negativity of the final states in all possible bipartitions. For a bipartite state ρ_{AB} of dimension $d_A \otimes d_B$, the logarithmic negativity is given by [54–57], $E_N^{\text{def}}(\rho_{AB}) = \log_2 \|\rho_{AB}^{T_B}\|_1$, where $\rho_{AB}^{T_B}$ is the partial transposition of the state ρ_{AB} with respect to subsystem B and $\|\cdot\|_1$ denotes the trace norm. We consider a normalized version of this measure in this paper, which is given as

$$E_N(\rho_{AB}) = \frac{E_N^{\text{def}}(\rho_{AB})}{\log_2 d_{\min}}. \quad (4)$$

Here, $d_{\min} = \min(d_A, d_B)$. Interestingly enough, we see that under dephasing noise, while for GHZ states the values of the logarithmic negativity of entanglement are the same for all bipartitions, for the W state, that is not the case. This is due to the fact that, for GHZ states, when they are written in any bipartition, they ultimately resemble the structure of a bipartite maximally entangled state. On the other hand, in the presence of depolarizing noise, GHZ states do not retain this feature of equivalence across all possible bipartitions. Intuitively, this could be an effect of the fact that the depolarizing noise takes into account the dimension of the system, which makes all bipartitions behave differently in the presence of the noise, even for the GHZ states.

Local dephasing noise. Let us start by considering that the qubits are locally interacting with some bosonic reservoirs and hence the total system-reservoir Hamiltonian for a single qubit is given by the sum of the system Hamiltonian H_S , the Hamiltonian of the reservoir H_B , and the interaction Hamiltonian H_I . The particular form that we assume is given by

$$H_D/\hbar = \omega_0 \sigma_z + \sum_i \omega_i a_i^\dagger a_i + \sum_i \sigma_z (g_i a_i + g_i^* a_i^\dagger), \quad (5)$$

where $2\hbar\omega_0$ is the energy gap of the qubit energy levels, ω_i 's are the frequencies of the reservoir modes, a_i ($a_i^\dagger$) is the annihilation (creation) operator of the i th mode of the reservoir, and g_i is the coupling constant between the system and the i th mode of the reservoir. Here, σ_z is the Pauli z matrix, with the local computational basis being assumed for the eigenstates of σ_z . In the interaction picture, the exact master equation corresponding to this Hamiltonian takes the simple form

$$\dot{\rho}(t) = \frac{\gamma(t)}{\omega_0} [\sigma_z \rho(t) \sigma_z - \rho(t)]. \quad (6)$$

We assume that the initial state of the total system is in product form in the qubit-to-environment partition, and the environment is assumed to be initialized in the thermal state at a temperature T , so that the initial state of the environment is $e^{-\beta H_B}/\text{Tr}[e^{-\beta H_B}]$, with $\beta = 1/(k_B T)$, with k_B being the Boltzmann constant. Following these assumptions the time-dependent dephasing rate takes the form [22]

$$\gamma(t) = \int d\omega S(\omega) \coth\left(\frac{\hbar\omega}{2k_B T}\right) \frac{\sin(\omega t)}{\omega}. \quad (7)$$

Here, $S(\omega)$ is the reservoir spectral density, which, in the continuum limit of the modal frequencies, is related to g_i as $\sum_i |g_i|^2 \rightarrow \int d\omega S(\omega) \delta(\omega_i - \omega)$. The dynamics given in Eq. (6) represents the usual dephasing channel with the usual Kraus operators $K_0 = \sqrt{1 - \eta(t)/2} \mathcal{I}$ and $K_1 = \sqrt{\eta(t)/2} \sigma_z$. The time-dependent dephasing factor $\eta(t)$ is related to $\gamma(t)$ as $\eta(t) = 1 - \exp[-2 \int_0^t \gamma(t') dt']$. Now let us consider that the reservoir is governed by the Ohmic spectral density function given by

$$S(\omega) = \frac{\omega^s}{\omega_c^{s-1}} e^{-\omega/\omega_c}, \quad (8)$$

where ω_c is the cutoff frequency of the reservoir and s is the Ohmicity parameter. Using this spectral density in Eq. (7), we can get the exact expression for the dephasing decay rate at a given temperature [58]. At zero temperature it takes the form

$$\gamma_{T=0}(t, s) = \omega_c [1 + (\omega_c t)^2]^{-\frac{s}{2}} \Gamma[s] \sin[s \arctan(\omega_c t)]. \quad (9)$$

Here, $\Gamma[\cdot]$ denotes the Euler gamma function. Note that for $s > 2$, the decay rate takes negative values for a given time range. Such negative values of the decay rate can be argued to bring memory effects into the system's dynamics. This, in turn, induces the features of non-Markovianity in the evolution because there is a possibility of the backflow of information from the reservoir to the attached qubit.

In this paper, we consider cases with different values of s and show how the entanglement of the final state saturates to a nonzero value with respect to time. We show that with an increasing number of qubits, although the value at which the final-state entanglement saturates decreases, the decrement converges with the number of qubits.

Local depolarizing noise. Let us now consider a further system-environment coupling in which the total Hamiltonian of the system and the bath is given by

$$H_{DP}/\hbar = \omega_0 \sigma_z + \sum_i \omega_i a_i^\dagger a_i + \sum_{j=x,y,z} \sum_i \sigma_j (g_i a_i + g_i^* a_i^\dagger). \quad (10)$$

With this Hamiltonian, the exact master equation takes the following form:

$$\dot{\rho}(t) = \sum_{j=x,y,z} \frac{\gamma_j(t)}{\omega_0} [\sigma_j \rho(t) \sigma_j - \rho(t)]. \quad (11)$$

Here, σ_x , σ_y , and σ_z are Pauli matrices. Note that, for $\gamma_x(t) = \gamma_y(t) = 0$, we get back the case corresponding to the dephasing channel. Now, in the previous case, the explicit form of the decay rate can be evaluated because the system Hamiltonian commutes with the total Hamiltonian H_D given in Eq. (5). Finding the explicit forms of the γ_j 's becomes very involved in this case because the interaction Hamiltonian has changed. Instead, here, we consider the forms of the decay rates while keeping certain constraints in mind. From [59,60], in the case of the above evolution, the dynamics is Markovian [i.e., has completely positive (CP) divisibility] if and only if $\gamma_x(t) \geq 0$, $\gamma_y(t) \geq 0$, and $\gamma_z(t) \geq 0$ for all time $t \geq 0$. In addition, only P divisibility is equivalent to the following conditions:

$$\begin{aligned} \gamma_x(t) + \gamma_y(t) &\geq 0, \\ \gamma_y(t) + \gamma_z(t) &\geq 0, \\ \gamma_z(t) + \gamma_x(t) &\geq 0 \end{aligned} \quad (12)$$

for all times $t \geq 0$. Violation of any of the inequalities in Eq. (12) implies that the dynamics is essentially non-Markovian. Note that breaking P divisibility definitely suggests CP divisibility, but the reverse is not true. This follows from the concepts of k -positive maps.

In this paper, we choose one of the γ 's to take negative values for a certain period of time t . Explicitly, we take $\gamma_z(t) = \alpha \sin(t)$, which definitely makes $\gamma_z(t) \leq 0$ for a certain range of t , breaking the CP divisibility. Evidently, the P divisibility will be broken depending on the choices of the values of the other two decay rates.

III. CHARACTERIZING THE FINAL-STATE ENTANGLEMENT UNDER NOISE

Results for dephasing noise. Let us start by describing the simplest case involving three qubits, i.e., $N = 3$ distributed among three parties A_1 , A_2 , and A_3 . First, let us assume that the system is initialized in a GHZ state. Hence, the initial state looks like $|\psi\rangle_{\text{GHZ}} = (|000\rangle + |111\rangle)/\sqrt{2}$. According to our protocol, each qubit interacts with identical baths (three baths in this case) at zero temperature at any instant of time t .

To obtain the evolution of the state of the system undergoing the environmental interaction, we solve the master equation. In this particular case, mathematically, we can write the dynamics of the state as

$$\begin{aligned} \dot{\rho}(t) = & \frac{\gamma(t)}{4\omega_0} [(\sigma_z \otimes \mathbb{I} \otimes \mathbb{I})\rho(t)(\sigma_z \otimes \mathbb{I} \otimes \mathbb{I}) \\ & + (\mathbb{I} \otimes \sigma_z \otimes \mathbb{I})\rho(t)(\mathbb{I} \otimes \sigma_z \otimes \mathbb{I}) \\ & + (\mathbb{I} \otimes \mathbb{I} \otimes \sigma_z)\rho(t)(\mathbb{I} \otimes \mathbb{I} \otimes \sigma_z) - \rho(t)]. \end{aligned} \quad (13)$$

In the above equation the dynamics is considered to be governed by the dephasing noise, as can be seen from the dissipative part of the equation. In the case of depolarizing noise, Eq. (13) is accordingly modified. As the number of qubits of the system increases, the above expression will be

modified in an iterative fashion. Now, we let the system evolve from $t = 0$ to $t = 100$ under the given dynamics, where the decay rate $\gamma(t)$ is given by Eq. (9). At every instant of time, we determine the state $\rho(t)$ by solving Eq. (13). For the particular case of a three-qubit system we evaluate the entanglement of the final state in any of the one vs two bipartitions, i.e., in A_1 vs A_2A_3 , A_2 vs A_3A_1 , or A_3 vs A_1A_2 , as the state remains symmetric across all such bipartitions. We evaluate the logarithmic negativity of the given bipartite system considering the above-mentioned bipartitions.

In Fig. 2, the variation of the logarithmic negativity of entanglement of the evolved state in any given bipartition is plotted against time. Note that, for different values of s , i.e., for a varying decay rate of the noise model, we obtain different saturations for the final-state entanglement. For $s < 2$ the decay rate γ remains positive, and hence, the dynamics is surely Markovian, implying that the information gets dissipated from the system to the environment due to the interaction. In this domain, the value of the logarithmic negativity of the evolved state reduces with time and reaches zero after a certain period of time; i.e., after the interaction with a Markovian environment for a significant amount of time, the system loses quantum correlation. However, the non-Markovian features in the dynamics for $s > 2$ induce information backflow from the environment to the system, which in turn helps the system to retain a nonzero value of entanglement at the output end, and the final value saturates with time. At the very beginning of the timescale, the entanglement falls and reaches the value where it finally saturates. As the number of qubits increases, the saturation of entanglement occurs at a lower value for any bipartition. This is due to the fact that the system interacts locally with the environment and we calculate the entanglement of the evolved state in a given bipartition. In any bipartition, the systems become a higher-dimensional bipartite one. With an increasing number of qubits, the dimensions of the individual system(s) when expressed in two-party form also increase, leading to more dissipation due to the environmental interaction. Hence, the value of the retained entanglement decreases with an increasing number of qubits [Fig. 3(c)]. For two particular cases with two different numbers of qubits, i.e., $N = 3$ and $N = 5$, the characterizations of the final-state entanglement with time are illustrated in Figs. 2(a) and 2(b), respectively. Numerically, it is evident that a similar nature of the entanglement of the output state can be observed for $N = 3$ to $N = 10$ in the non-Markovian domain of s for a long period of time.

Now, we concentrate only on the region where the decay rate takes negative values, i.e., $s > 2$. For these values of s , the final state retains the entanglement more than in the Markovian domain. From Fig. 3(a), it is evident that the final state retains the maximum amount of entanglement in the region between $s = 2.3$ and $s = 2.5$ for all N . Interestingly, we find that the final state takes the maximum saturating value of entanglement for $s = 2.47$, which is valid for any N . The value of s guides the nature of the dephasing parameter γ according to Eq. (9). The lower the value of the dephasing parameter is, the greater the amount of retained entanglement of the evolved system is. Figure 3(b) indicates how the saturating value of final-state entanglement falls off with an increasing number of qubits. From these plots, we can also notice

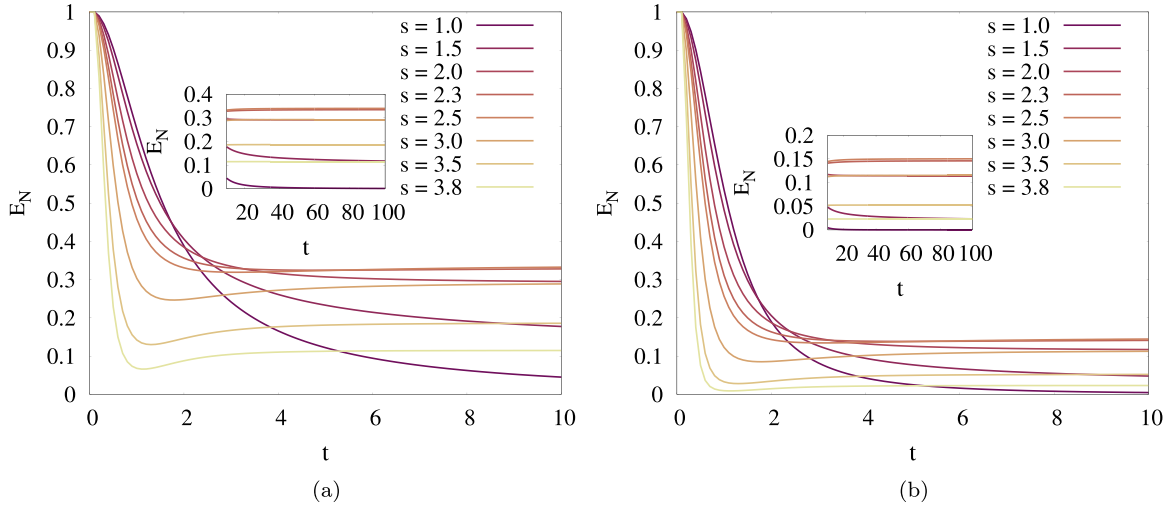


FIG. 2. The logarithmic negativity of entanglement E_N of the evolved state while interacting with local dephasing noise at time t for different values of decay parameter governed by s , where the input state is an N -qubit GHZ state. (a) Three-qubit GHZ state ($N = 3$). (b) Five-qubit GHZ state ($N = 5$). The vertical axes are in ebits, while the horizontal ones are dimensionless.

that the maximum improvement occurs in the same range of s . We plot the particular case corresponding to $s = 2.47$ in Fig. 3(c). Here, the value of entanglement falls with the number of qubits as $E_N = ae^{-c(N-3)} + b^2$, with $a = 0.3399$, $b = 7.0847 \times 10^{-5}$, and $c = 0.4167$. Here, a , b , and c are mathematical constants governing the exponential nature of the decay. Note that, when $N = 3$, at $t = 30$, the retained value of entanglement after the interaction with local dephasing noise is $E_3|_{t=30} \approx a = 0.3399$ (hence, a is the initial value of the retained entanglement). At the same time, c gives the rate of exponential decay with respect to the number of qubits, and b gives the retained entanglement in the asymptotic limit when $N \rightarrow \infty$. When we restrict ourselves to the non-Markovian domain of dephasing noise, i.e., for $s > 2$, from numerical analysis, it is evident that the decay of the logarithmic

entanglement of the evolved state follows the same pattern as the exponential function given above. But in this case, we need to choose a different parametrization of a , b , and c . From the extrapolation of E_N with respect to N , as illustrated in Fig. 3(c), after $N = 10$, we see that the final state retains a nonzero value of entanglement up to $N = 16$ in the case of dephasing noise before it is completely diminished by the presence of noise.

Next, we move our discussion to the case when the initial state is a W state with single- to $(N - 1)$ -mode excitation. The prescription for the interaction with the environment is the same as in the previous case, and the dynamics of the state under the noise is given in Eq. (13). In this case the results are more involved because the bipartitions are not equivalent except for $N = 3$. From $N = 4$ to $N = 10$, we explicitly

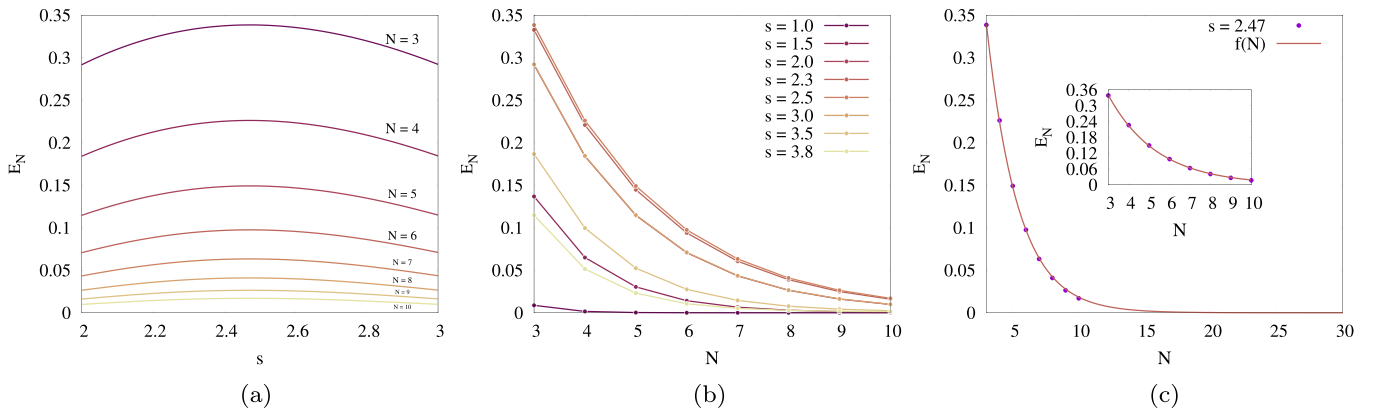


FIG. 3. Entanglement characterization of the evolved state obtained at a given point of time while the system interacts locally with dephasing noise. The initial system is considered to be an N -qubit GHZ state. The vertical axes are in ebits, while the horizontal ones are dimensionless in all the plots. (a) Variation of the logarithmic negativity of entanglement of the evolved state plotted against the decay rate governed by the quantity s for different numbers of qubits at a given time $t = 30$. (b) Decay of the value of logarithmic negativity of entanglement with the number of qubits for different values of the decay rate at a given time $t = 30$. (c) The decaying nature of the logarithmic negativity of entanglement of the evolved state under dephasing noise plotted against the number of qubits N for a given decay rate corresponding to $s = 2.47$ at a given time $t = 30$. Here, the dots indicate the exact data points, while $f(N)$ gives the extrapolated version of the same.

discuss two cases, (1) the “1-Rest” scenario, in which the bipartition is between *one* party standing alone and the rest of the parties standing together, and (2) the “highest-cut” scenario. In case of the highest-cut scenario, for even N the bipartition is between $N/2$ parties and the remaining $N/2$, whereas for odd N , the bipartition is between $(N-1)/2$ parties and the other $1 + (N-1)/2$ parties. For example, when $N = 7$ and the system is distributed among $A_1, A_2, A_3, A_4, A_5, A_6$, and A_7 , then the unique bipartitions possible are A_1 vs $A_2A_3A_4A_5A_6A_7$, A_1A_2 vs $A_3A_4A_5A_6A_7$, and $A_1A_2A_3$ vs $A_4A_5A_6A_7$. Any other bipartitions possible are merely permutations of the parties, under which the state remains symmetric, as discussed earlier. Hence, for $N = 7$, we call the A_1 vs $A_2A_3A_4A_5A_6A_7$ bipartition the 1-Rest case and the $A_1A_2A_3$ vs $A_4A_5A_6A_7$ bipartition the highest-cut case. Similar nomenclature is extended for any N . We let the system evolve under the dephasing noise from $t = 0$ to $t = 100$ and analyze the characterization of entanglement during the evolution and after a certain time has passed. As we know that the non-Markovian effects are prominent only for $s > 2$ in Eq. (9), here, we concentrate on that particular region. We see that similar to the case for GHZ states, the value of the logarithmic negativity of entanglement of the evolved state first decreases and then saturates at a particular value with increasing time for all the bipartitions. The value of the saturation of the final-state entanglement is much higher than in the GHZ case. While in the case of GHZ states the evolved states behave similarly for all possible bipartitions for a given N , for W states we notice the behaviors are different. But in both the extreme bipartitions discussed, the evolved states retain a higher value of entanglement than the GHZ states for a given N after the same amount of time of evolution. For example, at $t = 30$ and $s = 2.47$, while the entanglement saturates at $E_N \approx 0.225$ if the initial state is a GHZ state of four qubits, the retained entanglement of the evolved state obtained from an initial W state is $E_N = 0.44$ in the 1-Rest cut and $E_N = 0.4977$ in the highest cut. Now let us discuss how the entanglement of the output state E_N changes with an increasing number of qubits N in the region of saturation (we choose $t = 30$). Two corresponding situations, i.e., 1-Rest and highest cut, are plotted in Figs. 4(a) and 4(b), respectively. In the 1-Rest case, note that the decay of entanglement almost coincides for $s = 2$ and $s = 3$, while the best value is obtained again between $s = 2.3$ and $s = 2.5$. Now for the highest-cut scenario the entanglement of the output state shows a particularly interesting feature when the initial state is a W state. In this case, we see that for an even number of qubits, the saturation occurs at the same value, making the output-state entanglement independent of the number of qubits. However, that is not the case for an odd number of qubits. Instead the saturation value increases with N , as can be seen from Fig. 4(b). Moreover we find, like in the case of the GHZ states, that the best effect of non-Markovianity is achieved for $s = 2.47$. Note that, from Proposition 1 and the following proof, it is evident that for any bipartition, the GHZ states behave as bipartite maximally entangled states in higher dimensions, unlike the W states. While the GHZ states hold their symmetry across different bipartitions for a given N under dephasing noise, W states do not. On the other hand, in terms of the measure of entanglement that has been considered, i.e., the logarithmic negativity, the GHZ states are

maximally entangled for a given number of qubits. The considered noise model is symmetric in nature for each qubit; i.e., the interactions affect all the qubits in a similar fashion. Note that the GHZ states are more affected than W states in terms of the amount of entanglement retained for the same number of qubits. However, one should be careful about drawing a more generic conclusion from this result because, with respect to the entanglement measures considered, the robustness factor may vary.

Now let us concentrate on the particular case with $s = 2.47$ and time $t = 30$. In the 1-Rest case the decay of final-state entanglement takes a form similar to that for the GHZ state. We have $E_N = ae^{-c(N-3)} + b^2$ for all values of $s > 2$. For $s = 2.47$, the parameters take the following values: $a = 0.2329$, $b = 0.4906$, and $c = 0.1559$. For other values of s , the final-state entanglement has a similar structure with different parametrization. This scenario is illustrated in Fig. 4(c), where the inset shows the numerically obtained situation for $N = 3$ to $N = 10$. By extrapolating with respect to the given function, we see that the final-state entanglement in the 1-Rest case saturates at a nonzero value, specifically $E_N = 0.24$, as illustrated in Fig. 4(c), making W states 24% robust under dephasing noise in the 1-Rest cut. Now let us move on to the highest-cut case. The inset in Fig. 4(d) portrays the two distinct cases of final-state entanglement for even and odd numbers of qubits separately. For even N , the final-state entanglement saturates at $E_N = 0.4977$, and for the odd case, the output-state entanglement variation with N is governed by $E_N = 1/(ae^{-cN} + b^2)$, with $a = 0.2832$, $b = 1.4195$, and $c = 0.4546$. Extrapolating these data points, we can obtain the characterization of entanglement for large N as shown in Fig. 4(d). While for an even number of qubits the value of E_N does not change, for an odd number of qubits it increases and saturates at a value of $E_N = 0.4962$. This evidently indicates that a clear dichotomy exists between even and odd numbers of qubits in the case of a multipartite system when we are initiating any task with a W state. Note that, in the even qubit scenario, the W states are 49.77% robust, whereas in the odd qubit scenario they are 49.62% robust in the presence of dephasing noise.

From the detailed analysis of the GHZ and W states under dephasing evolution, it is clear that the W state is way more robust than the GHZ state under this type of noise, as the retained entanglement of the evolved state saturates at a higher value. While for the GHZ state entanglement vanishes with an increasing number of parties, the W state retains a nonzero amount of entanglement for any bipartition in the asymptotic limit. Also, while there is an information backflow from the environment to the system [when $\gamma(t)$ takes a negative value], evidently, the system retains more correlation in this noise model compared to the Markovian version of the same channel (i.e., for $s < 2$).

Results for depolarizing noise. As discussed in the previous section, in the case of the depolarizing noise, we choose either of the γ 's such that they take negative values for a certain interval of evolution time. First, we consider $\gamma_z(t)$ to have the form given in Eq. (9) because the corresponding part of the master equation in Eq. (11) is the same as the dephasing noise, along with $\gamma_x(t) = \gamma_y(t) = 0.1$. In this scenario, although the conditions given via Eq. (12) are fulfilled for the

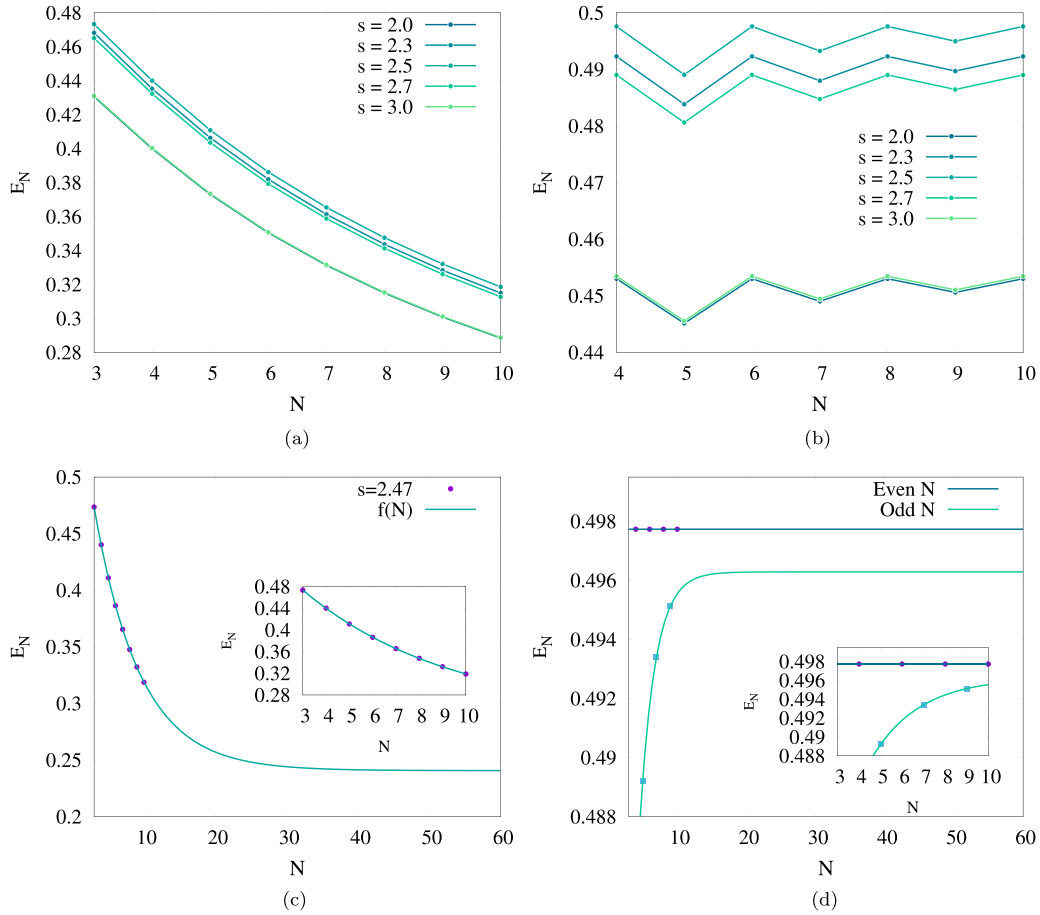


FIG. 4. Entanglement characterization of the evolved state obtained at a given point of time while the system interacts locally with dephasing noise. The initial system is considered to be an N -qubit W state. The vertical axes are in ebits, while the horizontal ones are dimensionless in all the plots. (a) The logarithmic negativity of entanglement of the evolved state in the “1-Rest” cut plotted against the number of qubits for different values of the decay rate governed by s at a given time $t = 30$. (b) The pattern of the logarithmic negativity of entanglement of the evolved state in the “highest cut” with the number of qubits plotted for different values of the decay rate governed by s at $t = 30$. (c) Decay of entanglement in the 1-Rest cut with the number of qubits is plotted for $s = 2.47$ at $t = 30$. The extrapolation shows that the value of the logarithmic negativity for the evolved state in this particular scenario decreases and saturates with an increasing number of qubits. (d) The entanglement pattern in the highest-cut scenario plotted against the number of qubits for $s = 2.47$ at $t = 30$. In this case, the evolved W state shows an even-odd dichotomy. Using extrapolation, we can see that while for an even number of qubits the value of entanglement saturates at a particular value for any number of qubits, for an odd number of qubits, the value grows slightly with an increasing number of qubits and then saturates.

dynamics, it does not show any significant improvement of final-state entanglement with time.

Next, we choose the case with $\gamma_z(t) = \alpha \sin(t)$, with $\alpha = 1.0$, along with $\gamma_x(t) = \gamma_y(t) = 0.1$. Let us consider the case when the initial state is a GHZ state. The corresponding scenario is depicted in Figs. 5(a) and 5(b). The prescription of the interaction with the environment at a given instant of time is similar to that in the previous case; i.e., each qubit interacts with identical reservoirs corresponding to the depolarizing noise. But unlike in the dephasing scenario, here, when the individual qubit undergoes interaction with the noise, the evolved state entanglement takes different values for different bipartitions even for the GHZ state. This is due to the breaking of symmetry due to the interaction with depolarizing noise. For better understanding, we refer to Proposition 1 and the following proof. Hence, as before, we show the entanglement

characterization for two cases, namely, 1-Rest and highest cut, in Figs. 5(a) and 5(b), respectively. We deal explicitly with qubits from $N = 3$ to $N = 10$. Note that, in both cases of bipartitions, the entanglement of the final state falls in the beginning and then returns to a nonzero value after a certain period of time after the collapse. In the highest-cut scenario, for $N \geq 7$ there is a second round of entanglement revival that occurs around $t = 12$ to $t = 13$, which is shown in the inset in Fig. 5(b). The same study was done for N -qubit systems when the initialization is done in the W state. Here, we portray the case of the W state with either single- or $(N - 1)$ -mode excitation. In the case of W states, the output-state-entanglement characterization is plotted in Figs. 5(c) and 5(d). Note that, in the case of depolarizing noise, GHZ states are more robust than W states with an increasing number of parties, which is the opposite situation compared to the dephasing noise.

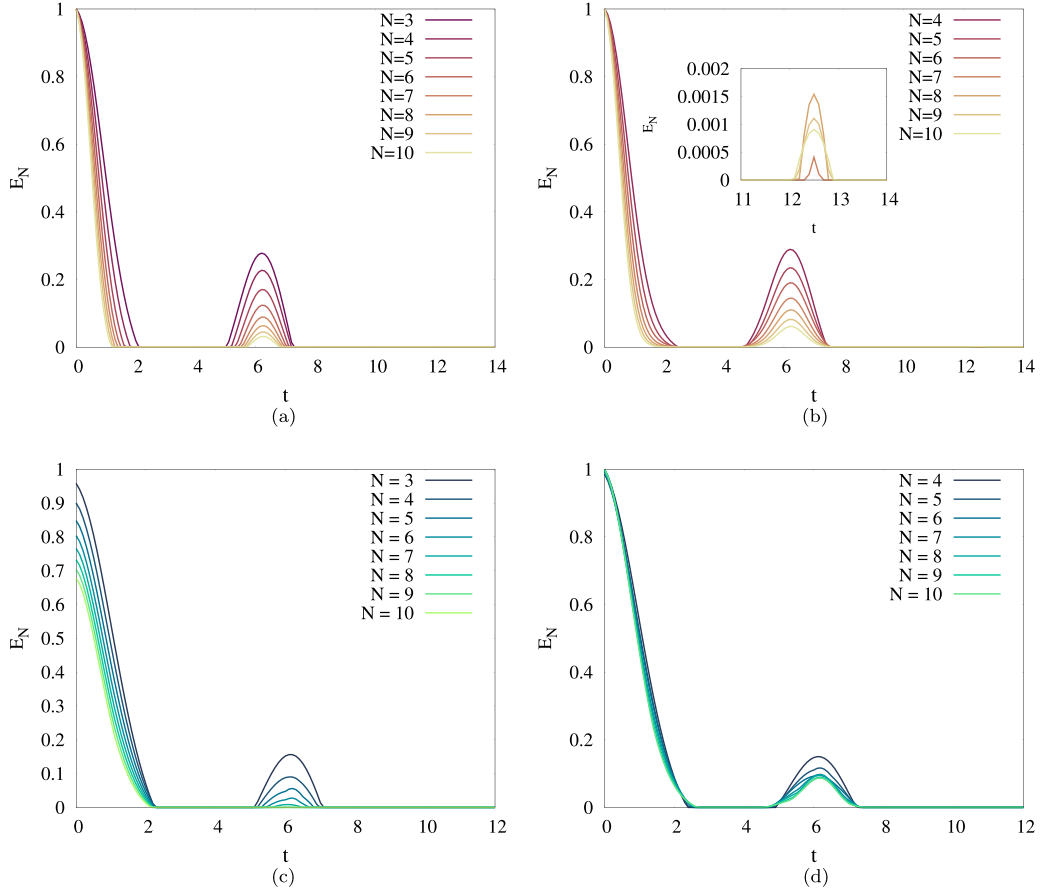


FIG. 5. Entanglement of the final state under depolarizing noise with time for GHZ and W initial states. The top row shows GHZ scenarios; the bottom row shows W scenarios. The vertical axes are in ebits, and the horizontal axes are dimensionless. (a) GHZ: “1-Rest” scenario. (b) GHZ: “highest-cut” scenario. (c) W : “1-Rest” scenario. (d) W : “highest-cut” scenario.

IV. CONCLUSION

In this article, we presented a detailed analysis of entanglement characterization of certain multipartite quantum states exposed to different noisy environments. We considered the standard N -party Greenberger-Horne-Zeilinger and W states as the multipartite states. The noise models taken into account were dephasing and depolarizing ones. The noisy evolution of the system was considered to be governed by the Gorini-Kossakowski-Sudarshan-Lindblad master equation. By imposing non-Markovian versions of these channels, we obtained the evolved states at arbitrary instants of time. A comparison of the channels in their Markovian and non-Markovian domains made it evident that the backflow of information from the environment to the system, whenever present, has an effect on the entanglement of the final states, making them more robust with respect to the Markovian case.

The dynamical picture indicated that the W states are more robust than GHZ states in the presence of dephasing noise. While we numerically obtained the exact data for 3-qubit to 10-qubit systems, we characterized the entanglement content of the final state for a larger number of qubits by extrapolation. While GHZ states lose the retained amount of entanglement with an increasing number of qubits, we showed that W states become robust, with a nonzero value of entanglement, in the presence of dephasing noise, and this is valid even in

the asymptotic regime. We predicted the limit of sustainable entanglement of the final states in both cases. In addition, we found that an even-odd dichotomy exists for the number of qubits in the case of W states exposed to dephasing noise. In the case of an even number of qubits, the W states were shown to be 49.77% robust, whereas in the case of an odd number of qubits, the remaining entanglement saturated at a particular value, making the states 49.62% robust under the particular noise in the asymptotic limit.

Next, in the presence of depolarizing noise, we found that when we consider the Born-Markov approximation, the dynamics governed by the Markovian master equation diminishes the presence of entanglement with time, and the remaining entanglement vanishes with an increase in the number of qubits. Considering the corresponding non-Markovian evolution, we showed that it is possible to regain the lost entanglement with time via a proper choice of decay parameters after the collapse.

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DATA AVAILABILITY

The data that supports the findings of the article can be generated from the codes written based on the theory described in the paper. The codes are openly available in [61].

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