

Quantum random walks on d -regular graphs with Haar-random coin operators

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With a unitary coin operator that is a random matrix drawn from a uniform distribution with respect to the Haar measure, we construct a variant of a discrete quantum random walk using a d -regular undirected connected simple graph. With each step of the walk, the coin operator random matrix is drawn independently from the distribution. The expectation value over the distribution of random unitaries for the associated quantum channel gives a depolarization channel for the coin subspace and resembles the associated classical random walk where the direction a walker steps depends upon the outcome of a fair coin or balanced d -sided dice. Remarkably, despite the fact that the averaged channel depolarizes the coin subspace, measurements in the vertex subspace can be designed that would reveal information about the initial quantum state, even after many iterations of the channel. We illustrate with examples of quantum walks on Cayley graphs of Abelian groups, such as the cycle and hypercube graphs. For Cayley graphs of Abelian groups, the averaged channel is dephasing in the Fourier basis. These quantum walks are examples of bipartite strongly interacting systems, where one subsystem is strongly perturbed, yet information about the initial state in the other subsystem is potentially measurable forever. Due to its decoherence, a quantum random walk with a Haar-random coin would not be useful in search algorithms but could aid in understanding quantum systems with strongly perturbed subsystems.

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I. INTRODUCTION

For quantum states that are linked via an undirected graph, a *quantum random walk* is often described as a product of two noncommuting local unitary operators that is then iterated, in analogy with a classical random walk [1–3]. One of the operators plays the role of a coin and the other conditionally causes transitions between quantum states, playing the role of a walker that chooses its direction based on the outcome of a tossed coin or the throw of a d -sided dice. The basis states of the quantum system describe the walker position and the state of the coin. Quantum walks can aid in search algorithms [4,5] and multiparticle quantum walks are capable of universal quantum computation [6]. For a review see Venegas-Andraca [7].

An *open quantum random walk* is similar to a quantum random walk as it also involves local quantum operations but instead of unitary operations applied to a state vector, more general types of iterated operations are applied to a density operator [8,9]. With a coin subspace that decoheres, an open random walk on the line can have position variance $\sigma^2 \propto t$

that grows linearly with time or walk iteration number t , like that of a classical random walk [10] and unlike that of a coherent quantum walk, where $\sigma^2 \propto t^2$ [1].

In quantum mechanics, values obtained through measurement have probability dependent on the measurement operator and state vector. A quantum random walk or open quantum random walk is deterministic, as the iteratively applied unitary operator or Lindbladian superoperator and resulting quantum state are not functions of a probability distribution. In contrast, a *quantum random circuit* consists of a product of local unitary operators, each operating on a couple of qubits or qudits, with some of them chosen identically and independently from a distribution [11,12]. If quantum random circuits are iterated, they can be described as *ergodic quantum processes* [13]. Ergodic quantum processes are a class of stochastic quantum operations that includes random and repeated quantum interactions [14] and collision models [15–17] which can describe interactions between a quantum system and one or more thermal reservoirs. If noise or coupling to a thermal bath is modeled as a specified completely positive trace preserving map, then it can be called a stochastic quantum channel even though the system is described with a fixed superoperator and is deterministic [18,19].

Quantum random walks that include variations in the choices of operators, or operators that are chosen from a distribution (e.g., Ref. [20]), can be described as having a *Markov coin* [21] or a Markov shift operator. A Markov coin usually causes decoherence [21], but certain types of variations in the operators can allow a quantum walk to illustrate the phenomenon known as Anderson localization [22,23].

In this manuscript we focus on discrete time quantum random walks where the applied coin unitary operators are

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independently drawn from a distribution during each iteration of the walk. We modify the Hadamard-coin quantum random walk on the line [1,24] and higher-dimensional ones [2,25] so that the coin operator is chosen identically and independently from a distribution. At each iteration of the walk, a different operator is chosen from this distribution. To facilitate analytical computation we choose a distribution that is uniform with respect to the Haar measure. This allows us to compute a mean or averaged quantum channel, which is a quantum channel computed from the expectation value of the map over the distribution of coin operators. A quantum random walk with a Haar-random coin operator can be considered a limiting and special case of a quantum system interacting with a high-temperature reservoir.

A. Some notation

We work with finite-dimensional Hilbert spaces. We use notation $\mathcal{L}(\mathbb{C}^d)$ to refer to the set of linear operators that act on the d -dimensional complex vector space $\mathbb{C}^d$. The associated group of unitary operators is $U(d)$. Many of our quantum walks use a bipartite quantum space $\mathbb{C}^d \otimes \mathbb{C}^N$ that consists of a tensor product of d -dimensional space, representing a coin, and an N -dimensional space, representing vertices of a graph. We refer to the d -dimensional subspace with subscript A and that in the N -dimensional subspace with subscript B . We use $\hat{I}_d$ to represent the identity operator on the d -dimensional subspace and $\hat{I}_N$ to represent the identity operator on the N -dimensional subspace.

The density operator $\rho \in \mathcal{L}(\mathbb{C}^{dN})$ in the space of linear operators operating on the bipartite space $\mathbb{C}^{dN}$ is a positive semidefinite Hermitian operator with a trace of 1. By *quantum channel* $\mathcal{E}(\rho)$ we mean a trace-preserving completely positive linear map operating on the space of density operators. The quantum channel itself is also a linear operator (often called a superoperator). Since we work with a bipartite tensor product space, we can take the partial trace of a density operator. When tracing over the d -dimensional subspace and giving a density operator in the N -dimensional subspace, we write $\rho_B = \text{tr}_A \rho$. When tracing over the N -dimensional subspace and giving a density operator in the d -dimensional subspace, we write $\rho_A = \text{tr}_B \rho$.

In the nomenclature of quantum channels, a channel is *ergodic* if it contains a unique eigenvector ρ_* where $\mathcal{E}(\rho_*) = \rho_*$ (a fixed point) with eigenvalue 1 and with ρ_* in the space of density operators (positive operators with trace of 1) [26].

When labeling quantum states in an N -dimensional space, it is sometimes convenient to index starting from 0, with indices ranging from 0 to $N - 1$, and other times it is convenient to index from 1, ranging from 1 to N . In this manuscript, we use both conventions. Vertices and edges of a graph are labeled with natural numbers (and in some cases including zero).

The symbol μ_H represents a distribution of unitary operators in $U(N)$ that is uniform with respect to the Haar measure. We refer to an operator $\hat{V} \in \mu_H$ as *Haar random*.

II. QUANTUM RANDOM WALK ON A d -REGULAR GRAPH

Following Aharonov *et al.* [24], we generate a quantum random walk from a d -regular undirected simple and

connected graph $G(V, E)$ with a set V of N vertices and a set E of edges. Here, d -regular means that each vertex or node has d edges, and simple means the graph lacks loops connecting a vertex to itself and lacks multiple edges connecting the same pair of vertices. The quantum walk is implemented on a bipartite quantum space $\mathbb{C}^d \otimes \mathbb{C}^N$. The quantum walk is described by iterating a unitary operator

$$\hat{U} = \Lambda_S(\hat{C} \otimes \hat{I}_N), \quad (1)$$

where $\Lambda_S \in \mathcal{L}(\mathbb{C}^{dN})$ is a controlled shift operator and $\hat{C} \in \mathcal{L}(\mathbb{C}^d)$ is a coin operator that operates on the d -dimensional subspace.

In this section we index starting from 1 in both d and N -dimensional subspaces. For each vertex $v \in V$ we assume that the d -regular undirected graph allows construction of a set of d different index functions $w(j, v)$ [3,24] for $j \in \{1, 2, \dots, d\}$, where j uniquely labels the edges that contain the vertex v . Specifically, the function $w(j, v)$ gives the vertex at the other end of the j th edge that is connected to the vertex v . For each j the function $w(j, v)$ should give a permutation that permutes all vertices. So as to generate a unitary transformation using these functions, we require that the set $\{w(j, v) : v \in V\} = V$ for all j so that $w(j, v) = w(j, v')$ iff $v = v'$. For each j we have a unique vertex $w(j, v) \in V$ such that $[v, w(j, v)] \in E$ is an edge of the graph and this is true for every vertex v . All edges in E should be present in the set $\{(v, w(j, v)) : v \in V, j \in \{1, 2, \dots, d\}\}$.

To represent traversal along the edge connecting vertex v to vertex $w(j, v)$, it is convenient to define the operator

$$\hat{W}_j = \sum_{v=1}^N |w(j, v)\rangle\langle v|, \quad (2)$$

where the sum is over all vertices and $j \in \{1, 2, \dots, d\}$. The conditions placed upon the graph and index functions $w(j, v)$ ensure that $\hat{W}_j$ is unitary.

With the $w(j, v)$ index functions and associated operators $\hat{W}_j$, we define the controlled shift unitary operator

$$\Lambda_S \equiv \sum_{j=1}^d |j\rangle\langle j| \otimes \sum_{v=1}^N |w(j, v)\rangle\langle v| = \sum_{j=1}^d |j\rangle\langle j| \otimes \hat{W}_j. \quad (3)$$

Each $|j\rangle\langle j|$ is a projection operator onto the j th edge in the coin subspace. The controlled shift operator permutes the vertices depending upon the edge that is selected with the projection operator $|j\rangle\langle j|$. Figure 1 illustrates the quantum walk as a quantum circuit. The controlled shift operator can also be written as a product of controlled operators as shown in the illustration: $\Lambda_S = \prod_{j=1}^d (|j\rangle\langle j| \otimes \hat{W}_j + (\hat{I}_d - |j\rangle\langle j|) \otimes \hat{I}_N)$.

Not all d -regular graphs allow construction of functions $w(j, v)$ that are permutations. An undirected Cayley graph of a finite Abelian group allows such a construction using generators of the group to give the edges of the graph [24]. A larger coin space is needed for construction of a quantum walk for any undirected d -regular graph [2].

A. Example 1: The Hadamard coin quantum walk on the cycle graph

We give an example similar to the quantum walk on the line [1]. Instead of an infinite dimensional line, the graph is

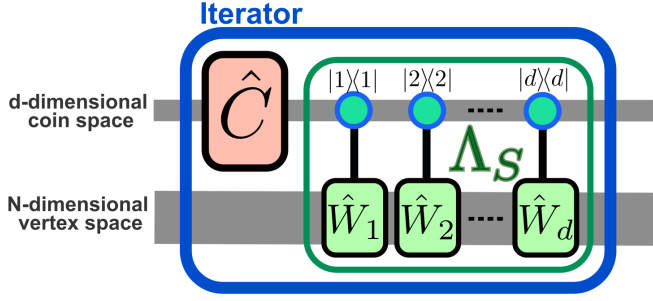


FIG. 1. A quantum circuit illustration of a discrete quantum random walk based on a d -regular graph that takes place on a quantum space that is the tensor product of a d -dimensional coin space and an N -dimensional vertex space. The controlled shift operator Λ_S [outlined in green and defined in Eq. (3)] permutes the vertices with operators $\hat{W}_j$ [defined in Eq. (2)] depending upon the graph edge that is selected via one of the states in the coined subspace. The coin operator $\hat{C}$ operates on the coin subspace, where it can induce superposition. The pair of operations is outlined in blue and is iterated to carry out the walk.

the cycle graph with N vertices (e.g., following Ref. [24]). Each vertex has two edges, so the graph degree is $d = 2$. The quantum space is $2 \times N$ dimensional. We label the vertices so that v is an integer in $\{0, 1, \dots, N-1\}$ so that we can use the modulo function in the conditional shift operator. The coin space consists of two states, $|0\rangle$ and $|1\rangle$. The index function

$$w(j, v)_{\text{cycle}} = \begin{cases} v + 1 \bmod N, & \text{if } j = 0, \\ v - 1 \bmod N, & \text{if } j = 1. \end{cases} \quad (4)$$

The operator associated with the edges of the graph

$$\hat{W}_{j,\text{cycle}} = \sum_{v=0}^{N-1} |v + 1 \bmod N\rangle \langle v| \quad (5)$$

is equivalent to a Heisenberg shift operator. The quantum walk on the cycle graph is given by the product

$$\hat{U}_{\text{cycle}} = \Lambda_{S,\text{cycle}}(\hat{C}_{\text{cycle}} \otimes \hat{I}_N) \quad (6)$$

with shift operator

$$\begin{aligned} \Lambda_{S,\text{cycle}} = & |0\rangle\langle 0| \otimes \sum_{v=0}^{N-1} |(v+1) \bmod N\rangle \langle v| \\ & + |1\rangle\langle 1| \otimes \sum_{v=0}^{N-1} |(v-1) \bmod N\rangle \langle v|. \end{aligned} \quad (7)$$

For this quantum walk, a popular coin operator is a Hadamard operator [1]

$$\hat{C}_{\text{cycle}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \sum_{j,k=0}^1 (-1)^{jk} |j\rangle \langle k|. \quad (8)$$

1. Example 2: The Grover coin quantum walk on the hypercube graph

We give an example for a quantum walk on the hypercube [3,27]. For the d -dimensional hypercube, with $N = 2^d$ vertices, the N -dimensional space describing vertices is equivalent to the tensor product of d two-state quantum systems or

d qubits. A vertex on the hypercube can be labeled with its binary string $v = v_1 v_2 v_3 \dots v_d$, where each digit in the string $v_j \in \{0, 1\}$ with index j ranging from 1 to d . Equivalently, v is an integer ranging from 0 to $2^d - 1$. Instead of a two-state coin, we have d possible coin states, so the coin is more like a d -sided dice. The full quantum space for the quantum walk is $\mathbb{C}^d \otimes \mathbb{C}^N$. For the coin state we let j range from 1 to d . The function $w(j, v)$ consists of applying a logical NOT operation applied to the j th digit of the binary string labeling a vertex v . In other words,

$$w(j, v_1 v_2 \dots v_d)_{\text{hypercube}} = v_1 v_2 \dots \text{NOT}(v_j) \dots v_d. \quad (9)$$

This implies that $|w(j, v)\rangle \langle v| = \hat{X}_j |v\rangle \langle v|$, where $\hat{X}_j$ is the Pauli X operator operating on the j th qubit.

The quantum walk on the hypercube is given by the product

$$\hat{U}_{\text{hypercube}} = \Lambda_{S,\text{hypercube}}(\hat{C}_{\text{hypercube}} \otimes \hat{I}_N) \quad (10)$$

with shift operator

$$\Lambda_{S,\text{hypercube}} = \sum_{j=1}^d |j\rangle \langle j| \otimes \hat{X}_j. \quad (11)$$

For the quantum walk on the hypercube, the coin operator $\hat{C}_{\text{hypercube}}$ can be a discrete Fourier transform operator [2] or a Grover operator,

$$\hat{C}_{\text{hypercube}} = 2|y\rangle \langle y| - \hat{I}_d \quad (12)$$

[3,27] with superposition state

$$|y\rangle \equiv \frac{1}{\sqrt{d}} \sum_{j=1}^d |j\rangle. \quad (13)$$

III. A QUANTUM WALK WITH A MARKOV COIN THAT IS HAAR RANDOM

We modify the quantum random walk by choosing a coin operator $\hat{V}$ from the distribution $\mu_H \in U(d)$ that is uniform with respect to the Haar measure. The coin operator

$$\hat{C}_{\tilde{V}} \equiv \hat{V} \otimes \hat{I}_N \quad \text{with} \quad \hat{V} \in \mu_H. \quad (14)$$

In each application of the $\hat{C}_{\tilde{V}}$ we choose a different and independent random unitary $\hat{V}$ in $U(d)$. We use a tilde superscript on V to denote an operator that is based on a distribution for $\hat{V}$. We refer to the coin operator as *Haar random* to denote that it is drawn from a distribution that is uniform with respect to the Haar measure.

The shift operator [Eq. (3)] can be implemented as a unitary quantum channel operating on a density operator $\rho \in \mathcal{L}(\mathbb{C}^{dN})$ via

$$\mathcal{E}_S(\rho) \equiv \Lambda_S \rho \Lambda_S^\dagger. \quad (15)$$

The coin random channel based on $\hat{C}_{\tilde{V}}$ in Eq. (14), and with $\hat{V}$ drawn from a distribution, is a unitary channel

$$\mathcal{E}_{\tilde{V}}(\rho) \equiv (\hat{V} \otimes \hat{I}_N) \rho (\hat{V}^\dagger \otimes \hat{I}_N) \quad \text{with} \quad \hat{V} \in \mu_H. \quad (16)$$

The random matrix coined quantum walk consists of iterating the consecutive application of the two

channels

$$\begin{aligned}\mathcal{E}_{S\hat{V}}(\rho) &\equiv \mathcal{E}_S[\mathcal{E}_{\hat{V}}(\rho)] \\ &= \Lambda_S(\hat{V} \otimes \hat{I}_N) \rho (\hat{V}^\dagger \otimes \hat{I}_N) \Lambda_S^\dagger\end{aligned}\quad (17)$$

with $\hat{V} \in \mu_H$.

A. The averaged quantum channel

To compute an expectation value integrating over possible values for coin operator $\hat{V} \in \mu_H$, we use an application of Corollary 13 by Mele [28] applied to a two state system: For any operator $O \in \mathcal{L}(\mathbb{C}^d)$ (the space of linear operators that operate on a d -state system) the expectation value

$$\mathbb{E}_{U \in \mu_H}[U O U^\dagger] = \text{tr} O \frac{\hat{I}_d}{d} \quad (18)$$

(via Eq. (50) by Mele [28]), where μ_H is the distribution of unitaries in $U(d)$ that is uniform according to the Haar measure.

We compute the expectation value of the Haar-random coin quantum channel [defined in Eq. (16)] by averaging over all possible unitary operators drawn from the Haar uniform distribution

$$\begin{aligned}\bar{\mathcal{E}}_V(\rho) &\equiv \mathbb{E}_{\hat{V} \in \mu_H} \mathcal{E}_{\hat{V}}(\rho) \\ &= \mathbb{E}_{\hat{V} \in \mu_H} [(\hat{V} \otimes \hat{I}_N) \rho (\hat{V}^\dagger \otimes \hat{I}_N)] \\ &= \frac{\hat{I}_d}{d} \otimes \text{tr}_A \rho.\end{aligned}\quad (19)$$

We have applied Eq. (18) to the d -dimensional subspace. We use an overline to denote an averaged channel. The averaged channel *depolarizes* within the coin space.

Taking the expectation value over the Haar-random component [applying Eq. (18) to the Haar-random coin quantum walk—the channel of Eq. (17)],

$$\bar{\mathcal{E}}_{SV}(\rho) = \mathbb{E}_{V \in \mu_H} [\mathcal{E}_{S\hat{V}}(\rho)] = \Lambda_S \left(\frac{\hat{I}_d}{d} \otimes \text{tr}_A(\rho) \right) \Lambda_S^\dagger. \quad (20)$$

We insert the shift operator Λ_S with Eq. (3), giving

$$\begin{aligned}\bar{\mathcal{E}}_{SV}(\rho) &= \frac{1}{d} \sum_{j=1}^d |j\rangle\langle j| \otimes \sum_{j=1}^d \sum_{v,v'=1}^N \langle v | \text{tr}_A(\rho) | v' \rangle \\ &\quad \times |w(j, v)\rangle\langle w(j, v')|. \end{aligned}\quad (21)$$

The averaged channel can also be written in terms of the operator $\hat{W}_j$ in Eq. (2) as

$$\bar{\mathcal{E}}_{SV}(\rho) = \frac{1}{d} \sum_{j=1}^d |j\rangle\langle j| \otimes \hat{W}_j \text{tr}_A(\rho) \hat{W}_j^\dagger. \quad (22)$$

Taking the trace of this equation over the coin space, we can construct a quantum channel operating on the N -dimensional subspace

$$\bar{\mathcal{E}}_{BSV}(\rho_B) = \text{tr}_A[\bar{\mathcal{E}}_{SV}(\rho)] = \frac{1}{d} \sum_{j=1}^d \hat{W}_j \rho_B \hat{W}_j^\dagger. \quad (23)$$

This quantum channel describes a quantum process with a probability of $1/d$ of transition along each edge of the graph and illustrates that the quantum channel constructed by taking

the average over the random coin is decoherent and resembles a classical random walk on the graph with a balanced coin. The averaged channel resembles the decoherent open quantum channels for quantum walks studied by Brun *et al.* [10].

B. A numerical illustration of the Haar-random coin quantum walk on the cycle graph

As previously described [1,3], due to interference caused by the unitary coin operator, a quantum random walk's behavior differs from that of a classical random walk. To show that a Haar-random coin gives behavior similar to the associated classical random walk, we compare probabilities of position measurements for the quantum walk on the cycle graph with a Hadamard coin (described in Sec. II A) and that on the same graph with a Haar-random coin (described in Sec. III). For both cases the dimension of the quantum space is $2 \times N$ with $N = 100$ and the system is initialized in the $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \otimes |N/2\rangle$ state. After t iterations of the walk, the system is in the $|\psi\rangle^{(t)}$ state. In both walks the system remains in a pure state. In Fig. 2, we plot the probability $p_{n,t} = |\langle 0| \otimes \langle n | \psi \rangle^{(t)}|^2 + |\langle 1| \otimes \langle n | \psi \rangle^{(t)}|^2$ as a function of vertex position $n \in \{0, 1, \dots, N-1\}$ at three different iteration times. The top panel shows a Hadamard coin quantum walk and the bottom panel shows a similar random walk that has a Haar-random or random matrix coin.

After a series of iterations, the distribution of walkers for the Haar-random coin resembles a Gaussian distribution and in that sense is similar to a classical random walk. In contrast, the quantum random walk has a probability distribution for position in the N -state subsystem that is much broader than the classical one. The difference between the two types of walks is due to interference that is present in the walk with the Hadamard coin and decoherence caused by the Haar-random coin in the quantum walk with the Haar-random coin.

C. Expectation values of the iterated Markov coin random walk

On a state ρ , we apply the random walk twice, each time with an independently chosen random matrix coin. With Eq. (17),

$$\begin{aligned}\mathcal{E}_{S\hat{V}_1}[\mathcal{E}_{S\hat{V}_2}(\rho)] &= \Lambda_S(\hat{V}_1 \otimes \hat{I}_N) \Lambda_S(\hat{V}_2 \otimes \hat{I}_N) \rho \\ &\quad \times (\hat{V}_2^\dagger \otimes \hat{I}_N) \Lambda_S^\dagger(\hat{V}_1^\dagger \otimes \hat{I}_N) \Lambda_S^\dagger\end{aligned}\quad (24)$$

with $\hat{V}_1, \hat{V}_2 \in \mu_H$ both uniformly distributed with respect to the Haar measure in $U(d)$. As $\hat{V}_1$ and $\hat{V}_2$ are independently drawn from the distribution μ_H , averaging of $\hat{V}_1$ and $\hat{V}_2$ can be computed in either order:

$$\begin{aligned}\mathbb{E}_{\hat{V}_1, \hat{V}_2 \in \mu_H} [\mathcal{E}_{S\hat{V}_1}(\mathcal{E}_{S\hat{V}_2}(\rho))] &= \mathbb{E}_{\hat{V}_1 \in \mu_H} [\mathcal{E}_{S\hat{V}_1}(\mathbb{E}_{\hat{V}_2 \in \mu_H} [\mathcal{E}_{S\hat{V}_2}(\rho)])] \\ &= \mathbb{E}_{\hat{V}_1 \in \mu_H} \left[\Lambda_S(\hat{V}_1 \otimes \hat{I}_N) \Lambda_S \left(\frac{\hat{I}_d}{d} \otimes \text{tr}_A \rho \right) \Lambda_S^\dagger(\hat{V}_1^\dagger \otimes \hat{I}_N) \Lambda_S^\dagger \right] \\ &= \Lambda_S \left(\frac{\hat{I}_d}{d} \otimes \text{tr}_A \left(\Lambda_S \left(\frac{\hat{I}_d}{d} \otimes \text{tr}_A \rho \right) \Lambda_S^\dagger \right) \right) \Lambda_S^\dagger \\ &= \bar{\mathcal{E}}_{SV}^{(2)}(\rho).\end{aligned}\quad (25)$$

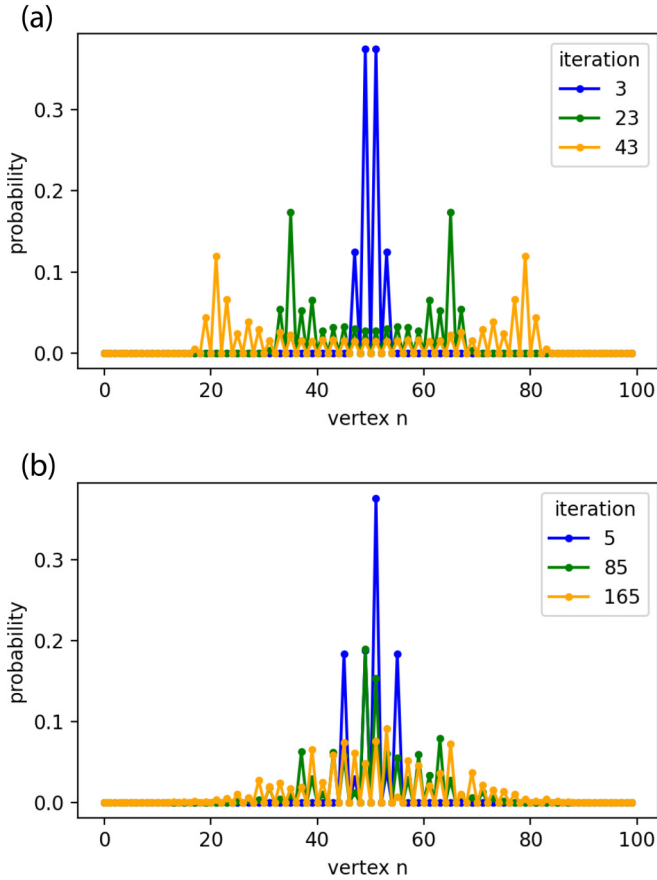


FIG. 2. (a) An illustration of the quantum random walk on the cycle graph with a Hadamard coin. The initial state was $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \otimes |50\rangle$ in a 2×100 -dimensional state space. Plotted are probabilities of measuring the system to be in the n th vertex state at iterations 3, 23, and 43. (b) Similar to (a) except we show the associated quantum walk with a Haar-random coin at iterations 5, 85, and 165. This quantum walk is begun with the same initial condition as that in (a). The Haar-random coined quantum walk does not spread rapidly like the Hadamard walk. It spreads with a distribution that is approximately Gaussian. Decoherence caused by the coin causes the vertex distribution to resemble that of a classical random walk on the same graph with a fair coin.

Taking the average over both $\hat{V}_1, \hat{V}_2 \in \mu_H$ is equivalent to two iterations of the averaged channel of Eq. (20). If the random walk is iterated t times and during each application an independent random matrix coin is chosen, then the expectation value of the iterated channel is equal to the averaged channel iterated t times. In other words,

$$\mathbb{E}_{\hat{V}_1, \hat{V}_2, \dots, \hat{V}_t \in \mu_H} [\mathcal{E}_{S\hat{V}_1}(\mathcal{E}_{S\hat{V}_2}(\dots \mathcal{E}_{S\hat{V}_t}(\rho)))] = \bar{\mathcal{E}}_{SV}^{(t)}(\rho). \quad (26)$$

D. The averaged channel is not ergodic

We consider a Haar-random coined quantum walk on a d -regular undirected connected and simple graph (not necessarily a Cayley graph). We show here that as long as we can construct a unitary shift operator in the form of Eq. (2), the averaged channel [Eq. (22)] of the quantum walk is not ergodic.

To construct the channels discussed in this manuscript, the d -regular undirected, simple, and connected graph must allow construction of the index labeling index functions $w(j, v)$ described in Sec. II. With the edge labeling index functions on a graph with N vertices it is convenient to compute

$$\frac{1}{N} \sum_{v=1}^N |w(j, v)\rangle \langle w(j, v)| = \frac{\hat{I}_N}{N}. \quad (27)$$

This follows because the set $\{w(j, v) : \forall \text{ vertices } v\}$ must contain all the vertices, and this must be true for all values of $j \in \{0, 1, \dots, d\}$. We also compute

$$\frac{1}{N} \sum_{v, v'=1}^N |w(j, v)\rangle \langle w(j, v')| = |y\rangle \langle y| \quad (28)$$

with superposition state $|y\rangle \equiv \frac{1}{\sqrt{N}} \sum_{v=1}^N |v\rangle$.

A basis for operators (with respect to the Frobenius inner product) in the vertex subspace is the set of elements $|v\rangle \langle v'|$ with $v, v' \in \{1, \dots, N\}$. The averaged channel is that given by Eq. (22). We compute

$$\bar{\mathcal{E}}_{SV} \left(\frac{\hat{I}_d}{d} \otimes |v\rangle \langle v'| \right) = \frac{1}{d} \sum_{j=1}^d |j\rangle \langle j| \otimes |w(j, v)\rangle \langle w(j, v')|. \quad (29)$$

We consider two states,

$$\frac{\hat{I}_d}{d} \otimes \sum_{v=1}^N |v\rangle \langle v| \frac{\hat{I}_d}{d} \otimes |y\rangle \langle y|. \quad (30)$$

When we operate on these states with the averaged channel, Eqs. (27)–(29) imply that they are both eigenstates of the averaged channel with eigenvalue 1. Since these states have trace of 1 and are positive, they are both in the space of density operators. The averaged channel has more than one fixed point (in the space of density operators) and consequently it is not ergodic. We find that the averaged channel of a quantum random walk with a Haar-random coin is not ergodic when constructed on any d -regular graph that allows ordering of edges giving unitary operators in the form of $\hat{W}_j$ [Eq. (2)].

The first of the states in Eq. (30) is proportional to the identity and is completely depolarized. However, because the averaged channel is not ergodic, iteration of the channel does not necessarily cause a density operator to approach the depolarized state. This implies that information about the initial density operator is retained even when the channel is iterated many times. In other words, $\bar{\mathcal{E}}_{SV}^{(t)}(\rho_0)$ retains information about ρ_0 even if t is large. In examples discussed in Sec. IV of Cayley graphs on Abelian groups (including the cycle and hypercube graphs) we find additional fixed points of the averaged channel.

E. The probability distribution of vertex measurements

Following Aharonov *et al.* [24], a quantum walk can be characterized with the probability distribution of vertex position measurements. The measurement of vertex or node position is a projective measurement with a set of vertex measurement operators $\{P_v : v \in \{1, \dots, N\}\}$ with each

projection operator associated with a single vertex

$$\hat{P}_v = \hat{I}_d \otimes |v\rangle\langle v|. \quad (31)$$

For a specific density operator $\rho \in \mathcal{L}(\mathbb{C}^{dN})$ describing a quantum state, we can compute the probability

$$p_v(\rho) = \text{tr}(\hat{P}_v \rho) \quad (32)$$

for the quantum system to be measured at a vertex v . As $\sum_{v=1}^N \hat{P}_v = \hat{I}_d \otimes \hat{I}_N$ is the identity, the set of non-negative real numbers $\{p_v : v \in \{1, \dots, N\}\}$ is a normalized probability distribution. After t iterations of a quantum walk, we can determine whether the vertex probability distribution $\{p_v(\rho) : \forall v\}$ approaches a stationary distribution in the limit of many iterations (large t). Equivalently, for initial density operator ρ_0 , if

$$\lim_{k \rightarrow \infty} p_v[\mathcal{E}^{(t)}(\rho_0)] = \frac{1}{N} \quad (33)$$

for iterations of the channel $\mathcal{E}$, then the vertex probability distribution approaches a stationary distribution.

For the quantum walk with a Haar-random coin and using Eq. (26),

$$\begin{aligned} \mathbb{E}_{\hat{V}_1, \hat{V}_2, \dots, \hat{V}_k \in \mu_H} [\text{tr}(\hat{P}_v \mathcal{E}_{S\hat{V}_1}(\mathcal{E}_{S\hat{V}_2}(\dots \mathcal{E}_{S\hat{V}_k}(\rho)))))] \\ = \text{tr}(\hat{P}_v \bar{\mathcal{E}}_{SV}^{(k)}(\rho)) = p_v(\bar{\mathcal{E}}_{SV}^{(k)}(\rho)). \end{aligned} \quad (34)$$

If the averaged channel obeys Eq. (33), then on average the vertex probability distribution approaches a stationary distribution for the iterated stochastic channel.

We show below in Sec. IV that after iteration of the averaged channel, a density operator can approach one with a stationary and uniform vertex probability distribution, while maintaining correlations that would allow measurements to reveal information about the initial density operator. This is possible because the averaged channel is not ergodic (as discussed in Sec. III D).

IV. EXAMPLES OF AVERAGED CHANNELS

We illustrate properties of averaged channels (taking the expectation value over the distribution of coin operators) for the quantum walk on the cycle graph and for the walk on the hypercube. In the last subsection below we consider the more general case of a Cayley graph of a finite Abelian group.

A. Example 1: Averaged channel for the quantum walk on the cycle graph

Using shift operator $\Lambda_{S, \text{cycle}}$ from Eq. (7) for the quantum walk on the cycle graph with a Haar-random coin, the averaged channel [Eq. (20) or (21)] becomes

$$\begin{aligned} \bar{\mathcal{E}}_{SV, \text{cycle}}(\rho) = \frac{1}{2} |0\rangle\langle 0| \otimes \sum_{v, v'=0}^{N-1} |v+1\rangle\langle v'+1| \langle v | \text{tr}_A(\rho) | v' \rangle \\ + \frac{1}{2} |1\rangle\langle 1| \otimes \sum_{v, v'=0}^{N-1} |v-1\rangle\langle v'-1| \langle v | \text{tr}_A(\rho) | v' \rangle. \end{aligned} \quad (35)$$

Here, we have omitted the modulo operator but it is still implied. This channel describes a walker that with 1/2 probability moves a step clockwise on the cycle graph and with 1/2 probability moves in the opposite direction. In this sense the averaged quantum walk recovers the classical random walk with a fair coin.

The Hadamard walk on the line is simpler with vertex space transformed to a Fourier basis [24]. For the cycle graph we use the discrete Fourier transform basis

$$|k\rangle_F = \frac{1}{\sqrt{N}} \sum_{v=0}^{N-1} \omega^{kv} |v\rangle \quad \text{for } k \in \{0, \dots, N-1\} \quad (36)$$

with

$$\omega \equiv e^{\frac{2\pi i}{N}}. \quad (37)$$

For calculations it is convenient to recognize that

$$\langle v | |k\rangle_F = \frac{\omega^{vk}}{\sqrt{N}}. \quad (38)$$

The shift operation is diagonal in the Fourier basis,

$$\sum_{v=0}^{N-1} |v+1\rangle\langle v| = \sum_{k=0}^{N-1} \omega^{-k} |k\rangle_F \langle k|_F. \quad (39)$$

The averaged channel of Eq. (35) is

$$\begin{aligned} \bar{\mathcal{E}}_{SV, \text{cycle}}(\rho) = \frac{1}{2} \sum_{k, k'=0}^{N-1} (|0\rangle\langle 0| \omega^{k'-k} + |1\rangle\langle 1| \omega^{k-k'}) \\ \otimes \rho_{k, k'} |k\rangle_F \langle k'|_F, \end{aligned} \quad (40)$$

where

$$\rho_{k, k'} = \langle k |_F \text{tr}_A \rho | k' \rangle_F. \quad (41)$$

Viewing the channel as a linear superoperator, a set of eigenstates that spans the space of linear operators on $\mathcal{L}(\mathbb{C}^2 \otimes \mathbb{C}^N)$ is the following:

$$\begin{aligned} \hat{e}_{0, kk'} &= \frac{1}{2} (|0\rangle\langle 0| \omega^{k'-k} + |1\rangle\langle 1| \omega^{k-k'}) \otimes |k\rangle_F \langle k'|_F \\ \hat{e}_{x, kk'} &= (|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes |k\rangle_F \langle k'|_F \\ \hat{e}_{y, kk'} &= (|0\rangle\langle 1| - |1\rangle\langle 0|) \otimes |k\rangle_F \langle k'|_F \\ \hat{e}_{z, kk'} &= (|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes |k\rangle_F \langle k'|_F \end{aligned} \quad (42)$$

for each of $k, k' \in \{0, 1, \dots, N-1\}$. We have labeled the eigenstates in analogy to Pauli operators. The eigenvectors that have nonzero trace are in the form of $\hat{e}_{0, kk}$. The eigenvector-eigenvalue pairs are

$$\begin{aligned} \left(\hat{e}_{0, kk'}, \cos \frac{2\pi(k-k')}{N} \right), (\hat{e}_{x, kk'}, 0), \\ (\hat{e}_{y, kk'}, 0), \text{ and } (\hat{e}_{z, kk'}, 0). \end{aligned} \quad (43)$$

We take the trace of $\hat{e}_{0, kk'}$

$$\text{tr}_A(\hat{e}_{0, kk'}) = \cos \frac{2\pi(k-k')}{N} |k\rangle_F \langle k'|_F. \quad (44)$$

Equations (44) and (43) imply that the eigenvectors $\hat{e}_{0, kk}$ that are diagonal in the Fourier basis have an eigenvalue of 1 and a trace of 1. They are fixed points of the channel; $\bar{\mathcal{E}}_{SV, \text{cycle}}(\hat{e}_{0, kk}) = \hat{e}_{0, kk}$. Due to the existence of multiple fixed

points (one for each value of k) in the space of density operators, the averaged channel $\bar{\mathcal{E}}_{SV, \text{cycle}}$ is not ergodic. The number of linearly independent fixed points is equal to the number of vertices N and is remarkably large.

The set of operators composed of the eigenvectors in Eq. (42) is a linearly independent set that spans the space of operators in $\mathcal{L}(\mathbb{C}^{dN})$. However, this set is not an orthogonal basis with respect to the Frobenius inner product [$\text{tr}(\hat{e}_{0,kk}^\dagger \hat{e}_{0,kk'}) \neq 0$]. Nevertheless, any density operator can be written as a vector using this set of eigenvectors as a basis. Iteration of the averaged channel leaves only a sum of eigenvectors with nonzero eigenvalues which only contains a sum of $\hat{e}_{0,kk'}$ terms. The set of eigenvectors $\{\hat{e}_{0,kk'} : \forall k, k'\}$ are orthogonal with respect to the Frobenius inner product, $\text{tr}(\hat{e}_{0,jj'}^\dagger \hat{e}_{0,kk'}) = \frac{1}{2} \delta_{jk} \delta_{j'k'}$.

The magnitude of the eigenvalues of $\hat{e}_{0,kk'}$ in Eq. (43) $|\lambda_{0,kk'}| \leq 1$, as expected for eigenvalues of a quantum channel. For eigenvalues satisfying $|\lambda_{0,kk'}| < 1$, iteration of the channel would cause a reduction in the amplitude of that particular eigenvector, and in the limit of many iterations only those with $|\lambda_{0,kk'}| = 1$ remain. Based on Eq. (42), the only eigenvectors with eigenvalues of magnitude 1 are

$$\begin{aligned} \hat{e}_{0,kk} &= \frac{\hat{I}}{2} \otimes |k\rangle_F \langle k|_F \\ \hat{e}_{0,k,k+N/2} &= \frac{\hat{I}}{2} \otimes |k\rangle_F \left\langle k + \frac{N}{2} \right|_F, \quad \text{if } N \text{ even,} \end{aligned} \quad (45)$$

where we omitted a global phase in the second expression. The eigenvalue of $\hat{e}_{0,kk}$ is 1 and the eigenvalue of $\hat{e}_{0,k,k+N/2}$ is $\lambda_{0,k,k+N/2} = -1$.

For an initial state ρ_0 , we compute the coefficients for each k

$$\begin{aligned} a_k &= 2\text{tr}(\rho_0 \hat{e}_{0,kk}) \\ b_{k,k+N/2} &= 2\text{tr}(\rho_0 \hat{e}_{0,k,k+N/2}) \\ b_{k+N/2,k} &= 2\text{tr}(\rho_0 \hat{e}_{0,k+N/2,k}). \end{aligned} \quad (46)$$

The b coefficients are only relevant for N even. Because the density operator is Hermitian, $b_{k,k+N/2} = b_{k+N/2,k}^*$, and because it has a trace of 1, $\sum_{k=0}^{N-1} a_k = 1$. After a larger number t of iterations, the resulting density operator

$$\begin{aligned} \rho^{(t)} &= \bar{\mathcal{E}}_{SV, \text{cycle}}^{(t)}(\rho_0) \sim \sum_{k=0}^{N-1} [a_k \hat{e}_{0,kk} + b_{k,k+N/2} \hat{e}_{0,k,k+N/2} (-1)^t \\ &\quad + b_{k+N/2,k} \hat{e}_{0,k+N/2,k} (-1)^t], \end{aligned} \quad (47)$$

where eigenvalues of the eigenvectors have been used and all decaying eigenstates have been neglected. This operator encodes information that can be retrieved from measurements that is retained upon iteration of the channel.

Using Eq. (38),

$$\langle v | \hat{e}_{0,kk} | v \rangle = \frac{1}{N} \langle v | \hat{e}_{0,k,k+N/2} | v \rangle = \frac{(-1)^v}{N}. \quad (48)$$

Using Eqs. (32) and (38) the probability of measuring the system to be at vertex v after a large number of iterations of the

averaged channel is

$$p_v(\rho^{(t)}) \sim \frac{1}{N} \left(1 - \sum_{k=0}^{N-1} (\text{Re}[b_{k,k+N/2}] (-1)^{t+v}) \right). \quad (49)$$

If N is odd, the probability $p_v = 1/N$ is independent of initial conditions and corresponds to a uniform vertex distribution, recovering the limiting distribution expected for the classical random walk. In this case vertex measurements do not reveal any information about the initial conditions. If N is even, then the probabilities are sensitive to $b_{k,k+N/2}$ which depends upon the initial conditions through Eq. (46). In this case vertex measurements can reveal information about the initial conditions.

What types of initial states would have an off-diagonal $k, k + N/2$ component? Assuming N even, consider a pure initial state that is a superposition state of the k and $k + N/2$ Fourier components that depend on the angle $\theta \in [0, \pi]$, and a relative phase $\phi \in [0, 2\pi]$,

$$|\psi_{0a}\rangle = |c\rangle \otimes \left(\cos \frac{\theta}{2} |k\rangle_F + e^{i\phi} \sin \frac{\theta}{2} |k + N/2\rangle_F \right), \quad (50)$$

and with $|c\rangle$ is any normalized state in the coin subspace. We use this superposition state to construct an initial density operator

$$\begin{aligned} \rho_{0a} &= |\psi_{0a}\rangle \langle \psi_{0a}| \\ &= |c\rangle \langle c| \otimes \left[\cos^2 \frac{\theta}{2} |k\rangle_F \langle k|_F + \sin^2 \frac{\theta}{2} \left| k + \frac{N}{2} \right\rangle_F \left\langle k + \frac{N}{2} \right|_F \right. \\ &\quad \left. + \frac{1}{2} \sin \theta \left(e^{-i\phi} |k\rangle_F \left\langle k + \frac{N}{2} \right|_F + e^{i\phi} \left| k + \frac{N}{2} \right\rangle_F \langle k|_F \right) \right]. \end{aligned} \quad (51)$$

Using Eq. (46), we compute coefficients for this initial density operator

$$\begin{aligned} a_k &= \cos^2 \frac{\theta}{2}, \quad a_{k+N/2} = \sin^2 \frac{\theta}{2}, \\ b_{k,k+N/2} &= \frac{e^{-i\phi}}{2} \sin \theta, \quad b_{k+N/2,k} = \frac{e^{i\phi}}{2} \sin \theta. \end{aligned} \quad (52)$$

After iteration of the channel t times, Eq. (47) gives a density operator

$$\begin{aligned} \rho_a^{(t)} &= \bar{\mathcal{E}}_{SV, \text{cycle}}^{(t)}(\rho_{0a}) \\ &= \frac{\hat{I}}{2} \otimes \cos^2 \frac{\theta}{2} |k\rangle_F \langle k|_F + \sin^2 \frac{\theta}{2} \left| k + \frac{N}{2} \right\rangle_F \left\langle k + \frac{N}{2} \right|_F \\ &\quad + \frac{(-1)^t}{2} \sin \theta \left(e^{-i\phi} |k\rangle_F \left\langle k + \frac{N}{2} \right|_F + e^{i\phi} \left| k + \frac{N}{2} \right\rangle_F \langle k|_F \right). \end{aligned} \quad (53)$$

Note the initial state $|\psi_{0a}\rangle$ is a pure state and the quantum walk with the Haar-random coin is unitary, giving pure states during each iteration. The averaged channel is an average over the possible randomly chosen coin operators. The density operator $\rho_a^{(t)}$ is a mixture of pure states, taking into account the averaging.

Using Eq. (49), the probability of measuring the system with density operator $\rho_a^{(t)}$ to be at vertex v after t iterations of the

the averaged channel

$$p_v(\rho_a^{(t)}) = \frac{1}{N} + \frac{(-1)^{t+v}}{N} \sin \theta \cos \phi. \quad (54)$$

The distribution of vertex position measurements would not approach a uniform distribution, and a series of measurements could reveal information about the initial state. A series of measurements after many iterations of the averaged channel would allow one to measure the product $\sin \theta \cos \phi$. Despite the fact that the coin operator is completely depolarizing, it is possible to learn something about the initial state, even after a large number of iterations (but only if N is even). With vertex position measurements alone you could not learn everything about the state, as you would not be able to constrain θ independent of ϕ and nothing is learned about the initial coin state $|c\rangle$. With other types of measurements (such as those in the Fourier basis or that involve correlations between vertices) it would be possible to constrain both ϕ and θ , as that information is retained in the density operator.

We now consider the setting where N is odd and all off-diagonal elements of the density operator decay upon iteration of the averaged channel. We have established that vertex position measurements approach a uniform distribution, but is it possible to design other types of measurements that can reveal information about the initial state, even after many iterations of the channel? Inspection of the density operator after many iterations [Eq. (47)] shows that only the information about the initial Fourier amplitudes a_k is retained. In this sense the channel is dephasing. While the amplitudes cannot be measured with vertex position measurements, they can be measured from measurements in the Fourier basis or from measurements of correlations between vertices. For example, in analogy to a Pauli X operator, we can carry out a measurement that measures correlations between vertex positions using the Hermitian operator

$$\hat{P}_{vv'} = \frac{1}{2} \hat{I}_d \otimes (|v'\rangle\langle v| + |v\rangle\langle v'|). \quad (55)$$

The expectation values for $\rho^{(t)}$ of Eq. (47) (but without b coefficients)

$$\text{tr}(\rho^{(t)} \hat{P}_{vv'}) = \frac{1}{N} \sum_k a_k \cos\left(\frac{2\pi(v' - v)k}{N}\right). \quad (56)$$

For $v \neq v'$, these measurements are sensitive to the initial Fourier amplitudes.

B. Example 2: Averaged channel for the walk on the hypercube graph

Using shift operator $\Lambda_{S, \text{hypercube}}$ from Eq. (11) for the quantum walk on the hypercube with Haar-random coin, the averaged channel [Eq. (21)] becomes

$$\bar{\mathcal{E}}_{SV, \text{hypercube}}(\rho) = \frac{1}{d} \sum_{j=1}^d |j\rangle\langle j| \otimes \hat{X}_j \rho_B \hat{X}_j. \quad (57)$$

This channel describes a walker with $1/d$ probability moving along each of the d edges on the hypercube. In this sense the averaged quantum walk recovers the classical walk on the hypercube with a fair d -sided dice.

From the averaged quantum channel, we can construct a quantum channel that operates on the density operator in the

N -dimensional subspace with Eq. (23),

$$\begin{aligned} \bar{\mathcal{E}}_{BSV}(\rho_B)_{\text{hypercube}} &= \text{tr}_A \left(\bar{\mathcal{E}}_{SV, \text{hypercube}} \left(\frac{\hat{I}_d}{d} \otimes \rho_B \right) \right) \\ &= \frac{1}{d} \sum_{j=1}^d \hat{X}_j \rho_B \hat{X}_j. \end{aligned} \quad (58)$$

In this form we can see that the set of operators $\{\hat{X}_j/\sqrt{d} : j \in \{1, \dots, d\}\}$ (proportional to the Pauli X operators) are a complete set of Kraus operators for an operator-sum decomposition of the averaged quantum channel restricted to the vertex subspace.

The relevant discrete Fourier transform would be that associated with the Abelian group $(\oplus \mathbb{Z}_2)^d$. We use a basis $|k\rangle_F$ with $k = k_1 k_2 \dots k_d \in \{0, 1\}^{\otimes d}$ written as a binary string

$$|k\rangle_F = \frac{1}{2^{d/2}} \sum_{v \in \{0, 1\}^{\otimes d}} (-1)^{k \cdot v} |v\rangle, \quad (59)$$

where $k \cdot v = \sum_{m=1}^d k_m v_m$ is the dot product of the two binary digit strings. It is convenient to compute

$$\langle v || k \rangle_F = \frac{1}{2^{d/2}} (-1)^{v \cdot k}. \quad (60)$$

We transform the Pauli operator $\hat{X}_j$ (operating on the j th qubit) into the Fourier basis

$$\hat{X}_j = \sum_{k \in \{0, 1\}^{\otimes d}} |k\rangle_F \langle k|_F (-1)^{k_j}. \quad (61)$$

Using $\rho_{kk'} = \langle k|_F \text{tr}_A(\rho) |k'\rangle_F$, and Eq. (61), the averaged channel of Eq. (58) in the Fourier basis is

$$\begin{aligned} \bar{\mathcal{E}}_{SV}(\rho)_{\text{hypercube}} &= \frac{1}{d} \sum_{j=1}^d |j\rangle\langle j| \otimes \sum_{k, k' \in \{0, 1\}^{\otimes d}} |k\rangle_F \langle k'|_F \\ &\quad \times \rho_{kk'} (-1)^{k_j + k'_j}. \end{aligned} \quad (62)$$

As done in the previous section for the circle graph (Sec. IV A), we construct a set of eigenstates of the channel's superoperator

$$\begin{aligned} \hat{e}_{0, kk'} &= \frac{1}{d} \left(\sum_{j=1}^d |j\rangle\langle j| (-1)^{k_j + k'_j} \right) \otimes |k\rangle_F \langle k'|_F \\ \hat{e}_{jj', kk'} &= |j\rangle\langle j'| \otimes |k\rangle_F \langle k'|_F \quad \text{for } j \neq j' \\ \hat{e}_{j, kk'} &= (|j\rangle\langle j| - |j+1\rangle\langle j+1|) \otimes |k\rangle_F \langle k'|_F \quad \text{for } j < d. \end{aligned} \quad (63)$$

The eigenvalues of the eigenvectors [listed in Eq. (63)] are zero except for those in the form of $\hat{e}_{0, kk'}$, in which case their eigenvalues are

$$\lambda_{0, kk'} = \frac{1}{d} \sum_{m=1}^d (-1)^{k_m + k'_m}. \quad (64)$$

Here, k_m, k'_m are the m th digits of labeling strings k, k' .

The eigenvectors listed in Eq. (63) are not orthogonal with respect to the Frobenius inner product, but they are linearly independent and span the space of operators in $\mathcal{L}(\mathbb{C}^{dN})$. The eigenvectors in the set $\{\hat{e}_{0, kk'} : k, k' \in \{0, 1\}^{\otimes d}\}$

are orthogonal with respect to the Frobenius inner product as $\text{tr}(\hat{e}_{0,kk'}^\dagger \hat{e}_{0,mm'}) = \frac{1}{d} \delta_{km} \delta_{k'm'}$. We can write an initial density operator as a vector sum of the eigenvectors listed in Eq. (63). After many iterations, only terms dependent on eigenvectors that have eigenvalues $|\lambda_{0,kk'}| = 1$ would remain.

With $k = k'$, Eq. (64) shows that eigenvector $\hat{e}_{0,kk}$ has eigenvalue $\lambda_{0,kk} = 1$. As the trace of $\hat{e}_{0,kk} = 1$ for any $k \in \{0, 1\}^{\otimes d}$, the eigenvectors $\hat{e}_{0,kk}$ are fixed points in the space of density operators of the averaged channel $\bar{\mathcal{E}}_{SV, \text{hypercube}}$. As there are multiple density operator fixed points, the averaged channel is not ergodic.

The only eigenstates that have an eigenvalue of 1 are $\hat{e}_{0,kk}$. We find that $\lambda_{0,kk'} = -1$ only if each digit of k is the logical NOT of each digit of k' . We denote these as $\hat{e}_{0,k\bar{k}}$. As we did in the example of the cycle graph (Sec. IV A), we can consider a pure superposition state that gives a component of $\hat{e}_{0,k\bar{k}}$. We denote $|\tilde{0}\rangle_F$ and $|\tilde{1}\rangle_F$ to be the state in the Fourier basis with all digits of k equal to 0 and 1, respectively. A normalized initial superposition state dependent upon angles θ and ϕ

$$|\psi_{0c}\rangle = |c\rangle \otimes \left(\cos \frac{\theta}{2} |\tilde{0}\rangle_F + e^{i\phi} \sin \frac{\theta}{2} |\tilde{1}\rangle_F \right) \quad (65)$$

for any state $|c\rangle$ in the coin space. The initial density operator for this state

$$\begin{aligned} \rho_{0c} &= |\psi_{0c}\rangle \langle \psi_{0c}| \\ &= |c\rangle \langle c| \otimes \left(\cos^2 \frac{\theta}{2} |\tilde{0}\rangle_F \langle \tilde{0}|_F + \sin^2 \frac{\theta}{2} |\tilde{1}\rangle_F \langle \tilde{1}|_F \right. \\ &\quad \left. + \frac{1}{2} \sin \theta (e^{-i\phi} |\tilde{0}\rangle_F \langle \tilde{1}|_F + e^{i\phi} |\tilde{1}\rangle_F \langle \tilde{0}|_F) \right). \end{aligned} \quad (66)$$

We compute the strength of the coefficients for a vector expansion in terms of eigenstates

$$\begin{aligned} a_0 &= d \text{tr}(\rho_{0c} \hat{e}_{0,\tilde{0}\tilde{0}}) = \cos^2 \frac{\theta}{2} \\ a_1 &= d \text{tr}(\rho_{0c} \hat{e}_{0,\tilde{1}\tilde{1}}) = \sin^2 \frac{\theta}{2} \\ b_{01} &= d \text{tr}(\rho_{0c} \hat{e}_{0,\tilde{0}\tilde{1}}) = -\frac{e^{i\phi}}{2} \sin \theta \\ b_{10} &= d \text{tr}(\rho_{0c} \hat{e}_{0,\tilde{1}\tilde{0}}) = -\frac{e^{-i\phi}}{2} \sin \theta. \end{aligned} \quad (67)$$

We compute the probability of measuring the system at vertex v for the channel's eigenvectors using Eq. (63),

$$\begin{aligned} p_v(\hat{e}_{0,kk'}) &= \text{tr}(\hat{P}_v \hat{e}_{0,kk'}) \\ &= \frac{1}{d} \sum_{m=1}^d (-1)^{k_m + k'_m} \langle v | |k\rangle_F \langle k' |_F | v \rangle \\ &= \frac{2^{-d}}{d} \sum_{m=1}^d (-1)^{k_m + k'_m} (-1)^{v \cdot (k + k')}. \end{aligned} \quad (68)$$

If $k = k'$,

$$p_v(\hat{e}_{0,kk}) = \frac{1}{2^d}, \quad (69)$$

and the probability is independent of the vertex v . However, if k and k' differ in all their digits ($k' = \bar{k}$), then

$$p_v(\hat{e}_{0,k\bar{k}}) = -\frac{1}{2^d} (-1)^{\text{parity}(v)}, \quad (70)$$

where the parity function $\text{parity}(v) = 1$ if the sum of the digits of v is odd and is 0 otherwise.

For initial state ρ_{0c} given in Eq. (66) and iteration number t large,

$$\begin{aligned} \rho_c^{(t)} &= \bar{\mathcal{E}}_{SV, \text{hypercube}}^{(t)}(\rho_0) \\ &\sim \cos^2 \frac{\theta}{2} \hat{e}_{0,\tilde{0}\tilde{0}} + \sin^2 \frac{\theta}{2} \hat{e}_{0,\tilde{1}\tilde{1}} \\ &\quad - (-1)^t \frac{\sin \theta}{2} (e^{-i\phi} \hat{e}_{0,\tilde{0}\tilde{1}} + e^{i\phi} \hat{e}_{0,\tilde{1}\tilde{0}}). \end{aligned} \quad (71)$$

Note that the information describing the initial superposition state (in terms of angles θ, ϕ) is present in the density operator, which implies that it is measurable. A measurement of the vertex position gives probability

$$\begin{aligned} p_v(\rho_c^{(t)}) &= \text{tr}(\rho_c^{(t)} \hat{P}_v) \\ &\sim \frac{1}{2^d} (1 + (-1)^t \sin \theta \cos \phi (-1)^{\text{parity}(v)}). \end{aligned} \quad (72)$$

The measurement probability depends on the angles θ, ϕ and the parity of v . From measurements of different parity vertex positions it would be possible to determine $\sin \theta \cos \phi$ giving a constraint on the angles. With additional types of measurements, such as in the Fourier basis or with correlated vertex position measurements, it would be possible to constrain both angles as the density operator contains both phase and amplitude information about the initial superposition state.

The averaged quantum channel corresponding to the quantum walk with a depolarized coin on the hypercube exhibits phenomena similar to the average quantum channel on the cycle graph that was discussed in Sec. IV A. However, for the hypercube graph, the averaged channel always has traceless off-diagonal eigenstates with eigenvalue -1 that do not decay upon iteration of the channel. The off-diagonal eigenstates are present in an initial density operator if the initial state is a superposition of states $|k\rangle_F$ and $|\bar{k}\rangle_F$ in the Fourier basis, for any d -length bit string k . Measurements can be devised that would reveal information about the initial superposition even after many iterations of the quantum channel. Measurement of the vertex positions only approaches a uniform distribution if the initial density operator does not contain these nondecaying off-diagonal terms.

C. Example 3: A quantum walk on an undirected Cayley graph of an Abelian group and its averaged channel

The examples of the cycle and hypercube graphs are in the class of undirected Cayley graphs of Abelian groups. In this section we add a third example that is a more general case of a quantum walk on an undirected Cayley graph of a finite Abelian group.

According to the fundamental theorem of finite Abelian groups, any finite Abelian group can be written as a direct sum of cyclic groups. We denote the set of group generators of group G as the set H . Any element of $g \in G$ in the

Abelian group G can be written as a product of powers of the group's generators $g = \prod_{j=1}^{|H|} s_j^{v_j}$ for group generator $s_j \in H$, and where each power $v_j \in \{0, 1, \dots, |s_j| - 1\}$. With an order chosen for the generators, we can use a string v to specify group elements or, equivalently, vertices of the Cayley graph. The string v has $|H|$ digits, and each digit is an integer that takes values $v_j \in \{0, 1, \dots, |s_j| - 1\}$. With this notation in mind, the associated quantum state in the vertex subspace is $|v\rangle$ and the dimension of the vertex quantum subspace $N = |G| = \prod_{j=1}^{|H|} |s_j|$ is the product of the orders of the group's generators.

A quantum walk can be constructed from a d -regular graph that is formed from a group G using a Cayley graph [24,29]. A Cayley graph $\Gamma(G, S)$ is generated from a group G and a set S of group elements. Each element of the group is associated with a graph vertex, and the set S specifies the edges of the graph. By convention, the set S does not include the identity element of the group. The graph's edges are vertex pairs (g, sg) , where $g \in G$ is an element of the group, $s \in S$, and sg denotes group multiplication of s and g . If the set S satisfies $s, s^{-1} \in S \forall s \in S$, then the Cayley graph is undirected. The number of edges per vertex (d , the degree) of a Cayley graph depends on the number of elements in the edge generating set S .

The cycle graph is an example of the Cayley graph $\Gamma(\mathbb{Z}_N, S)$ of the cyclic group $G = \mathbb{Z}_N$ with set $S = \{s, s^{-1}\}$, where s is an element that generates the group. The set of group generators $H = \{s\}$, but as long as $N > 1$, the number of elements in S is two because the set S contains both the group generator and its inverse. In this case $d = |S| = 2|H|$.

The hypercube is also an example of the Cayley graph $\Gamma((\oplus \mathbb{Z}_2)^d, S)$ of the group $G = (\oplus \mathbb{Z}_2)^d$, a direct sum of d copies of $\mathbb{Z}_2$ (the Abelian group with two elements). There are d generators of the group in the set S and every generator is its own inverse. In this case $S = H$ the set of generators of the group and the degree of the graph $d = |S| = |H|$.

So that the set S contains its own inverse (giving an undirected graph) and so that it generates the whole group G (giving a connected graph), we require that the set S contain the generators of each cyclic group of its decomposition into cyclic groups and their inverses.

The set S is used to generate graph edges and the operators for edge transitions [Eq. (2)], but the Fourier transform and its associated basis only depend on the number of group generators.

Suppose the set H of generators of the group has a number a_s of generators of order 2. We order the elements of the edge set S beginning with the generators of order 2,

$$S \equiv \{s_1, s_2, \dots, s_{a_s}, s_{a_s+1}, s_{a_s+1}^{-1}, s_{a_s+2}, s_{a_s+2}^{-1}, \dots, s_{|H|}, s_{|H|}^{-1}\}. \quad (73)$$

The length of this list is equal to the degree d of each vertex of the Cayley graph

$$d = |S| = 2|H| - a_s. \quad (74)$$

For the Cayley graph generated by a finite Abelian group G , the ordering of the list in Eq. (73) based on the generators gives an ordering for edges and ensures that the operators $\hat{W}_m \in \mathcal{L}(\mathbb{C}^{|G|})$ [of Eq. (2) and for $m \in \{1, \dots, d\}$] are unitary.

Specifically, the unitary operators use for constructing a shift operator

$$\hat{W}_m = \sum_{g \in G} |r_m g\rangle \langle g| \quad \text{for } r_m \in S, \quad (75)$$

where r_m represents the generator or inverse generator in the ordered list S . The shift operator used to construct a quantum walk is the same as we discussed previously [Eq. (3)],

$$\Lambda_S = \sum_{m=1}^d |m\rangle \langle m| \otimes \hat{W}_m, \quad (76)$$

where $\Lambda_S \in \mathcal{L}(\mathbb{C}^{dN})$ with degree d given in Eq. (74) and $N = |G|$.

For a finite Abelian group G , a discrete Fourier transform [30] can be defined using the group $\hat{G}$ of its multiplicative characters (which are the one-dimensional irreducible representations). A character $\chi_k(\cdot) : G \rightarrow \mathbb{C}$ (labeled by index k) generates a complex number $\chi_k(g)$ that is a root of unity for each element g in the group. Multiplication of the characters obey group multiplication; $\chi_k(g)\chi_k(h) = \chi_k(gh)$ for $g, h \in G$. For a finite Abelian group, the group of characters contains the same number of elements as the group itself. A discrete Fourier transform and an orthonormal Fourier basis is constructed from the characters with

$$|k\rangle_F = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \chi_k(g) |g\rangle. \quad (77)$$

For each generator $s_j \in H$ of the group we construct a complex root of unity

$$\omega_j = e^{\frac{2\pi i}{|s_j|}}. \quad (78)$$

The discrete Fourier transform of a cyclic group is particularly straightforward as it is in the form of Eq. (36). A Fourier basis for a finite Abelian group can be described as a product of discrete Fourier transforms in the form of Eq. (36). Each discrete Fourier transform in the product is associated with a generator $s_j \in H$ (the generating set for the group) and depends on a phase factor ω_j as defined in Eq. (78).

The associated Fourier vector is likewise a string k , where each digit $k_j \in \{0, 1, \dots, |s_j| - 1\}$. We define a Fourier basis

$$|k\rangle_F = \frac{1}{\sqrt{|G|}} \sum_{v \in V} \left(\prod_{j=1}^{|H|} \omega_j^{v_j k_j} \right) |v\rangle. \quad (79)$$

It is convenient to compute

$$\langle v || k \rangle_F = \frac{1}{\sqrt{|G|}} \prod_{j=1}^{|H|} \omega_j^{v_j k_j}. \quad (80)$$

Here, v_j, k_j are the j th digits of the labeling strings v, k .

To find the operators $\hat{W}_m$ for edge transitions [defined in Eq. (2)] in the Fourier basis, we compute

$$\langle k |_F \hat{W}_m | k' \rangle_F \quad (81)$$

for each of the elements of the ordered list S of Eq. (73), with index m specifying the specific element of the set S . We find

that

$$\hat{W}_m = \sum_k \omega_m^{k_{j(m)}(-1)^{x_m}} |k\rangle_F \langle k|_F \quad (82)$$

with

$$\begin{aligned} j(m) &= \begin{cases} m & \text{for } 1 < m \leq a_s \\ \text{int}[(m - a_s)/2] & \text{for } a_s < m \leq d \end{cases} \\ x_m &= \begin{cases} 0 & \text{for } 1 < m \leq a_s \\ (m - a_s) \bmod 2 & \text{for } a_s < m \leq d \end{cases} \\ \omega_m &= e^{\frac{2\pi i}{|S|}}. \end{aligned} \quad (83)$$

Here, $k_{j(m)}$ is the j th digit in k , written as a string, and s_m is the m th group element in the ordered list of Eq. (73). Here, x_m serves to identify the inverse elements for elements in S that have an order greater than 2. The sum over k is over all Fourier basis elements. Equation (82) shows that in the Fourier basis, the unitary operators $\hat{W}_m \in \mathcal{L}(\mathbb{C}^N)$ are diagonal.

As before, we construct a unitary quantum random walk with a Haar-random coin. The coin space has dimension $d = |S|$. The averaged quantum channel $\bar{\mathcal{E}}_{SV}$ is again given via Eq. (22) and corresponds to an open quantum channel with a depolarizing coin operator. We transform the channel so that the vertex space is in the Fourier basis, giving

$$\begin{aligned} \bar{\mathcal{E}}_{SV}(\rho) &= \frac{1}{d} \sum_{m=1}^d |m\rangle \langle m| \otimes \hat{W}_m \text{tr}_A(\rho) \hat{W}_m^\dagger \\ &= \frac{1}{d} \sum_{m=1}^d |m\rangle \langle m| \otimes \sum_{kk'} \omega_m^{k_{j(m)}(-1)^{x_m}} \rho_{kk'} |k\rangle_F \langle k'|_F \end{aligned} \quad (84)$$

with $\rho_{kk'} = \langle k|_F \text{tr}_A(\rho) |k'\rangle$ and notation given in Eqs. (82) and (83). The sum in Eq. (84) is over all strings k, k' specifying states in the Fourier basis.

The only eigenvectors of the averaged channel that do not have eigenvalue of zero are in the class

$$\hat{e}_{0,kk'} = \frac{1}{d} \sum_{m=1}^d |m\rangle \langle m| \otimes \omega_m^{(k_{j(m)} - k'_{j(m)})(-1)^{x_m}} |k\rangle_F \langle k'|_F. \quad (85)$$

Here, k, k' can be that of any state in the Fourier basis. Recall that these are strings with a number of digits equal to $|H|$, the size of the generating set, and with digit possible values dependent on the order of the corresponding group generator element. The associated eigenvalues for the eigenvectors of the channel are

$$\lambda_{0,kk'} = \frac{1}{|H|} \sum_{j=1}^{|H|} \cos \frac{2\pi(k_j - k'_j)}{|s_j|}. \quad (86)$$

Eigenvectors of the averaged channel that are not in the form of Eq. (85) have an eigenvalue of zero.

As was true for the eigenvectors of the averaged channel on the cycle and hypercube graphs, the eigenvalues of the averaged channel for a Cayley graph [Eq. (86)] are 1 only if $k = k'$. In this case the associated eigenstate $\hat{e}_{0,kk}$ has a trace of 1 so it is in the space of density operators. For each k the eigenstate $\hat{e}_{0,kk}$ gives a fixed point of the channel, hence the averaged channel is not ergodic. Again, the number of linearly

independent fixed points is equal to the number of vertices or $|G|$ and is remarkably large.

As we did in Secs. IV A and IV B, we can write a density operator as a vector (in the space of linear operators) with a sum of its eigenvectors. The only components that remain after many iterations of the channel are those with an eigenvalue of 1 or -1 . Following the arguments presented in the previous sections, a measurement of vertex position can reveal whether or not the initial state contains a component of an eigenstate with eigenvalue -1 .

To have an eigenvalue that has a magnitude of -1 , Eq. (86) implies that all generators must have even order. For eigenvalue $\lambda_{0,kk'} = -1$ the following must be satisfied:

$$k'_j = k_j + |s_j|/2 \bmod |s_j| \quad \text{for } j \in \{1, \dots, |H|\}. \quad (87)$$

Here, k_j and k'_j are the j th digits of k, k' corresponding to the j th generator s_j with order $|s_j|$, and the condition of (87) must be satisfied for all digits. As we showed in Secs. IV A and IV B, with a superposition state, we can construct a density operator that has an off-diagonal component that would not decay after iteration of the channel. Furthermore, measurements of the vertex position would be able to determine whether the initial density operator contained these off-diagonal states, even after the channel is iterated.

What if there are no eigenstates in the form of $\hat{e}_{0,kk'}$ in Eq. (85) that have an eigenvalue of -1 ? In this case all eigenstates that have an eigenvalue with magnitude 1 are diagonal in the Fourier basis. For an initial density operator ρ_0 decomposed into eigenstates, we again denote $\rho^{(t)} = \bar{\mathcal{E}}_{SV}^{(t)}(\rho_0)$. After iteration of the channel, only eigenstates in the form of $\hat{e}_{0,kk}$ would remain in the density operator, giving

$$\lim_{t \rightarrow \infty} \text{tr}_A(\rho^{(t)}) = \sum_k \rho_{0,k} |k\rangle_F \langle k|_F, \quad (88)$$

where the positive and real coefficients $\rho_{0,k} = \text{tr}(\rho_0 \hat{f}_d \otimes |k\rangle \langle k|_F)$ depend only on the initial Fourier amplitudes. The averaged channel is described as *dephasing*, as information about superposition is lost. However, the channel is dephasing in the Fourier basis, not in the vertex basis.

We compute the probability of measuring a state to be at a particular vertex

$$p_v(\hat{e}_{0,kk}) = \text{tr}(\hat{P}_v \hat{e}_{0,kk}) = |\langle v|k\rangle_F|^2 = \frac{1}{N} \quad (89)$$

with $\hat{P}_v$ defined in Eq. (31). If there are no eigenstates with eigenvalue -1 , upon iteration of the channel, the probability of vertex position measurements approaches a uniform distribution, and in this sense the averaged quantum walk resembles a classical random walk. This is expected because the vertex and Fourier bases are mutually unbiased; $|\langle v|k\rangle_F|$ is independent of v and k , for all v, k . After iteration of the channel, vertex position measurements would not reveal any information about the initial conditions. However, the initial Fourier amplitudes are retained in the density operator and can be measured with other types of measurements such as measurements in the Fourier basis or those that measure correlations between vertex positions, as illustrated in more detail in Sec. IV A.

In summary, the cycle graph and hypercube are examples of a Cayley graph generated from an Abelian group. For other

finite Abelian groups, we can similarly generate a quantum random walk and a random walk with a Haar-random coin using its Cayley graph. The averaged channel for the Haar-random coined quantum walk on the Cayley graph of the Abelian group (corresponding to a depolarized coin) could have off-diagonal eigenstates (in the Fourier basis of the vertex subspace) with an eigenvalue of -1 and that persist upon iteration of the channel. However, these nondecaying off-diagonal eigenvectors of the averaged channel are only present if all group elements (other than the identity) have even order. The indices of the Fourier components for the nondecaying off-diagonal eigenvectors [in the form of Eq. (85)] of Fourier indices k, k' must satisfy Eq. (87) for all digits of k, k' .

If the averaged channel lacks eigenvalues that are -1 , then the averaged channel is dephasing in the Fourier basis (of the vertex subspace), as all off-diagonal components in this basis decay under iteration of the channel. In this case iteration of the averaged channel (of the quantum walk with a Haar-random coin operator) gives a uniform distribution in the probabilities of vertex measurements. The fully depolarized state $\frac{1}{dN} \hat{I}_d \otimes \hat{I}_N$ is a fixed point of the channel. However, because the channel is not ergodic, the iteration of the channel does not necessarily cause the resulting density operator to approach this depolarized state. The amount of information about the initial conditions that is retained upon iteration of the channel is related to the number of fixed points of the channel. A measurement can reveal information about the relative amplitudes of the original state in the Fourier basis but not their original relative phases.

V. VARIANCES OF VERTEX POSITION MEASUREMENTS

In this section we estimate the size of fluctuations for measurements in the vertex subspace in a quantum walk with a Haar-random coin. Given a Hermitian operator $\hat{B}$ associated with a measurement, and a quantum state ρ , the conventional expectation value

$$\langle \hat{B} \rangle = \text{tr}(\rho \hat{B}). \quad (90)$$

The density operator after applications of the Markov coin quantum channel t times, starting from the state ρ_0 , is

$$\rho_V^{(t)} = \mathcal{E}_{SV}^{(t)}(\rho_0). \quad (91)$$

Here, $\rho_V^{(t)}$ is a specific instance of the density operator resulting from t different choices for the Haar-random coins used during each step of the walk. The density operator $\rho_V^{(t)}$ depends on $\hat{V}_1, \hat{V}_2, \dots, \hat{V}_t$, the specific choices of the coin operators. We use the subscript V with a tilde to make it clear that it depends on these choices. If we carry out a particular quantum walk (specifying $\hat{V}_1, \hat{V}_2, \dots, \hat{V}_t$) to give a density operator $\rho_V^{(t)}$, we would require multiple experimental copies of this same operator $\rho_V^{(t)}$ to measure the expectation value $\langle \hat{B} \rangle = \text{tr}(\rho_V^{(t)} \hat{B})$ in the laboratory. If we generate an ensemble of $\rho_V^{(t)}$ states using independently chosen coin operators, then we can compute the expectation value

$$\overline{\langle \hat{B} \rangle^{(t)}} = \mathbb{E}_{V_1, V_2, \dots, V_t \in \mu_H} [\text{tr}(\rho_V^{(t)} \hat{B})]. \quad (92)$$

This is the expectation value for measurements of $\hat{B}$ from an ensemble of random walks after t iterations.

Following Eq. (26), the expectation value of Eq. (92), taking into account the probability distribution of the different randomly selected coin operators and after t iterations of the quantum walk, is

$$\overline{\langle \hat{B} \rangle^{(t)}} = \text{tr}(\bar{\mathcal{E}}_{SV}^{(t)}(\rho_0) \hat{B}). \quad (93)$$

This depends on the averaged channel that was discussed in Sec. III A.

How do we describe the scatter or variance of the $\hat{B}$ measurements for an ensemble of walks? Given quantum state ρ and Hermitian operator $\hat{B}$, the variance of measurements

$$\sigma_B^2 = \langle \hat{B}^2 \rangle - (\langle \hat{B} \rangle)^2 = \text{tr}(\rho \hat{B}^2) - (\text{tr}(\rho \hat{B}))^2. \quad (94)$$

When averaging over a distribution of different walks, we need to keep in mind that

$$(\mathbb{E}_{\hat{V}_1, \hat{V}_2, \dots, \hat{V}_t \in \mu_H} [\text{tr}(\hat{B} \rho^{(t)})])^2 \neq \mathbb{E}_{\hat{V}_1, \hat{V}_2, \dots, \hat{V}_t \in \mu_H} [(\text{tr}(\hat{B} \rho^{(t)}))^2]. \quad (95)$$

While the left-hand side of Eq. (95) is straightforward to compute using the averaged channel with Eq. (93), the right-hand side is difficult to compute as it involves a second moment of the distribution of random unitaries. To get around this problem, we estimate the change in expectation values caused by one iteration of the quantum walk,

$$\begin{aligned} \delta \langle \hat{B} \rangle &= \text{tr}((\mathcal{E}_{SV}(\rho) - \rho) \hat{B}) \\ \delta \langle \hat{B}^2 \rangle &= \text{tr}((\mathcal{E}_{SV}(\rho) - \rho) \hat{B}^2). \end{aligned} \quad (96)$$

With these changes we estimate a change in variance by linearizing Eq. (94),

$$\delta \sigma_B^2 \approx \delta \langle \hat{B}^2 \rangle - 2 \langle \hat{B} \rangle \delta \langle \hat{B} \rangle, \quad (97)$$

for a difference caused by one application of the channel. Inserting Eq. (96) into Eq. (97), we estimate the change in the scatter of $\hat{B}$ measurements as

$$\delta \sigma_B^2 \approx \text{tr}((\mathcal{E}_{SV}(\rho) - \rho) \hat{B}^2) - 2 \text{tr}(\rho \hat{B}) \text{tr}((\mathcal{E}_{SV}(\rho) - \rho) \hat{B}). \quad (98)$$

We compute the expectation value of the change in the variance as

$$\begin{aligned} \overline{\delta \sigma_B^2} &= \mathbb{E}_{V \in \mu_H} [\delta \sigma_B^2] \approx \text{tr}((\bar{\mathcal{E}}_{SV}(\rho) - \rho) \hat{B}^2) \\ &\quad - 2 \text{tr}(\rho \hat{B}) \text{tr}((\bar{\mathcal{E}}_{SV}(\rho) - \rho) \hat{B}). \end{aligned} \quad (99)$$

This depends only on the averaged channel and is more easily computed than the expression on the right-hand side of Eq. (95).

A. The variance of vertex position estimates for the quantum walk on the cycle graph

A property of the quantum walk with the Hadamard coin on the line is that the variance in position measurements increases more rapidly than for the classical random walk [1]. In this subsection, we apply Eq. (99) to estimate the variance

of position measurements in the quantum walk on the cycle graph with a Haar-random coin.

For a coined quantum walk on a line, an operator $\hat{I}_2 \otimes \sum_{v=-\infty}^{\infty} v|v\rangle\langle v|$ is used to measure the expectation value of position and its variance. On the cycle graph with N vertices, we consider a similar operator that is Hermitian and diagonal in the vertex basis, and periodic so that it has no large gaps in its spectrum,

$$\hat{B} = \hat{I}_2 \otimes \sum_{v=0}^{N-1} \frac{N}{2\pi} \sin \frac{2\pi v}{N} |v\rangle\langle v|. \quad (100)$$

For localized states, the expectation value of this operator gives an estimate for the mean position index. We use a sine to be consistent with the cyclic nature of the graph, but keep in mind that states are labeled modulo N , so we allow vertex label v to be negative. For small v , the value $\frac{N}{2\pi} \sin \frac{2\pi v}{N} \sim v$. If N is large and the system is begun in an initial state that is localized near $v = 0$, then we can use $\hat{B}$ and the small angle

approximation to measure the width of the vertex position probability distribution.

Expectation values for state ρ and operator $\hat{B}$ given in Eq. (100) are

$$\begin{aligned} \text{tr}(\rho \hat{B}) &= \sum_{v=0}^{N-1} \rho_{vv} \frac{N}{2\pi} \sin \frac{2\pi v}{N} \\ \text{tr}(\rho \hat{B}^2) &= \sum_{v=0}^{N-1} \rho_{vv} \left(\frac{N}{2\pi} \right)^2 \sin^2 \frac{2\pi v}{N} \end{aligned} \quad (101)$$

with notation

$$\rho_{vv} \equiv \langle v | \text{tr}_A(\rho) | v \rangle. \quad (102)$$

With the averaged channel for the quantum walk on the cycle graph with Haar-random coin in the form of (35), we compute the expectation values

$$\begin{aligned} \text{tr}(\bar{\mathcal{E}}_{SV, \text{cycle}}(\rho) \hat{B}) &= \text{tr} \left[\frac{1}{2} \sum_{vv'} (|0\rangle\langle 0| \otimes |v+1\rangle\langle v'+1| + |1\rangle\langle 1| \otimes |v-1\rangle\langle v'-1|) \rho_{vv'} \hat{I}_2 \otimes \sum_{v''} \frac{N}{2\pi} \sin \frac{2\pi v''}{N} |v''\rangle\langle v''| \right] \\ &= \frac{N}{4\pi} \sum_v \left(\sin \frac{2\pi(v+1)}{N} + \sin \frac{2\pi(v-1)}{N} \right) \rho_{vv} \\ \text{tr}(\bar{\mathcal{E}}_{SV, \text{cycle}}(\rho) \hat{B}^2) &= \frac{1}{2} \left(\frac{N}{2\pi} \right)^2 \sum_v \left(\sin^2 \frac{2\pi(v+1)}{N} + \sin^2 \frac{2\pi(v-1)}{N} \right) \rho_{vv}. \end{aligned} \quad (103)$$

Using Eqs. (101) and (103),

$$\begin{aligned} \text{tr}((\bar{\mathcal{E}}_{SV, \text{cycle}}(\rho) - \rho) \hat{B}) &= \frac{N}{4\pi} \sum_v \left(\sin \frac{2\pi(v+1)}{N} + \sin \frac{2\pi(v-1)}{N} - 2 \sin \frac{2\pi v}{N} \right) \rho_{vv} \\ \text{tr}((\bar{\mathcal{E}}_{SV, \text{cycle}}(\rho) - \rho) \hat{B}^2) &= \frac{1}{2} \left(\frac{N}{2\pi} \right)^2 \sum_v \left(\sin^2 \frac{2\pi(v+1)}{N} + \sin^2 \frac{2\pi(v-1)}{N} - 2 \sin^2 \frac{2\pi v}{N} \right) \rho_{vv}. \end{aligned} \quad (104)$$

If state ρ mostly has power at small v , then $\sin \frac{2\pi v}{N} \sim \frac{2\pi v}{N}$ and

$$\begin{aligned} \text{tr}((\bar{\mathcal{E}}_{SV, \text{cycle}}(\rho) - \rho) \hat{B}) &\sim 0 \\ \text{tr}((\bar{\mathcal{E}}_{SV, \text{cycle}}(\rho) - \rho) \hat{B}^2) &\sim 1. \end{aligned} \quad (105)$$

We insert these relations into Eq. (99) to estimate the increase in variance

$$\mathbb{E}_{\hat{V} \in \mu_h} [\delta \sigma_{\hat{B}}^2] \approx 1. \quad (106)$$

For an initially localized state, each iteration of the channel would on average increase the variance of vertex position measurements by 1. After t iterations, we expect the variance of vertex position measurements to be

$$\overline{\sigma_{\hat{B}}^2(t)} \sim \sigma_{\hat{B}}^2(0) + t. \quad (107)$$

For the classical random walk on the line and walkers initially begun at a particular location, the distribution of positions at time t is Gaussian with a variance that is linearly dependent on t . We recover the linear dependence of variance on iteration number for the quantum walk with the Haar-random coin on the cycle graph, but describing positive

measurements for an ensemble of quantum walks. The linear dependence of variance on iteration number is the same as expected for the classical random walk. The linear dependence of the variance is consistent with the recovery of classical behavior for a quantum walk on a cycle graph with a decoherent coin [10].

Even though information about initial conditions can be learned through measurements of correlations, decoherence caused by the Haar-random coin causes the quantum walk to spread in the same way as the classical one. The vertex position probability distribution for the Hadamard quantum walk and one instance of the Haar-random coined quantum walk on the cycle graph is illustrated in Fig. 2 (see Sec. III B). Figure 2(b) shows that the variance in vertex position is approximately Gaussian for the quantum walk with the Haar random, as would be expected from Eq. (107).

VI. SUMMARY AND CONCLUSION

In this manuscript we have explored a generalization of a quantum random walk on a d -regular connected simple graph,

replacing the coin operator with a random matrix $\in U(d)$ that is independently drawn from a uniform distribution with respect to the Haar measure. The random matrix is chosen independently each step of the quantum walk. Moments of Haar-random matrices (e.g., Ref. [28]) aid in computing statistical properties of the Markov coin random quantum walk model. We find that the averaged channel, computed with the expectation value over randomly chosen operators operating on the coin subsystem, depolarizes the coin subspace. Nevertheless, the channel itself is a unitary channel and an initially pure state remains a pure state after each iteration of the walk. Expectation values of operators for ensembles of walkers can be computed with the averaged quantum channel and its iterates.

Even though the coined space is on average depolarized, we show that in some special cases, measurements of vertex positions can detect information about the initial conditions of the quantum state, even after repeated iteration of the averaged channel. The cases where this is possible depend upon the graph. For a Cayley graph of an Abelian group, the initial state must contain off-diagonal terms in the Fourier basis, associated with eigenstates of the channel's superoperator with eigenvalue -1 , and these only exist if the order of all group elements (other than the identity) is even. Examples of such states are superpositions of two states of the Fourier basis. The existence of such eigenstates is related to conditions placed on a coined quantum random walk on the Cayley graph of an Abelian group for the vertex position distribution to approach a uniform distribution upon iteration of the walk [24].

We find that the averaged quantum channel of the Haar-random coined quantum walk has multiple fixed points and so is not ergodic. For quantum walks constructed on a Cayley graph of an Abelian group, the number of linearly independent fixed points of the averaged channel is equal to the number of elements in the group. Because the fixed points do not decay upon iteration of the channel, information about an initial state is potentially measurable even after multiple iterations of the averaged channel. If there is no off-diagonal eigenstate with eigenvalue -1 , then the averaged channel for the walk on a Cayley graph is *dephasing* in the Fourier basis, and the vertex position measurements approach a uniform distribution. However, measurements of correlations between pairs of vertices can reveal information about the initial state, even after repeated iteration of the depolarizing averaged channel. For example, the relative amplitudes of a state that is initially a superposition of two Fourier basis states can be detected via measurement of correlations between two vertex positions in the averaged quantum channel.

Even though the Haar-random coined quantum walk is a unitary quantum channel, an initially localized particle does

not spread ballistically as is true in the Hadamard quantum random walk based on the cycle graph. We show this by estimating the scatter in position measurements for the Haar-random quantum walk on the cycle graph. This result is consistent with prior studies of an open quantum walk with a decoherent coin operator [10]. Decoherence is introduced by the independently chosen random matrices that are applied consecutively at each iteration of the quantum walk.

Our Markov coin, because it is uniformly distributed according to the Haar measure, is straightforward and gives a nice example of correspondence between a quantum and classical walk and decoherence. Except for the special cases discussed above, the vertex distribution approaches the uniform distribution, as would a classical random walk with a fair coin. Behavior associated with the classical random walk on a graph is embedded within the quantum walk via averaging over the Haar-random matrix distribution. In the case of quantum walks on Cayley graphs, the classical behavior is associated with dephasing with respect to the Fourier basis.

Due to its decoherence, a quantum random walk with a Haar-random coin would probably not be useful in quantum search algorithms which are aided by ballistic spreading of the quantum state vector [4,5]. However, they may be useful in other contexts such as in measuring transport properties (e.g., Ref. [31]) or modeling quantum systems with noisy subsystems or subsystems that interact with a thermal reservoir (e.g., Refs. [15–17]).

For future work, the case of a Haar-random coin could be generalized to coin operators drawn from other distributions. More complex Markov models that perturb the system in position or coin or both spaces (e.g., Refs. [14,23]) could be studied. It might be possible to develop a theory for fluctuations exhibited by this simple model, improving upon our rough estimates in Sec. V.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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