Quantum dynamical entropy and non-Markovianity: A collisional model perspective

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Accessing the physical mechanisms behind non-Markovian phenomena in open quantum dynamics requires the study of the full multitime statistical properties of the joint system-environment dynamics. This is impossible at the level of the reduced dynamics of the open system alone as the latter is obtained by suitably eliminating the environment. The task is instead made possible by considering multitime correlation functions involving observables of the open system, only: the open system-environment interactions turn them into global ones, thus building up correlations between the two systems. Multitime correlations form the basis of both the theory of quantum stochastic processes and of the Alicki-Lindblad-Fannes dynamical entropy (ALF entropy for short). This latter quantity provides for quantum systems a measure of the dynamical entropy production as the Kolmogorov-Sinai entropy does for classical systems. Indeed, in the following we propose the open-system ALF entropy as a measure of non-Markovianity signaling deviations from the so-called quantum regression regime. In turn, the operational interpretation of the open-system ALF entropy makes it a suitable witness of backflow of information from the environment to the open system. In the case of a collisional model whereby the dissipative dynamics of a finite-level system is obtained by its coupling to an infinite classical spin chain, the ALF entropy can be explicitly computed. It turns out to depend on the parameters characterizing the statistical properties of the environment and can be related to the activation and superactivation of memory effects in the open quantum system.

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I. INTRODUCTION

Due to their importance in quantum information technologies, the non-Markovian features of open quantum dynamics have recently become the focus of much research and have been addressed by several, nonequivalent, techniques resulting in an intricate hierarchy [1]. As an instance, typically the time evolution of an open quantum system is provided by a family $\{\Lambda_t^\ddagger\}_{t\geqslant 0}$ of completely positive and trace-preserving (CPTP) linear maps $\rho \longmapsto \Lambda_t^\ddagger[\rho]$ on the space of states ρ that result from tracing over the environment degrees of freedom. Then, in the absence of further knowledge of the system-environment state, one may identify Markovianity with the completely positive divisibility (CP-divisibility) of the evolution [2–4]. Namely, with the complete positivity of intermediate-time propagators $\Lambda_{t,s}^\ddagger$, $t \geqslant s \geqslant 0$, composing as $\Lambda_t^\ddagger = \Lambda_{t,s}^\ddagger \circ \Lambda_s^\ddagger$. In such a scenario, the lack of positivity preservation and thus of contractivity of the propagators (lack of P-divisibility) gives rise to the increase, instead of the decrease, of several information distances, a phenomenon generally interpreted as a physical backflow of information from the environment into the open system [5]. Considering instead a classical discrete-time scenario as a benchmark for possible quantum extensions of the notion of Markovianity, there one deals with multitime probabilities $\pi_{[0,n-1]}$ of a stochastic variable X having, say, discrete values x_j at subsequent times:

$$p^{[0,n-1]}_{\boldsymbol{x}_{[0,n-1]}} := P(X_0 = x_0, \ldots, X_{n-1} = x_{n-1}),$$

$$\pi_{[0,n-1]} = \left\{p^{[0,n-1]}_{\boldsymbol{x}_{[0,n-1]}}\right\}_{\boldsymbol{x}_{[0,n-1]}}.$$

Markovianity is then identified with lack of memory and mathematically characterized by multitime conditional probabilities depending only on the immediately preceding outcome:

$$p^{(n-1|n-2,\ldots,0)}_{\boldsymbol{x}_{[0,n-1]}} := \frac{p^{[0,n-1]}_{\boldsymbol{x}_{[0,n-1]}}}{p^{[0,n-2]}_{\boldsymbol{x}_{[0,n-2]}}} = p^{(n-1|n-2)}_{x_{n-1}\,x_{n-2}}, \tag{1}$$

where $\boldsymbol{x}_{[0,n-1]} := x_0 \ldots x_{n-1}$. Evidently, for classical Markovian processes, multitime probabilities factorize into products of conditional probabilities,

$$p^{[0,n-1]}_{\boldsymbol{x}_{[0,n-1]}} = \prod_{j=1}^{n-1} p^{(j|j-1)}_{x_j x_{j-1}}\; p^{(0)}_{x_0}, \tag{2}$$

and satisfy the Chapman-Kolmogorov equations; namely, for all $j \geqslant \ell \geqslant k$,

$$p^{(j|k)}_{x_j x_k} = \sum_{x_\ell} p^{(j|\ell)}_{x_j x_\ell}\; p^{(\ell|k)}_{x_\ell x_k}, \tag{3}$$

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where $p^{(j|k)}_{x_j x_k} = \frac{\sum_{x_i \neq x_j, x_k} p^{[0,j]}_{\boldsymbol{x}_{[0,j]}}}{\sum_{x_i \neq x_k} p^{[0,k]}_{\boldsymbol{x}_{[0,k]}}}$. Therefore, the transition matrices $T(j,k) = [p^{(j|k)}_{x_j x_k}]$ are stochastic matrices composing as $T(j,k) = T(j,\ell)T(\ell,k)$ and play the same role as the discrete-time quantum propagators $\Lambda^{\ddagger}_{j,k}$, $j \geqslant k$. However, this composition law does not imply classical Markovianity as embodied by (1), which is stronger. Instead, it is equivalently identified by the so-called *classical regression* (CR). This latter property holds if multitime correlation functions $\langle F^{(0)} F^{(1)} \dots F^{(n-1)} \rangle_\pi$ are expressible in terms of transition probabilities, as follows:

$$\langle F^{(0)} F^{(1)} \dots F^{(n-1)} \rangle_\pi = \sum_{\boldsymbol{a}_{[0,n-1]}} p^{[0,n-1]}_{\boldsymbol{x}_{[0,n-1]}} F^{(0)}_{x_0} F^{(1)}_{x_1} \dots F^{(n-1)}_{x_n}$$

$$= \sum_{\boldsymbol{x}_{[0,n-1]}} \prod_{j=2}^{n} T_{x_j x_{j-1}}(j, j-1) F^{(0)}_{x_0} F^{(1)}_{x_1} \dots F^{(n-1)}_{x_n}, \tag{4}$$

for all functions $F^{(j)}$ of the random variable X. In the quantum realm, in order to properly discuss non-Markovian issues, two are the main reasons for choosing the noncommutative analog of (4), namely, *quantum regression* (QR), as a starting point, instead of the reduced dynamics [2,6–8]: the first one is that the latter in practice involves two time-correlation functions losing precious information coming from the multitime statistics. The second reason is that multitime correlations provide a powerful tool to access the dynamics of information exactly when a given open quantum dynamics at discrete time n, $\Lambda^{\ddagger}_n$, is derived from the global reversible dynamics $\Theta^{\ddagger}_n$ of system-environment factorized states $\rho_S \otimes \rho_E$ after tracing away the environment, $\Lambda^{\ddagger}_n[\rho_S] = \mathrm{Tr}_E(\Theta^{\ddagger}_n[\rho_S \otimes \rho_E])$. Indeed, QR regards multitime correlations $\mathrm{Tr}_{S+E}(\rho_S \otimes \rho_E(\widetilde{B}^{(n)\dagger}\widetilde{A}^{(n)}))$ of observables A_j, B_j of the open system only, evolving according to the Heisenberg joint reversible dynamics Θ_n, associated to $\Theta^{\ddagger}_n$, where

$$\widetilde{A}^{(n)} := \Theta_{n-1}[A_{n-1} \otimes \mathbb{1}_E] \dots \Theta_1[A_1 \otimes \mathbb{1}_E] A_0 \otimes \mathbb{1}_E,$$

and amounts to the possibility of expressing them in terms of CPU (completely positive and unity-preserving) propagators $\Lambda_{n,n-1}$ [1,2] acting on S:

$$\mathrm{Tr}_{S+E}(\rho_S \otimes \rho_E(\widetilde{B}^{(n)\dagger}\widetilde{A}^{(n)}))$$

$$= \mathrm{Tr}_S(\rho_S B_0^\dagger \Lambda_1[B_1^\dagger \Lambda_{2,1}[\dots \Lambda_{n-1,n-2}[B_{n-1}^\dagger A_{n-1}] \dots]A_1]A_0),$$

for all possible choices of observables of S. Another approach to non-Markovian quantum issues is thus to check whether QR is violated.

The proper quantum extension of non-Markovian stochastic processes, including the multitime statistics, is rooted in a long history dating back to pioneering works [9–13] and still much debated in recent years. In particular, including the multitime statistics in the description involves repeated quantum measurements that interfere with the dynamics and generally change the quantum state (see [14–16] and the recent review [17]). Among the most recent proposals concerning such operational approach, particularly prominent ones include the so-called process tensor formalism [18–20], quantum combs (see [16,21] and references therein), conditional past-future independence [22,23], and temporal entanglement (see the recent work [24]). A shared idea among most of these approaches is to map multitime correlations into spatial ones encoded in a quantum many-body state. Furthermore, in such a noncommutative scenario, Lindblad [11,25] proposed to look at Markovianity using a quantum version of the Kolmogorov-Sinai (KS) entropy, this latter concept measuring the amount of entropy, that is of ignorance before measuring or, equivalently, of information gathered afterward, per unit time step provided by the dynamics. Such a notion was later fully developed by Alicki and Fannes in [26], with applications mainly to the reversible dynamics of quantum many-body systems [14]. Differently from other proposals of quantum dynamical entropy, the Alicki-Lindblad-Fannes (ALF) entropy takes into consideration the fact that information about quantum systems is obtained by measuring schemes that, unlike classical phase-space localization with respect to a suitable coarse graining, interfere with the given dynamics. Notice that such an approach contrasts with the standard concept of information backflow [5] which instead focuses on the dynamics of the system S only, without the need of any measurement and so, effectively, on marginal probabilities.

In the following, we adapt the ALF entropy to the open quantum system scenario, and propose it as a measure of information flowing back to the open system, capturing the full multitime statistics. The proposal is presented first in a rather general setting and then applied to the concrete case of so-called collisional models [27,28]. Namely, of finite-level systems coupled to an infinite spin chain serving as their environment, whereby the open quantum system interacts locally with a single-site algebra at each tick of time. In particular, explicit results will be obtained with the environment being a correlated classical spin chain. Nevertheless, despite the memory-storing mechanisms of our model being entirely classical, the memory effects arising in the open quantum evolution might have no classical counterpart [29]. Moreover, the multitime structure upon which, taking inspiration from the ALF entropy, our approach is based, has not yet been fully explored even with classical environments and certainly provides a necessary first step toward genuinely quantum environments. In such a context, we are able to explicitly compute the open-system ALF entropy of the irreversible qubit dynamics and relate its behavior to the activation and superactivation of memory effects, the latter being a pure quantum phenomenon whereby backflow of information shows up putting together a system with no backflow of information with a copy of itself.

This work expands on the results presented in our companion paper [30] by establishing a general theoretical framework for the open-system ALF entropy and providing detailed derivations of the results. The remainder of the paper is organized as follows. In Sec. II, we review the main algebraic and conceptual tools necessary in the remaining sections, such as the algebraic approach to collisional models, the QR formula, and the ALF entropy for reversible systems. In Sec. III, we introduce the concept of open-system ALF entropy for generic open quantum systems and compare and contrast its typical behavior for closed finite systems with Markovian open ones. Moreover, we discuss its general physical interpretation in Sec. IV in the context of the so-called Gelfand, Neimark, and Segal (GNS) construction.

Section V is devoted to the main concrete results of the work that are based on a collisional model derivation of the dynamics.

Results

The main results of the paper are as follows. First, in Sec. III, we introduce the general framework for an open-system generalization of the ALF entropy. Specifically, we give an independent new proof of the results of [11]; namely, we show that, by looking at specific positive operator value measurement (POVM) dependent entropy rates, one can ascertain the property of QR-Markovianity.

Then, we propose the decrease of such rates with respect to the ones holding in the QR-Markovian setting as a measure of how much information has flown back from environment to system.

In Sec. IV, the operational interpretation of the previous proposal is discussed relatively to the state purification scheme known as GNS construction.

Section V is dedicated to concrete applications by framing collisional models within the algebraic approach. First, we provide a general upper bound to the ALF entropy of a finite level system coupled to a quantum spin chain. Then, we focus on the concrete model of a finite-level system collisionally interacting with a classical spin chain, for which we explicitly compute the open-system ALF entropy. In particular, we show that a dissipative, highly non-Markovian dynamics can achieve vanishing entropy production, as a reversible dynamics would. Finally, through the GNS construction, we introduce dissipative dilations of the reduced dynamics, that allow us to assess memory effects that are not detectable by the reduced dynamics only: in particular, the so-called superactivation of memory effects that may characterize.

II. PRELIMINARIES: QUANTUM REGRESSION AND ALF ENTROPY

In the following we shall identify finite or infinite discrete-time quantum dynamical systems by means of triples $(\mathcal{A}, \omega, \Theta)$ [31] where $\mathcal{A}$ is a suitable algebra of observables, ω a state that is a positive, normalized linear functional on $\mathcal{A}$, and Θ denotes the dynamics which generically consists of a countable family $\{\Theta_n\}_n$ of completely positive and unital (CPU) maps, not necessarily satisfying the (semi)group property $\Theta_n = \Theta^n = \Theta \circ \Theta \cdots \circ \Theta$, n times. As a simple instance, consider a finite d level system; then, $\mathcal{A} = M_d(\mathbb{C})$ is the algebra of $d \times d$ matrices, $\omega(A) = \mathrm{Tr}(\rho A)$ with ρ a density matrix, while Θ_n may be the unitary evolution of a closed, but not necessarily autonomous qubit $\Theta_n[A] = U_n^\dagger A U_n$, with U_n unitary, or a dissipative channel $\Theta_n[A] = \sum_k L_k^\dagger(n) A L_k(n)$, with $\sum_k L_k^\dagger(n) L_k(n) = \mathbb{1}$, describing an open d-level system in interaction with its environment. In the first case, one talks of an automorphic dynamics as the adjoint and the algebraic relations are preserved; namely, $(\Theta_n[X])^\dagger = \Theta_n[X^\dagger]$ and $\Theta_n[XY] = \Theta_n[X]\Theta_n[Y]$ for all $X, Y \in \mathcal{A}$, $n \in \mathbb{N}$. Moreover, the algebraic inverse Θ_n^{-1} is an automorphism itself. In the second case, only the first relation holds.

As an infinite-dimensional instance, consider a quantum spin chain whose algebra $\mathcal{A}$ is generated by all tensor products $X_{\boldsymbol{i}_{[a,b]}}^{[a,b]} = \bigotimes_{k=a}^b X_{i_k}^{(k)}$, where the upper index refers to the site at which the operator X_{i_k} is located. These tensor products form strictly local algebras $\mathcal{A}^{[a,b]}$ supported by the site interval $[a, b]$. States ω on $\mathcal{A}$ are specified by a set of density matrices $\rho^{[a,b]} \in \mathcal{A}^{[a,b]}$ such that

$$\omega\left(\bigotimes_{k=a}^b X_{i_k}^{(k)}\right) = \mathrm{Tr}\left(\rho^{[a,b]} \bigotimes_{k=a}^b X_{i_k}^{(k)}\right), \tag{5}$$

$$\mathrm{Tr}_b(\rho^{[a,b]}) = \rho^{[a,b-1]}, \tag{6}$$

where Tr_k defines the partial trace over the kth site. Furthermore, if $\mathrm{Tr}_a \rho^{[a,b]} = \rho^{[a+1,b]} = \rho^{[a,b-1]}$, then the state ω is invariant under the (right) shift $\omega \circ \sigma = \omega$, where $\sigma : \mathcal{A} \to \mathcal{A}$ moves all single-site observables one step to the right:

$$\sigma\left[X_{i_a}^{(a)} \otimes \cdots \otimes X_{i_b}^{(b)}\right] = X_{i_a}^{(a+1)} \otimes \cdots \otimes X_{i_b}^{(b+1)}. \tag{7}$$

In such a case, the local states $\rho^{[1,n]}$ have von Neumann entropy $S(\rho^{[1,n]}) = -\mathrm{Tr}(\rho^{[1,n]} \ln \rho^{[1,n]})$ and the chain is characterized by a mean entropy

$$\mathfrak{S}_\omega := \lim_n \frac{1}{n} S(\rho^{[1,n]}). \tag{8}$$

The limit exists because of the strong subadditivity of the von Neumann entropy and of the assumed shift invariance of ω [14]; actually, for the same reasons the following limit also holds:

$$\mathfrak{S}_\omega = \lim_n \left[S(\rho^{[1,n]}) - S(\rho^{[1,n-1]})\right]. \tag{9}$$

A. Collisional models

The reduced dynamics of open quantum systems emerges from suitably handling their non-negligible interactions with the environment: collisional models prove a most convenient way of doing that, in a way that allows several types of reduced dynamics to emerge. They consist of a suitable quantum spin-chain environment E with algebra $\mathcal{A}_E$ generated by strictly local matrix algebras $M_D^{[-a,b]}(\mathbb{C}) = \bigotimes_{k=-a}^b M_D^{(k)}(\mathbb{C})$. The chain is coupled to a d-level system S, $\mathcal{A}_S = M_d(\mathbb{C})$, and the tensor algebra $\mathcal{A}_S \otimes \mathcal{A}_E$ describing the compound system $S + E$ is endowed with a factorized state $\omega_{SE} = \omega_S \otimes \omega_E$, represented on each strictly local algebra $\mathcal{A}_S \otimes \mathcal{A}_E^{[-a,b]}$ by a factorized density matrix $\Omega_{S[-a,b]} = \rho_S \otimes \rho_E^{[-a,b]}$.

The SE collisional coupling is constructed by means of an automorphism Φ on $\mathcal{A}_S \otimes \mathcal{A}_E^{(0)}$ that describes the local interaction of S with the chain 0th site and extends to $\mathcal{A}_S \otimes \mathcal{A}_E$ as

$$\Phi\left[X_S \otimes X_{\boldsymbol{i}_{[-a,-1]}}^{[-a,-1]} \otimes X_{i_0}^{(0)} \otimes X_{\boldsymbol{i}_{[1,b]}}^{[1,b]}\right] = X_{\boldsymbol{i}_{[-a,-1]}}^{[-a,-1]} \otimes \Phi\left[X_S \otimes X_{i_0}^{(0)}\right] \otimes X_{\boldsymbol{i}_{[1,b]}}^{[1,b]}. \tag{10}$$

The compound automorphic dynamics Θ of system S and chain E is finally given by composing Φ with the shift dynamics σ_E on $\mathcal{A}_E$, yielding a discrete group of automorphisms $\{\Theta_n = \Theta_{\mathrm{coll}}^n\}_{n \in \mathbb{N}}$, where

$$\Theta_{\mathrm{coll}} := (\mathrm{id}_S \otimes \sigma_E) \circ \Phi. \tag{11}$$

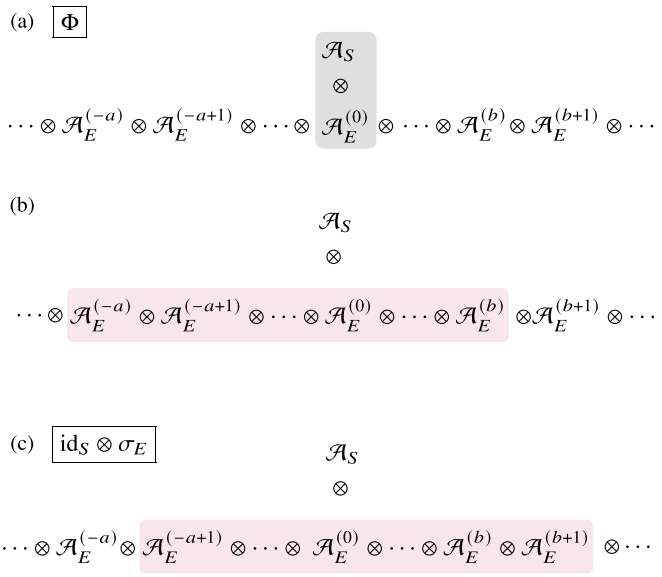


FIG. 1. One step Θ_{coll} of the algebraic collisional dynamics: (a) Φ acts on the algebra of the system $\mathcal{A}_S$ and on the 0th site of the chain. (b) An operator $A^{[-a,b]}$, localized in $\mathcal{A}_E^{[-a,b]}$, is then translated to the right in (c) by the shift automorphism σ_E.

The algebraic implementation of (11) is illustrated in Fig. 1. Collisional models will be exploited in Sec. V, where the open-system ALF entropy will be worked out explicitly in a concrete example.

B. Quantum regression

Non-Markovianity is intuitively associated with the presence of memory effects; as already anticipated in the Introduction, given the reduced dynamics $\{\Lambda_n^\ddagger\}_{n\geqslant 0}$ of an open quantum system, a standard approach is via the notion of *divisibility*. Namely, one looks at propagators

$$\Lambda_n^\ddagger = \Lambda_{n,m}^\ddagger \circ \Lambda_m^\ddagger, \quad n \geqslant m \tag{12}$$

and calls the reduced evolution (C)P-divisible if $\Lambda_{n,m}^\ddagger$ are (completely) positive and trace preserving for all $n \geqslant m$. In such a setting, where only the reduced dynamics of the system is available, CP-divisibility represents a standard notion of Markovianity [1,2]. An associated remarkable feature of non-Markovianity is the so-called backflow of information [5,32], that manifests itself when the evolution is not even P-divisible. For algebraically invertible dynamics, in fact, lack of P-divisibility equals lack of trace-norm contractivity with the consequence that the distinguishability of two density matrices ρ, σ might increase in time. Such time revivals of trace norms $||\rho - \sigma||_1$, as well as of other distances or entropic divergences, is accordingly interpreted as backflow of information from the environment into the system, a clear signature of memory effects. We further stress that, although both rely only on the reduced dynamics, CP-divisibility and the lack of backflow of information are inequivalent notions of Markovianity. Indeed, CP-divisibility implies the absence of backflow of information, whereas the converse does not hold [1]. Backflow of information can also occur in a commutative setting when the transition matrices from time step m to time step n are not positive [33,34]; instead, a purely quantum phenomenon, impossible in a commutative scenario, is the superactivation of backflow of information [35,36]. Indeed, going from a single-system reduced dynamics $\Lambda_n^\ddagger$ to a bipartite one of the form $\Lambda_n^\ddagger \otimes \Lambda_n^\ddagger$, lack of CP-divisibility of the former might generally cause superactivation of backflow of information in the latter, while the subsystems do not display backflow in terms of distinguishability revivals.

Evidently, the physical mechanisms behind a possible backflow of information cannot be appreciated sticking to the open quantum system only, as it asks to take into account the full joint system-environment dynamics Θ which originates that particular dissipative open quantum dynamics $\{\Lambda_n\}_{n\geqslant 0}$. For the moment, we shall not assume that the family $\{\Theta_n\}_n$ forms a group, namely, the automorphic Heisenberg propagators

$$\Theta_{n,m} = \Theta_m^{-1} \circ \Theta_n \tag{13}$$

between subsequent times $n \geqslant m$ might be genuinely time dependent. As emphasized in the Introduction, by focusing upon multitime correlation functions, one can substantially refine the definition of Markovian open quantum behavior by identifying it with QR [37,38], that we now reformulate. QR requires measuring observables of the open system only as the environment is typically neither accessible nor under control. Using the algebraic notation just introduced, let then $\{A_k\}_{k=0}^{n-1} \in \mathcal{A}_S$, $n \geqslant 1$, define

$$\widetilde{A}^{(n)} := \Theta_{n-1}[A_{n-1} \otimes \mathbb{1}_E] \ldots \Theta_1[A_1 \otimes \mathbb{1}_E] A_0 \otimes \mathbb{1}_E, \tag{14}$$

belonging to $\mathcal{A}_S \otimes \mathcal{A}_E$ and evaluate the multitime correlation functions $\omega_S \otimes \omega_E(\widetilde{B}^{(n)\dagger}\widetilde{A}^{(n)})$ given by

$$\omega_S \otimes \omega_E(B_0^\dagger \otimes \mathbb{1}_E\, \Theta_1[B_1^\dagger \otimes \mathbb{1}_E] \cdots \Theta_{n-1}[B_{n-1}^\dagger A_{n-1} \otimes \mathbb{1}_E] \ldots \Theta_1[A_1 \otimes \mathbb{1}_E] A_0 \otimes \mathbb{1}_E). \tag{15}$$

Definition 1. Let the triple $(\mathcal{A}_S \otimes \mathcal{A}_E, \widetilde{\omega}_S \otimes \omega_E, \Theta)$ define an open dynamical system, with $\widetilde{\omega}_S$ a generic state of the system. The open quantum system S will be called *QR-Markovian* if it satisfies QR, namely if, for arbitrary $\{A_k\}_{k=0}^{n-1}$, $\{B_k\}_{k=0}^{n-1}$ in $\mathcal{A}_S$, $n \geqslant 1$,

$$\widetilde{\omega}_S \otimes \omega_E(\widetilde{B}^{(n)\dagger}\widetilde{A}^{(n)}) = \widetilde{\omega}_S(B_0^\dagger \Lambda_1[B_1^\dagger \Lambda_{2,1}[\ldots \Lambda_{n-1,n-2}[B_{n-1}^\dagger A_{n-1}] \ldots] A_1] A_0), \tag{16}$$

with $\Lambda_{n,n-1}$, $n \in \mathbb{N}$ completely positive, unital maps.

For $n = j+1$ and $A_i = B_i = \mathbb{1}_S$, for $i = 0, 1, \ldots, j-1$, QR yields

$$\begin{aligned} &\omega_S \otimes \omega_E(\Theta_j[B_j^\dagger A_j \otimes \mathbb{1}_E]) \\ &\quad = \omega_S(\Lambda_1 \circ \Lambda_{2,1} \circ \cdots \circ \Lambda_{j,j-1}[B_j^\dagger A_j]) \\ &\quad = \mathrm{Tr}(\Lambda_{j,j-1}^\ddagger \circ \cdots \circ \Lambda_{2,1}^\ddagger \circ \Lambda_1^\ddagger[\rho_S] B_j^\dagger A_j). \end{aligned} \tag{17}$$

The left-hand side above defines the reduced dynamics of the open system S at time step j so that, in the Schrödinger

picture, one retrieves the composition law

$$\Lambda_j^{\ddagger} = \Lambda_{j,j-1}^{\ddagger} \circ \cdots \circ \Lambda_{2,1}^{\ddagger} \circ \Lambda_1^{\ddagger}. \tag{18}$$

Furthermore, since in QR scenario the one-step propagators $\Lambda_{j,j-1}^{\ddagger}$ must be CPTP, the reduced dynamics is Markovian in the divisibility sense of [39] (see Remark 6). This is only a necessary condition for QR-Markovianity; indeed, as we shall show later in a concrete example, CPTP propagators need not imply QR.

Remark 1. Suppose that the system algebra is maximally commutative, namely, spanned by one-dimensional orthogonal projectors P_i, $P_iP_j = \delta_{ij}P_j$. Then, QR condition reduces to CR. Indeed, let $C_k = B_k^{\dagger}A_k = \sum_j c_j^{(k)}P_j$; due to commutativity, using (13), we can rewrite (15) as

$$\omega_S \otimes \omega_E(\Theta_1[\Theta_{2,1}[\dots\Theta_{n-1,n-2}(C_{n-1}\otimes\mathbb{1}_E)\dots]C_1\otimes\mathbb{1}_E]C_0\otimes\mathbb{1}_E) = \sum_{i_0\dots i_{n-1}} c_{i_0}^{(0)}c_{i_1}^{(1)}\dots c_{i_{n-1}}^{(n-1)}\, p_{i_0 i_1\dots i_{n-1}}^{[0,n-1]},$$

where we let

$$p_{i_0 i_1\dots i_{n-1}}^{[0,n-1]} := \omega_S\otimes\omega_E\big(\Theta_1\big[\Theta_{2,1}\dots\big[\Theta_{n-1,n-2}\big[P_{i_{n-1}}\otimes\mathbb{1}_E\big]\dots\big]P_{i_1}\otimes\mathbb{1}_E\big]P_{i_0}\otimes\mathbb{1}_E\big). \tag{19}$$

On the other hand, (16) can be rewritten as

$$\begin{aligned}\omega_S(\Lambda_1[\Lambda_{2,1}[\dots\Lambda_{n-1,n-2}[C_{n-1}]\dots\dots]C_1]C_0) &= \sum_{i_0\dots i_{n-1}} c_{i_0}^{(0)}c_{i_1}^{(1)}\dots c_{i_{n-1}}^{(n-1)}\,\omega_S\big(P_{i_0}\Lambda_1\big[P_{i_1}\Lambda_{2,1}\big[\dots P_{i_{n-2}}\Lambda_{n-1,n-2}\big[P_{i_{n-1}}\big]P_{i_{n-2}}\dots\big]P_{i_1}\big]P_{i_0}\big)\\ &= \sum_{i_0\dots i_{n-1}} c_{i_0}^{(0)}c_{i_1}^{(1)}\dots c_{i_{n-1}}^{(n-1)}\prod_{k=0}^{n-1} T_{i_k i_{k-1}}(k,k-1)\,p_{i_0},\end{aligned} \tag{20}$$

where $p_j = \omega_S(P_j)$, we defined $T_{ij}(k,k-1) := \mathrm{Tr}(P_j\Lambda[P_i])$ and used that $P_{i_{k-1}}\Lambda_{k,k-1}[P_{i_k}]P_{i_{k-1}} = T_{i_k i_{k-1}}(k,k-1)P_{i_{k-1}}$. By equating (19) and (20), we recover (4).

C. ALF entropy

The degree of instability of a classical dynamics, for instance, the existence of a positive Lyapounov exponent, can be associated to dynamical entropy production by means of the Kolmogorov-Sinai entropy [40]. The basic idea is to reduce the dynamics to the shift along sequences of symbols labeling the cells of a finite partition of the phase space that are visited by the time-evolving system at each time step. Any time-invariant classical state assigns probabilities to the sequences with Shannon entropies tending to a well-defined asymptotic rate. The Kolmogorov-Sinai entropy is the largest among such rates. The ALF entropy provides a quantum counterpart to it in that the coarse graining of the phase space is replaced by a POVM at successive time steps of the state dynamics. Let a quantum dynamical system $(\mathcal{A},\omega,\Theta)$ be endowed with a reversible automorphic dynamics Θ and with a Θ-invariant state ω on $\mathcal{A}$, $\omega\circ\Theta_n = \omega$. Then, any finite POVM $\mathcal{X}$ consisting of $|\mathcal{X}|$ operators $\mathcal{X} := \{X_a\}_{a=1}^{|\mathcal{X}|} \subseteq \mathcal{A}$ such that $\sum_{a=1}^{|\mathcal{X}|} X_a^{\dagger}X_a = \mathbb{1}_{\mathcal{A}}$ gives rise to a family of time-dependent POVM $\mathcal{X}^{(n)} = \{X_{\boldsymbol{a}}^{(n)}\}_{\boldsymbol{a}}$, where $\boldsymbol{a} = a_0a_1\dots a_{n-1}$ and

$$X_{\boldsymbol{a}}^{(n)} := \Theta_{n-1}\big[X_{a_{n-1}}\big]\dots\Theta_1\big[X_{a_1}\big]X_{a_0}. \tag{21}$$

Notice that this is not true if Θ_n is dissipative (see Remark 3). For such a reversible dynamical system, it follows that

$$\rho[\mathcal{X}^{(n)}] = \sum_{\boldsymbol{a},\boldsymbol{b}} \omega\big(X_{\boldsymbol{b}}^{(n)\dagger}X_{\boldsymbol{a}}^{(n)}\big)|\boldsymbol{a}_{[0,n-1]}\rangle\langle\boldsymbol{b}_{[0,n-1]}| \tag{22}$$

is a density matrix, that is self-adjoint, positive, and normalized with multitime correlation functions as matrix elements:

$$\begin{aligned}(\rho[\mathcal{X}^{(n)}])_{\boldsymbol{a},\boldsymbol{b}} &= \omega\big(X_{\boldsymbol{b}}^{(n)\dagger}X_{\boldsymbol{a}}^{(n)}\big)\\ &= \omega\big(X_{b_0}^{\dagger}\Theta_1\big[X_{b_1}^{\dagger}\big]\Theta_2\big[X_{b_2}^{\dagger}\big]\dots\Theta_{n-1}\big[X_{b_{n-1}}^{\dagger}X_{a_{n-1}}\big]\dots\\ &\qquad \Theta_2\big[X_{a_2}\big]\Theta_1\big[X_{a_1}\big]X_{a_0}\big).\end{aligned} \tag{23}$$

Using (13) and the fact that $\Theta_{n,m}[XY] = \Theta_{n,m}[X]\,\Theta_{n,m}[Y]$, along with (13), one can recast (23) in a "nested" fashion as in Remark 1:

$$\begin{aligned}(\rho[\mathcal{X}^{(n)}])_{\boldsymbol{a},\boldsymbol{b}} = \omega\big(X_{b_0}^{\dagger}\Theta_1\big[X_{b_1}^{\dagger}\Theta_{2,1}\big[X_{b_2}^{\dagger}\dots\\ \Theta_{n-1,n-2}\big[X_{b_{n-1}}^{\dagger}X_{a_{n-1}}\big]\dots X_{a_2}\big]X_{a_1}\big]X_{a_0}\big).\end{aligned} \tag{24}$$

The state $\rho[\mathcal{X}^{(n)}]$ provides a description of the statistics of the evolving state as reconstructed by measuring a fixed POVM.

Remark 2. By passing from the Heisenberg to the Schrödinger picture via the following dual maps on the space of linear functionals ω over the algebra $\mathcal{A}$ of the system, $\omega\mapsto\mathbb{X}_{ab}^{\ddagger}[\omega]$ and $\omega\mapsto\Theta_{k,k-1}^{\ddagger}[\omega]$ such that

$$\mathbb{X}_{ab}^{\ddagger}[\omega](X) = \omega(X_b^{\dagger}\,X\,X_a), \qquad \Theta_{k,k-1}^{\ddagger}[\omega](X) = \omega(\Theta_{k,k-1}[X]),$$

one rewrites

$$\begin{aligned}(\rho[\mathcal{X}^{(n)}])_{\boldsymbol{a},\boldsymbol{b}} = \mathbb{X}_{a_{n-1}b_{n-1}}^{\ddagger}\circ\Theta_{n-1,n-2}^{\ddagger}\circ\\ \cdots\circ\mathbb{X}_{a_1b_1}^{\ddagger}\circ\Theta_1^{\ddagger}\circ\mathbb{X}_{a_0b_0}^{\ddagger}[\omega](\mathbb{1}).\end{aligned} \tag{25}$$

The diagonal entries of $\rho[\mathcal{X}^{(n)}]$ are then the probabilities of sequences of selective measurements of the various POVM terms followed by the one-step reversible dynamics.

The information gathered in the long run can then be characterized by the so-called ALF entropy of the given POVM [26], by means of the following entropy rate:

$$\mathfrak{h}_{\omega}(\Theta,\mathcal{X}) := \limsup_n \frac{1}{n}S(\rho[\mathcal{X}^{(n)}]), \tag{26}$$

where $S(\rho)$ is the von Neumann entropy. One has to do with the lim sup since the limit of the ratio could not exist. Indeed, even if the family of automorphisms $\{\Theta_n\}_n$ form a group, the family of density matrices $\rho[\mathcal{X}^{(n)}]$ is not generally stationary since measuring the POVM changes in general the state on which the POVM measurement is performed. The entropy rate is finally made independent from the chosen POVM by maximizing over all of them chosen from a suitable subalgebra of physically admissible partitions $\mathcal{B}$, typically assumed to be globally Θ invariant and dense in $\mathcal{A}$:

$$\mathfrak{h}_\omega^{\mathcal{B}}(\Theta) := \sup_{\mathcal{X}\subseteq\mathcal{B}} \mathfrak{h}_\omega(\Theta,\mathcal{X}). \tag{27}$$

A useful control of the ALF entropy is in some cases provided by the following bound whose proof can be found in [14].

Proposition 1. Let $A \in \mathcal{A}$ and $\omega(A) = \mathrm{Tr}(\rho A)$ for some density matrix ρ and let $\mathcal{X} = \{X_a\}_{a=1}^{|\mathcal{X}|}$ be a POVM. Then,

$$S(\rho[\mathcal{X}]) \leqslant S(\rho) + S\left(\sum_{a=1}^{|\mathcal{X}|} X_a\,\rho X_a^\dagger\right). \tag{28}$$

Remark 3. If the quantum dynamical system $(\mathcal{A},\omega,\Theta)$ is dissipative, the dynamics Θ stands for a family of CPTP channels Λ_n. Then, given a POVM $\mathcal{X} = \{X_i\}_{i=1}^{|\mathcal{X}|}$, it is no longer true that $\mathcal{X}^{(n)}$ as in (21) is a POVM as required by the ALF construction. Indeed, for general CPU maps the Schwarz inequality yields

$$\sum_{a_{n-1}} \Lambda_{n-1}\big[X_{a_{n-1}}^\dagger\big]\Lambda_{n-1}\big[X_{a_{n-1}}\big] \leqslant \sum_{a_{n-1}} \Lambda_{n-1}\big[X_{a_{n-1}}^\dagger X_{a_{n-1}}\big] = \mathbb{1},$$

so that the quantity

$$\sum_{\boldsymbol{a}^{(n)}} X_{\boldsymbol{a}^{(n)}}^\dagger X_{\boldsymbol{a}^{(n)}} = \sum_{\boldsymbol{a}^{(n)}} X_{a_0}\Lambda_1\big(X_{a_1}^\dagger\big) \ldots \Lambda_{n-1}\big[X_{a_{n-1}}^\dagger\big]\Lambda_{n-1}\big[X_{a_{n-1}}\big]\ldots\Lambda_1\big[X_{a_1}\big]X_{a_0}$$

does not generally sum to the identity. Nevertheless, consider the nested form (24) of the coarse-grained density matrix; it then appears natural to replace the automorphism Θ with CPU intertwiners $\Lambda_{n,m}$:

$$(\rho[\mathcal{X}^{(n)}])_{\boldsymbol{a},\boldsymbol{b}} = \omega\big(X_{b_0}^\dagger\Lambda_1\big[X_{b_1}^\dagger\Lambda_{2,1}\big[X_{b_2}^\dagger \ldots \Lambda_{n-1,n-2}\big[X_{b_{n-1}}^\dagger X_{a_{n-1}}\big]X_{a_2}\big]X_{a_1}\big]X_{a_0}\big). \tag{29}$$

One thus obtains a compatible family of legitimate density matrices $\mathrm{Tr}_{n-1}(\rho[\mathcal{X}^{(n)}]) = \rho[\mathcal{X}^{(n-1)}]$. As seen in the quantum regression formula (16), (29) has the precise physical meaning of a coarse-grained density matrix of an open system undergoing a QR-Markovian dynamics.

III. OPEN-SYSTEM ALF ENTROPY

In this section, we introduce a generalization of the ALF framework to study quantum dissipative evolutions. We will consider concrete applications to collisional models later, in Sec. V.

As seen in Sec. II B, rather than through the reduced dynamics, the statistical properties of the dissipative behavior are better accessed by using multitime correlation functions of the form $\omega_S\otimes\omega_E(\widetilde{B}^{(n)\dagger}\widetilde{A}^{(n)})$ in (15). These involve the compound automorphic dynamics Θ of system S and environment E together and operators $X_S\otimes\mathbb{1}_E$ of the system S only.

Within the ALF construction applied to the compound system $S+E$ restricting to such particular operators is justified by the experimental inaccessibility of the environment: we shall thus focus on POVMs $\mathcal{X}$ belonging to the subalgebra $\subseteq M_d(\mathbb{C})\otimes\mathbb{1}_E$.

In practice, we shall then focus upon the dynamical triple $(\mathcal{A}_S\otimes\mathcal{A}_E, \omega_S\otimes\omega_E, \Theta)$ where $\mathcal{A}_S$ is a finite matrix algebra of S operators, $\mathcal{A}_E$ a quantum spin chain algebra, evolving under a family Θ of automorphisms, while $\omega_S\otimes\omega_E$ is a Θ-invariant state over $\mathcal{A}_S\otimes\mathcal{A}_E$.

Following the construction described in Sec. II C, the elements of the coarse-grained density matrix are thus of the form (15):

$$(\rho_S[\mathcal{X}^{(n)}])_{\boldsymbol{a},\boldsymbol{b}} = \omega_S\otimes\omega_E\big(X_{b_0}^\dagger\otimes\mathbb{1}_E\,\Theta_1\big[X_{b_1}^\dagger\otimes\mathbb{1}_E\big]\ldots \Theta_{n-1}\big[X_{b_{n-1}}^\dagger X_{a_{n-1}}\otimes\mathbb{1}_E\big]\ldots \Theta_1\big[X_{a_1}\otimes\mathbb{1}_E\big]X_{a_0}\otimes\mathbb{1}_E\big). \tag{30}$$

Here, the subscript S stressed that we are restricting all possible POVMs to the subalgebra $\mathcal{A}_S\otimes\mathbb{1}_E$, only. One can thus compute the entropy rate

$$\mathfrak{h}_S(\Theta,\mathcal{X}) := \limsup_n \frac{1}{n} S(\rho_S[\mathcal{X}^{(n)}]), \quad \mathcal{X}\subseteq\mathcal{A}_S\otimes\mathbb{1}_E, \tag{31}$$

and maximize it over the POVMs of the open system to obtain the asymptotic entropy rate

$$\mathfrak{h}_S(\Theta) := \sup_{\mathcal{X}\subseteq\mathcal{A}_S\otimes\mathbb{1}_E} \mathfrak{h}_S(\Theta,\mathcal{X}). \tag{32}$$

Note that, differently from what is usually done in most applications of the ALF entropy, the optimization is now over the subalgebra $\mathcal{A}_S\otimes\mathbb{1}_E$, which is neither time invariant nor, clearly, dense in the full $S+E$ algebra. Rather, (32) represents the information rate due to repeated measurements on S only, intertwining the $S+E$ unitary evolution (see Sec. IV). Accordingly, we shall call it the open-system ALF entropy.

Remark 4. Here, following the Alicki-Fannes approach [14,26] and unlike previous operational approaches to quantum dynamical entropies [25,41], we do not impose any restrictions on the measurements performed on the system, which may therefore generally interfere with both the dynamics and the state.

A. Quantum correlation maps

The coarse-grained density matrices (22) define a many-body quantum state on a quantum spin chain that encodes the multitime correlation functions of the original open quantum system S. In principle, the structure of such state is highly nontrivial and depends on the choice of the POVM. Suppose that the d-level open system S is endowed with a faithful state ω_S, namely, represented by a density matrix ρ_S without zero eigenvalues. Select then the following special POVM from $\mathcal{A}_S$:

$$\mathcal{F} := \{F_{a,a'}\otimes\mathbb{1}_E\}_{a,a'=1}^d, \quad F_{a,a'} = \sqrt{r_a}|r_a\rangle\langle r_{a'}|, \tag{33}$$

where $r_a > 0$, $|r_a\rangle$ are the eigenvalues, respectively, eigenvectors of ρ_S. It preserves ρ_S: $\sum_{a,a'=1}^d F_{a,a'}\,\rho_S\,F_{a,a'}^\dagger = \rho_S$. The

speciality of such POVM is twofold: (i) the structure of the coarse-grained matrix $\rho_S[\mathcal{F}^{(n)}]$ can be explicitly worked out and (ii) it allows to clearly identify the QR-Markovian regime in terms of entropy rates, as we shall see in the following. Note, however, that this POVM might well not be the one achieving the supremum in (32).

First, we show that the coarse-grained density matrix corresponding to the POVM (33) has, remarkably, a closed form. To this end, it proves useful to consider, together with our original d-level open system S, the auxiliary multipartite system $M_d^{\mathbb{Z}}(\mathbb{C}) = \bigotimes_{k=-\infty}^{\infty} \mathcal{A}_S$; namely, a quantum spin chain algebra, each site of which being a matrix algebra identical to the system algebra $\mathcal{A}_S$. Denote by σ_S the right shift on such infinite algebra. Then, as done in Sec. II A, an operator $\bigotimes_{k=0}^{n-1} A_k \in M_d^{\otimes n}(\mathbb{C})$ can be embedded into the chain algebra $M_d^{\mathbb{Z}}(\mathbb{C})$ by localizing it within the interval $[0, n-1]$.

Assume also that the Heisenberg propagators $\Theta_{j,j-1}$, defined in (13), act nontrivially only on $M_d^{(0)}(\mathbb{C}) \otimes \mathcal{A}_E$. With respect to a time-invariant state $\omega_S \otimes \omega_E$, consider now the CPU map $\mathbb{T}_n : M_d^{\otimes n}(\mathbb{C}) \to M_d^{\otimes n}(\mathbb{C})$, defined by

$$\mathbb{T}_n\left[\bigotimes_{k=0}^{n-1} A_k\right] := \omega_E\left(\bigcirc_{j=1}^{n} (\Theta_{j,j-1} \circ (\sigma_S \otimes \mathrm{id}_E))\left[\bigotimes_{k=0}^{n-1} A_k^{(-n+k)} \otimes \mathbb{1}_E\right]\right), \tag{34}$$

where, to compactly denote the ordered composition of maps Φ_j, $j = 1, \ldots, n$, we introduced the symbol

$$\bigcirc_{j=1}^{n} \Phi_j := \Phi_1 \circ \Phi_2 \circ \cdots \circ \Phi_n.$$

We shall refer to (34) as *quantum correlation maps*. We are now ready to state the following

Proposition 2. The system coarse-grained density matrix with respect to the POVM $\mathcal{F}$ is given, up to a rearrangement of tensor factors, by

$$\rho_S[\mathcal{F}^{(n+1)}] = \rho_S \otimes \rho_S \otimes \left(\mathbb{T}_n^{\ddagger} \otimes \mathrm{id}\left[\left|\sqrt{\rho_S^{\otimes n}}\right\rangle\!\left\langle\sqrt{\rho_S^{\otimes n}}\right|\right]\right), \tag{35}$$

where $\mathbb{T}_n^{\ddagger}$ is the CPTP correlation map dual of $\mathbb{T}_n$ and $|\sqrt{\rho}\rangle$ denotes the standard purification of a faithful density matrix $\rho = \sum_a r_a |r_a\rangle\langle r_a|$,

$$|\sqrt{\rho}\rangle = \sum_a \sqrt{r_a} |r_a \otimes r_a\rangle, \quad 0 < r_a < 1.$$

Remark 5. The interaction-shift automorphism appearing in (34) is characteristic of the collisional approach of Sec. II A, wherein the open system S iteratively interacts with a new site of a spin-chain environment. In (34) the emerging picture is, in a sense, dual, in that it is the same environment E that interacts with a fresh copy of the system at each time step. Such dynamics is depicted in Fig. 2.

An independent and self-contained proof of the above Proposition 2 based on the formalism of quantum spin chains is reported in Appendix A.

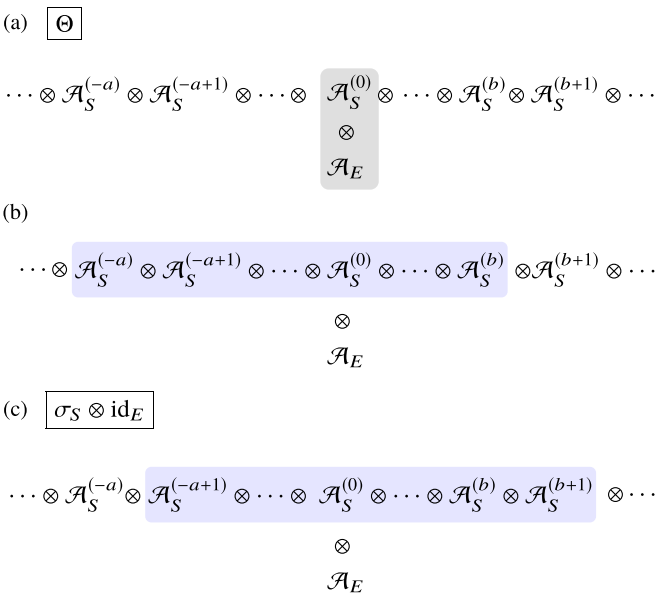


FIG. 2. Dynamical system underlying the construction of the map (34). (a) Θ acts non trivially on the 0-th copy of the system $\mathcal{A}_S^{(0)} = M_d^{(0)}(\mathbb{C})$ and on the algebra of the environment. (b) An operator, initially localized in $\mathcal{A}_S^{[-a,b]}$, is then translated to the right in (c) by the shift automorphism σ_S. Note the similarity with the algebraic collisional model of Fig. 1.

B. Open-system ALF entropy and QR-Markovianity

Let us consider again the open-system symbolic dynamics in the quantum regression regime. This means that the entries of the coarse-grained density matrix

$$(\rho_S[\mathcal{X}^{(n)}])_{\boldsymbol{a},\boldsymbol{b}} = \omega_S \otimes \omega_E\left(X_{\boldsymbol{b}}^{(n)\dagger} X_{\boldsymbol{a}}^{(n)}\right)$$

become, for all $n \geqslant 1$,

$$(\rho_S[\mathcal{X}^{(n)}])_{\boldsymbol{a},\boldsymbol{b}} = \omega_S\big(X_{b_0}^{\dagger}\, \Lambda_1\big[X_{b_1}^{\dagger}\, \Lambda_{2,1}\big[\ldots\, \Lambda_{n-1,n-2}\big[X_{b_{n-1}}^{\dagger} X_{a_{n-1}}\big]\ldots\big] X_{a_1}\big] X_{a_0}\big). \tag{36}$$

By choosing the POVM as in (33), one is led to the following alternative characterization of QR-Markovianity, thereby recovering the results of Lindblad [11].

Proposition 3. The dynamical system $(\mathcal{A}_S \otimes \mathcal{A}_E, \omega_S \otimes \omega_E, \Theta)$ satisfies QR-Markovianity if and only if

$$\mathbb{T}_n = \bigotimes_{j=1}^{n} \Lambda_{j,j-1}, \tag{37}$$

with $\Lambda_{j,j-1}$ CPU maps on $\mathcal{A}_S = M_d(\mathbb{C})$.

The proof is reported in Appendix B.

Remark 6. From Eq. (34), map $\mathbb{T}_n$ is a CPU map on $M_d^{\otimes n}(\mathbb{C})$. Complete positivity of $\Lambda_{j,j-1}$ in (37) then follows from that of $\mathbb{T}_n$. Moreover, this explicitly shows that QR-Markovianity implies CP-divisibility of the reduced dynamics Λ_n, thus its P-divisibility from which follows that there is no backflow of information as distinguishability revivals. In Sec. V, we will present explicit examples showing that the converse implication does not hold: a (C)P-divisible dynamics need not be QR-Markovian.

Let us now focus upon the case in which the system-environment dynamics is given by a one-parameter group of automorphisms $\Theta_n = \Theta^n$. Under such fairly general conditions, the QR condition actually implies a semigroup dynamics.

Proposition 4. Suppose that the QR condition (16) holds for a Θ-invariant state $\omega_S \otimes \omega_E$, with ω_S represented by a faithful density matrix. Then, the dissipative Heisenberg propagators are all the same:

$$\Lambda_{n+1,n} = \Lambda_1 \equiv \Lambda, \quad \forall\, n \geqslant 0. \tag{38}$$

The proof is reported in Appendix C. In this scenario, it is instructive to compare the behavior of the ALF entropy under reversible and nonreversible dynamics.

Finite, closed quantum system. For a closed system S not interacting with its environment, the dynamics is such that $\Theta = \Theta_S \otimes \Theta_E$, with Θ_S and Θ_E automorphisms of $\mathcal{A}_S$ and $\mathcal{A}_E$. In this case QR holds trivially with $\Lambda_{j,j-1} = \Theta_S$ and (30) becomes

$$(\rho_S[\mathcal{X}^{(n)}])_{\boldsymbol{a},\boldsymbol{b}} = \mathrm{Tr}\big(\rho_S X_{b_0}^\dagger \Theta_S\big[X_{b_1}^\dagger \ldots \Theta_S\big[X_{b_n}^\dagger X_{a_n}\big] \ldots X_{a_1}\big] X_{a_0}\big). \tag{39}$$

Then, the dynamics maps the system matrix algebra into itself and $\mathfrak{h}_S(\Theta) = 0$ as for all closed finite-level systems [14].

Finite, open quantum system in the QR regime. From the above Proposition 2, the following result allows to ascertain QR-Markovianity by looking at POVM-specific entropy rates.

Corollary 1. The ALF entropy associated with the POVM $\mathcal{F}$ in (33) is bounded from above by

$$\mathfrak{h}_S(\Theta, \mathcal{F}) \;\leqslant\; S(\Lambda^\ddagger \otimes \mathrm{id}_d[|\sqrt{\rho_S}\rangle\langle\sqrt{\rho_S}|]). \tag{40}$$

Furthermore, the bound saturates if and only if $\mathbb{T}_n = \bigotimes_{k=1}^n \Lambda$ or, equivalently, if and only if QR condition holds.

See Appendix D for the proof. From its definition in (32), the POVM dependent ALF entropy (40) provides a lower bound to the open-system ALF entropy.

Corollary 2. In the QR regime one has

$$\mathfrak{h}_S(\Theta) \geqslant \mathfrak{h}_S(\Theta, \mathcal{F}) = S(\Lambda^\ddagger \otimes \mathrm{id}_d[|\sqrt{\rho_S}\rangle\langle\sqrt{\rho_S}|]), \tag{41}$$

which are strictly positive quantities unless Λ is an isometry, $\Lambda[X] = V^\dagger X V$, $V^\dagger V = \mathbb{1}_d$. Thus, for finite quantum systems, the dynamical entropy vanishes in all reversible dynamical settings, while it is strictly positive for an irreversible QR-Markovian one.

Proof. The inequality is evident by definition of $\mathfrak{h}_S(\Theta)$. On the other hand, the lower bound vanishes if and only if $\xi_\Lambda := \Lambda^\ddagger \otimes \mathrm{id}_d[|\sqrt{\rho_S}\rangle\langle\sqrt{\rho_S}|]$ is a one-dimensional projector, which is obvious if $\Lambda^\ddagger$ is an isometry. To prove the converse, let $\rho_S = \sum_a r_a |r_a\rangle\langle r_a|$, $0 < r_a < 1$. Notice that the action of the dual map Λ (Heisenberg picture) can be written as

$$\Lambda[|\phi\rangle\langle\psi|] = \sum_{ab} \frac{1}{\sqrt{r_a r_b}} \langle\psi \otimes r_b|\xi_\Lambda|\phi \otimes r_a\rangle |r_a\rangle\langle r_b|. \tag{42}$$

If $\xi_\Lambda = |\psi_\Lambda\rangle\langle\psi_\Lambda|$, we have $\Lambda[|\psi\rangle\langle\phi|] = V^\dagger|\psi\rangle\langle\phi|V$ with $V^\dagger : |\phi\rangle \mapsto \langle\psi_\Lambda|\phi \otimes a\rangle|a\rangle$. Finally, unity preservation yields $V^\dagger V = \mathbb{1}_d$. ∎

In light of the previous results, we propose the following:

Definition 2 (Entropic backflow of information measure). We identify the difference

$$\Delta\mathfrak{h}_S(\Theta, \mathcal{F}) := S(\Lambda^\ddagger \otimes \mathrm{id}_d[|\sqrt{\rho_S}\rangle\langle\sqrt{\rho_S}|]) - \mathfrak{h}_S(\Theta, \mathcal{F}) \tag{43}$$

between the entropy rate corresponding to QR-Markovianity and the POVM dependent entropy rate $\mathfrak{h}_S(\Theta, \mathcal{F})$, with $\mathcal{F}$ being the POVM in (33), as a measure of the backflow of information from environment to the open system that accounts for the multitime statistics.

The previous definition is motivated by the fact that, as will become clear in the next section, the decrease of the residual ignorance about the system dynamics signals information coming back from the environment.

Crucially, such depletion of gathered information per unit time step is a stronger witness of backflow of information than lack of contractivity; indeed, if the distinguishability of evolving states increases, then there cannot be P-divisibility (contractivity) of propagators, therefore, no QR is possible which, in turn, implies a dynamical entropy rate strictly smaller than the QR rate. On the other hand, as will be explicitly shown in Sec. V, $\Delta\mathfrak{h}_S(\Theta, \mathcal{F})$ can be positive even when other standard measures of non-Markovianity vanish, in particular the Breuer-Laine-Piilo (BLP) measure [5], which identifies backflow of information with revivals of state distinguishability.

IV. ALF ENTROPY: OPERATIONAL INTERPRETATION

We now discuss a natural physical interpretation of the ALF entropy provided by the so-called GNS representation.

Given an operator algebra $\mathcal{A}$ equipped with a state ω, the familiar Hilbert space setting of quantum mechanics emerges through the approach developed by Gelfand, Neimark, and Segal (GNS construction, see, e.g., [42]). It consists in associating to the pair $(\mathcal{A}, \omega)$ a triple $(\mathcal{H}_\omega, \pi_\omega, |\Omega_\omega\rangle)$ consisting of a Hilbert space $\mathcal{H}_\omega$, a representation $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$ of observables as bounded operators on $\mathcal{H}_\omega$, and a state vector $|\Omega_\omega\rangle$ enjoying the cyclic property, i.e., such that the subspace

$$\pi_\omega(\mathcal{A})|\Omega_\omega\rangle := \{\pi_\omega(X)|\Omega_\omega\rangle,\ X \in \mathcal{A}\}$$

is dense in $\mathcal{H}_\omega$. The expectation value of any observable $X \in \mathcal{A}$ can be then expressed as

$$\omega(X) = \langle\Omega_\omega|\pi_\omega(X)|\Omega_\omega\rangle, \quad X \in \mathcal{A}. \tag{44}$$

The GNS triple $(\mathcal{H}_\omega, \pi_\omega, |\Omega_\omega\rangle)$ is unique up to a unitary equivalence. From this fact, one deduces that every automorphism $\Theta : \mathcal{A} \to \mathcal{A}$ that leaves ω invariant, namely, such that $\omega \circ \Theta = \omega$, can be implemented by a unitary operator

$$\mathbb{U}_\omega[\pi_\omega(X)] := U_\omega^\dagger \pi_\omega(X)\, U_\omega = \pi_\omega[\Theta(X)], \quad X \in \mathcal{A} \tag{45}$$

where $U_\omega^\dagger$ is defined on the dense subspace $\pi_\omega(\mathcal{A})|\Omega_\omega\rangle$ as

$$U_\omega^\dagger \pi_\omega(X)|\Omega_\omega\rangle := \pi_\omega \circ \Theta(X)|\Omega_\omega\rangle, \quad U_\omega^\dagger|\Omega_\omega\rangle = |\Omega_\omega\rangle. \tag{46}$$

Example 1. Finite d-level systems are described by the C^* algebra of $d \times d$ matrices $X \in M_d(\mathbb{C})$ (endowed with the matrix norm) and by states given by density matrices

$$\omega(X) = \mathrm{Tr}(\rho X), \quad M_d(\mathbb{C}) \ni \rho \geqslant 0,\ \mathrm{Tr}(\rho) = 1. \tag{47}$$

For faithful density matrices $\rho = \sum_\alpha r_\alpha |\alpha\rangle\langle\alpha|$, $0 < r_\alpha < 1$, the GNS construction is achieved by state purification. Indeed, through the Hilbert-Schmidt scalar product $\mathrm{Tr}(X^\dagger Y) = \langle X|Y\rangle$ operators $X \in M_d(\mathbb{C})$ are readily identified as vectors in $\mathbb{C}^d \otimes \mathbb{C}^d$. One then introduces the following representations of $M_d(\mathbb{C})$:

$$\pi(A)|B\rangle = |AB\rangle, \quad \pi'(A)|B\rangle = |BA\rangle. \tag{48}$$

By vectorizing matrices $A = \sum_{\alpha\beta} A_{\alpha\beta}|\alpha\rangle\langle\beta| \in M_d(\mathbb{C})$ as $|A\rangle = \sum_{\alpha\beta} A_{\alpha\beta}|\alpha \otimes \beta\rangle$ the above ones become factor representations:

$$\pi(A) = A \otimes \mathbb{1}_d, \quad \pi'(A) = \mathbb{1}_d \otimes \overline{A}, \quad A \in M_d(\mathbb{C}), \tag{49}$$

where $\langle\alpha|\overline{A}|\beta\rangle \equiv \overline{\langle\alpha|A|\beta\rangle}$. For faithful density matrices $\rho > 0$, we have

$$\mathrm{Tr}(\rho A) = \mathrm{Tr}(\sqrt{\rho}\, A \sqrt{\rho}) = \langle\sqrt{\rho}|\pi(A)|\sqrt{\rho}\rangle, \tag{50}$$

where the purification of ρ,

$$|\sqrt{\rho}\rangle = \sum_\alpha \sqrt{r_\alpha}|\alpha \otimes \alpha\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d, \tag{51}$$

is a cyclic vector for $\pi(\mathcal{A})$. The latter is also separating, namely, $\mathrm{Tr}(\rho X^\dagger X) = 0$ if and only if $X = 0$, thus, equivalently, cyclic for the commutant $\pi'(\mathcal{A})$. Let the automorphism of $M_d(\mathbb{C})$ that leaves ρ invariant be $\Theta(A) = V^\dagger A V$ with dual map $\Theta^\ddagger$ such that $\Theta^\ddagger[\rho] = V\rho V^\dagger = \rho$. Then its GNS implementation reads as $U_\rho = V \otimes \overline{V}$. Indeed, as V commutes with ρ,

$$\begin{aligned} U_\rho^\dagger \pi_\rho(A)|\sqrt{\rho}\rangle &= \sum_\alpha \sqrt{r_\alpha}\, V^\dagger A\,|\alpha\rangle \otimes \overline{V}^\dagger|\alpha\rangle \\ &= \sum_{\alpha,\beta} \sqrt{r_\alpha}\, V^\dagger A|\alpha\rangle\langle\alpha|V|\beta\rangle \otimes |\beta\rangle \\ &= \sum_\beta V^\dagger A\sqrt{\rho}\, V|\beta\rangle \otimes |\beta\rangle \\ &= V^\dagger A\, V \otimes \mathbb{1}_d|\sqrt{\rho}\rangle \\ &= \pi_\rho(\Theta(A))|\sqrt{\rho}\rangle. \end{aligned}$$

Setting $A = \mathbb{1}_d$ yields $U_\rho^\dagger|\sqrt{\rho}\rangle = U_\rho|\sqrt{\rho}\rangle = |\sqrt{\rho}\rangle$. As a concrete instance, let $V = e^{iH}$ with $H = \sum h_\alpha|\alpha\rangle\langle\alpha|$, that is H and V are diagonal in the eigenbasis $\{|\alpha\rangle\}_\alpha$ of any fixed faithful ρ. Then, $\overline{V} = V^\dagger$.

If a time-invariant state $\omega \circ \Theta = \omega$ is considered, the physical interpretation of the ALF entropy (27) is neatly obtained in the GNS representation as an alternating sequence of measurement processes and reversible one-step dynamics. This possibility already emerged in Remark 2; however, there the interpretation is only valid for the diagonal entries of $\rho[\mathcal{X}^{(n)}]$. One indeed proves that the von Neumann entropy of the coarse-grained density matrix can equivalently be computed, in the GNS representation, as [14,43]

$$S(\rho[\mathcal{X}^{(n)}]) = S((\mathbb{U}_\omega^\ddagger \circ \widetilde{\mathbb{X}}^\ddagger)^n[|\Omega_\omega\rangle\langle\Omega_\omega|]), \tag{52}$$

where $\mathbb{U}_\omega[\pi_\omega(X)] = U_\omega^\dagger \pi_\omega(X) U_\omega = \pi_\omega(\Theta[X])$ is the unitary implementation of the dynamics in the GNS representation and, for any given POVM $\mathcal{X} = \{X_a\}_{a=1}^{|\mathcal{X}|}$,

$$\widetilde{\mathbb{X}}[A] := \sum_a \pi_\omega(X_a)^\dagger\, A\, \pi_\omega(X_a), \quad A \in \mathcal{B}(\mathcal{H}_\omega) \tag{53}$$

while $\mathbb{U}_\omega^\ddagger$ and $\widetilde{\mathbb{X}}^\ddagger$ are the dual maps of $\mathbb{U}_\omega$ and $\widetilde{\mathbb{X}}$ in the Schrödinger picture. The derivation of (52) is reported in Appendix E and its structure supports an interpretation of the ALF entropy similar to that of the classical KS entropy that is as the maximal information per time step extracted from repeated measurements on the system intertwined with the time evolution. Only measurements and dynamics do not commute in a quantum context.

Remark 7. The completely positive map associated to an POVM $\mathcal{X}$ as in (53) can be dilated as to comprehend the measurement apparatus [44]. In such case, the von Neumann entropy of $\rho[\mathcal{X}] = [\omega(X_b^\dagger X_a)]$ can be also seen as the entropy exchanged with the apparatus; this puts a bound upon the possible reduction of the ignorance about the initial state ω due to the measurement [45,46]. Similarly, through the GNS construction, (52) can also be seen as the entropy exchanged with the measurement apparatus due to repeated measurements. More details can be found in Appendix E.

An analogous operational interpretation can be developed for the open-system ALF entropy (32). When the state ω_S corresponds to a faithful density matrix $\rho_S > 0$, the GNS representation associated with the state $\omega_{SE} = \omega_S \otimes \omega_E$ is the tensor product $(\mathcal{H}_S \otimes \mathcal{H}_E, \pi_S \otimes \pi_E, |\sqrt{\rho_S} \otimes \Omega_E\rangle)$ of the GNS representations of S and E, respectively ([42], Theorem 6.4.7). Here, $|\sqrt{\rho_S}\rangle$ is the canonical purification of ρ_S as in Example 1. Assuming a semigroup global evolution $\Theta_n = \Theta^n$ as in the previous Sec. III B, with Θ leaving the state $\omega_S \otimes \omega_E$ invariant, we can implement the joint $S + E$ evolution by a unitary

$$\begin{aligned} &U_\Theta^\dagger \pi_S \otimes \pi_E(X_S \otimes X_E)|\sqrt{\rho_S} \otimes \Omega_E\rangle \\ &\quad := \pi_S \otimes \pi_E(\Theta[X_S \otimes X_E])|\sqrt{\rho_S} \otimes \Omega_E\rangle, \\ &\mathbb{U}_\Theta[\pi_S \otimes \pi_E(X_S \otimes X_E)] \\ &\quad := U_\Theta^\dagger \pi_S \otimes \pi_E(X_S \otimes X_E) U_\Theta \\ &\quad = \pi_S \otimes \pi_E(\Theta[X_S \otimes X_E]), \\ &U_\Theta|\sqrt{\rho_S} \otimes \Omega_E\rangle = |\sqrt{\rho_S} \otimes \Omega_E\rangle. \end{aligned} \tag{54}$$

Then, by (52), the entropy of $\rho_S[\mathcal{X}^{(n)}]$ turns out to be the same as that of

$$(\mathbb{U}_\Theta^\ddagger \circ (\widetilde{\mathbb{X}}^\ddagger \otimes \mathrm{id}_E))^n[|\sqrt{\rho_S} \otimes \Omega_E\rangle\langle\sqrt{\rho_S} \otimes \Omega_E|], \tag{55}$$

where

$$\widetilde{\mathbb{X}} = \mathbb{X} \otimes \mathrm{id}_d \quad \text{and} \quad \mathbb{X}[Y] = \sum_{a=1}^{|\mathcal{X}|} X_a^\dagger\, Y\, X_a, \quad Y \in M_d(\mathbb{C}).$$

$\mathfrak{h}_S(\Theta)$ is then the entropy production rate due to the discrete semigroup evolution (55), that describes repeated measurements on S intertwining the $S + E$ unitary evolution: it quantifies the maximal rate of information about the dynamics that is extractable by measurements on the open system S only.

Moreover, the above discussion justifies interpreting (43) as a measure of backflow of information that relies on the full multitime statistics of the open dynamics. Indeed, asymptotic entropy rates quantify the information gained from repeated observations of the open dynamics. If the POVM dependent rate $\mathfrak{h}(\Theta,\mathcal{F})$ decreases relative to its value in the QR-Markovian regime, the dynamics becomes less unpredictable, indicating that information has flowed back into the open system.

V. DYNAMICAL ENTROPY AND COLLISIONAL MODELS

In the following, we exploit the collisional approach to open systems to get more insight regarding the behavior of $\mathfrak{h}_S(\Theta)$ beyond the Markovian regime, i.e., beyond quantum regression. We now begin with a general upper bound to the ALF entropy $\mathfrak{h}_S(\Theta)$ of the collisional model discussed in Sec. II.

Proposition 5. Let $\omega_S\otimes\omega_E$ be a Θ-invariant state. Then,

$$\mathfrak{h}_S(\Theta) \leqslant \mathfrak{S}_{\omega_E} + \ln(D), \tag{56}$$

where $\mathfrak{S}_{\omega_E}$ is the mean von Neumann entropy of the chain (8).

Proof. Consider a POVM involving operators of the system S only, as in (30):

$$\mathcal{X} = \{X_a\otimes\mathbb{1}_E\}_a, \quad X_a\in\mathcal{A}_S, \quad \sum_{a=1}^{d} X_a^\dagger X_a = \mathbb{1}_d.$$

After n steps of the dynamics the algebra of the system is mapped into the local algebras $\mathcal{A}_S\otimes\mathcal{A}_E^{[1,n]}$, due to the action of the shift [see (11)]: $\Theta^n(\mathcal{A}_S\otimes\mathbb{1}_E)\subseteq\mathcal{A}_S\otimes\mathcal{A}_E^{[1,n]}$. With respect to a Θ-invariant state, the correlation matrix reads as

$$(\rho[\mathcal{X}^{(n)}])_{\boldsymbol{a},\boldsymbol{b}} = \omega_{SE}\big(X_{\boldsymbol{b}}^{(n)\dagger}X_{\boldsymbol{a}}^{(n)}\big) = \mathrm{Tr}\big(\Omega_{S[1,n]}X_{\boldsymbol{b}}^{(n)\dagger}X_{\boldsymbol{a}}^{(n)}\big),$$

with $\Omega_{S[1,n]}=\rho_S\otimes\rho_E^{[1,n]}\in M_d(\mathbb{C})\otimes M_D^{[1,n]}(\mathbb{C})$. Hence, from Proposition 1, together with the fact that $S(\Omega_{S[1,n]}) = S(\rho_S)+S(\rho_E^{[1,n]})$ and

$$S\left(\sum_{\boldsymbol{a}} X_{\boldsymbol{a}}^{(n)}\,\Omega_{S[1,n]}\,X_{\boldsymbol{a}}^{(n)\dagger}\right)\leqslant\ln(dD^n)\,,$$

the following upper bound ensues:

$$S(\rho[\mathcal{X}^{(n)}])\leqslant S(\rho_S)+S\big(\rho_E^{[1,n]}\big)+\ln(d)+n\ln(D).$$

Dividing both sides by n and taking the lim sup, one gets inequality (56). ∎

Remark 8. The so-called quantum mutual information

$$I_n(\omega_E) := S\big(\rho_E^{[1,n]}\big)+S\big(\rho_E^{\{1\}}\big)-S\big(\rho_E^{[1,n+1]}\big)$$

quantifies the correlations between the first n sites of the spin chain and the $(n+1)$th. From (8), the large-n limit yields

$$\mathfrak{I}_{\omega_E} := \lim_n I_n(\omega_E) = S\big(\rho_E^{\{1\}}\big)-\mathfrak{S}_{\omega_E}\geqslant 0, \tag{57}$$

the inequality saturating for a Bernoulli source. Thus, the stronger the correlations in the environment, the lower results its mean entropy and thus the tighter becomes the bound in (56) on the maximal entropy production of the open system S. Notice that the right-hand side of bound (56) amounts to the ALF entropy of the shift on a quantum spin chain as proved in [26]. However, that quantum bound is obtained using all possible POVMs of the chain, whereas here we restricted to those of the open system appended to the chain. It is an open question whether for special system POVMs bound (56) could be saturated.

A. Classical spin-chain environment

In this section, we compute the open-system ALF entropy $\mathfrak{h}_S(\Theta)$ for the collisional model studied in [29] which in its simplicity exhibits different types of memory effects directly related to the "microscopic" parameters of the environment E. The latter is taken to be a classical stationary source; namely, each site of the chain carries a same commutative, that is diagonal, matrix algebra $\mathcal{A}=D_D(\mathbb{C})$, spanned by one-dimensional orthogonal projections $\{\Pi_k\}_{k=0}^{D-1}$, $\sum_{k=0}^{D-1}\Pi_k=\mathbb{1}_D$. Furthermore, the chain is endowed with a shift-invariant state ω_E whose restrictions to any local algebra $\mathcal{A}_E^{[1,2]}$ are diagonal density matrices:

$$\rho_E^{[a,b]} = \sum_{\boldsymbol{i}_{[a,b]}} p_{\boldsymbol{i}_{[a,b]}}\,\Pi_{\boldsymbol{i}_{[a,b]}}^{[a,b]}, \quad \Pi_{\boldsymbol{i}_{[a,b]}}^{[a,b]} = \bigotimes_{k=a}^{b}\Pi_{i_k}^{(k)}\in\mathcal{A}_E^{[a,b]}, \tag{58}$$

with the family of probabilities $\pi_{[1,n]}=\{p_{\boldsymbol{i}_{[1,n]}}\}_{\boldsymbol{i}_{[1,n]}}$ satisfying the consistency and stationarity conditions

$$\sum_{i_b} p_{\boldsymbol{i}_{[a,b]}} = p_{\boldsymbol{i}_{[a,b-1]}} \quad\text{and}\quad \sum_{i_a} p_{\boldsymbol{i}_{[a,b]}} = p_{\boldsymbol{i}_{[a-1,b]}}.$$

A finite d-level open quantum system is coupled to such a classical environment by means of an interaction with the chain 0th site via a controlled-unitary type,

$$\begin{aligned}\Phi\big[X_S\otimes A_{i_0}^{(0)}\big] &= \sum_{k=0}^{D-1}\phi_k[X_S]\otimes\Pi_k^{(0)}A_{i_0}^{(0)}\Pi_k^{(0)}\\ &= \sum_{k=0}^{D-1}\phi_k[X_S]\otimes A_{i_0}^{(0)}\Pi_k^{(0)},\\ \phi_k[X_S] &:= U_k^\dagger X_S\,U_k,\end{aligned} \tag{59}$$

where $U_kU_k^\dagger=U_k^\dagger U_k=\mathbb{1}_d$ so that the dual maps on the states of S, $\phi_k^\ddagger[\rho_S]=U_k\,\rho_S\,U_k^\dagger=\phi_k^{-1}[\rho_S]$, coincide with the inverse maps as well as the the dual map of $\Phi^\ddagger$ extended to $\mathcal{A}_S\otimes\mathcal{A}_E$:

$$\begin{aligned}\Phi^\ddagger\big[X_S\otimes A_{i_0}^{(0)}\big] &= \sum_{k=0}^{D-1}\phi_k^\ddagger[X_S]\otimes\Pi_k^{(0)}A_{i_0}^{(0)}\Pi_k^{(0)}\\ &= \Phi^{-1}\big[X_S\otimes A_{i_0}^{(0)}\big].\end{aligned}$$

Finally, the compound SE-automorphic dynamics is given by powers of $\Theta=\Theta_{\mathrm{coll}}=(\mathrm{id}_S\otimes\sigma_E)\circ\Phi$ obtained by acting with the chain right shift after the interaction of the system S with the 0th site of the classical chain [see (11)].

We then take as state of the system S the maximally mixed state $\omega_S(X_S) = \mathrm{Tr}(\rho_S X_S)$, where $\rho_S = \mathbb{1}_d/d$, thus enforcing the time invariance of $\omega_S \otimes \omega_E$. Indeed, leveraging the shift invariance of ω_E and the commutativity of the algebra $\mathcal{A}_E$, one gets, for all $X_S \in \mathcal{A}_S$ and $X_E \in \mathcal{A}_E$,

$$\begin{aligned}&\omega_S \otimes \omega_E(\Theta(X_S \otimes X_E))\\ &= \frac{1}{d}\sum_{k=0}^{D-1} \mathrm{Tr}(\phi_k[X_S])\,\omega_E\big(\sigma_E\big(\Pi_k^{(0)} X_E\, \Pi_k^{(0)}\big)\big)\\ &= \frac{\mathrm{Tr}(X_S)}{d}\omega_E\left(\sum_k \Pi_k^{(0)} X_E \Pi_k^{(0)}\right) = \omega_S \otimes \omega_E(X_S \otimes X_E).\end{aligned}$$

Remark 9. The classically controlled coupling (59) is still actively debated in the literature [47–49]. For sure, despite the memory-storing mechanism of this model being entirely classical, it can give rise to memory effects in the open quantum dynamics that have no classical counterpart. This is the case of the so-called superactivation of backflow of information [29], occurring in the tensor product dynamics of a bipartite system, each copy undergoing the same dissipative dynamics. Such phenomenon occurs when the reduced dynamics is positive but not completely positive divisible.

B. Structure of the coarse-grained density matrix

The $(n+1)$th step coarse-grained density matrix based upon a system S POVM $\mathcal{X} = \{X_a \otimes \mathbb{1}_E\}_{a=1}^{|\mathcal{X}|}$, $X_a \in \mathcal{A}_S = M_d(\mathbb{C})$, and the Θ-invariant state $\omega_{SE} = \omega_S \otimes \omega_E$ takes the form

$$\rho_S[\mathcal{X}^{(n+1)}] \;=\; \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}}\; \rho_S\big[\mathcal{X}_{\boldsymbol{i}_{[1,n]}}\big], \tag{60}$$

where $\boldsymbol{i}_{[1,n]} = i_1 \dots i_n$; namely, of a convex combination of sub-coarse-grained matrices $\rho_S[\mathcal{X}_{\boldsymbol{i}_{[1,n]}}]$ with entries

$$\begin{aligned}\big(\rho_S\big[\mathcal{X}_{\boldsymbol{i}_{[1,n]}}\big]\big)_{\boldsymbol{a},\boldsymbol{b}} &= \omega_S\big(X_{b_0}^\dagger \phi_{\boldsymbol{i}_{\{1\}}}\big[X_{b_1}^\dagger\big]\phi_{\boldsymbol{i}_{[1,2]}}\big[X_{b_2}^\dagger\big]\dots\\ &\qquad \phi_{\boldsymbol{i}_{[1,n]}}\big[X_{b_n}^\dagger X_{a_n}\big]\dots\phi_{\boldsymbol{i}_{[1,2]}}\big[X_{a_1}\big]\phi_{\boldsymbol{i}_{\{1\}}}\big[X_{a_1}\big]X_{a_0}\big)\\ &= \omega_S\big(X_{b_0}^\dagger \phi_{i_1}\big[X_{b_1}^\dagger \phi_{i_2}\big[X_{b_2^\dagger}\dots\\ &\qquad \phi_{i_n}\big[X_{b_n}^\dagger X_{a_n}\big]\dots X_{a_2}\big]X_{a_1}\big]X_{a_0}\big)\end{aligned} \tag{61}$$

specified by the automorphisms $\phi_{\boldsymbol{i}_{[1,k]}} = \phi_{i_1} \circ \cdots \circ \phi_{i_k}$ and by the n-step POVM $\mathcal{X}_{\boldsymbol{i}_{[1,n]}}$ from $\mathcal{A}_S$ with components $X^{\boldsymbol{a}}_{\boldsymbol{i}_{[1,n]}} := \phi_{\boldsymbol{i}_{[1,n]}}[X_{a_n}]\dots\phi_{\boldsymbol{i}_1}[X_{a_1}]X_{a_0} \in \mathcal{A}_S$.

Proof. Consider $X_{a_1} \in \mathcal{A}_S$; then,

$$\Theta\big[X_{a_1} \otimes \mathbb{1}_E\big] = \sum_{k_1} \phi_{k_1}\big[X_{a_1}\big] \otimes \Pi_{k_1}^{(1)},$$

$$\Theta^2\big[X_{a_2} \otimes \mathbb{1}_E\big] = \sum_{k_1,k_2} \phi_{k_1} \circ \phi_{k_2}\big[X_{a_2}\big] \otimes \Pi_{k_1}^{(1)} \otimes \Pi_{k_2}^{(2)},$$

so that the two-step POVM consists of operators

$$\begin{aligned}&\Theta^2\big[X_{a_2} \otimes \mathbb{1}_E\big]\Theta\big[X_{a_1} \otimes \mathbb{1}_E\big]X_{a_0} \otimes \mathbb{1}_E\\ &\quad = \sum_{k_1,k_2} \phi_{k_1} \circ \phi_{k_2}\big[X_{a_2}\big]\phi_{k_1}\big[X_{a_1}\big]X_{a_0} \otimes \Pi_{k_1}^{(1)} \otimes \Pi_{k_2}^{(2)}.\end{aligned}$$

Thus, one has

$$\begin{aligned}X_{\boldsymbol{a}}^{(n+1)} &= \Theta^n\big[X_{a_n} \otimes \mathbb{1}_E\big]\dots\Theta\big[X_{a_1} \otimes \mathbb{1}_E\big]X_{a_0} \otimes \mathbb{1}_E\\ &= \sum_{\boldsymbol{k}_{[1,n]}} \phi_{\boldsymbol{k}_{[1,n]}}\big[X_{a_n}\big]\phi_{\boldsymbol{k}_{[1,n-1]}}\big[X_{a_{n-1}}\big]\dots\phi_{\boldsymbol{k}_{\{1\}}}\big[X_{a_1}\big]X_{a_0}\\ &\qquad \otimes \bigotimes_{j=1}^{n} \Pi_{k_j}^{(j)}.\end{aligned}$$

The elements of the coarse-grained density matrix then become

$$\begin{aligned}\omega_{SE}\big(X_{\boldsymbol{b}}^{(n+1)\dagger} X_{\boldsymbol{a}}^{(n+1)}\big) = \sum_{\boldsymbol{k}_{[1,n]}} p_{\boldsymbol{k}_{[1,n]}}\, \omega_S\big(X_{b_0}^\dagger \phi_{k_1}\big[X_{b_1}^\dagger\big]\\ \dots\phi_{\boldsymbol{k}_{[1,n]}}\big[X_{b_n}^\dagger X_{a_n}\big]\dots\phi_{k_1}\big[X_{a_1}\big]X_{a_0}\big).\end{aligned}$$

Given a fixed multi-index $\boldsymbol{k}_{[1,n]}$, the corresponding operators $X^{\boldsymbol{a}}_{\boldsymbol{k}_{[1,n]}} := \phi_{\boldsymbol{k}_{[1,n]}}[X_{a_n}]\dots\phi_{\boldsymbol{k}_{[1,2]}}[X_{a_2}]\phi_{k_1}[X_{a_1}]X_{a_0} \in \mathcal{A}_S$ form a POVM. Indeed, the maps $\phi_{\boldsymbol{k}_{[1,n-1]}}$ are automorphisms; namely, they are invertible and such that $\phi_{\boldsymbol{k}_{[1,n]}}[XY] = \phi_{\boldsymbol{k}_{[1,n]}}[X]\phi_{\boldsymbol{k}_{[1,n]}}[Y]$ and $\phi_{\boldsymbol{k}_{[1,n]}}[X^\dagger] = (\phi_{\boldsymbol{k}_{[1,n]}}[X])^\dagger$. In the case $k = 2$, for example,

$$\begin{aligned}&\sum_{\boldsymbol{a}} \big(X^{\boldsymbol{a}}_{\boldsymbol{k}_{[1,2]}}\big)^\dagger X^{\boldsymbol{a}}_{\boldsymbol{k}_{[1,2]}}\\ &\quad = \sum_{a_0,a_1} X_{a_0}^\dagger \phi_{k_1}\big[X_{a_1}^\dagger\big]\phi_{\boldsymbol{k}_{[1,2]}}\left[\sum_{a_2} X_{a_2}^\dagger X_{a_2}\right]\phi_{k_1}\big[X_{a_1}\big]X_{a_0}\\ &\quad = \sum_{a_0} X_{a_0}^\dagger \phi_{k_1}\left[\sum_{a_1} X_{a_1}^\dagger X_{a_1}\right]X_{a_0} = \sum_{a_0} X_{a_0}^\dagger X_{a_0} = \mathbb{1}_S.\end{aligned}$$

■

Remark 10. Similarly, the reduced Heisenberg dynamics $\Lambda_n[X_S] = \mathrm{id}_S \otimes \omega_E(\Theta^n[X_S \otimes 1_E])$ is implemented by the CPU map

$$M_d(\mathbb{C}) \ni X \;\longmapsto\; \Lambda_n[X] = \sum_{\boldsymbol{k}_{[1,n]}} p_{\boldsymbol{k}_{[1,n]}}\; \phi_{\boldsymbol{k}_{[1,n]}}[X]. \tag{62}$$

If the classical chain represents a Bernoulli process, then $p_{\boldsymbol{k}_{[1,n]}} = \prod_{j=1}^n p_{k_j}$. The collisional environment is fully uncorrelated and the reduced dynamics is a semigroup $\Lambda_n = \Lambda_1^n$ that is the reduced dynamics at time step n is the nth power of the single time-step reduced dynamics $\Lambda_1 = \sum_k p_k \phi_k$. Further, QR holds. For example, after two iterations of the dynamics,

$$\begin{aligned}&(\rho_S[\mathcal{X}^{(2)}])_{\boldsymbol{a},\boldsymbol{b}}\\ &\quad = \sum_{k_2,k_1} p_{k_1} p_{k_2}\; \omega_S\big(X_{b_0}^\dagger \phi_{k_1}\big[X_{b_1}^\dagger \phi_{k_2}\big[X_{b_2}^\dagger X_{a_2}\big]X_{a_1}\big]X_{a_0}\big)\\ &\quad = \omega_S\left(X_{b_0}^\dagger \sum_{k_1} p_{k_1}\phi_{k_1}\left[X_{b_1}^\dagger \sum_{k_2} p_{k_2}\phi_{k_2}\big[X_{b_2}^\dagger X_{a_2}\big]X_{a_1}\right]\right)\\ &\quad = \omega_S\big(X_{b_0}^\dagger \Lambda_1\big[X_{b_1}^\dagger \Lambda_1\big[X_{b_2}^\dagger X_{a_2}\big]X_{a_1}\big]X_{a_0}\big),\end{aligned}$$

so that the knowledge of Λ_1 suffices to reconstruct the full coarse-grained density matrix.

C. Upper and lower bounds to the ALF entropy

Given the convex combination (60), its von Neumann entropy can then be bounded from above as

$$S(\rho_S[\mathcal{X}^{(n+1)}]) \leqslant H(\pi_{[1,n]}) + \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} S\big(\rho_S[\mathcal{X}_{\boldsymbol{i}_{[1,n]}}]\big) \leqslant H(\pi_{[1,n]}) + 2\,\ln(d), \tag{63}$$

where we used the well-known inequality for the entropy of a convex mixture ([44], Theorem 11.10), along with Proposition 1 and the fact that, for all states of d-level systems, $S(\rho) \leqslant \ln d$. Dividing both sides of (63) by n and taking the lim sup, one gets that the dynamical entropy of the open system is upper bounded by the entropy rate of the chain,

$$\mathfrak{h}_S(\Theta) \leqslant \lim_n \frac{1}{n} H(\pi_{[1,n]}) = \mathfrak{S}_{\omega_E}. \tag{64}$$

We now show that inequality (64) can be saturated.

Proposition 6. Consider a d-level system collisionally interacting with a classical stationary spin chain through (59). With respect to the invariant state $\mathbb{1}_d/d \otimes \omega_E$ and the special POVM $\mathcal{F}$ in (33), the coarse-grained density matrix takes the form

$$\rho_S[\mathcal{F}^{(n+1)}] = \frac{\mathbb{1}_d}{d} \otimes \frac{\mathbb{1}_d}{d} \otimes \left(\mathbb{T}_n^{\ddagger} \otimes \mathrm{id}_d^{\otimes n} \left[\left| \sqrt{\frac{\mathbb{1}_d}{d}}^{\otimes n} \right\rangle \left\langle \sqrt{\frac{\mathbb{1}_d}{d}}^{\otimes n} \right| \right] \right), \tag{65}$$

with correlation maps

$$\mathbb{T}_n = \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} \phi_{\boldsymbol{i}_{[1,n]}}^{\otimes[1,n]}, \quad \phi_{\boldsymbol{i}_{[1,n]}}^{\otimes[1,n]} := \bigotimes_{k=1}^{n} \phi_{i_k}. \tag{66}$$

Moreover, if the unitaries in (59) are such that

$$\mathrm{Tr}(U_j^{\dagger} U_k) = d\,\delta_{jk},$$

one has

$$\mathfrak{h}_S(\Theta) = \mathfrak{h}_S(\Theta, \mathcal{F}) = \lim_n \frac{1}{n} H(\pi_{[1,n]}) = \mathfrak{S}_{\omega_E}, \tag{67}$$

that is, the ALF entropy of the system is equal to the chain entropy rate.

The proof is reported in Appendix F. There, it is shown that the saturation of the bound (64) relies on the condition $\mathrm{Tr}(U_j^{\dagger} U_k) = d\,\delta_{jk}$. If the latter is not satisfied, the POVM $\mathcal{F}$ is not guaranteed to be optimal. A natural example is obtained for $D = d^2$ and by choosing U_{i_k} as the discrete Weyl operators,

$$U_{a,b} = \sum_{k=0}^{d-1} \omega^{kb} |a+k\rangle\langle k|, \quad \omega = e^{2\pi i/d}, \tag{68}$$

which are unitary and satisfy $\mathrm{Tr}(U_{a,b}^{\dagger} U_{c,d}) = d\,\delta_{ac}\delta_{bd}$.

Remark 11. Suppose that $\mathrm{Tr}(U_j^{\dagger} U_k) = d\,\delta_{jk}$, $D = d^2$. As shown in Remark 8,

$$\mathfrak{h}_S(\Theta) = \mathfrak{S}_{\omega_E} = S\big(\rho_E^{(1)}\big) - \mathfrak{I}_{\omega_E}.$$

If the environment is a Bernoulli source, one has simply

$$\mathfrak{h}_S(\Theta) = S\big(\rho_E^{(1)}\big) = H(\pi_{\{1\}}).$$

Note also that $|\sqrt{\rho_S}\rangle = |\sqrt{\mathbb{1}_d/d}\rangle = d^{-1/2} \sum_i |i \otimes i\rangle =: |\psi_+^{(d)}\rangle$, namely, the fully symmetric maximally entangled state. Then,

$$\Lambda_1^{\ddagger} \otimes \mathrm{id}_d[|\psi_+^{(d)}\rangle\langle\psi_+^{(d)}|] = \sum_i p_i\, U_i \otimes \mathbb{1}_d |\psi_+^{(d)}\rangle\langle\psi_+^{(d)}| U_i^{\dagger} \otimes \mathbb{1}_d.$$

Since $U_i \otimes \mathbb{1}_d |\psi_+^{(d)}\rangle$, $i = 1, \ldots, d^2$, form a basis of $\mathbb{C}^d \otimes \mathbb{C}^d$, the von Neumann entropy of the latter state amounts to $H(\pi_{\{1\}}) = -\sum_i p_i \ln p_i$ and, thus, consistently, equals to $\mathfrak{h}_S(\Theta)$ since QR holds (compare Corollary 1).

D. Markov chain environment

We now let the classical environment be a stationary Markov chain; namely, we choose the probabilities $\pi_{[1,n]} = \{p_{\boldsymbol{i}_{[1,n]}}\}_{\boldsymbol{i}_{[1,n]}}$ to form a stationary Markov process,

$$p_{\boldsymbol{i}_{[1,n]}} = \prod_{k=2}^{n} T_{i_k\, i_{k-1}}\; p_{i_1}, \tag{69}$$

defined by the stochastic matrix $T = [T_{ij}]$, $T_{ij} \geqslant 0$, with $\sum_i T_{ij} = 1$, $\sum_j T_{ij}\, p_j = p_i$. The mean entropy rate of the Markov source is equal to the two-site conditional entropy of the chain [40],

$$\mathfrak{S}_{\omega_E} = -\sum_{ij} p_j\, T_{ij} \ln(T_{ij}) = H(\pi_1) - I(\pi_1; \pi_2), \tag{70}$$

where

$$I(\pi_1; \pi_2) = H(\pi_1) + H(\pi_2) - H(\pi_{[1,2]}) \tag{71}$$

denotes the mutual information between the first two subsequent sites of the classical chain. Because of stationarity, the latter quantity does not depend on where the chosen pair is located along the chain and measures the correlations between any two subsequent sites. Then, one sees that the entropy rate decreases with increasing correlations between subsequent sites. In the following, we construct a process with U_k as in Remark 11 and for which $T_{ij} \in \{0, 1\}$, so that $\mathfrak{h}_S(\Theta) = \mathfrak{S}_{\omega_E} = 0$ as for reversible time evolutions.

1. *Pauli dynamics: Divisibility*

We now let the open quantum system S to be a qubit ($d = 2$), we couple it to a Markov chain with 4×4 diagonal matrices at each site and choose the local probability distributions to be generated by the following 4×4 stochastic matrix:

$$T = \begin{pmatrix} p_0 & p_0 & p_0 & p_0 \\ p & p+\Delta & p-\Delta & p \\ p & p-\Delta & p+\Delta & p \\ r & r & r & r \end{pmatrix}, \quad 0 \leqslant \Delta \leqslant p \leqslant \frac{1}{2}, \quad p_0 + 2p + r = 1, \tag{72}$$

with invariant probability vector $\boldsymbol{p} = (p_0, p, p, r)$, guaranteeing stationarity $T\boldsymbol{p} = \boldsymbol{p}$. Let the coupling be as in (59) with

$$\phi_k[X] = \sigma_k\, X\, \sigma_k, \quad k = 0, \ldots, 3 \tag{73}$$

$\sigma_0 = \mathbb{1}_2$ and σ_k, $k = 1, 2, 3$, being the Pauli matrices so that the reduced dynamics corresponds to a Pauli map

$$\Lambda_n^{\ddagger} = \sum_{\boldsymbol{i}_{[1,n]}} p_{i_{[1,n]}}\, \phi_{\boldsymbol{i}_{[1,n]}} = \Lambda_n, \quad \phi_{\boldsymbol{k}_{[1,n]}} := \phi_{k_1} \circ \cdots \circ \phi_{k_n}. \tag{74}$$

The latter can be explicitly computed for the Markov chain generated by (72). Indeed, for all $n \in \mathbb{N}$, (74) is diagonal in the Pauli basis,

$$\Lambda_n^{\ddagger}[\sigma_\alpha] = \lambda_n^{(\alpha)} \sigma_\alpha, \quad \alpha = 0, \ldots, 3 \tag{75}$$

where $\lambda_n^{(0)} = 1$ follows from the preservation of unity and the other eigenvalues evolving according to recursive equations

$$\begin{aligned}\lambda_n^{(\alpha)} &= \big(p_0 + p\big(\mu_1^{(\alpha)} + \mu_2^{(\alpha)}\big) + r\mu_3^{(\alpha)}\big)\lambda_{n-1}^{(\alpha)} \\ &\quad + \Delta\, p\big(\mu_1^{(\alpha)} - \mu_2^{(\alpha)}\big)^2 \lambda_{n-2}^{(\alpha)}, \quad \alpha = 0, \ldots, 3 \end{aligned} \tag{76}$$

where $\phi_\beta[\sigma_\alpha] = \mu_\beta^{(\alpha)} \sigma_\alpha$, $\mu_0^{(\alpha)} = \mu_\beta^{(0)} = 1$, and $\mu_\beta^{(\alpha)} = -1$ for $\beta \neq \alpha \in \{1, 2, 3\}$. Accordingly, (76) is rewritten as

$$\begin{aligned}\lambda_n^{(1,2)} &= (1 - 2(p + r))\lambda_{n-1}^{(1,2)} + 4\, p\, \Delta\, \lambda_{n-2}^{(1,2)}, \\ \lambda_n^{(3)} &= (1 - 4p)\, \lambda_{n-1}^{(3)},\end{aligned}$$

which are easily solvable, yielding

$$\begin{aligned}\lambda_n := \lambda_n^{(1,2)} &= \left(\frac{B_{p,r,\Delta} + A_{p,r}}{2\, B_{p,r,\Delta}}\right)\left(\frac{A_{p,r} + B_{p,r,\Delta}}{2}\right)^n \\ &\quad + \left(\frac{B_{p,r,\Delta} - A_{p,r}}{2\, B_{p,r,\Delta}}\right)\left(\frac{A_{p,r} - B_{p,r,\Delta}}{2}\right)^n, \\ \lambda_n^{(3)} &= (1 - 4p)^n, \end{aligned} \tag{77}$$

where we set $A_{p,r} = 1 - 2(p + r)$ and $B_{p,r,\Delta} = \sqrt{A_{p,r}^2 + 16\, p\, \Delta}$. In [29], it was proved that, for $A_{p,r} > 0$, the divisibility features of the one-qubit dynamical map $\Lambda_n^{\ddagger}$ are controlled by the parameter Δ as follows:

$$\begin{aligned}&\Lambda_n^{\ddagger} \text{ is CP divisible iff} \quad \frac{\Delta}{A_{p,r}} \leqslant \frac{r}{2p}, \\ &\Lambda_n^{\ddagger} \otimes \Lambda_n^{\ddagger} \text{ is P divisible iff} \\ &\frac{\Delta}{A_{p,r}} \leqslant \frac{r}{2p} + \frac{1}{2} - \frac{1 - \sqrt{1 - 4p(1 - 2p)}}{4p}, \\ &\Lambda_n^{\ddagger} \text{ is P divisible iff} \quad \frac{\Delta}{A_{p,r}} \leqslant \frac{r}{2p} + \frac{1}{2}. \end{aligned} \tag{78}$$

Remark 12. Note that we discuss divisibility of the dynamical map only in the Schrödinger picture. Recently, an interesting dependence of the divisibility properties of the dynamics on the chosen picture was discussed in [50]. However, for Pauli maps, which are functionally commutative in time, the divisibility regimes in the Schrödinger and Heisenberg pictures coincide.

Hence, by tuning Δ and keeping fixed the other parameters, conditions (78) establish that one can change the divisibility degree of the dynamics by increasing the correlations of the environment. Indeed, by increasing Δ one first loses CP-divisibility, then the P-divisibility of the second tensor power $\Lambda_n^{\ddagger} \otimes \Lambda_n^{\ddagger}$, and finally the P-divisibility of the one-qubit dynamics $\{\Lambda_n^{\ddagger}\}_{n \geqslant 0}$. Notice that only in the latter case one can detect backflow of information as distinguishability revivals for one qubit. Indeed, lack of P-divisibility means lack of contractivity of some intertwiner $\Lambda_{n,n-1}^{\ddagger}$ and thus the possibility that the trace distances of suitable states increase from time step $n - 1$ to time step n.

Remark 13. Recall that, for continuous-time dynamics, the behavior of integrated measures of non-Markovianity is closely related to their divisibility properties. One of the most widely used measures of non-Markovianity is the Rivas-Huelga-Plenio (RHP) measure [3],

$$\mathcal{N}_{\mathrm{RHP}} = \int_0^\infty \mathrm{d}t\, g_t, \quad g_t = \lim_{\epsilon \to 0} \frac{||\Lambda_{t+\epsilon,t}^{\ddagger} \otimes \mathrm{id}_d[P_+^{(d)}]||_1 - 1}{\epsilon}, \tag{79}$$

with $P_+^{(d)} = |\psi_+^{(d)}\rangle\langle\psi_+^{(d)}|$. Then, $\mathcal{N}_{\mathrm{RHP}}$ vanishes if and only if the dynamics is CP-divisible. On the other hand, the measure proposed by Breuer-Laine-Piilo (BLP) [5] rather focuses on state-distinguishability revivals

$$\begin{aligned}\mathcal{N}_{\mathrm{BLP}} &= \max_{\rho,\sigma} \int_{\delta_t > 0} \mathrm{d}t\, \delta_t(\rho, \sigma), \\ \delta_t(\rho, \sigma) &= \frac{\mathrm{d}}{\mathrm{d}t}\left|\left|\Lambda_t^{\ddagger}\left[\frac{\rho - \sigma}{2}\right]\right|\right|_1. \end{aligned} \tag{80}$$

Note that P-divisibility of $\Lambda_t^{\ddagger}$ yields $\mathcal{N}_{\mathrm{BLP}} = 0$. Moreover, $\mathcal{N}_{\mathrm{RHP}} = 0 \Rightarrow \mathcal{N}_{\mathrm{BLP}} = 0$, while the converse is obviously not true.

2. *Pauli dynamics: Non-Markovianity measures*

We further illustrate that the standard measures of Markovianity of Remark 13, when nonzero, exhibit a monotonic dependence on the chain correlations Δ, analogous to the behavior observed for the divisibility degree in (78). We first need to suitably adapt the aforementioned measures to our discrete-time scenario. Let us fix $N \in \mathbb{N}$ and consider the following discrete-time generalization of the RHP measure:

$$\mathcal{N}_{\mathrm{RHP}}^{(N)} = \sum_{n=1}^{N} g_n, \quad g_n = ||\Lambda_{n,n-1}^{\ddagger} \otimes \mathrm{id}_d[P_+^{(d)}]||_1 - 1. \tag{81}$$

Letting $E_{n,n-1}^{(i)}$ be the eigenvalues of $\Lambda_{n,n-1}^{\ddagger} \otimes \mathrm{id}_d[P_+^{(d)}]$, one has $||\Lambda_{n,n-1}^{\ddagger} \otimes \mathrm{id}_d[P_+^{(d)}]||_1 - 1 = 2\sum_{i:\, E_{n,n-1}^{(i)} < 0} |E_{n,n-1}^{(i)}|$. For the one-qubit Pauli dynamics at hand, with $A_{p,r} > 0$, propagators $\Lambda_{n,n-1}^{\ddagger}$ might develop at most one negative eigenvalue $E_{n,n-1}^{(3)} = (1 - 2p - \lambda_{n,n-1})/2$ (see also [29]), where $\lambda_{n,n-1} := \lambda_n/\lambda_{n-1}$; in such case

$$g_n = \lambda_{n,n-1} - (1 - 2p).$$

We then claim that

$$\frac{\partial g_n}{\partial \Delta} = \frac{\partial \lambda_{n,n-1}}{\partial \Delta} \geqslant 0, \quad \forall\, n \in \mathbb{N} \tag{82}$$

namely, g_n increases monotonically with Δ, thereby ensuring that $\mathcal{N}_{\mathrm{RHP}}^{(N)}$ likewise increases monotonically with the correlations Δ. The proof is reported in Appendix G. Similarly, for the discrete-time analog of the BLP measure, we fix $N \in \mathbb{N}$

and consider

$$\mathcal{N}_{\mathrm{BLP}}^{(N)} = \max_{\rho,\sigma} \sum_{\substack{n=1 \\ d_n - d_{n-1} > 0}}^{N} (d_n(\rho,\sigma) - d_{n-1}(\rho,\sigma)), \tag{83}$$

where $d_n(\rho,\sigma) := ||\Lambda_n^{\ddagger}[\frac{\rho-\sigma}{2}]||_1$ and the optimization is performed on all pairs of density matrices ρ, σ. Then, one proves that for states ρ, σ, and $n \in \mathbb{N}$ such that $d_n(\rho,\sigma) > d_{n-1}(\rho,\sigma)$,

$$\frac{\partial}{\partial \Delta}(d_n(\rho,\sigma) - d_{n-1}(\rho,\sigma)) \geqslant 0. \tag{84}$$

The detailed derivation is again reported in Appendix G. As a consequence, both discretized measures (81) and (83) are monotonically increasingly with the correlations Δ.

3. *Pauli dynamics: Correlation maps and ALF entropy*

In the context of the qubit dynamics described in Sec. V D 1, the correlation maps $\mathbb{T}_n : M_2^{\otimes n}(\mathbb{C}) \to M_2^{\otimes n}(\mathbb{C})$, introduced in (34), are n-qubit Pauli channels:

$$\mathbb{T}_n[X] = \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} \sigma_{\boldsymbol{i}_{[1,n]}}^{[1,n]} X \sigma_{\boldsymbol{i}_{[1,n]}}^{[1,n]}, \quad \sigma_{\boldsymbol{i}_{[1,n]}}^{[1,n]} := \bigotimes_{k=1}^{n} \sigma_{i_k}, \tag{85}$$

so that $\mathbb{T}_n = \mathbb{T}_n^{\ddagger}$. The purification of $(\mathbb{1}_2/2)^{\otimes n}$ corresponds to the maximally entangled state

$$|\psi_+^{[1,n]}\rangle = 2^{-n/2} \sum_{\boldsymbol{k}_{[1,n]}} |\boldsymbol{k}_{[1,n]} \otimes \boldsymbol{k}_{[1,n]}\rangle.$$

Set then $|\psi_{\boldsymbol{i}_{[1,n]}}\rangle := (\bigotimes_{k=1}^{n} \sigma_{i_k}) \otimes \mathbb{1}_2^{\otimes n} |\psi_+^{[1,n]}\rangle$. It is easy to check that such vectors are orthonormal, so that the state

$$\mathbb{T}_n^{\ddagger} \otimes \mathrm{id}_d^{\otimes n}\big[|\psi_+^{[1,n]}\rangle\langle\psi_+^{[1,n]}|\big] = \sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}} |\psi_{\boldsymbol{i}_{[1,n]}}\rangle\langle\psi_{\boldsymbol{i}_{[1,n]}}|$$

is diagonal with eigenvalues $p_{\boldsymbol{i}_{[1,n]}}$ and associated eigenvectors $|\psi_{\boldsymbol{i}_{[1,n]}}\rangle$. Then, as in (67), the dynamical entropy of the system equals the entropy rate of the Markov source. With T parametrized as in (72), one gets

$$\begin{aligned} \mathfrak{h}_S(\Theta) &= \eta(p_0) + \eta(r) \\ &\quad + 2\left(2p\,\frac{\eta(p+\Delta)+\eta(p-\Delta)}{2} + (1-2p)\eta(p)\right), \end{aligned} \tag{86}$$

where $\eta(x) = -x \ln x$, $\eta(0) := 0$. The associated two-site mutual information reads as

$$\begin{aligned} I(\pi_1;\pi_2) &= 2p(2\eta(p) - \eta(p+\Delta) - \eta(p-\Delta)) \\ &= 4p^2\left(\ln 2 - h\left(\frac{1}{2} + \frac{\Delta}{2p}\right)\right), \end{aligned} \tag{87}$$

where $h(x) := \eta(x) + \eta(1-x)$, $0 \leqslant x \leqslant 1$ is the Shannon binary entropy, which is a decreasing function of x for $\frac{1}{2} \leqslant x \leqslant 1$. Therefore, the two-site correlations increase with Δ and are maximal at $p = \Delta$. Correspondingly, the open-system ALF entropy (86) is a monotonically decreasing function of the chain correlations Δ.

Notice that the conditions (78) show that the reduced dynamics can be P or even CP-divisible while, for $\Delta > 0$, QR does not hold (as a consequence of Proposition 4, see also the following Remark 15). In such case, standard measures of backflow of information based on the dynamical map [3,5], such as the BLP one, would vanish, while the quantity $\Delta\mathfrak{h}_S(\Theta,\mathcal{F}) > 0$, introduced in Definition 2, namely, the ALF entropy results diminished with respect to its maximal value corresponding to the QR regime. Such behavior is depicted in Fig. 3, where we superimpose the regions of parameters corresponding to the QR, CP-divisible, P-divisible, and non-P-divisible regimes. Notice also that the transition between the various regimes is continuous in the parameter Δ, as already observed in the literature on several occasions [36,51,52]. Consider now $\Delta = p = \frac{1}{2}$, namely, the parameters for which the two-site correlations (87) are maximal. Also, from (72) it follows that $r = p_0 = 0$. Then, (86) yields

$$p = \Delta = \tfrac{1}{2} \implies \mathfrak{h}_S(\Theta) = 0, \tag{88}$$

as one would have for a closed finite system. It is instructive to study the dynamical map $\Lambda_n^{\ddagger}$ in the same regime. One obtains the spectrum by setting $p = \Delta = \frac{1}{2}$ in (77), so that $A_{\frac{1}{2},0} = 0$, $B_{\frac{1}{2},0,\frac{1}{2}} = 2$. The eigenvalues are

$$\lambda_n = \frac{1+(-1)^n}{2}, \quad \lambda_n^{(3)} = (-1)^n. \tag{89}$$

They correspond to the dynamical map

$$\Lambda_n^{\ddagger} = \begin{cases} \mathrm{id}_2 & n \text{ even}, \\ \Lambda_1^{\ddagger} & n \text{ odd}, \end{cases} \quad \Lambda_1^{\ddagger} = \frac{\phi_1 + \phi_2}{2}. \tag{90}$$

Note that the dynamics is not algebraically invertible since $\lambda_{2m+1} = 0$. Nevertheless, we can explicitly study the contractivity of the trace norm under Λ_n. In particular, the trace norm $||X||_1 = \mathrm{Tr}\sqrt{X^\dagger X}$ is always contractive between step $2m$ and

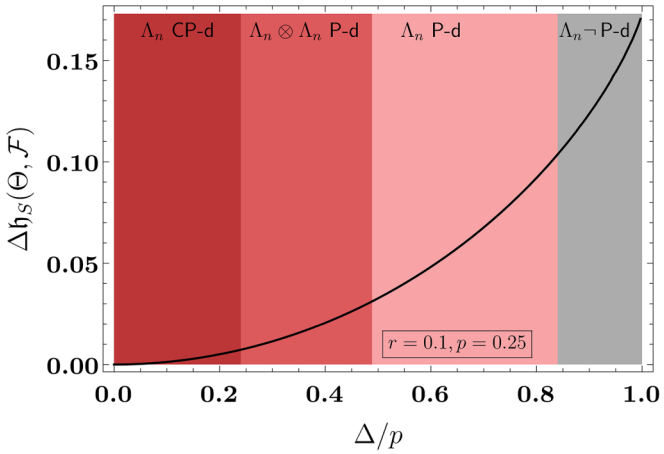


FIG. 3. Entropic measure (43) for the model of Sec. V D 1 as a function of Δ/p for fixed values $r = 0.1$ and $p = 0.25$. The colored regions correspond to different divisibility degrees of the reduced dynamics $\Lambda_n^{\ddagger}$, as determined by conditions (78): in the gray region $\Lambda_n^{\ddagger}$ is not P-divisible and shows distinguishability revivals; in the pink region $\Lambda_n^{\ddagger}$ is P-divisible and does not show revivals, while they can superactivate for $\Lambda_n^{\ddagger} \otimes \Lambda_n^{\ddagger}$; the two-tensor power achieves P-divisibility in the red region, while in the dark-red region $\Lambda_n^{\ddagger}$ is CP-divisible. Notice that QR-Markovianity can be achieved only at $\Delta = 0$, which yields a semigroup evolution.

$2m+1$,

$$||\Lambda^{\ddagger}_{2m+1}[X]||_1 - ||\Lambda^{\ddagger}_{2m}[X]||_1 = ||\Lambda^{\ddagger}_{1}[X]||_1 - ||X||_1 \leqslant 0, \tag{91}$$

since Λ_1 is CPTP. Between steps $2m-1$ and $2m$, instead, we have the converse inequality:

$$||\Lambda^{\ddagger}_{2m}[X]||_1 - ||\Lambda^{\ddagger}_{2m-1}[X]||_1 = ||X||_1 - ||\Lambda^{\ddagger}_{1}[X]||_1 \geqslant 0, \tag{92}$$

so that the trace norm always revives at odd times, signaling backflow of information.

Remark 14. The operational interpretation of Sec. IV allows to better interpret the monotonically decreasing behavior of the open-system ALF entropy (86) as function of the bath correlations Δ. Note that such behavior is also reflected by the divisibility degree of the reduced dynamics. Remarkably, the dynamical map (90) exhibits an extreme backflow of information since the trace norm revives after each odd time, while the dynamical entropy is zero, as one would have for a reversible evolution (see Sec. III). This means that asymptotically, no new information can be gathered by further probing the open system. Such an interpretation can be generalized through inequality (56): for a strongly correlated environment, we expect the dynamics of the system to be affected by strong memory effects. On the other hand, the occurrence of a low, possibly zero, entropy production for a dissipative system is compatible with the interpretation of information having flown back to the open system since, asymptotically, repeated measurements on the system become little informative.

Remark 15. When the classical environment is a Markov process we can write the coarse-grained density matrix as (compare with Remark 10)

$$(\rho_S[\mathcal{X}^{(2)}])_{\boldsymbol{a},\boldsymbol{b}} = \sum_{i_2 i_1} T_{i_2 i_1}\, p_{i_1}\, \omega_S\big(X^{\dagger}_{b_0}\phi_{i_1}\big[X^{\dagger}_{b_1}\big]\phi_{i_1} \circ \phi_{i_2}\big[X^{\dagger}_{b_2}X_{a_2}\big]\phi_{i_1}\big[X_{a_1}\big]X_{a_0}\big).$$

For the concrete case of $T_{i_2 i_1}$ as in (72), we have

$$\begin{aligned}(\rho_S[\mathcal{X}^{(2)}])_{\boldsymbol{a},\boldsymbol{b}} &= \omega_S\big(X^{\dagger}_{b_0}\Lambda_1\big[X^{\dagger}_{b_1}\Lambda_1\big[X^{\dagger}_{b_2}X_{a_2}\big]X_{a_1}\big]X_{a_0}\big) \\ &+ p\,\Delta\,\big(\omega_S\big(X^{\dagger}_{b_0}\phi_1\big[X^{\dagger}_{b_1}\big](\mathrm{id}_2-\phi_3)\big[X^{\dagger}_{b_2}X_{a_2}\big]\phi_1[X_{a_1}]X_{a_0}\big) \\ &+ \omega_S\big(X^{\dagger}_{b_0}\phi_2\big[X^{\dagger}_{b_1}\big](\mathrm{id}_2-\phi_3)\big[X^{\dagger}_{b_2}X_{a_2}\big]\phi_2\big[X_{a_1}\big]X_{a_0}\big)\big).\end{aligned}$$

Thus, Δ, that is the strength of the environment correlations, signals the deviation of the evolution from the QR (and semigroup) regime. Nevertheless, for $\Delta > 0$, the reduced dynamics could be (C)P-divisible, as displayed in Fig. 3.

E. Reduced dynamics in the GNS representation

In the GNS approach to the open-system ALF entropy, discussed in Sec. IV, the measurements affect only π_S and not the commutant π'_S, while the dynamics affects nontrivially also the commutant, as emerges from (55). We now remove the degrees of freedom of the environment by the expectation

$$\begin{aligned}&\langle\sqrt{\rho_S}\otimes\Omega_E|\pi_S\otimes\pi_E(X^{\dagger}\otimes\mathbb{1}_E)[(\widetilde{\mathbb{X}}\otimes\mathrm{id}_E)\circ\mathbb{U}_\Theta]^n[\pi_S\otimes\pi_E(A\otimes\mathbb{1}_E)\pi'_S\otimes\pi_E(B\otimes\mathbb{1}_E)]\,\pi_S\otimes\pi_E(X\otimes\mathbb{1}_E)|\sqrt{\rho_S}\otimes\Omega_E\rangle \\ &\quad= \mathrm{Tr}\big(\Gamma^{\mathbb{X}}_n[(X\otimes\mathbb{1}_d)|\sqrt{\rho_S}\rangle\langle\sqrt{\rho_S}|(X^{\dagger}\otimes\mathbb{1}_d)]A\otimes B\big),\end{aligned} \tag{93}$$

where $A, B, X \in M_d(\mathbb{C})$, $\widetilde{\mathbb{X}} = \mathbb{X}\otimes\mathrm{id}_d$ and where we introduced a CPTP map $\Gamma^{\mathbb{X}}_n$ on $M_d(\mathbb{C})\otimes M_d(\mathbb{C})$, describing the joint dissipative evolution of π_S and π'_S together. From (93), such map is well defined by the choice of a reference state of the form $\omega_S\otimes\omega_E$. Accordingly, as in the previous section, the GNS triple can be identified with the triple $(\pi_S\otimes\pi_E, \mathcal{H}_S\otimes\mathcal{H}_E, |\sqrt{\rho_S}\otimes\Omega_E\rangle)$. In particular, the representations π_S and π'_S amount to factor representations; namely, for $A\in M_d(\mathbb{C})$ (see Example 1)

$$\pi_S(A) = A\otimes\mathbb{1}_d, \quad \pi'_S(A) = \mathbb{1}_d\otimes\overline{A}.$$

The factorization of the $S+E$ cyclic vector, along with the complete positivity of $(\mathbb{U}^{\ddagger}_\Theta\circ(\widetilde{\mathbb{X}}^{\ddagger}\otimes\mathrm{id}_E))^n$, ultimately ensure the complete positivity of $\Gamma^{\mathbb{X}}_n$. We expect that such dissipative bipartite dynamics could help in assessing the lack of Markovianity of the stochastic process, for example, by revealing memory effects that cannot be simply ascertained through the reduced dynamics $\Lambda^{\ddagger}_n$. In the case of the collisional model considered in Sec. V, we fix the system S reference state to be the maximally mixed one $\rho_S = \mathbb{1}/d$. The system GNS vector then corresponds to the maximally entangled state $|\sqrt{\rho_S}\rangle\equiv|\psi^{(d)}_+\rangle = \sum_{i=1}^d |i\otimes i\rangle/\sqrt{d}$. The tensor shift on the chain is implemented by the unitary operator defined by

$$U^{\dagger}_\sigma\,\pi_E\big(\Pi^{[a,b]}_{\boldsymbol{i}^{[a,b]}}\big)|\Omega_E\rangle := \pi_E\big(\Pi^{[a+1,b+1]}_{\boldsymbol{i}^{[a,b]}}\big)|\Omega_E\rangle, \tag{94}$$

and such that $U^{\dagger}_\sigma|\Omega_E\rangle = U_\sigma|\Omega_E\rangle = |\Omega_E\rangle$. Furthermore, we implement unitarily the automorphism Θ by the operator

$$\begin{aligned}&U^{\dagger}_\Theta\,\pi_S(A)\pi'_S(B)\otimes\pi_E\big(\Pi^{[a,b]}_{\boldsymbol{i}^{[a,b]}}\big)|\psi^{(d)}_+\otimes\Omega_E\rangle \\ &\quad:= \sum_k U^{\dagger}_k A\otimes\overline{U_k}^{\dagger}B\otimes U^{\dagger}_\sigma\pi_E\big(\Pi^{(0)}_k\Pi^{[a,b]}_{\boldsymbol{i}^{[a,b]}}\big)|\psi^{(d)}_+\otimes\Omega_E\rangle.\end{aligned} \tag{95}$$

We then explicitly consider the evolution intertwined by measurements on the open system as appears in the coarse-grained density matrix, and evaluate explicitly (93):

$$\begin{aligned}&\langle\psi^{(d)}_+\otimes\Omega_E|\pi_S\otimes\pi_E(X^{\dagger}\otimes\mathbb{1}_E)((\widetilde{\mathbb{X}}\otimes\mathrm{id}_E)\circ\mathbb{U}_\Theta)^n \\ &\quad[\pi_S\otimes\pi_E(A\otimes\mathbb{1}_E)\pi'_S\otimes\pi_E(B\otimes\mathbb{1}_E)]\pi_S \\ &\quad\otimes\,\pi_E(X\otimes\mathbb{1}_E)|\psi^{(d)}_+\otimes\Omega_E\rangle\end{aligned}$$

$$= \sum_{k_1\ldots k_n} p_{k_1\ldots k_n} \mathrm{Tr}\big(A\otimes \overline{B}\,\phi^{\ddagger}_{k_n}\otimes\overline{\phi}^{\ddagger}_{k_n}\circ\widetilde{\mathbb{X}}^{\ddagger}\circ\cdots\circ\phi^{\ddagger}_{k_1}$$
$$\otimes\,\overline{\phi}^{\ddagger}_{k_1}\circ\widetilde{\mathbb{X}}^{\ddagger}[X\otimes\mathbb{1}_d P^{(d)}_{+}X^{\dagger}\otimes\mathbb{1}_d]\big), \tag{96}$$

where $P^{(d)}_{+}=|\psi^{(d)}_{+}\rangle\langle\psi^{(d)}_{+}|$. Accordingly, the dissipative GNS dynamics in the Schrödinger picture, resulting from the iterated measurement scheme and unitary evolution, is described by the following CPTP map:

$$\Gamma^{\mathbb{X}}_n := \sum_{k_1\ldots k_n} p_{k_1\ldots k_n}\phi^{\ddagger}_{k_n}\otimes\overline{\phi}^{\ddagger}_{k_n}\circ\widetilde{\mathbb{X}}^{\ddagger}\circ\cdots\circ\phi^{\ddagger}_{k_1}\otimes\overline{\phi}^{\ddagger}_{k_1}\circ\widetilde{\mathbb{X}}^{\ddagger}, \tag{97}$$

where again $\widetilde{\mathbb{X}}^{\ddagger}=\mathbb{X}^{\ddagger}\otimes\mathrm{id}_d$. The detailed derivation of (97) is reported in Appendix H. The choice $\mathbb{X}=\mathrm{id}_d$, in particular, yields

$$\Gamma^{\mathrm{id}}_n = \sum_{\boldsymbol{k}_{[1,n]}} p_{\boldsymbol{k}_{[1,n]}}\phi^{\ddagger}_{\boldsymbol{k}_{[1,n]}}\otimes\overline{\phi}^{\ddagger}_{\boldsymbol{k}_{[1,n]}}, \tag{98}$$

which is the intrinsic reduced evolution in the GNS space, having traced away the environment degrees of freedom. Formally, (98) corresponds to the joint evolution of two subsystems coupled in a common environment. Notice that the partial trace over the commutant gives

$$\mathrm{Tr}_{II}\big(\Gamma^{\mathrm{id}}_n[X\otimes Y]\big) = \sum_{\boldsymbol{k}_{[1,n]}} p_{\boldsymbol{k}_{[1,n]}}\phi^{\ddagger}_{\boldsymbol{k}_{[1,n]}}[X]\mathrm{Tr}(Y)$$
$$= \Lambda^{\ddagger}_n[\mathrm{Tr}(Y)X] = \Lambda^{\ddagger}_n[\mathrm{Tr}_{II}(X\otimes Y)].$$

Hence, consistently, Γ^{id}_n provides a dissipative CP dilation of the dynamical map $\Lambda^{\ddagger}_n$ of the system.

F. Pauli GNS reduced dynamics

We now consider the environment to be a classical Markov chain, as in Sec. V D 1:

$$p_{\boldsymbol{k}_{[1,n]}} = T_{k_n k_{n-1}}\ldots T_{k_2 k_1}p_{k_1}\ ,$$

the open quantum system as a qubit $d=2$ and the environment as consisting of a one-dimensional lattice of classical four-level systems $D=d^2=4$. Furthermore, we shall choose the stochastic matrix T as in (72), while setting the unitaries $U_k=\sigma_k$, $k=0,\ldots,3$. Accordingly, $\Gamma^{\mathrm{id}}_n:M_4(\mathbb{C})\to M_4(\mathbb{C})$ defined in (98) becomes a two-qubit Pauli map, namely, for all $n\geqslant 0$ it is diagonal in the Pauli matrices

$$\Gamma^{\mathrm{id}}_n[\sigma_\alpha\otimes\sigma_\beta]=\gamma^{(\alpha,\beta)}_n\sigma_\alpha\otimes\sigma_\beta,\quad \alpha,\beta=0,\ldots,3. \tag{99}$$

The spectrum of Γ^{id}_n follows a similar rule than that of $\Lambda^{\ddagger}_n$:

$$\gamma^{(\alpha,\beta)}_n = \big(p_0+p\big(\mu^{(\alpha,\beta)}_1+\mu^{(\alpha,\beta)}_2\big)+r\mu^{(\alpha,\beta)}_3\big)\gamma^{(\alpha,\beta)}_{n-1}$$
$$+\ \Delta\, p\big(\mu^{(\alpha,\beta)}_1-\mu^{(\alpha,\beta)}_2\big)^2\gamma^{(\alpha,\beta)}_{n-2}, \tag{100}$$

where $\phi_k\otimes\phi_k[\sigma_\alpha\otimes\sigma_\beta]=\mu^{(\alpha,\beta)}_k\sigma_\alpha\otimes\sigma_\beta$, $\mu^{(\alpha,\beta)}_k:=\mu^{(\alpha)}_k\mu^{(\beta)}_k$. The solutions of (100) are the same as that of (76). The eigenvalues are associated to their respective eigenvectors $\sigma_\alpha\otimes\sigma_\beta$ according to the following 4×4 matrix:

$$G_n := \big[\gamma^{(\alpha,\beta)}_n\big] = \begin{pmatrix} 1 & \lambda_n & \lambda_n & \lambda^{(3)}_n \\ \lambda_n & 1 & \lambda^{(3)}_n & \lambda_n \\ \lambda_n & \lambda^{(3)}_n & 1 & \lambda_n \\ \lambda^{(3)}_n & \lambda_n & \lambda_n & 1 \end{pmatrix}. \tag{101}$$

Note that all the Bell-diagonal states, which are linear combinations of $\sigma_\alpha\otimes\sigma_\alpha$, as well as the GNS reduced initial state $P^{(d)}_{+}$, are invariant under Γ^{id}_n. The spectrum of the intertwining propagators $\Gamma^{\mathrm{id}}_{n,n-1}=\Gamma^{\mathrm{id}}_n\circ(\Gamma^{\mathrm{id}}_{n-1})^{-1}$ is determined multiplicatively,

$$\Gamma^{\mathrm{id}}_{n,n-1}[\sigma_\alpha\otimes\sigma_\beta]=\gamma^{(\alpha,\beta)}_{n,n-1}\sigma_\alpha\otimes\sigma_\beta,\quad \gamma^{(\alpha,\beta)}_{n,n-1}=\frac{\gamma^{(\alpha,\beta)}_n}{\gamma^{(\alpha,\beta)}_{n-1}},$$

and given by

$$G_{n,n-1} := \big[\gamma^{(\alpha,\beta)}_{n,n-1}\big]$$
$$= \begin{pmatrix} 1 & \lambda_{n,n-1} & \lambda_{n,n-1} & \lambda^{(3)}_{n,n-1} \\ \lambda_{n,n-1} & 1 & \lambda^{(3)}_{n,n-1} & \lambda_{n,n-1} \\ \lambda_{n,n-1} & \lambda^{(3)}_{n,n-1} & 1 & \lambda_{n,n-1} \\ \lambda^{(3)}_{n,n-1} & \lambda_{n,n-1} & \lambda_{n,n-1} & 1 \end{pmatrix},$$

where $\lambda_{n,n-1}=\lambda_n/\lambda_{n-1}$, $\lambda^{(3)}_{n,n-1}=\lambda^{(3)}_n/\lambda^{(3)}_{n-1}$ are the eigenvalues of $\Lambda_{n,n-1}$. Then, as derived in Appendix I, the action of Γ^{id}_n can be recast as

$$\Gamma^{\mathrm{id}}_n[X]=\sum_{\alpha} q^{(\alpha)}_n\,\sigma_\alpha\otimes\sigma_\alpha\,X\,\sigma_\alpha\otimes\sigma_\alpha \tag{102}$$

with

$$q^{(0)}_n=\tfrac{1}{4}\big(1+2\lambda_n+\lambda^{(3)}_n\big),\quad q^{(1)}_n=q^{(2)}_n=\tfrac{1}{4}\big(1-\lambda^{(3)}_n\big),$$
$$q^{(3)}_n=\tfrac{1}{4}\big(1+2\lambda_n-\lambda^{(3)}_n\big).$$

We now aim at characterizing the divisibility properties of the GNS discrete dynamics Γ^{id}_n.

Proposition 7. Let $\Gamma_n:M_4(\mathbb{C})\to M_4(\mathbb{C})$ be the Pauli map given by

$$\Gamma_n[X]=\sum_{\alpha=0}^{3} q^{(\alpha)}_n\,\sigma_\alpha\otimes\sigma_\alpha\,X\,\sigma_\alpha\otimes\sigma_\alpha,\quad X\in M_4(\mathbb{C}).$$

Then, consider the one-qubit CPU map obtained as

$$\Lambda_n\circ\mathrm{Tr}_{II}[X]=\mathrm{Tr}_{II}(\Gamma_n[X]).$$

where Tr_{II} is the partial trace over the second qubit. Then, the following conditions are equivalent:

(i) Λ_n is CP-divisible;
(ii) Γ_n is P-divisible;
(iii) Γ_n is CP-divisible.

Proof. Take $X\in M_4(\mathbb{C})$ and let $Y=\Gamma_n[X]$. The partial trace of X over the second qubit reads as $\mathrm{Tr}_{II}(X)=\mathrm{Tr}_{II}(\Gamma^{-1}_n[Y])$ but also

$$\mathrm{Tr}_{II}(X)=\Lambda^{-1}_n\circ\Lambda_n[\mathrm{Tr}_{II}(X)]=\Lambda^{-1}_n[\mathrm{Tr}_{II}(\Gamma_n[X])]$$
$$=\Lambda^{-1}_n[\mathrm{Tr}_{II}(Y)].$$

Hence, for all $Y \in M_4(\mathbb{C})$, $\Lambda_n^{-1}[\mathrm{Tr}_{II}(Y)] = \mathrm{Tr}_{II}(\Gamma_n^{-1}[Y])$. Applying Λ_m, $m \geqslant n$, to both sides, yields

$$\Lambda_{m,n}[\mathrm{Tr}_{II}(Y)] = \mathrm{Tr}_{II}(\Gamma_{m,n}[Y]), \quad m \geqslant n. \tag{103}$$

Intertwiners of Pauli maps are Pauli maps, so that

$$\Gamma_{n,n-1}[X] = \sum_{\alpha=0}^{3} q_{n,n-1}^{(\alpha)} \sigma_\alpha \otimes \sigma_\alpha X \sigma_\alpha \otimes \sigma_\alpha, \tag{104}$$

and $\mathrm{Tr}_{II}(\Gamma_{n,n-1}[X]) = \Lambda_{n,n-1}[\mathrm{Tr}_{II} X]$, $\forall\, n \geqslant 1$, with

$$M_2(\mathbb{C}) \ni Y \longmapsto \Lambda_{n,n-1}[Y] = \sum_{\alpha=0}^{3} q_{n,n-1}^{(\alpha)} \sigma_\alpha\, Y\, \sigma_\alpha. \tag{105}$$

Positivity of $\Gamma_{n,n-1}$, $n \geqslant 1$, is expressed by the condition

$$\mathrm{Tr}(Q\, \Gamma_{n,n-1}[P]) \geqslant 0, \quad \forall\, P, Q \geqslant 0 \tag{106}$$

where P, Q can be chosen as projectors in $M_4(\mathbb{C})$. Pick separable projectors

$$P = Q = |+\rangle\langle+| \otimes |0\rangle\langle 0|, \quad |\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}, \tag{107}$$

with $\sigma_3|0\rangle = |0\rangle$, $\sigma_3|1\rangle = -|1\rangle$. One readily finds

$$\mathrm{Tr}(Q\, \Gamma_{n,n-1}[P]) = q_{n,n-1}^{(0)} \geqslant 0. \tag{108}$$

The choice

$$P = |+\rangle\langle+| \otimes |0\rangle\langle 0|, \quad Q = |-\rangle\langle-| \otimes |0\rangle\langle 0|, \quad PQ = 0, \tag{109}$$

yields instead the condition

$$\mathrm{Tr}(Q\, \Gamma_{n,n-1}[P]) = q_{n,n-1}^{(3)} \geqslant 0. \tag{110}$$

Similarly, one finds $q_{n,n-1}^{(\alpha)} \geqslant 0$, $\alpha = 0, \ldots, 3$, so that $(ii) \Rightarrow (i) \Rightarrow (iii)$. Clearly, $(iii) \Rightarrow (ii)$. ■

Interestingly, for the model under consideration, the intrinsic evolution Γ_n^{id} in the GNS construction can display memory effects not present in the one-qubit reduced dynamics Λ_n.

Corollary 3. Let Δ be such that $\Lambda_n^{\ddagger}$ is P-divisible but not CP-divisible. Then, the evolution of the system does not display backflow of information in terms of revivals. On the other hand, as the two-qubit dilation Γ_n^{id} is not P divisible, it will show it.

Proof. P-divisibility of the reduced qubit dynamics ensures that

$$||\Lambda_n^{\ddagger}[X]||_1 \leqslant ||\Lambda_m^{\ddagger}[X]||_1, \quad \forall\, n \geqslant m,\ \forall\, X \in M_2(\mathbb{C}).$$

On the other hand, as the two-qubit dilation Γ_n^{id} is not P-divisible and invertible, it certainly breaks monotonicity, namely, there exists $X \in M_4(\mathbb{C})$ and $n > m \in \mathbb{N}$ such that $\|\Gamma_n^{\mathrm{id}}[X]\|_1 > \|\Gamma_m^{\mathrm{id}}[X]\|_1$. ■

VI. CONCLUSIONS

In this work, we have extended the ALF entropy which has mainly been applied in reversible, quantum many-body settings to dissipative finite-dimensional quantum systems and named it *open-system ALF entropy*. The first part of the paper was devoted to presenting the general framework for the newly extended open-system ALF entropy and its operational interpretation. In Sec. III, we introduced the positive quantity (43) signaling the deviation of the entropy rate from the QR regime. We proposed the latter as an entropic measure of information flowing from the environment back into the open quantum system which takes into account the full multitime statistics. The latter operational interpretation is justified by the analysis in the GNS representation. In the second part of the paper, we have explicitly computed the open-system ALF entropy for a finite-level quantum system coupled to a classical stationary chain; in particular, we found conditions under which the ALF entropy equals the mean entropy of the environment. This result has a simple but clear information-theoretic interpretation: the ALF entropy measures the average amount of information that can be extracted by probing the open system alone. Such an information rate decreases as the environment becomes more correlated. In the qubit case, this physical effect has been compared to memory effects arising in the reduced qubit dynamics and associated with the lack of (C)P-divisibility, which are likewise governed by the strength of environmental correlations. This also allows for an explicit comparison of our proposal with other standard measures of non-Markovianity.

The open-system ALF entropy, obtained from the coarse-grained matrix $\rho_S[\mathcal{X}^{(n)}]$ built from the system multitime correlation functions, is neatly interpreted when going to the GNS representation which involves a purifying ancillary system. Accordingly, we studied the GNS reduced dynamics obtained by tracing away the environment for the model of Sec. V. Interestingly, the GNS dilation is affected by memory effects that are seen by the reduced dynamics. In particular, it can show superactivation of backflow of information in terms of revivals, as it typically occurs when a tensor product dilation of the reduced dynamics is considered [29,35,36].

In a more general scenario of a system coupled to a quantum spin chain, the above interpretation is supported by an upper bound to the ALF entropy of the open system by the mean entropy of the environment (plus a quantum correction). For a stationary chain, the more correlated it is, the tighter is such bound. This is only a partial result; a thorough study of the ALF entropy in the case of a genuinely quantum environment is still to be developed.

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

APPENDIX A: PROOF OF PROPOSITION 2

We need to recast the system coarse-grained matrix $\rho_S[\mathcal{F}^{(n+1)}]$ for the POVM $\mathcal{F}$ defined in (33) in the form

$$\rho_S[\mathcal{F}^{(n+1)}] = \rho_S \otimes \rho_S \otimes \left(\mathbb{T}_n^{\ddagger} \otimes \mathrm{id}\big[\big|\sqrt{\rho_S^{\otimes n}}\big\rangle\big\langle\sqrt{\rho_S^{\otimes n}}\big|\big]\right).$$

By definition,

$$\mathbb{T}_n\Big[\bigotimes_{k=0}^{n-1} A_k\Big] = \omega_E\Big(\widetilde{\Theta}_n^{\mathbb{Z}E}\Big[\bigotimes_{k=0}^{n-1} A_k^{(k-n)} \otimes \mathbb{1}_E\Big]\Big) \quad \text{(A1)}$$

$$= \omega_E\big(\Theta_1\big[A_0^{(0)} \otimes \Theta_{2,1}\big[A_1^{(1)} \otimes \cdots \Theta_{n,n-1}\big[A_{n-1}^{(n-1)} \otimes \mathbb{1}_E\big]\big]\big]\big), \quad \text{(A2)}$$

where we set

$$\widetilde{\Theta}_n^{\mathbb{Z}E} := \prod_{j=1}^{n} (\Theta_{j,j-1} \circ \sigma_S \otimes \mathrm{id}_E), \quad \text{(A3)}$$

and σ_S denotes the shift on $M_d^{\mathbb{Z}}(\mathbb{C})$. From (A1), $\mathbb{T}_n$ is manifestly completely positive and unital. Notice that $\mathbb{T}_1[A_0] = \omega_E(\Theta_1(A_0^{(0)} \otimes \mathbb{1}_E)) = \Lambda_1[A_0]$. Moreover, for sake of brevity, the positioning (upper indices) of system operators along the S chain in (A2) has been chosen in accordance with how many times the shift has acted after the automorphism Θ made the 0th site system operators interact with the environment; indeed,

$$\Theta_{n,n-1} \circ (\sigma_S \otimes \mathrm{id}_E)\big[A_0^{(-n)} \otimes \cdots \otimes A_{n-1}^{(-1)}\big] := A_0^{(-n+1)} \otimes \cdots \otimes \Theta_{n,n-1}\big[A_{n-1}^{(0)} \otimes \mathbb{1}_E\big].$$

Then, the action of the automorphism $\Theta_{n,n-1}[A_{n-1}^{(0)} \otimes \mathbb{1}_E] = \widetilde{A}^{(0)} \otimes B_E$ is such that

$$\sigma_S \otimes \mathrm{id}_E\big[A_0^{(-n+1)} \otimes \cdots \otimes \Theta_{n,n-1}\big[A_{n-1}^{(0)} \otimes \mathbb{1}_E\big]\big] = A_0^{(-n+2)} \otimes \cdots \otimes A_{n-2}^{(0)} \otimes \widetilde{A}_{n-1}^{(1)} \otimes B_E = A_0^{(-n+2)} \otimes \cdots \otimes A_{n-2}^{(0)} \otimes \Theta_{n,n-1}\big[A_{n-1}^{(1)} \otimes \mathrm{id}_E\big].$$

One thus gets

$$(\Theta_{n-1,n-2} \circ (\sigma_S \otimes \mathrm{id}_E)) \circ (\Theta_{n,n-1} \circ (\sigma_S \otimes \mathrm{id}_E))\big[A_0^{(-n)} \otimes \cdots \otimes A_{n-1}^{(-1)}\big] = A_0^{(-n+2)} \otimes \cdots \Theta_{n-1,n-2}\big[A_{n-2}^{(0)} \otimes \Theta_{n,n-1}\big[A_{n-1}^{(1)} \otimes \mathrm{id}_E\big]\big].$$

The elements of the coarse-grained density matrix with respect to the POVM (33) read as

$$\rho_S[\mathcal{F}^{(n+1)}]_{\boldsymbol{a},\boldsymbol{b}} = \omega_S \otimes \omega_E\big(F_{b_0b_0'}^{\dagger} \otimes \mathbb{1}_E\, \Theta_1\big[F_{b_1b_1'}^{\dagger} \otimes \mathbb{1}_E \ldots \Theta_{n,n-1}\big[F_{b_nb_n'}^{\dagger} F_{a_na_n'} \otimes \mathbb{1}_E\big] \ldots F_{a_1a_1'} \otimes \mathbb{1}_E\big] F_{a_0a_0'} \otimes \mathbb{1}_E\big)$$

$$= r_{a_n}\delta_{a_nb_n} r_{a_0'}\delta_{a_0'b_0'} \Big(\prod_{m=0}^{n-1} \sqrt{r_{a_m} r_{b_m}}\Big) \mathrm{Tr}\big(R_{a_0b_0}\, \omega_E\big(\Theta_1\big[R_{b_1b_1'}^{\dagger} \otimes \mathbb{1}_E \ldots \Theta_{n,n-1}\big[R_{a_n'b_n'}^{\dagger} \otimes \mathbb{1}_E\big] \ldots R_{a_1a_1'} \otimes \mathbb{1}_E\big]\big)\big), \quad \text{(A4)}$$

with multi-indices $\boldsymbol{a} = (a_0a_0', \ldots, a_na_n')$, $\boldsymbol{b} = (b_0b_0', \ldots, b_nb_n')$ and where $R_{ab} = |r_a\rangle\langle r_b|$ are the matrix units corresponding to the eigenbasis of ρ_S; they satisfy $R_{ab}^{\dagger} = R_{ba}$, $R_{ab}R_{cd} = \delta_{bc}R_{ad}$. We shall now obtain another expression for (A4) in terms of the CPU map (A1). To this end, we split the proof in several steps.

Step 1. First, we introduce a state functional $\Omega_S^{\mathbb{Z}\mathbb{Z}} : M_d^{\mathbb{Z}}(\mathbb{C}) \otimes M_d^{\mathbb{Z}}(\mathbb{C}) \to \mathbb{C}$ on the doubled S chain, whose local restrictions are the purified vector states corresponding to the tensor products of the system S state; namely,

$$\Omega_S^{\mathbb{Z}\mathbb{Z}}(X^{[a,a+n]}) = \big\langle \sqrt{\rho_S^{\otimes[1,n]}} \big| X^{[a,a+n]} \big| \sqrt{\rho_S^{\otimes[1,n]}} \big\rangle, \quad \text{(A5)}$$

$$X^{[a,a+n]} \in M_d^{[a,a+n]}(\mathbb{C}) \otimes M_d^{[a,a+n]}(\mathbb{C}),$$

$$\big|\sqrt{\rho_S^{\otimes[1,n]}}\big\rangle = \sum_{\boldsymbol{a}_{[1,n]}} \sqrt{r_{\boldsymbol{a}_{[1,n]}}}\, |r_{\boldsymbol{a}_{[1,n]}} \otimes r_{\boldsymbol{a}_{[1,n]}}\rangle, \quad \text{(A6)}$$

where $r_{\boldsymbol{a}_{[1,n]}} = \prod_{k=1}^{n} r_{a_k}$. Consider then the compound dynamical system

$$\big(M_d^{\mathbb{Z}}(\mathbb{C}) \otimes M_d^{\mathbb{Z}}(\mathbb{C}) \otimes \mathcal{A}_E,\ \Omega_S^{\mathbb{Z}\mathbb{Z}} \otimes \omega_E,\ \widetilde{\Theta}_n^{\mathbb{Z}\mathbb{Z}E}\big),$$

$$\text{where} \quad \widetilde{\Theta}_n^{\mathbb{Z}\mathbb{Z}E} := \prod_{k=1}^{n} (\Theta_{k,k-1} \otimes \mathrm{id}_{\mathbb{Z}} \circ \sigma_S \otimes \sigma_S \otimes \mathrm{id}_E). \quad \text{(A7)}$$

Introducing the operators

$$Z_{LR}^{[-n,-1]} \otimes \mathbb{1}_E, \quad Z_{LR}^{[-n,-1]} := Z_L^{[-n,-1]} \otimes Z_R^{[-n,-1]}, \quad \text{(A8)}$$

$$Z_L^{[-n,-1]} := \bigotimes_{k=1}^{n} R_{a_k'b_k'}^{\dagger(k-n-1)}, \quad Z_R^{[-n,-1]} := \bigotimes_{l=0}^{n-1} R_{a_lb_l}^{\dagger(l-n)}, \quad \text{(A9)}$$

the definition in (A2) of the CPU map $\mathbb{T}_n$ yields

$$\Omega_S^{\mathbb{Z}\mathbb{Z}} \otimes \omega_E\big(\widetilde{\Theta}_n^{\mathbb{Z}\mathbb{Z}E}\big[Z_{LR}^{[-n,-1]} \otimes \mathbb{1}_E\big]\big) = \Omega_S^{\mathbb{Z}\mathbb{Z}}\big(\mathbb{T}_n\big[Z_L^{[-n,-1]}\big] \otimes Z_R^{[0,n-1]}\big). \quad \text{(A10)}$$

Step 2. Since $Z_{LR}^{[-n,-1]}$ is supported by the double-chain sites $-n \leqslant j \leqslant -1$, the action of the nth power of the shift $\sigma_S \otimes \sigma_S$ makes the operator $\mathbb{T}_n[Z_L^{[-n,-1]}] \otimes Z_R^{[0,n-1]}$ supported by the interval $[0, n-1]$. Then, in the Schrödinger picture, using the dual $\mathbb{T}_n^{\ddagger}$, one can rewrite the expectation on the right-hand side of (A10) as

$$\Omega_S^{\mathbb{Z}\mathbb{Z}}\big(\mathbb{T}_n\big[Z_L^{[-n,-1]}\big] \otimes Z_R^{[0,n-1]}\big) = \mathrm{Tr}_{[-n,-1]}\big(\mathbb{T}_n^{\ddagger} \otimes \mathrm{id}_d^{\otimes n}\big[\big|\sqrt{\rho_S^{\otimes[0,n-1]}}\big\rangle\big\langle\sqrt{\rho_S^{\otimes[0,n-1]}}\big|\big] Z_R^{[-n,-1]} \otimes Z_R^{[0,n-1]}\big). \quad \text{(A11)}$$

On the other hand, the left-hand side of (A10) reads as

$$\Omega_S^{\mathbb{Z}\mathbb{Z}} \otimes \omega_E\big(\widetilde{\Theta}_n^{\mathbb{Z}\mathbb{Z}E}\big[Z_{LR}^{[-n,-1]} \otimes \mathbb{1}_E\big]\big) = \Omega_S^{\mathbb{Z}\mathbb{Z}} \otimes \omega_E\big(\Theta_n^{\mathbb{Z}E}\big[Z_L^{[-n,-1]} \otimes \mathbb{1}_E\big] \otimes Z_R^{[0,n-1]}\big)$$
$$= \big\langle \sqrt{\rho_S^{\otimes[0,n-1]}} \big| \omega_E\big(\widetilde{\Theta}_n^{\mathbb{Z}E}\big[Z_L^{[-n,-1]} \otimes \mathbb{1}_E\big]\big) \otimes Z_R^{[0,n-1]} \big| \sqrt{\rho_S^{\otimes[0,n-1]}} \big\rangle.$$

By explicitly using the spectral form of the vector state in (A6), one gets

$$\Omega_S^{\mathbb{Z}\mathbb{Z}} \otimes \omega_E\big(\widetilde{\Theta}_n^{\mathbb{Z}\mathbb{Z}E}\big[Z_{LR}^{[-n,-1]} \otimes \mathbb{1}_E\big]\big) = \sum_{\substack{\boldsymbol{i}_{[0,n-1]} \\ \boldsymbol{j}_{[0,n-1]}}} \prod_{m=0}^{n-1} \sqrt{r_{i_m} r_{j_m}}\, \mathrm{Tr}\left(\bigotimes_{q=0}^{n-1} R^{(q)}_{i_q j_q}\, \omega_E\big(\widetilde{\Theta}_n^{\mathbb{Z}E}\big[Z_L^{[-n,-1]} \otimes \mathbb{1}_E\big]\big)\right) \mathrm{Tr}\left(\bigotimes_{p=0}^{n-1} R^{(p)}_{i_p j_p}\, Z_R^{[0,n-1]}\right).$$

Using the expression for $Z_R^{[0,n-1]}$ in (A9) yields $\mathrm{Tr}(\bigotimes_{p=0}^{n-1} R^{(p)}_{i_p j_p} Z_R^{[0,n-1]}) = \prod_{p=0}^{n-1} \delta_{i_p a_p}\, \delta_{j_p b_p}$, so that

$$\Omega_S^{\mathbb{Z}\mathbb{Z}} \otimes \omega_E\big(\widetilde{\Theta}_n^{\mathbb{Z}\mathbb{Z}E}\big(Z_{LR}^{[-n,-1]} \otimes \mathbb{1}_E\big)\big) = \left(\prod_{m=0}^{n-1} \sqrt{r_{a_m} r_{b_m}}\right) \mathrm{Tr}_{[0,n-1]}\left(\bigotimes_{l=0}^{n-1} R^{(l)}_{a_l b_l}\, \omega_E\big(\Theta_1\big[R^{\dagger(0)}_{a'_1,b'_1}\right.$$
$$\left. \otimes\, \Theta_{2,1}\big[R^{\dagger(1)}_{a'_2,b'_2} \ldots \Theta_{n,n-1}\big[R^{\dagger(n-1)}_{a'_n b'_n} \otimes \mathbb{1}_E\big]\big]\big]\big)\right). \tag{A12}$$

Step 3. The trace over the local subalgebras $M_d^{\otimes n}(\mathbb{C})$ in (A12) can be converted into a trace over the single algebra $M_d(\mathbb{C})$ at time step 0; namely,

$$\Omega_S^{\mathbb{Z}\mathbb{Z}} \otimes \omega_E\big(\widetilde{\Theta}_n^{\mathbb{Z}\mathbb{Z}E}\big[Z_{LR}^{[-n,-1]} \otimes \mathbb{1}_E\big]\big) = \left(\prod_{m=0}^{n-1} \sqrt{r_{a_m} r_{b_m}}\right) \mathrm{Tr}\big(R_{a_0 b_0}\, \omega_E\big(\Theta_1\big[R^{\dagger}_{b_1 b'_1} \otimes \mathbb{1}_E\, \Theta_{2,1}\big[\ldots \Theta_{n-1,n-2}\big[R^{\dagger}_{b_{n-1} b'_{n-1}}$$
$$\otimes\, \mathbb{1}_E\, \Theta_{n,n-1}\big(R^{\dagger}_{a'_n b'_n} \otimes \mathbb{1}_E\big) R_{a_{n-1} a'_{n-1}} \otimes \mathbb{1}_E\big] \ldots\big] R_{a_1 a'_1} \otimes \mathbb{1}_E\big]\big)\big).$$

This fact is a consequence of the following equality:

$$\mathrm{Tr}_{[0,n-1]}\left(\bigotimes_{l=0}^{n-1} R^{(l)}_{a_l b_l}\, \Theta_1\big[R^{\dagger(0)}_{a'_1,b'_1} \otimes \Theta_{2,1}\big[R^{\dagger(1)}_{a'_2,b'_2} \ldots \otimes \Theta_{n,n-1}\big[R^{\dagger(n-1)}_{a'_n b'_n} \otimes E\big]\big]\big]\right) \tag{A13}$$
$$= \mathrm{Tr}\big(R_{a_0 b_0}\, \Theta_1\big[R^{\dagger}_{b_1 b'_1} \otimes \mathbb{1}_E\, \Theta_{2,1}\big[R^{\dagger}_{b_2 b'_2} \otimes \mathbb{1}_E\, \Theta_{3,2}[\ldots \Theta_{n-1,n-2}\big[R^{\dagger}_{b_{n-1} b'_{n-1}} \otimes \mathbb{1}_E\, \Theta_{n,n-1}\big[R^{\dagger}_{a'_n b'_n} \otimes E\big] R_{a_{n-1} a'_{n-1}} \otimes \mathbb{1}_E\big]$$
$$\ldots R_{a_2 a'_2} \otimes \mathbb{1}_E\big] R_{a_1 a'_1} \otimes \mathbb{1}_E\big]\big]\big), \tag{A14}$$

that holds for any environment operator E and which can be proved by recursion. Indeed, the equality is true for $n = 1$; assume it holds for up to $n = k$ and consider its left-hand side for $n = k + 1$:

$$\mathrm{Tr}_{[0,k]}\left(\bigotimes_{l=0}^{k} R^{(l)}_{a_l b_l}\, \Theta_1\big[R^{\dagger(0)}_{a'_1,b'_1} \otimes \Theta_{2,1}\big[R^{\dagger(1)}_{a'_2,b'_2} \ldots \otimes \Theta_{k,k-1}\big[R^{\dagger(k)}_{a'_k b'_k} \otimes \Theta_{k+1,k}\big[R^{\dagger(k)}_{a'_{k+1} b'_{k+1}} \otimes E\big]\big]\big]\big]\right). \tag{A15}$$

Note that, expanding with respect to the system matrix units, one can always cast the action of the automorphism $\Theta_{k+1,k}$ on $M_d^{(0)}(\mathbb{C}) \otimes \mathcal{A}_E$ as

$$\Theta_{k+1,k}\big[R^{\dagger}_{a'_{k+1} b'_{k+1}} \otimes E\big] = \sum_{ij} R_{ij} \otimes E_{ij}^{a'_{k+1} b'_{k+1}}, \tag{A16}$$

with suitable environment operators $E_{ij}^{a'_{k+1} b'_{k+1}}$ and then shift k times to the right the system operators, so that

$$\Theta_{k+1,k}\big[R^{\dagger(k)}_{a'_{k+1} b'_{k+1}} \otimes E\big] = \sum_{ij} R^{(k)}_{ij} \otimes E_{ij}^{a'_{k+1} b'_{k+1}}. \tag{A17}$$

Tracing over the site k then yields $\mathrm{Tr}_{\{k\}}(R^{(k)}_{a_k b_k} R^{\dagger(k)}_{a'_{k+1} b'_{k+1}} \otimes E) = E_{b_k a_k}^{a'_{k+1} b'_{k+1}}$. Plugging this result in the expression (A15), it becomes

$$\mathrm{Tr}_{[0,k-1]}\left(\bigotimes_{l=0}^{k} R^{(l)}_{a_l b_l}\, \Theta_1\big[R^{\dagger(0)}_{a'_1,b'_1} \otimes \Theta_{2,1}\big[R^{\dagger(1)}_{a'_2,b'_2} \ldots \otimes \Theta_{k,k-1}\big[R^{\dagger(k)}_{a'_k b'_k} \otimes E^{a'_{k+1} b'_{k+1}}_{b_k a_k}\big]\big]\big]\right). \tag{A18}$$

By the recursion assumption, this expression equals

$$\mathrm{Tr}\big(R_{a_0b_0}\Theta_1\big[R^\dagger_{b_1b'_1}\otimes\mathbb{1}_E\;\Theta_{2,1}\big[\dots\Theta_{k-1,k-2}\big[R^\dagger_{b_{k-1}b'_{k-1}}\otimes\mathbb{1}_E\Theta_{k,k-1}\big[R^\dagger_{a'_kb'_k}\otimes E^{a'_{k+1}b'_{k+1}}_{b_ka_k}\big]R_{a_{k-1}a'_{k-1}}\otimes\mathbb{1}_E\big]\dots\big]R_{a_1a'_1}\otimes\mathbb{1}_E\big]\big). \tag{A19}$$

The proof of equality (A14) is concluded by using (A16) with $E=\mathbb{1}_E$ and by observing that

$$R^\dagger_{a'_kb'_k}\otimes E^{a'_{k+1}b'_{k+1}}_{b_ka_k}=R^\dagger_{b_kb'_k}\otimes\mathbb{1}_E\;\Theta_{k+1,k}\big[R^\dagger_{a'_{k+1}b'_{k+1}}\otimes\mathbb{1}_E\big]R_{a_ka'_k}\otimes\mathbb{1}_E. \tag{A20}$$

Step 4. Putting together the equalities (A10), (A11), and (A13) one gets

$$\begin{aligned}\Big(\prod_{m=0}^{n-1}\sqrt{r_{a_m}r_{b_m}}\Big)\,\mathrm{Tr}\big(R_{a_0b_0}\omega_E\big(\Theta_1\big[R^\dagger_{b_1b'_1}\otimes\mathbb{1}_E\;\Theta_{2,1}\big[R^\dagger_{b_2b'_2}\otimes\mathbb{1}_E\Theta_{3,2}\big[\dots\Theta_{n-1,n-2}\big[R^\dagger_{b_{n-1}b'_{n-1}}\otimes\mathbb{1}_E\;\Theta_{n,n-1}\\ \big[R^\dagger_{a'_nb'_n}\otimes\mathbb{1}_E\big]R_{a_{n-1}a'_{n-1}}\otimes\mathbb{1}_E\big]\dots R_{a_2a'_2}\otimes\mathbb{1}_E\big]R_{a_1a'_1}\otimes\mathbb{1}_E\big]\big)\big)\\ =\mathrm{Tr}_{[-n,-1]}\big(\mathbb{T}^\ddagger_n\otimes\mathrm{id}_d^{\otimes n}\big[\big|\sqrt{\rho_S^{\otimes[0,n-1]}}\big\rangle\big\langle\sqrt{\rho_S^{\otimes[0,n-1]}}\big|\big]\,Z_R^{[-n,-1]}\otimes Z_R^{[0,n-1]}\big).\end{aligned} \tag{A21}$$

Plugging this relation in (A4), one gets

$$\rho_S[\mathcal{F}^{(n+1)}]_{\boldsymbol{a},\boldsymbol{b}}=r_{a'_0}\delta_{a'_0b'_0}\,r_{a_n}\delta_{a_nb_n}\,\mathrm{Tr}_{[-n,-1]}\Bigg(\bigotimes_{k=1}^{n}R^{\dagger(k-n-1)}_{a'_kb'_k}\otimes\bigotimes_{l=0}^{n-1}R^{\dagger(l-n)}_{a_lb_l}\,\mathbb{T}^\ddagger_n\otimes\mathrm{id}_d^{\otimes n}\big[\big|\sqrt{\rho_S^{\otimes[0,n-1]}}\big\rangle\big\langle\sqrt{\rho_S^{\otimes[0,n-1]}}\big|\big]\Bigg). \tag{A22}$$

Finally, a more convenient spectrally equivalent expression for the coarse-grained matrix $\rho_S[\mathcal{F}^{(n+1)}]$ can be obtained by unitarily swapping factors in the following way:

$$\rho_S[\mathcal{F}^{(n+1)}]=\sum_{\boldsymbol{a}\boldsymbol{b}}\rho_S[\mathcal{F}^{(n+1)}]_{\boldsymbol{a},\boldsymbol{b}}\,R_{a'_0b'_0}\otimes R_{a_nb_n}\otimes\bigotimes_{k=1}^{n}R^{(k-n-1)}_{a'_kb'_k}\otimes\bigotimes_{l=0}^{n-1}R^{(l-n)}_{a_lb_l}, \tag{A23}$$

so that, by substituting (A22) into the latter, one obtains

$$\rho_S[\mathcal{F}^{(n+1)}]=\rho_S\otimes\rho_S\otimes\big(\mathbb{T}^\ddagger_n\otimes\mathrm{id}_d^{\otimes n}\big[\big|\sqrt{\rho_S^{\otimes n}}\big\rangle\big\langle\sqrt{\rho_S^{\otimes n}}\big|\big]\big).$$

APPENDIX B: PROOF OF PROPOSITION 3

We need to prove that QR $\iff$ $\mathbb{T}_n=\bigotimes_{k=1}^n\Lambda_{k,k-1}$. Consider generic operators $\{A_k\}_{k=0}^n$ and $\{B_k\}_{k=0}^n$ in $M_d(\mathbb{C})$ and evaluate

$$\sum_{\boldsymbol{i}_{[1,n-1]},\,\boldsymbol{j}_{[1,n-1]}}\mathrm{Tr}_{[1,n-1]}\Bigg(\bigotimes_{k=1}^{n-1}R^{\dagger(k)}_{i_kj_k}\;\mathbb{T}_n\big[B_1^\dagger R_{i_1j_1}A_1\otimes B_2^\dagger R_{i_2j_2}A_2\otimes\dots\otimes B_{n-1}^\dagger R_{i_{n-1}j_{n-1}}A_{n-1}\otimes B_n^\dagger A_n\big]\Bigg) \tag{B1}$$

$$\begin{aligned}=\sum_{\boldsymbol{i}_{[1,n-1]},\,\boldsymbol{j}_{[1,n-1]}}\mathrm{Tr}_{[1,n-1]}\Bigg(\bigotimes_{k=1}^{n-1}R^{\dagger(k)}_{i_kj_k}\;\omega_E\big(\Theta_1\big[B_1^\dagger R^{(0)}_{i_1j_1}A_1\otimes\Theta_{2,1}\big[B_2^\dagger R^{(1)}_{i_2j_2}A_2\dots\\ \Theta_{n-1,n-2}\big[B_{n-1}^\dagger R^{(n-2)}_{i_{n-1}j_{n-1}}A_{n-1}\otimes\Theta_{n,n-1}[B_n^\dagger A_n^{(n-1)}\otimes\mathbb{1}_E]\big]\big]\big]\big)\Bigg)\\ =\omega_E\big(\Theta_1\big[B_1^\dagger\otimes\mathbb{1}_E\;\Theta_{2,1}[B_2^\dagger\otimes\mathbb{1}_E\dots\Theta_{n,n-1}[B_n^\dagger A_n\otimes\mathbb{1}_E]\dots A_2\otimes\mathbb{1}_E]A_1\otimes\mathbb{1}_E\big]\big).\end{aligned} \tag{B2}$$

Similarly to (A14), we used the fact that, for a generic operator $Z^{(k)}_{SE}=\sum_{ij}R^{(k)}_{ij}\otimes Z^E_{ab}$, with the system part localized in $M_d^{(k)}(\mathbb{C})$,

$$\sum_{i_kj_k}B_k^\dagger R^{(k-1)}_{i_kj_k}A_k\otimes\mathrm{Tr}_k\big(R^{\dagger(k)}_{i_kj_k}Z^{(k)}_{SE}\big)=\sum_{i_kj_k}B_k^\dagger R^{(k-1)}_{i_kj_k}A_k\otimes Z^E_{i_kj_k}=B_k^\dagger\otimes\mathbb{1}_E\Big(\sum_{i_kj_k}R^{(k-1)}_{i_kj_k}\otimes Z^E_{i_kj_k}\Big)A_k\otimes\mathbb{1}_E=B_k^\dagger\otimes\mathbb{1}_E\;Z^{(k-1)}_{SE}\;A_k\otimes\mathbb{1}_E.$$

In particular, if $\mathbb{T}_n=\bigotimes_k\Lambda_{k,k-1}$, (B1) is also equal to

$$\sum_{\boldsymbol{i}_{[1,n-1]},\,\boldsymbol{j}_{[1,n-1]}}\mathrm{Tr}_{[1,n-1]}\Bigg(\bigotimes_{k=1}^{n-1}R^{\dagger(k)}_{i_kj_k}\;\Lambda_1\big[B_1^\dagger R^{(0)}_{i_1j_1}A_1\big]\otimes\Lambda_{2,1}\big[B_2^\dagger R^{(1)}_{i_2j_2}A_2\big]\cdots\otimes\Lambda_{n-1,n-2}\big[B_{n-1}^\dagger R^{(n-2)}_{i_{n-1}j_{n-1}}A_{n-1}\big]\otimes\Lambda_{n,n-1}\big[B_n^\dagger A_n^{(n-1)}\big]\Bigg) \tag{B3}$$

$$=\;\Lambda_1[B_1^\dagger\,\Lambda_{2,1}[B_2^\dagger\dots\Lambda_{n-1,n-2}[B_{n-1}^\dagger\,\Lambda_{n,n-1}[B_n^\dagger A_n]A_{n-1}]\dots A_2]A_1], \tag{B4}$$

so that, by equating (B2) and (B4), one obtains the QR condition. Conversely, choose $B_k = R^\dagger_{b_k b'_k}$ and $A_k = R_{a_k a'_k}$, $k = 1, \ldots, n$, in (B2) and (B1). Then, set $a_n = b_n$ and consider

$$\begin{aligned}&\mathrm{Tr}\big(R_{a_0 b_0}\omega_E\big(\Theta_1\big[R^\dagger_{b_1 b'_1}\otimes \mathbb{1}_E\ \Theta_{2,1}\left[R^\dagger_{b_2 b'_2}\otimes \mathbb{1}_E \ldots \Theta_{n,n-1}\big[R^\dagger_{a_n b'_n} R_{a_n a'_n}\otimes \mathbb{1}_E\big]\ldots R_{a_2 a'_2}\otimes \mathbb{1}_E\right] R_{a_1 a'_1}\otimes \mathbb{1}_E\big]\big)\big)\\ &\quad = \mathrm{Tr}_{[0,n-1]}\left(\bigotimes_{k=0}^{n-1} R^{\dagger(k)}_{b_k a_k}\ \mathbb{T}_n\left[\bigotimes_{l=1}^{n} R_{b'_l a'_l}\right]\right),\end{aligned} \tag{B5}$$

and, choosing the same operators in (B4) and (B3),

$$\mathrm{Tr}\big(R_{a_0 b_0}\Lambda_1\big[R^\dagger_{b_1 b'_1}\Lambda_{2,1}\left[R^\dagger_{b_2 b'_2}\ldots\Lambda_{n,n-1}\big[R^\dagger_{a_n b'_n} R_{a_n a'_n}\big]\ldots R_{a_2 a'_2}\right] R_{a_1 a'_1}\big]\big) = \mathrm{Tr}_{[0,n-1]}\left(\bigotimes_{k=0}^{n-1} R^{\dagger(k)}_{b_k a_k}\ \bigotimes_{j=1}^{n}\Lambda_{k,k-1}\left[\bigotimes_{l=1}^{n} R_{b'_l a'_l}\right]\right). \tag{B6}$$

If QR holds, (B5) and (B6) are equal. This implies that $\mathbb{T}_n = \bigotimes_{j=1}^n \Lambda_{k,k-1}$. ■

APPENDIX C: PROOF OF PROPOSITION 4

We need to prove that $\Lambda_{k,k-1} = \Lambda_1 =: \Lambda$, $\forall\, k \geqslant 1$. We proceed by induction and first show it for $n = 2$. Suppose that QR holds and consider again the special POVM $\mathcal{F}$ defined in (33), with associated CPU map $\mathbb{F}[X] = \sum_{ij} r_i R_{ji} X R_{ij} = \mathrm{Tr}(\rho_S X)\mathbb{1}_d$. Accordingly, for $n = 2$, one has

$$\rho_S[\mathcal{F}^{(3)}] = \omega_S\otimes\omega_E\big(F^\dagger_{a_0 a'_0}\otimes\mathbb{1}_E\ \Theta_1\big[F^\dagger_{b_1 b'_1}\otimes\mathbb{1}_E\ \Theta_{2,1}\big[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\otimes\mathbb{1}_E\big]F_{a_1 a'_1}\otimes\mathbb{1}_E\big]F_{a_0 a'_0}\otimes\mathbb{1}_E\big) \tag{C1}$$

$$= \omega_S\big(F^\dagger_{a_0 a'_0}\Lambda_1\big[F^\dagger_{b_1 b'_1}\ \Lambda_{2,1}\big[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\big]F_{a_1 a'_1}\big]F_{a_0 a'_0}\big); \tag{C2}$$

the latter is a density matrix acting on $\mathbb{C}^{d^2}\otimes\mathbb{C}^{d^2}\otimes\mathbb{C}^{d^2}$. Consider then the marginal density matrix obtained from by tracing over the first Hilbert space from expression (C1). This amounts to setting $a_0 = b_0$, $a'_0 = b'_0$ and to summing over a_0 and a'_0. Due to invariance of the state under both Θ and the POVM,

$$\begin{aligned}(\mathrm{Tr}_I\rho[\mathcal{F}^{(3)}])_{a_1 a'_1, b_1 b'_1} &= \sum_{a_0 a'_0}\omega_S\otimes\omega_E\big(F^\dagger_{a_0 a'_0}\otimes\mathbb{1}_E\ \Theta_1\big[F^\dagger_{b_1 b'_1}\otimes\mathbb{1}_E\ \Theta_{2,1}\big[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\otimes\mathbb{1}_E\big]F_{a_1 a'_1}\otimes\mathbb{1}_E\big]F_{a_0 a'_0}\otimes\mathbb{1}_E\big)\\ &= \omega_S\otimes\omega_E\big(F^\dagger_{b_1 b'_1}\otimes\mathbb{1}_E\ \Theta_{2,1}\big[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\otimes\mathbb{1}_E\big]F_{a_1 a'_1}\otimes\mathbb{1}_E\big)\\ &= \omega_S\otimes\omega_E\big(F^\dagger_{b_1 b'_1}\otimes\mathbb{1}_E\ \Theta_1\big[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\otimes\mathbb{1}_E\big]F_{a_1 a'_1}\otimes\mathbb{1}_E\big),\end{aligned}$$

where, in the last equality, we used the group assumption $\Theta_{2,1} = \Theta_1 = \Theta$. Thus, we have $\mathrm{Tr}_I(\rho[\mathcal{F}^{(3)}]) = \rho[\mathcal{F}^{(2)}]$. Taking instead the partial trace over the first Hilbert space from expression (C2), we have

$$\begin{aligned}(\mathrm{Tr}_I\rho[\mathcal{F}^{(3)}])_{a_1 a'_1, b_1 b'_1} &= \sum_{a_0, a'_0}\mathrm{Tr}\big(\rho_S\, F^\dagger_{a_0, a'_0}\Lambda_1\big[F^\dagger_{b_1, b'_1}\ \Lambda_{2,1}\big[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\big]F_{a_1 a'_1}\big]F_{a_0 a'_0}\big)\\ &= \mathrm{Tr}\big(\Lambda_1^\ddagger\circ\mathbb{F}^\ddagger[\rho_S]\,F^\dagger_{b_1 b'_1}\ \Lambda_{2,1}\big[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\big]F_{a_1 a'_1}\big) = \mathrm{Tr}\big(\rho_S\, F^\dagger_{b_1, b'_1}\ \Lambda_{2,1}\big[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\big]F_{a_1 a'_1}\big),\end{aligned}$$

where, in the last equality, we used invariance of ρ_S under $\mathbb{F}^\ddagger$ and $\Lambda_1^\ddagger$. Then, $\mathrm{Tr}_I(\rho[\mathcal{F}^{(3)}]) = \rho[\mathcal{F}^{(2)}]$ yields

$$\mathrm{Tr}\big(\rho_S\, F^\dagger_{b_1 b'_1}\ \Lambda_{2,1}\big[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\big]F_{a_1 a'_1}\big) = \mathrm{Tr}\big(\rho_S\, F^\dagger_{b_1 b'_1}\ \Lambda_1\big[F^\dagger_{b_2 b'_2}F_{a_2 a'_2}\big]F_{a_1 a'_1}\big).$$

Taking $a_2 = b_2$, $a'_1 = b'_1$ yields, in particular,

$$r_{a_2} r_{a'_1}\sqrt{r_{a_1} r_{b_1}}\langle r_{a'_2}|\Lambda^\ddagger_{2,1}\big[|r_{a_1}\rangle\langle r_{b_1}|\big]|r_{b'_2}\rangle = r_{a_2} r_{a'_1}\sqrt{r_{a_1} r_{b_1}}\langle r_{a'_2}|\Lambda^\ddagger_1\big[|r_{a_1}\rangle\langle r_{b_1}|\big]|r_{b'_2}\rangle,$$

which implies $\Lambda_{2,1} = \Lambda_1$. Now, suppose $\Lambda_{k,k-1} = \Lambda_1 =: \Lambda$ holds for $k \leqslant n-1$. Choose again the POVM $\mathcal{F}$ as in (33). Due to the assumed invariance of the $\omega_S\otimes\omega_E$ under both Θ and the chosen POVM, the state on the half-spin chain defined through the coarse-grained density matrices $\rho_S[\mathcal{F}^{(n+1)}]$, $n \geqslant 1$, is invariant under the shift to the right. On the other hand, explicit evaluation of $\rho[\mathcal{F}^{(n+1)}]$ yields

$$\rho[\mathcal{F}^{(n+1)}] = \sum_{\substack{a_0 b_0 \ldots a_n b_n\\ a'_0 b'_0 \ldots a'_n b'_n}}\delta_{b_n a_n} r_{a_n}\delta_{b'_0 a'_0} r_{a'_0}\prod_{k=1}^{n}\sqrt{r_{a_{k-1}} r_{b_{k-1}}}\langle r_{a'_k}|\Lambda^\ddagger_{k,k-1}\big[|r_{a_{k-1}}\rangle\langle r_{b_{k-1}}|\big]|r_{b'_k}\rangle\big|r_{a_0} r_{a'_0}\ldots r_{a_n} r_{a'_n}\big\rangle\big\langle r_{b_0} r_{b'_0}\ldots r_{b_n} r_{b'_n}\big|. \tag{C3}$$

Shift invariance means that $\mathrm{Tr}_I \rho[\mathcal{F}^{(n+1)}] = \rho[\mathcal{F}^{(n)}]$. Here, Tr_I denotes the trace on the first two tensor factors in (C3); this amounts to setting $a_0 = b_0$ and $a'_0 = b'_0$ and then summing over a_0, a'_0. Consider thus the entries

$$\begin{aligned}\mathrm{Tr}_I(\rho[\mathcal{F}^{(n+1)}])_{a_1a'_1a_na'_n\ldots b_1a'_1a_nb'_n} &= r_{a_n}\sum_{a'_0} r_{a'_0}\langle r_{a'_1}|\Lambda_1^\ddagger[\rho_S]|r_{b'_1}\rangle\prod_{k=2}^{n}\sqrt{r_{a_{k-1}}r_{b_{k-1}}}\langle r_{a'_k}|\Lambda_{k,k-1}^\ddagger\big[|r_{a_{k-1}}\rangle\langle r_{b_{k-1}}|\big]|r_{b'_k}\rangle\\ &= r_{a_n}r_{a'_1}\prod_{k=2}^{n-1}\sqrt{r_{a_{k-1}}r_{b_{k-1}}}\langle r_{a'_k}|\Lambda^\ddagger\big[|r_{a_{k-1}}\rangle\langle r_{b_{k-1}}|\big]|r_{b'_k}\rangle\langle r_{a'_n}|\Lambda_{n,n-1}^\ddagger\big[|r_{a_{n-1}}\rangle\langle r_{b_{n-1}}|\big]|r_{b'_n}\rangle.\end{aligned}$$

The latter must be equal to

$$\rho[\mathcal{F}^{(n)}]_{a_1a'_1a_na'_n\ldots b_1a'_1a_nb'_n} = r_{a_n}r_{a'_1}\prod_{k=2}^{n}\sqrt{r_{a_{k-1}}r_{b_{k-1}}}\langle r_{a'_k}|\Lambda^\ddagger\big[|r_{a_{k-1}}\rangle\langle r_{b_{k-1}}|\big]|r_{b'_k}\rangle,$$

which yields $\langle r_{a'_n}|\Lambda_{n,n-1}^\ddagger[|r_{a_{n-1}}\rangle\langle r_{b_{n-1}}|]|r_{b'_n}\rangle = \langle r_{a'_n}|\Lambda^\ddagger[|r_{a_{n-1}}\rangle\langle r_{b_{n-1}}|]|r_{b'_n}\rangle$ and, consequently, $\Lambda_{n,n-1} = \Lambda$. ■

APPENDIX D: PROOF OF COROLLARY 1

Consider again the dynamical system (A7) as in the proof of Proposition 2, now for a group of automorphisms $\Theta_n = \Theta^n$. Note that setting $A_{n-1} = \mathbb{1}_d$ in (A2),

$$\mathbb{T}_n[A_0\otimes\cdots\otimes A_{n-2}\otimes\mathbb{1}_d] = \mathbb{T}_{n-1}[A_0\otimes\cdots\otimes A_{n-2}]\otimes\mathbb{1}_d, \tag{D1}$$

yields

$$\begin{aligned}&\Omega_S^{\mathbb{Z}\mathbb{Z}}\otimes\omega_E\big(\mathbb{T}_n\otimes\mathrm{id}_d^{\otimes n}[A_0\otimes\cdots\otimes A_{n-2}\otimes\mathbb{1}_d\otimes A'_0\otimes\cdots\otimes A'_{n-2}\otimes\mathbb{1}_d]\big)\\ &\quad= \mathrm{Tr}_{[-n,-2]}\left(\mathrm{Tr}_{\{n-1\}}\big(\mathbb{T}_n^\ddagger\otimes\mathrm{id}_d^{\otimes n}\big[\big|\sqrt{\rho_S^{[0,n-1]}}\big\rangle\big\langle\sqrt{\rho_S^{[0,n-1]}}\big|\big]\big)\bigotimes_{k=0}^{n-2}A_k\bigotimes_{l=0}^{n-2}A'_l\right)\\ &\quad= \mathrm{Tr}_{[-n,-2]}\left(\mathbb{T}_{n-1}^\ddagger\otimes\mathrm{id}_d^{\otimes n-1}\big[\big|\sqrt{\rho_S^{[0,n-2]}}\big\rangle\big\langle\sqrt{\rho_S^{[0,n-2]}}\big|\big]\bigotimes_{k=0}^{n-2}A_k\bigotimes_{l=0}^{n-2}A'_l\right)\end{aligned}$$

from which

$$\mathrm{Tr}_{\{-1\}}\big(\mathbb{T}_n^\ddagger\otimes\mathrm{id}_d^{\otimes n}\big[\big|\sqrt{\rho_S^{[0,n-1]}}\big\rangle\big\langle\sqrt{\rho_S^{[0,n-1]}}\big|\big]\big) = \mathbb{T}_{n-1}^\ddagger\otimes\mathrm{id}_d^{\otimes n-1}\big[\big|\sqrt{\rho_S^{[0,n-2]}}\big\rangle\big\langle\sqrt{\rho_S^{[0,n-2]}}\big|\big]. \tag{D2}$$

On the other hand, one also notices that

$$\begin{aligned}\mathrm{Tr}_{\{0\}}\left(\rho_S\,\mathbb{T}_n\left[\mathbb{1}_d\otimes\bigotimes_{k=1}^{n-1}A_k\right]\right) &= \omega_S\otimes\omega_E\big(\Theta\big[\mathbb{1}_d^{(0)}\otimes\Theta\big[A_1^{(1)}\otimes\ldots\Theta\big[A_{n-1}^{(n-1)}\otimes\mathbb{1}_E\big]\big]\big]\big)\\ &= \omega_E\big(\Theta\big[A_1^{(1)}\otimes\ldots\Theta\big[A_{n-1}^{(n-1)}\otimes\mathbb{1}_E\big]\big]\big) = \mathbb{T}_{n-1}[A_1\otimes\cdots\otimes A_{n-1}].\end{aligned}$$

Hence,

$$\begin{aligned}&\Omega_S^{\mathbb{Z}\mathbb{Z}}\otimes\omega_E\big(\mathbb{T}_n\otimes\mathrm{id}_d^{\otimes n}[\mathbb{1}_d\otimes A_1\ldots\otimes A_{n-1}\otimes\mathbb{1}_d\otimes A'_1\otimes\cdots\otimes A'_{n-1}]\big)\\ &\quad= \mathrm{Tr}_{[1,n-1]}\left(\big|\sqrt{\rho_S^{[1,n-1]}}\big\rangle\big\langle\sqrt{\rho_S^{[1,n-1]}}\big|\mathrm{Tr}_{\{0\}}\left(\rho_S\,\mathbb{T}_n\left[\mathbb{1}_d\otimes\bigotimes_{k=1}^{n-1}A_k\right]\right)\otimes\bigotimes_{l=1}^{n-1}A'_l\right)\\ &\quad= \mathrm{Tr}_{[1,n-1]}\big(\big|\sqrt{\rho_S^{[1,n-1]}}\big\rangle\big\langle\sqrt{\rho_S^{[1,n-1]}}\big|\mathbb{T}_{n-1}\otimes\mathrm{id}_d^{\otimes n-1}\big[A_1\otimes\cdots\otimes A_{n-1}\otimes A'_1\otimes\cdots\otimes A'_{n-1}\big]\big)\\ &\quad= \mathrm{Tr}_{[-n+1,-1]}\big(\mathbb{T}_{n-1}^\ddagger\otimes\mathrm{id}_d^{\otimes n-1}\big[\big|\sqrt{\rho_S^{[1,n-1]}}\big\rangle\big\langle\sqrt{\rho_S^{[1,n-1]}}\big|\big]A_1\otimes\cdots\otimes A_{n-1}\otimes A'_1\otimes\cdots\otimes A'_{n-1}\big),\end{aligned}$$

from which one has

$$\mathrm{Tr}_{\{-n\}}\big(\mathbb{T}_n^\ddagger\otimes\mathrm{id}_d^{\otimes n}\big[\big|\sqrt{\rho_S^{\otimes[0,n-1]}}\big\rangle\big\langle\sqrt{\rho_S^{\otimes[0,n-1]}}\big|\big]\big) = \mathbb{T}_{n-1}^\ddagger\otimes\mathrm{id}_d^{\otimes n-1}\big[\big|\sqrt{\rho_S^{\otimes[1,n-1]}}\big\rangle\big\langle\sqrt{\rho_S^{\otimes[1,n-1]}}\big|\big]. \tag{D3}$$

We can now prove (40). By exploiting subadditivity of the von Neumann entropy along with (D2) and (D3), one has

$$\begin{aligned}S(\rho[\mathcal{F}^{(n+1)}]) &= 2\,S(\rho_S) + S\big(\mathbb{T}_n^\ddagger\otimes\mathrm{id}_d^{\otimes n}\big[\big|\sqrt{\rho_S^{\otimes n}}\big\rangle\big\langle\sqrt{\rho_S^{\otimes n}}\big|\big]\big) \leqslant 2\,S(\rho_S) + n\,S(\mathbb{T}_1^\ddagger\otimes\mathrm{id}_d[|\sqrt{\rho_S}\rangle\langle\sqrt{\rho_S}|])\\ &= 2\,S(\rho_S) + n\,S\big(\Lambda^\ddagger\otimes\mathrm{id}_d[|\sqrt{\rho_S}\rangle\langle\sqrt{\rho_S}|]\big).\end{aligned} \tag{D4}$$

Hence, by dividing both sides of the inequality by n and taking the limit

$$\lim_n \frac{1}{n+1} S(\rho_S[\mathcal{F}^{(n+1)}]) \leqslant S(\Lambda^\ddagger \otimes \mathrm{id}_d[|\sqrt{\rho_S}\rangle\langle\sqrt{\rho_S}|]). \quad \text{(D5)}$$

Equality is achieved in (D4), and consequently in (D5), if and only if $\mathbb{T}_n = \bigotimes_{k=1}^n \mathbb{T}_1 = \bigotimes_{k=1}^n \Lambda$, corresponding to the QR regime by Propositions 3 and 4. ■

APPENDIX E: ALF ENTROPY IN THE GNS REPRESENTATION

Measurement model and entropy exchange

Consider a POVM on a d-level quantum system

$$\mathcal{X} = \{X_a\}_{a=1}^{|\mathcal{X}|} \subseteq M_d(\mathbb{C}), \quad \text{(E1)}$$

and the associated CPTP map $\mathbb{X}^\ddagger[\rho] = \sum_a X_a \rho X_a^\dagger$. To a POVM $\mathcal{P} = \{P_a\}_a$ made of orthogonal projections $P_a P_b = \delta_{a,b} P_a$, there corresponds a so-called projective (nonselective) measurement

$$\mathbb{P}[Y] = \sum_a P_a Y P_a, \quad \mathbb{P} \circ \mathbb{P} = \mathbb{P}, \quad \text{(E2)}$$

which is itself a projector on $M_d(\mathbb{C})$. Measuring orthogonal projectors P_a can be reduced to distinguishing the positions of a pointer on a grade. Less straightforward is sorting out the different labels a in the case of measurements described by a generic POVM $\mathcal{X} = \{X_a\}_a$, whereby they must be read by measuring generically nonorthogonal components X_a. For this one needs first dilating to a larger Hilbert space over which performing again an orthogonal measurement. This can be seen by means of the following simple model [44]. Let $|0\rangle$ a fixed vector in $\mathbb{C}^{|\mathcal{X}|}$ and define the isometry $V : \mathbb{C}^d \otimes |0\rangle \to \mathbb{C}^d \otimes \mathbb{C}^{|\mathcal{X}|}$,

$$V|\psi\rangle \otimes |0\rangle = \sum_{a=1}^{|\mathcal{X}|} X_a|\psi\rangle \otimes |a\rangle,$$

$$\rho' := V\rho \otimes |0\rangle\langle 0|V^\dagger = \sum_{a,b} X_a \rho X_b^\dagger \otimes |a\rangle\langle b|,$$

$$\mathrm{Tr}_2 \rho' = \mathbb{X}^\ddagger[\rho]. \quad \text{(E3)}$$

The isometry V provides a Stinespring-type dilation of the map $\mathbb{X}^\ddagger$ and can be extended to a unitary on the whole $\mathbb{C}^d \otimes \mathbb{C}^{|\mathcal{X}|}$, so that (E3) can be seen as a joint evolution of system and measurement apparatus [44]. Performing then a projective measurement $\mathbb{P}$, with $P_a = |a\rangle\langle a|$, on the apparatus (which does not influence the state of the system) yields

$$\mathrm{id}_d \otimes \mathbb{P}[\rho'] = \sum_a X_a \rho X_a^\dagger \otimes |a\rangle\langle a|. \quad \text{(E4)}$$

Thus, measuring the projectors $\mathbb{1} \otimes |a\rangle\langle a|$, the postmeasurement state of the system reads as

$$\rho_a' = \frac{X_a \rho X_a^\dagger}{\mathrm{Tr}(\rho X_a^\dagger X_a)},$$

with probability $p_a := \mathrm{Tr}(\rho X_a^\dagger X_a)$. The nonselective version of the measurement corresponds indeed to the action of the CPTP map $\rho \mapsto \mathbb{X}^\ddagger[\rho] = \mathrm{Tr}_2(\rho')$. Instead, tracing over the degrees of freedom of the apparatus yields

$$\rho_2' = \mathrm{Tr}_1(\rho') = \sum_{a,b=1}^{|\mathcal{X}|} \mathrm{Tr}(\rho X_b^\dagger X_a)|a\rangle\langle b| =: \rho[\mathcal{X}]. \quad \text{(E5)}$$

Its diagonal entries correspond to the outcome probabilities relative to the projective measurement performed on the apparatus p_a, while the off-diagonal ones are correlation terms containing further information about the apparatus. In fact, the von Neumann entropy of a density matrix is always less or equal to the Shannon entropy of its diagonal, so that

$$S(\rho[\mathcal{X}]) \leqslant H\big(\{p_a\}_{a=1}^{|\mathcal{X}|}\big), \quad \text{(E6)}$$

the inequality being saturated when the POVM $\mathcal{X}$ is made of orthogonal projections. The quantity $S(\rho[\mathcal{X}])$ is sometimes called *entropy exchange* [44,46]. It represents the entropy increase in the apparatus, initially in the pure state $|0\rangle\langle 0|$ and thus at zero entropy, due to the measurement process that sends it into the mixed state $\rho[\mathcal{X}]$. When the state ρ of the system is pure, so is the compound state ρ' in (E3) emerging from the measuring unitary interaction; then, its two marginals $\mathrm{Tr}_2\rho' = \mathbb{X}^\ddagger[\rho]$ [see (E3)] and $\mathrm{Tr}_1\rho' = \rho[\mathcal{X}]$ [see (E5)] have the same von Neumann entropy, hence the energy growth in the apparatus $S(\rho[\mathcal{X}])$ equals the entropy growth in the system $S(\mathbb{X}^\ddagger[\rho])$ due to the nonselective measurement process $\rho \mapsto \mathbb{X}^\ddagger[\rho]$. We now show, following [45], that the entropy gained by the measurement apparatus is at least as large as the information gained during the measurement. Recalling the definition of quantum relative entropy $S(\rho\|\sigma) = \mathrm{Tr}(\rho\,(\ln\rho - \ln\sigma))$, consider

$$S(\rho'\|\rho_1' \otimes \rho_2') = S(\mathbb{X}^\ddagger[\rho]) + S(\rho[\mathcal{X}]) - S(\rho), \quad \text{(E7)}$$

where we used the fact that, due to (E3), $S(\rho') = S(\rho)$. Further, consider the action of the full decoherence map $\mathbb{P}$ on the second party,

$$\mathrm{id} \otimes \mathbb{P}[\rho'] = \sum_{a=1}^{|\mathcal{X}|} p_a\, \rho_a' \otimes |a\rangle\langle a|, \quad \rho_a' = \frac{X_a \rho X_a^\dagger}{p_a}, \quad \text{(E8)}$$

so that

$$S(\mathrm{id} \otimes \mathbb{P}[\rho']\|\mathrm{id} \otimes \mathbb{P}[\rho_1' \otimes \rho_2']) = S(\mathbb{X}^\ddagger[\rho]) - \sum_a p_a S(\rho_a'). \quad \text{(E9)}$$

Hence, due to the monotonicity of the relative entropy under CPTP maps, the difference

$$S(\mathrm{id} \otimes \mathbb{P}[\rho']\|\mathrm{id} \otimes \mathbb{P}[\rho_1' \otimes \rho_2']) - S(\rho'\|\rho_1' \otimes \rho_2')$$
$$= S(\rho) - \sum_a p_a S(\rho_a') - S(\rho[\mathcal{X}]) \leqslant 0$$

or, equivalently,

$$S(\rho) - \sum_a p_a S(\rho_a') \leqslant S(\rho[\mathcal{X}]). \quad \text{(E10)}$$

On the left-hand side, one has the average information gain about the state ρ when the measurement associated with $\mathcal{X}$ is performed. Hence, the entropy exchange $S(\rho[\mathcal{X}])$ provides a universal upper bound for the average information gain.

APPENDIX F: PROOF OF PROPOSITION 6

From its definition (35) in Proposition 2, the map $\mathbb{T}_n^{\ddagger}$ reads as

$$\mathbb{T}_n\left[\bigotimes_{k=0}^{n-1} A_k^{(k-n)}\right]=\omega_E\left((\Theta\circ\sigma_S\otimes \mathrm{id}_E)^n\left[\bigotimes_{k=0}^{n-1} A_k^{(k-n)}\otimes \mathbb{1}_E\right]\right),$$

with $\Theta=\mathrm{id}_S\otimes\sigma_E\circ\Phi$.Note that in the one-step automorphism $\Theta\circ\sigma_S\otimes\mathrm{id}_E$ the shift appears twice: the rightmost one, σ_S, acts on the chain $M_d^{\mathbb{Z}}(\mathbb{C})$ of infinite copies of the open system (as in Proposition 2), while σ_E is the collisional shift on the spin-chain environment. Then, let us first evaluate

$$\begin{aligned}(\Theta\circ\sigma_S\otimes\mathrm{id}_E)&\left[\bigotimes_{k=0}^{n-1} A_k^{(k-n)}\otimes\mathbb{1}_E\right]\\ &=\bigotimes_{k=0}^{n-2} A_k^{(k-n+1)}\otimes\sum_{i_n}\phi_{i_n}\left[A_{n-1}^{(0)}\right]\otimes\Pi_{i_n}^{(1)},\end{aligned}$$

where Π_{i_n} is localized in the first site of the environment chain due to the action of σ_E. The second iteration leads to

$$\begin{aligned}(\Theta\circ\sigma_S\otimes\mathrm{id}_E)^2&\left[\bigotimes_{k=0}^{n-1} A_k^{(k-n)}\otimes\mathbb{1}_E\right]\\ &=\bigotimes_{k=0}^{n-3} A_k^{(k-n+2)}\otimes\sum_{i_{n-1}i_n}\phi_{i_{n-1}}\otimes\phi_{i_n}\left[A_{n-2}^{(0)}\otimes A_{n-1}^{(1)}\right]\\ &\quad\otimes\Pi_{i_{n-1}}^{(1)}\otimes\Pi_{i_n}^{(2)}.\end{aligned}$$

Hence, after n iterations,

$$\begin{aligned}(\Theta\circ\sigma_S\otimes\mathrm{id}_E)^n&\left[\bigotimes_{k=0}^{n-1} A_k^{(k-n)}\otimes\mathbb{1}_E\right]\\ &=\sum_{\boldsymbol{i}_{[1,n]}}\phi_{\boldsymbol{i}_{[1,n]}}^{\otimes[1,n]}\left[\bigotimes_{k=0}^{n-1} A_k^{(k)}\right]\otimes\Pi_{\boldsymbol{i}_{[1,n]}}^{[1,n]}.\end{aligned}\tag{F1}$$

Acting on the latter with the conditional expectation ω_E, we get

$$\begin{aligned}\mathbb{T}_n\left[\bigotimes_{k=0}^{n-1} A_k^{(k)}\right]&=\sum_{\boldsymbol{i}_{[1,n]}}\phi_{\boldsymbol{i}_{[1,n]}}^{\otimes[1,n]}\left[\bigotimes_{k=0}^{n-1} A_k^{(k)}\right]\omega_E\left(\Pi_{\boldsymbol{i}_{[1,n]}}^{[1,n]}\right)\\ &=\sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}}\,\phi_{\boldsymbol{i}_{[1,n]}}^{\otimes[1,n]}\left[\bigotimes_{k=0}^{n-1} A_k^{(k)}\right].\end{aligned}$$

With respect to the tracial state $\rho_S=\mathbb{1}_d/d$, the coarse-grained density matrix then reads as

$$\rho_S[\mathcal{F}^{(n+1)}]=\frac{\mathbb{1}_d}{d}\otimes\frac{\mathbb{1}_d}{d}\otimes\left(\mathbb{T}_n^{\ddagger}\otimes\mathrm{id}_d^{\otimes n}[|\psi_+^{[1,n]}\rangle\langle\psi_+^{[1,n]}|]\right),$$

where $|\psi_+^{[1,n]}\rangle=\left|\sqrt{\frac{\mathbb{1}}{d}}^{\otimes n}\right\rangle=d^{-\frac{n}{2}}\sum_{\boldsymbol{k}_{[1,n]}}|\boldsymbol{k}_{[1,n]}\otimes\boldsymbol{k}_{[1,n]}\rangle$

and where $\mathbb{T}_n^{\ddagger}$ is the dual of $\mathbb{T}_n$ in the Schrödinger picture, namely, the map

$$\mathbb{T}_n^{\ddagger}=\sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}}\,\phi_{\boldsymbol{i}_{[1,n]}}^{\otimes[1,n]\,\ddagger},\quad \phi_k^{\ddagger}[\,\cdot\,]=U_k\,\cdot\,U_k^{\dagger}.\tag{F2}$$

We now prove (67). First, by setting $|\psi_{\boldsymbol{i}_{[1,n]}}\rangle=(\bigotimes_{k=1}^n U_{i_k})\otimes\mathbb{1}_d^{\otimes n}|\psi_+^{[1,n]}\rangle$ one has

$$\mathbb{T}_n^{\ddagger}\otimes\mathrm{id}_d^{\otimes n}[|\psi_+^{[1,n]}\rangle\langle\psi_+^{[1,n]}|]=\sum_{\boldsymbol{i}_{[1,n]}} p_{\boldsymbol{i}_{[1,n]}}\left|\psi_{\boldsymbol{i}_{[1,n]}}\right\rangle\!\left\langle\psi_{\boldsymbol{i}_{[1,n]}}\right|.\tag{F3}$$

In particular, since the vector states $|\psi_{\boldsymbol{i}_{[1,n]}}\rangle$ are normalized,

$$\begin{aligned}S(\rho[\mathcal{F}^{(n+1)}])&=2\,S(\rho_S)+S\left(\mathbb{T}_n^{\ddagger}\otimes\mathrm{id}_d^{\otimes n}[|\psi_+^{[1,n]}\rangle\langle\psi_+^{[1,n]}|]\right)\\ &\leqslant 2\,S(\rho_S)+H(\pi_{[1,n]}),\end{aligned}\tag{F4}$$

with equality holding if and only if $|\psi_{\boldsymbol{i}_{[1,n]}}\rangle$ form an orthonormal basis in $\mathbb{C}_d^{\otimes n}$, that is, when (F3) is the spectral decomposition of $\mathbb{T}_n^{\ddagger}\otimes\mathrm{id}_d^{\otimes n}[|\psi_+^{[1,n]}\rangle\langle\psi_+^{[1,n]}|]$. Equivalently, the equality holds if and only if the unitary operators U_k satisfy the orthogonality relations $\mathrm{Tr}(U_j^{\dagger}U_k)=d\,\delta_{jk}$. In such case, dividing both sides of (F4) by $n+1$ and taking the limit, one gets

$$\mathfrak{h}_S(\Theta,\mathcal{F})=\lim_n\frac{1}{n}H(\pi_{[1,n]})=\mathfrak{S}_{\omega_E},\tag{F5}$$

that, along with (64), implies (67). ■

Derivation of Eq. (52)

We now derive Eq. (52). For a generic POVM $\mathcal{X}$, let $\{|a\rangle\}_{a=1}^{|\mathcal{X}|}$ be a basis of $\mathbb{C}^{|\mathcal{X}|}$ and define, similarly to Appendix E 1, an isometry $V:\mathcal{H}_\omega\otimes|0\rangle\longrightarrow\mathcal{H}_\omega\otimes\mathbb{C}^{|\mathcal{X}|}$:

$$V|\Omega_\omega\rangle\otimes|0\rangle=\sum_{a=1}^{|\mathcal{X}|}\pi_\omega(X_a)|\Omega_\omega\rangle\otimes|a\rangle=:|\psi\rangle\ \in\ \mathcal{H}_\omega\otimes\mathbb{C}^{|\mathcal{X}|},$$

corresponding to the projector

$$\begin{aligned}|\psi\rangle\langle\psi|&=V|\Omega_\omega\rangle\langle\Omega_\omega|\otimes|0\rangle\langle 0|V^{\dagger}\\ &=\sum_{ab=1}^{|\mathcal{X}|}\pi_\omega(X_a)|\Omega_\omega\rangle\langle\Omega_\omega|\pi_\omega(X_b)^{\dagger}\otimes|a\rangle\langle b|.\end{aligned}\tag{F6}$$

The marginals of $|\psi\rangle\langle\psi|$ yield

$$\mathrm{Tr}_I(|\psi\rangle\langle\psi|)=\sum_{a,b}\omega(X_b^{\dagger}X_a)|a\rangle\langle b|=\rho[\mathcal{X}]\tag{F7}$$

and

$$\begin{aligned}\mathrm{Tr}_{II}(|\psi\rangle\langle\psi|)&=\sum_a\pi_\omega(X_a)|\Omega_\omega\rangle\langle\Omega_\omega|\pi(X_a)^{\dagger}\\ &=:\widetilde{\mathbb{X}}^{\ddagger}[|\Omega_\omega\rangle\langle\Omega_\omega|].\end{aligned}\tag{F8}$$

The marginals (F7) and (F8) of the pure state $|\psi\rangle\langle\psi|$ have the same spectrum, multiplicities included, a part from the zero eigenvalue; thus, they have the same von Neumann entropy $S(\rho[\mathcal{X}])=S(\widetilde{\mathbb{X}}^{\ddagger}[|\Omega_\omega\rangle\langle\Omega_\omega|])$. From the latter expression, it is also evident how the entropy exchange discussed in Appendix E 1 does depend on the chosen POVM, but not on

its specific Kraus representation. Repeating the same steps for the time-refined POVM $\mathcal{X}^{(n)}$, one extends (F6) to

$$\sum_{\boldsymbol{a},\boldsymbol{b}} \pi_\omega\big(X_{\boldsymbol{a}}^{(n)}\big)|\Omega_\omega\rangle\langle\Omega_\omega|\pi_\omega\big(X_{\boldsymbol{b}}^{(n)\dagger}\big) \otimes |\boldsymbol{a}_{[0,n-1]}\rangle\langle\boldsymbol{b}_{[0,n-1]}| \quad \text{(F9)}$$

so that

$$S(\rho[\mathcal{X}^{(n)}]) = S\bigg(\sum_{\boldsymbol{a}} \pi\big(X_{\boldsymbol{a}}^{(n)}\big)|\Omega_\omega\rangle\langle\Omega_\omega|\big(X_{\boldsymbol{a}}^{(n)}\big)^\dagger\bigg). \quad \text{(F10)}$$

From (21),

$$\begin{aligned}\pi\big(X_{\boldsymbol{a}}^{(n)}\big)|\Omega_\omega\rangle &= \pi_\omega\big(\Theta^{n-1}\big(X_{a_{n-1}}\big)\big)\ \ldots\ \pi_\omega\big(\Theta\big(X_{a_1}\big)\big)\pi\big(X_{a_0}\big)\big|\Omega_\omega\big\rangle \\ &= (U^\dagger)^n\big(U\pi_\omega\big(X_{a_{n-1}}\big)\ldots U\pi_\omega\big(X_{a_0}\big)\big)|\Omega_\omega\rangle.\end{aligned}$$

Then,

$$\sum_{\boldsymbol{a}} \pi\big(X_{\boldsymbol{a}}^{(n)}\big)|\Omega_\omega\rangle\langle\Omega_\omega|\big(X_{\boldsymbol{a}}^{(n)}\big)^\dagger = \mathbb{U}^n \circ (\mathbb{U}^\ddagger \circ \widetilde{\mathbb{X}}^\ddagger)^n[|\Omega_\omega\rangle\langle\Omega_\omega|],$$

so that by the invariance of the von Neumann entropy under unitary maps and (F10),

$$S(\rho[\mathcal{X}^{(n)}]) = S((\mathbb{U}^\ddagger \circ \widetilde{\mathbb{X}}^\ddagger)^n\,[|\Omega_\omega\rangle\langle\Omega_\omega|]). \quad \text{(F11)}$$

Following the model of measurement process described in Appendix E 1, (F9) can be then interpreted as a dilation of the POVM process $(\mathbb{U}^\ddagger \circ \widetilde{\mathbb{X}}^\ddagger)^n$ where the apparatus is explicitly taken into account. The only difference is that, now, multiple copies of the apparatus are considered due to the repeated measurements. Accordingly, (F11) can be also seen as the entropy exchange with the apparatus due to iterated measurements.

APPENDIX G: Δ MONOTONICITY OF RHP AND BLP MEASURES

Discretized RHP measure. To obtain a meaningful analog of the RHP measure in the our discrete setting, we set a reference timescale $N \in \mathbb{N}$ and consider the following discretized version of the RHP measure:

$$\begin{aligned}\mathcal{N}_{\mathrm{RHP}}^{(N)} &= \sum_{n=1}^{N} g_n, \quad g_n = ||\Lambda_{n,n-1}^\ddagger \otimes \mathrm{id}_d[P_+^{(d)}]||_1 - 1 \\ &= \sum_i \big|E_{n,n-1}^{(i)}\big| - 1,\end{aligned}$$

where $E_{n,n-1}^{(i)}$ are the eigenvalues of $\Lambda_{n,n-1}^\ddagger \otimes \mathrm{id}_d[P_+^{(d)}]$. For the one-qubit model at hand, one has

$$g_n = \lambda_{n,n-1} - (1-2p),$$

where $\lambda_{n,n-1} = \lambda_n/\lambda_{n-1}$. We shall consider, as in the main text and [29], the case $A_{p,r} > 0$. Then, λ_n is positive and is given by

$$\lambda_{n,n-1} = \frac{A_{p,r}}{2}\frac{(1+x_{p,r,\Delta})^{n+1} - (1-x_{p,r,\Delta})^{n+1}}{(1+x_{p,r,\Delta})^n - (1-x_{p,r,\Delta})^n},$$

where we let $x_{p,r,\Delta} := B_{p,r,\Delta}/A_{p,r} \geqslant 1$ to prove that

$$\frac{\partial g_n}{\partial\Delta} = \frac{\partial\lambda_{n,n-1}}{\partial\Delta} = \frac{\partial\lambda_{n,n-1}}{\partial x}\frac{\partial x_{p,r,\Delta}}{\partial\Delta} \geqslant 0, \quad \forall\, n \in \mathbb{N}. \quad \text{(G1)}$$

Note that $\frac{\partial x_{p,r,\Delta}}{\partial\Delta} = \frac{8p}{A_{p,r}B_{p,r,\Delta}} \geqslant 0$. Then, letting $y := x_{p,r,\Delta}^{-1}$, $0 \leqslant y \leqslant 1$, one has

$$\frac{\partial\lambda_{n,n-1}}{\partial x} = \frac{(1+y)^{2n} - (1-y)^{2n} + (-)^n 4ny(1-y^2)^{n-1}}{[(1+y)^n - (1-y)^n]^2}. \quad \text{(G2)}$$

To verify the positivity of (G2), it is sufficient to check the positivity of

$$f_n(y) := (1+y)^{2n} - (1-y)^{2n} - 4ny(1-y^2)^{n-1}. \quad \text{(G3)}$$

By the binomial expansion, one rewrites

$$f_n(y) = 4\,n\,y(1-(1-y^2)^{n-1}) + 2\sum_{k=1}^{n-1}\binom{2n}{2k+1}y^{2k+1}.$$

Since $0 \leqslant y \leqslant 1$, it follows that $f_n(y) \geqslant 0$ for all $n \in \mathbb{N}$. This suffices to establish the monotonicity of $\lambda_{n,n-1}$, g_n, and $\mathcal{N}_{\mathrm{RHP}}^{(N)}$ as functions of Δ.

Discretized BLP measure. Consider the following discretized version of the BLP measure:

$$\mathcal{N}_{\mathrm{BLP}}^{(N)} = \max_{\rho,\,\sigma} \sum_{\substack{n=1 \\ d_n - d_{n-1} > 0}}^{N} (d_n(\rho,\sigma) - d_{n-1}(\rho,\sigma)),$$

$$\text{where} \quad d_n(\rho,\sigma) := ||\Lambda_n^\ddagger\bigg[\frac{\rho-\sigma}{2}\bigg]||_1. \quad \text{(G4)}$$

With respect to our qubit model, we shall now show that $d_n(\rho,\sigma) - d_{n-1}(\rho,\sigma)$ is a nondecreasing function of Δ for all $n \in \mathbb{N}$ and all ρ, σ. Let $X = (\rho-\sigma)/2$. Since $\mathrm{Tr}(X) = 0$, we can expand it as $X = \sum_{i=1}^3 x_i\,\sigma_i$. Then,

$$d_n(\rho,\sigma) = \sqrt{\lambda_n^2\,(x_1^2 + x_2^2) + \mu_n^2\,x_3^2}, \quad \text{(G5)}$$

where, for the sake of brevity, we set $\lambda_n := \lambda_n^{(i)}$, $i = 1, 2$, $\mu_n := \lambda_n^{(3)}$. Since $d_n(\rho,\sigma)$ depends on Δ only through λ_n, we have

$$\frac{\partial}{\partial\Delta}d_n(\rho,\sigma) = \big(x_1^2 + x_2^2\big)\frac{\lambda_n'\lambda_n}{d_n}, \quad \lambda_n' = \frac{\partial\lambda_n}{\partial\Delta},$$

and, accordingly,

$$\begin{aligned}&\frac{\partial}{\partial\Delta}(d_n(\rho,\sigma) - d_{n-1}(\rho,\sigma)) \\ &\quad = \big(x_1^2 + x_2^2\big)\bigg(\frac{\lambda_n'\lambda_n}{d_n} - \frac{\lambda_{n-1}'\lambda_{n-1}}{d_{n-1}}\bigg).\end{aligned}$$

The latter is positive if and only if

$$\lambda_{n,n-1}\frac{\lambda_n'}{\lambda_{n-1}'} \geqslant \frac{d_n}{d_{n-1}}. \quad \text{(G6)}$$

For the model at hand, lack of contractivity of $\Lambda_{n,n-1}^\ddagger$ occurs only if $\lambda_{n,n-1} > 1$. We previously proved that $\lambda_{n,n-1}' > 0$. These facts are sufficient to show that

$$\lambda_n' - \lambda_{n-1}' = (\lambda_{n,n-1} - 1)\lambda_{n-1}' + \lambda_{n,n-1}'\lambda_{n-1} \geqslant 0. \quad \text{(G7)}$$

From $\mu_n \leqslant \mu_{n-1}$ one further deduces that

$$d_n \leqslant \sqrt{\lambda_n^2\big(x_1^2+x_2^2\big)+\mu_{n-1}^2 x_3^2} < \sqrt{\lambda_{n,n-1}^2\lambda_{n-1}^2\big(x_1^2+x_2^2\big)+\lambda_{n,n-1}^2\mu_{n-1}^2 x_3^2} = \lambda_{n,n-1}\, d_{n-1}. \tag{G8}$$

Inequalities (G7) and (G8) yield, finally,

$$\lambda_{n,n-1}\frac{\lambda'_n}{\lambda'_{n-1}} \geqslant \lambda_{n,n-1} > \frac{d_n}{d_{n-1}},$$

thus verifying condition (G6) and ensuing monotonicity of $d_n(\rho,\sigma)-d_{n-1}(\rho,\sigma)$ as a function of Δ.

APPENDIX H: DERIVATION OF GNS REDUCED DYNAMICS

We check that (95) is a bona fide GNS implementation of the collisional dynamics. First notice that, in (95), the action on $\pi_S(\mathcal{A}_S)'$ is such that the GNS cyclic vector is invariant; namely,

$$\begin{aligned} U_\Theta^\dagger|\psi_+^{(d)}\otimes\Omega_E\rangle &= \sum_k U_k^\dagger\otimes\overline{U_k}^\dagger|\psi_+^{(d)}\rangle\otimes U_\sigma^\dagger\pi_E\big(\Pi_k^{(0)}\big)|\Omega_E\rangle \\ &= |\psi_+^{(d)}\rangle\otimes U_\sigma^\dagger\,\pi_E\Big(\sum_k\Pi_k^{(0)}\Big)|\Omega_E\rangle \\ &= |\psi_+^{(d)}\otimes\Omega_E\rangle, \end{aligned} \tag{H1}$$

where we used the fact that, along with (94), $V\otimes\overline{V}|\psi_+^{(d)}\rangle = |\psi_+^{(d)}\rangle$ for any unitary $V\in M_d(\mathbb{C})$. Also, it suffices to define U_Θ only on $\pi_E(\mathcal{A}_E)$. Indeed, because the environment is a classical chain, this dense subalgebra $\pi_E(\mathcal{A}_E)$ is contained in its commutant $(\pi_E(\mathcal{A}_E))'$. By taking the adjoint with respect to the scalar product in the GNS Hilbert space, the action of U_Θ can be also inferred,

$$\begin{aligned} &U_\Theta\,\pi_S(A)\pi'_S(B)\otimes\pi_E\big(\Pi_{\boldsymbol{i}^{[a,b]}}^{[a,b]}\big)|\psi_+^{(d)}\otimes\Omega_E\rangle \\ &\quad= \sum_k U_kA\otimes\overline{U_k}\,\overline{B}\otimes\pi_E\big(\Pi_k^{(0)}\big)U_\sigma\,\pi_E\big(\Pi_{\boldsymbol{i}^{[a,b]}}^{[a,b]}\big)|\psi_+^{(d)}\otimes\Omega_E\rangle \end{aligned} \tag{H2}$$

from which it also follows that $U_\Theta|\psi_+^{(d)}\otimes\Omega_E\rangle = |\psi_+^{(d)}\otimes\Omega_E\rangle$. By acting on local operators as in (95) and (H2), one checks explicitly that $U_\Theta U_\Theta^\dagger = U_\Theta^\dagger U_\Theta = \mathbb{1}$. Moreover, the automorphism Θ is correctly implemented. Indeed, from (95),

$$\begin{aligned} &U_\Theta^\dagger\,\pi_S(A)\otimes\pi_E\big(\Pi_{\boldsymbol{i}^{[a,b]}}^{[a,b]}\big)U_\Theta|\psi_+^{(d)}\otimes\Omega_E\rangle \\ &\quad= \sum_{k,l} U_k^\dagger AU_l\otimes\overline{U_k}^\dagger\overline{U_l}\otimes U_\sigma^\dagger\,\pi_E\big(\Pi_k^{(0)}\Pi_{\boldsymbol{i}^{[a,b]}}^{[a,b]}\Pi_l^{(0)}\big)\mathbb{1}_d\otimes\mathbb{1}_d\otimes U_\sigma|\psi_+^{(d)}\otimes\Omega_E\rangle \\ &\quad= \mathbb{1}_d\otimes\mathbb{1}_d\otimes U_\sigma^\dagger\Big(\sum_k U_k^\dagger AU_k\otimes\mathbb{1}_d\otimes\pi_E\big(\Pi_k^{(0)}\Pi_{\boldsymbol{i}^{[a,b]}}^{[a,b]}\Pi_k^{(0)}\big)\Big)\mathbb{1}_d\otimes\mathbb{1}_d\otimes U_\sigma|\psi_+^{(d)}\otimes\Omega_E\rangle \\ &\quad= \mathbb{1}_{d^2}\otimes U_\sigma^\dagger\pi_S\otimes\pi_E\big(\Phi\big[A\otimes\Pi_{\boldsymbol{i}^{[a,b]}}^{[a,b]}\big]\big)\mathbb{1}_{d^2}\otimes U_\sigma|\psi_+^{(d)}\otimes\Omega_E\rangle \\ &\quad= \pi_S\otimes\pi_E\big(\sigma_E\circ\Phi\big[A\otimes\Pi_{\boldsymbol{i}^{[a,b]}}^{[a,b]}\big]\big)|\psi_+^{(d)}\otimes\Omega_E\rangle = \pi_S\otimes\pi_E\big(\Theta\big[A\otimes\Pi_{\boldsymbol{i}^{[a,b]}}^{[a,b]}\big]\big)|\psi_+^{(d)}\otimes\Omega_E\rangle. \end{aligned}$$

Recalling that $\widetilde{\mathbb{X}} = \mathbb{X}\otimes\mathrm{id}_d$, consider the operator

$$\mathbb{Y}_m := ((\widetilde{\mathbb{X}}\otimes\mathrm{id}_E)\circ\mathbb{U}_\Theta)^m[\pi_S\otimes\pi_E(A\otimes\mathbb{1}_E)\pi'_S\otimes\pi_E(B\otimes\mathbb{1}_E)] = ((\widetilde{\mathbb{X}}\otimes\mathrm{id}_E)\circ\mathbb{U}_\Theta)^m[A\otimes\overline{B}\otimes\pi_E(\mathbb{1}_E)], \quad m\in\mathbb{N}.$$

We now show by induction that

$$\mathbb{Y}_m = \sum_{k_1\ldots k_m}\widetilde{\mathbb{X}}\circ\phi_{k_1}\otimes\overline{\phi}_{k_1}\circ\widetilde{\mathbb{X}}\circ\phi_{k_2}\otimes\overline{\phi}_{k_2}\circ\cdots\circ\widetilde{\mathbb{X}}\circ\phi_{k_m}\otimes\overline{\phi}_{k_m}[A\otimes\overline{B}]\otimes\pi_E\big(\Pi_{i_1}^{(1)}\otimes\cdots\otimes\Pi_{i_m}^{(m)}\big). \tag{H3}$$

For $m=1$ we have indeed

$$(\widetilde{\mathbb{X}}\otimes\mathrm{id}_E)\circ\mathbb{U}_\Theta[A\otimes\overline{B}\otimes\pi_E(\mathbb{1}_E)] = \sum_{jk}\widetilde{\mathbb{X}}[U_k^\dagger AU_j\otimes\overline{U}_k^\dagger\overline{BU}_j]\otimes\pi_E\big(\Pi_k^{(1)}\Pi_j^{(1)}\big) = \sum_k(\widetilde{\mathbb{X}}\circ\phi_k\otimes\overline{\phi}_k[A\otimes\overline{B}])\otimes\pi_E\big(\Pi_k^{(1)}\big).$$

Suppose that (H3) holds for $m\leqslant n$ and evaluate $\mathbb{Y}_{n+1}$:

$$\begin{aligned} \mathbb{Y}_{n+1} &= ((\widetilde{\mathbb{X}}\otimes\mathrm{id}_E)\circ\mathbb{U}_\Theta)^{n+1}[A\otimes\overline{B}\otimes\pi_E(\mathbb{1}_E)] \\ &= ((\widetilde{\mathbb{X}}\otimes\mathrm{id}_E)\circ\mathbb{U}_\Theta)\Big[\sum_{k_2\ldots k_{n+1}}\widetilde{\mathbb{X}}\circ\phi_{k_2}\otimes\overline{\phi}_{k_2}\circ\cdots\circ\widetilde{\mathbb{X}}\circ\phi_{k_{n+1}}\otimes\overline{\phi}_{k_{n+1}}[A\otimes\overline{B}]\otimes\pi_E\big(\mathbb{1}_D^{(0)}\otimes\Pi_{k_2}^{(1)}\otimes\cdots\otimes\Pi_{k_{n+1}}^{(n)}\big)\Big] \\ &= (\widetilde{\mathbb{X}}\otimes\mathrm{id}_E)\Big[\sum_{j_1k_1k_2\ldots k_{n+1}}U_{k_1}^\dagger\otimes\overline{U}_{k_1}^\dagger\big(\widetilde{\mathbb{X}}\circ\phi_{k_2}\otimes\overline{\phi}_{k_2}\circ\cdots\circ\widetilde{\mathbb{X}}\circ\phi_{k_{n+1}}\otimes\overline{\phi}_{k_{n+1}}[A\otimes\overline{B}]\big)\Big] \end{aligned}$$

$$U_{j_1}\otimes\overline{U}_{j_1}\otimes\pi_E\big(\Pi^{(1)}_{j_1}\Pi^{(1)}_{k_1}\otimes\Pi^{(2)}_{k_2}\otimes\cdots\otimes\Pi^{(n+1)}_{k_{n+1}}\big)\Big]$$
$$=\sum_{k_1\ldots k_{n+1}}\widetilde{\mathbb{X}}\circ\phi_{k_1}\otimes\overline{\phi}_{k_1}\circ\cdots\circ\widetilde{\mathbb{X}}\circ\phi_{k_{n+1}}\otimes\overline{\phi}_{k_{n+1}}[A\otimes\overline{B}]\otimes\pi_E\big(\Pi^{(1)}_{k_1}\otimes\Pi^{(2)}_{k_2}\otimes\cdots\otimes\Pi^{(n+1)}_{k_{n+1}}\big).$$

Then, we now consider expectation (96):

$$\langle\psi^{(d)}_+\otimes\Omega_E|\pi_S\otimes\pi_E(X^\dagger\otimes\mathbb{1}_E)((\widetilde{\mathbb{X}}\otimes\mathrm{id}_E)\circ\mathbb{U}_\Theta)^n(\pi_S\otimes\pi_E(A\otimes\mathbb{1}_E)\pi'_S\otimes\pi_E(B\otimes\mathbb{1}_E))\pi_S\otimes\pi_E(X\otimes\mathbb{1}_E)|\psi^{(d)}_+\otimes\Omega_E\rangle$$
$$=\sum_{k_1\ldots k_n}p_{k_1\ldots k_n}\mathrm{Tr}\big(A\otimes\overline{B}\,\phi^\ddagger_{k_n}\otimes\overline{\phi}^\ddagger_{k_n}\circ\widetilde{\mathbb{X}}^\ddagger\circ\cdots\circ\phi^\ddagger_{k_1}\otimes\overline{\phi}^\ddagger_{k_1}\circ\widetilde{\mathbb{X}}^\ddagger[X\otimes\mathbb{1}_dP^{(d)}_+X^\dagger\otimes\mathbb{1}_d]\big)=\mathrm{Tr}\big(A\otimes\overline{B}\,\Gamma^{\mathbb{X}}_n[X\otimes\mathbb{1}_dP^{(d)}_+X^\dagger\otimes\mathbb{1}_d]\big),$$

where, in the second line, we moved by duality to Schrödinger picture, so to finally deduce (97).

APPENDIX I: TWO-QUBIT PAULI MAPS

A one-qubit Pauli map has a spectral decomposition

$$\Lambda[X]=\frac{1}{2}\sum_{\alpha=0}^{3}\lambda^{(\alpha)}\mathrm{Tr}(\sigma_\alpha X)\sigma_\alpha,\tag{I1}$$

and can be equivalently rewritten as

$$\Lambda[X]=\sum_{\alpha=0}^{3}q^{(\alpha)}\sigma_\alpha X\sigma_\alpha,\qquad\sum_{\alpha=0}^{3}q^{(\alpha)}=1.\tag{I2}$$

The spectrum $\lambda^{(\alpha)}$ and the coefficients $q^{(\alpha)}$ are related by a linear transformation

$$q^{(\alpha)}=\frac{1}{4}\sum_{\alpha\beta=0}^{3}H_{\alpha\beta}\lambda^{(\beta)},\tag{I3}$$

given by an Hadamard matrix H, $H_{\alpha\beta}=\mathrm{Tr}(\sigma_\alpha\sigma_\beta\sigma_\alpha\sigma_\beta)$. Complete positivity of Λ is equivalent to $q^{(\alpha)}\geqslant 0$. Similarly, a two-qubit Pauli map is defined by its spectral decomposition

$$\Gamma[X]=\frac{1}{4}\sum_{\alpha\beta=0}^{3}\lambda^{(\alpha,\beta)}\mathrm{Tr}(\sigma_\alpha\otimes\sigma_\beta X)\sigma_\alpha\otimes\sigma_\beta,\tag{I4}$$

and can be recast in the form

$$\Gamma[X]=\sum_{\alpha\beta=0}^{3}q^{(\alpha,\beta)}\sigma_\alpha\otimes\sigma_\beta X\sigma_\alpha\otimes\sigma_\beta,\qquad\sum_{\alpha\beta=0}^{3}q^{(\alpha,\beta)}=1,\tag{I5}$$

where the matrices describing the spectrum $G:=[\gamma^{(\alpha,\beta)}]$, respectively, the coefficients $Q:=[q^{(\alpha,\beta)}]$ are similar and related through the Hadamard transformation

$$Q=\frac{HGH}{16}.\tag{I6}$$

Suppose now that the spectrum has the special form

$$G=\begin{pmatrix}1&\lambda&\lambda&\lambda^{(3)}\\ \lambda&1&\lambda^{(3)}&\lambda\\ \lambda&\lambda^{(3)}&1&\lambda\\ \lambda^{(3)}&\lambda&\lambda&1\end{pmatrix}.\tag{I7}$$

Then, using (I6), the matrix coefficient is diagonal,

$$Q=\frac{1}{4}\begin{pmatrix}1+\lambda^{(3)}+2\lambda&0&0&0\\ 0&1-\lambda^{(3)}&0&0\\ 0&0&1-\lambda^{(3)}&0\\ 0&0&0&1+\lambda^{(3)}-2\lambda\end{pmatrix},$$

and hence (I5) has the form

$$\Gamma[X]=\sum_{\alpha=0}^{3}q^{(\alpha,\alpha)}\sigma_\alpha\otimes\sigma_\alpha X\sigma_\alpha\otimes\sigma_\alpha.$$

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