

Temperature change can solve the Deutsch-Jozsa problem: An exploration of thermodynamic query complexity

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We demonstrate how a single heat exchange between a probe thermal qubit and multiqubit thermal machine encoding a Boolean function can determine whether the function is balanced or constant, thus providing a thermodynamic solution to the Deutsch-Jozsa problem. We introduce a thermodynamic model of quantum query complexity, showing how qubit thermal machines can act as oracles, queried via heat exchange with a probe. While the Deutsch-Jozsa problem requires an exponential encoding in the number of oracle bits, we also explore a restricted Bernstein-Vazirani problem, which admits a linear thermal oracle and a single thermal query solution. We establish bounds on the number of samples needed to determine the probe temperature encoding the solution for the Deutsch-Jozsa problem, showing that it remains constant with problem size. Additionally, we propose a proof-of-principle experimental implementation to solve the three-bit Bernstein-Vazirani problem via thermal kickback. This work bridges thermodynamics and complexity theory, suggesting that quantum thermodynamics could provide an unconventional route to computing beyond classical computation.

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I. INTRODUCTION

Quantum decision problems and the development of the quantum query complexity model [1–5] served as seminal points in quantum computation, exemplifying clearly the difference between the computational power of quantum and classical physics. These models, which eventually led to the development of quantum algorithms, made a number of assumptions in what constitutes a *quantum query*: (1) Qubits are initialized and used in pure states, and (2) unitaries create coherence as a resource for querying.

In quantum thermodynamics [6–8] these assumptions are often impossible to satisfy (pure states require infinite energy to obtain with certainty [9] or are unnecessary (a system may be cooled efficiently without the generation of coherences [10]. Indeed, limited models of quantum computation like deterministic quantum computation with one quantum bit (DQC1) circuits [11–14] are able to solve classically hard problems [15–18] without satisfying some of these assumptions. Recently, even computational models that make use of solely stochastic thermodynamics have been shown to provide an advantage over conventional classical models of computation in some tasks [19,20].

This provokes us to wonder, Can a classically hard quantum decision problem be solved in a quantum thermodynamic scenario? In this work we answer this question in the

affirmative. We show that an agent with access to a thermal probe qubit can learn the properties of a Boolean function, encoded by an oracle into the energetic gaps of a set of qubits forming a thermal machine [21–24], via a single heat exchange. We begin by examining how oracles for Boolean functions can be embedded into thermal machines and queried via different inputs which result in different heat exchanges with the probe. We investigate three different queries possible via heat exchange in our model (Fig. 1) for the Deutsch-Jozsa [1] problem. In particular we present the *thermal kickback* subroutine which imprints global properties of a function in the temperature of a probe qubit after a single heat exchange with the oracle's machine. We also show how the thermal kickback can detect the Hamming weight of an n -bit secret string. Following this, we explore the thermodynamic properties the probe must have to be sensitive to a thermal kickback, giving insight into the thermodynamic cost of querying quantum oracles. Importantly, we examine the number of copies of the probe that are required to determine its temperature and show that for the Deutsch-Jozsa problem the number of samples required is constant, with n being the size of the problem. Finally, we discuss the implications of our work and present a schematic of an experimental implementation of our model for the three-bit Bernstein-Vazirani problem in the Appendix.

II. FROM UNITARY ORACLES TO THERMAL MACHINE ORACLES

In the traditional quantum query complexity setting [5] an oracle with access to a Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ constructs a black-box unitary [Fig. 1, diagram (i)] U_f which maps an $(n + 1)$ -qubit input string $|x a\rangle$ to $|x a \oplus f(x)\rangle$, where the ancilla qubit's state now encodes the output of the

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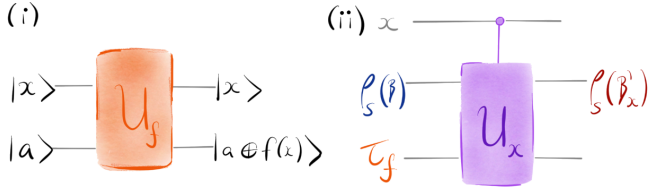


FIG. 1. (i) A unitary binary oracle takes as input an n -qubit string $|x\rangle : x \in \{0, 1\}^{\times n}$ and a single ancilla qubit $|a\rangle : a \in \{0, 1\}$ imprinting the output of the Boolean function encoded by the oracle into U_f onto the ancilla as $|a \oplus f(x)\rangle$. (ii) A thermal machine oracle τ_f is a collection of qubits whose energetic structure encodes f . An agent with access to a *probe* $\rho_S(\beta)$ can exchange heat with τ_f via U_x depending on an input n -bit string x . The probe state changes to $\rho_S(\beta'_x)$, where the temperature of S encodes the output $f(x)$.

query. Let us instead allow the oracle the ability to thermalize qubits by constructing a thermal machine whose energetic structure encodes information about the output of the function. For each input string x the oracle calls the function and prepares the state

$$\tau_x = \frac{1}{\mathcal{Z}_x} (|0\rangle\langle 0| + e^{-\beta_M(f(x)E_1 + (f(x) \oplus 1)E_2)} |1\rangle\langle 1|). \quad (1)$$

Using the notation $E(x) = f(x)E_1 + [f(x) \oplus 1]E_2$, we have that the qubit Hamiltonian $H_x = 0|0\rangle\langle 0| + E(x)|1\rangle\langle 1|$, the partition function $\mathcal{Z}_x = (1 + e^{-\beta_M E(x)})$, and the ground-state energy of the qubit is set to zero. A detailed protocol for the oracle to prepare the thermal machine using conditional thermalizations is given in the Appendix. Using a machine gap vector $\Gamma = (E(0^N), E(0^{N-1}1), E(0^{N-2}10), \dots, E(1^N))$ with entries corresponding to the energetic gap of the x th oracle qubit $E(x)$, the state of the *thermal machine oracle* is then given by

$$\tau_f = \bigotimes_{x \in \{0, 1\}^{\times n}} \tau_x = \sum_{i_M \in \{0, 1\}^{\times n}} \frac{e^{-\beta_M i_M \cdot \Gamma}}{\mathcal{Z}_f} |i_M\rangle\langle i_M|, \quad (2)$$

where $i_M \cdot \Gamma$ is the inner product between the energy level i_M and the gap vector Γ giving the energy value of i_M . This construction allows for a thermodynamic query model [Fig. 1, diagram (ii)] in which, conditioned on an input bit string x , a heat exchange U_x between a *probe qubit* $\rho_S(\beta)$ initially at temperature $\beta = 1/T$ and the thermal machine oracle τ_f is carried out, altering the temperature of the probe to $\rho_S(\beta'_x)$, which encodes the output $f(x)$. A *query* within

this model is then a heat exchange between $\rho_S(\beta)$ and τ_f carried out by U_x . The output of this query is encoded in the temperature of a thermal state and so is *probabilistic*, unlike the unitary query model, in which an eigenket is assumed to be the output, allowing for a *deterministic* output. This adds an additional complexity to the setting: the *sample complexity*, i.e., the number of samples required to obtain an answer with a given error. Finally, note that τ_f has 2^n qubits in the size of the n -bit Boolean function. This is a simple oracle encoding, a classical truth table that can be queried quantum thermodynamically as a first toy model. Better oracle encodings likely exist; e.g., the oracle could have set $H_M = \sum_{x \in \{0, 1\}^{\times n}} f(x)|x\rangle\langle x|$, leading to an *unphysical* linear encoding, but we leave investigating oracle encodings with low complexity for future work, focusing on this initial case. Despite its simplicity, the presented encoding is still linear for some problems, e.g., Hamming weight detection, which we will present.

III. THE DEUTSCH-JOZSA PROBLEM

To understand the core ideas behind the proposed thermodynamic query complexity model we begin by examining the Deutsch-Jozsa (DJ) problem and solve it via a thermodynamic kickback rather than a phase kickback. The n -bit Deutsch-Jozsa problem asks one to identify whether a Boolean function $f(x) : \{0, 1\}^{\times n} \rightarrow \{0, 1\}$ is constant, meaning for every $x \in \{0, 1\}^{\times n}$ the output is always 0 or always 1, or balanced, meaning the output is 0 for half the domain and 1 for the rest. To solve this problem, Deutsch and Jozsa [1] encoded the Boolean function $f(x)$ into a unitary oracle as in diagram (i) in Fig. 1. The problem is then solved using the circuit given in Fig. 2 for the two-bit DJ problem, in which two register qubits are put in a superposition over all possible strings and an ancilla qubit is made to interfere with this equal superposition of inputs while querying U_f . This imprints global information about $f(x)$ in terms of a *phase* on the ancilla qubit, often referred to as a *phase kickback*.

Consider, instead, a thermal oracle encoding τ_f as in Eq. (2); here, we will investigate the number of queries and samples required to learn whether f is balanced or constant using different heat exchanges as depicted in Fig. 3. (1) For an input n -bit string x a simple heat exchange $\text{SWAP}_{S,x} = \text{SWAP} \otimes (\bigotimes_{y \in \{0, 1\}^{\times n} \setminus x} \mathbb{1})$ acts on $\rho_S(\beta) \otimes \tau_f$ to swap the probe qubit ρ_S for qubit τ_x in the machine. The agent can then collect samples of each τ_x and carry out energy measurements for each collection, solving the problem with finite error given $2^{n-1} + 1$

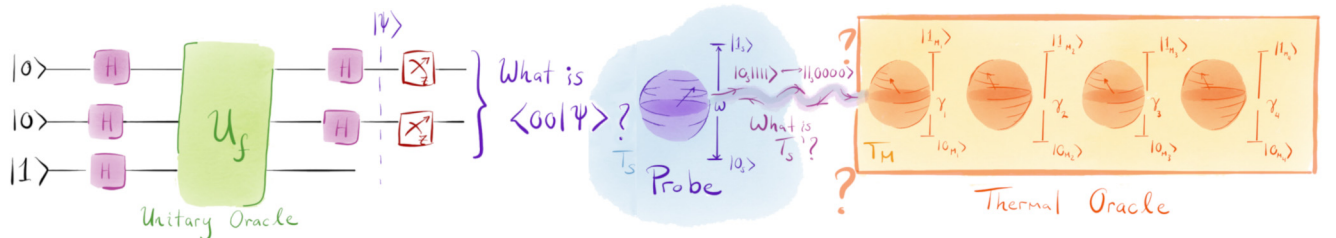


FIG. 2. A visualization of an illustrative example. At the top we see the standard three-qubit Deutsch-Jozsa circuit in which an oracle of the two-bit function $f(x)$ is implemented as a unitary and the decision problem is solved by evaluating $\langle 00 | \psi \rangle$. To the left, we see a four-qubit thermal machine at temperature T_M with gaps $\Gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ whose energy level structure is used by an oracle to encode $f(x)$; the global properties of $f(x)$ can be determined in a single heat exchange with a probe at temperature T_S with energetic gap ω .

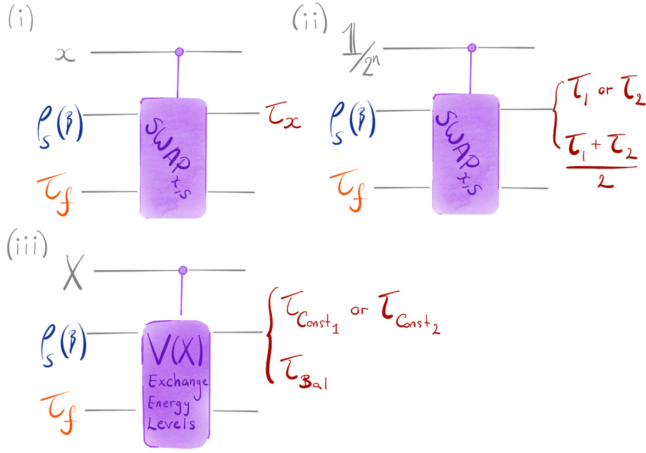


FIG. 3. Three circuit diagrams depicting different heat exchanges across the probe qubit and thermal machine oracle which result in three different queries which are examined in text for the Deutsch-Jozsa problem.

heat exchanges or queries and a constant number of samples per query [25]. (2) A classical mixture over all n -bit strings $\mathbb{1}/2^n = 1/2^n \sum_{x \in \{0,1\}^n} |x\rangle\langle x|$ can, instead, be input, resulting in a probabilistic $\text{SWAP}_{S,x}$ over every possible input in a single query. Letting $\tau_1 = 1/(1 + e^{-\beta_M E_1})(|0\rangle\langle 0| + e^{-\beta_M E_1}|1\rangle\langle 1|)$ denote τ_x in the case where $f(x) = 1$ and similarly using τ_2 for the case where $f(x) = 0$, we see that τ_f can take one of three forms, $\tau_f \in \{\tau_1^{\otimes N}, \tau_2^{\otimes N}, \pi(\tau_1^{\otimes N/2} \otimes \tau_2^{\otimes N/2})\}$, where $\pi \in \mathcal{S}_N$ is a permutation for different balanced functions. As such, the result of $\text{SWAP}_{S,S}$ on $\rho_S \otimes \tau_f$ conditioned on $\mathbb{1}/2^n$ is to return the probe in the state τ_1 or τ_2 in the constant cases and $(\tau_1 + \tau_2)/2$ for any balanced case. In a single query we obtain the desired information, but we require a constant number of samples for an error δ of $\Theta[\ln(1/\delta)/D[\tau_1||(\tau_1 + \tau_2)/2]]$. At best, in this model, i.e., for $\tau_1 = |0\rangle\langle 0|$ and $\tau_2 = \mathbb{1}/2$, the quantity $\Theta[\ln(1/\delta)/\ln(4/3)]$ is slightly worse than classical random sampling (See Appendix C). Classical randomness and bipartite quantum operations on thermal states fail to beat classical probabilistic computation; can stronger thermodynamic operations do better?

Thermal kickback. We will now and for the remainder of the paper consider a stronger heat exchange between the system and the oracle's thermal machine, i.e., the largest change in population in S given by a single energy level exchange, which can for some parameters beat random sampling. Diagram (iii) in Fig. 3 considers a vector-valued function $X(y_0, y_1, \dots, y_{2^n})$ with 2^n entries, each corresponding to an n -bit string, which maps the vector of strings to a 2^n -bit string, where it is mapped to a 1 in the 2^n -bit string if x_i is present in the input vector and a 0 if not, e.g., $(x_0, x_1) \rightarrow 110\dots 0$. The heat exchange $V(X) = |0_S X\rangle\langle 1_S X| \oplus |1_S X\rangle\langle 0_S X| + \mathbb{1}_{\text{Rest}}$ is then carried out, where the energy level $|0_S X\rangle$, featuring no excitations in the probe and excitations in the thermal machine oracle related to input bit strings x_i represented by 1's in X , is exchanged with its Hamming weight conjugate $|1_S X \oplus 1\rangle$, featuring opposite excitations. The case where all n -bit strings are input, i.e., $X = 1^N$, corresponds to the *virtual qubit subspace swap* [21,23,26] a mechanism often considered in quantum thermodynamics

because it is the optimal heat exchange for asymptotic cooling [27] and has been used in the design of quantum heat engines, clocks, thermometers, and, recently, artificial neuron circuits [20,21,28,29]. We will focus on $X = 1^N$ as the input because it will output global information about $f(x)$. The interaction $V(1^N)$ between a probe with gap ω and inverse temperature β_S and a thermal machine oracle at temperature β_M exchanges $|0_S 1^N\rangle$ with population $e^{-\beta_M |\Gamma|}/Z_S Z_f$ and $|1_S 0^N\rangle$ with population $e^{-\beta_S \omega}/Z_S Z_f$, leading to a ground-state population (GSP) in S of

$$p'_0 = \langle 0| \rho'_S |0\rangle = \frac{1 + Z_f^{-1}(e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|})}{Z_S}, \quad (3)$$

where $|\Gamma| = \text{tr}\{\Gamma\}$. Recalling that the GSP of a qubit needs to take the form $p'_0 = 1/(1 + e^{-\beta'_S \omega})$, where $\beta'_S = 1/T'_S$ is the inverse temperature after heat is exchanged, we can solve for $\beta'_S = \omega^{-1} \ln(\frac{p'_0}{1-p'_0})$ to find

$$\beta'_S = \frac{1}{\omega} \ln \left(\frac{1 + Z_f^{-1}(e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|})}{e^{-\beta_S \omega} - Z_f^{-1}(e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|})} \right), \quad (4)$$

as detailed in the Appendix, where $|\Gamma| = NE_1$ or NE_2 if $f(x)$ is constant and $|\Gamma| = N/2(E_1 + E_2)$ if $f(x)$ is balanced, allowing us to solve the Deutsch-Jozsa problem by measuring the temperature of a qubit after a single heat exchange. The sample complexity of this measurement is investigated in the next section, where, for some parameters, advantage over random sampling is possible. In the Appendix, expressions for the GSP and temperature of the probe after the query for the general input X that is not $X = 1^N$ is provided.

IV. A LINEAR THERMAL ORACLE: HAMMING WEIGHT DETECTION OF SECRET STRINGS

In the Bernstein-Vazirani problem [4] an agent is given access to an oracle encoding a Boolean function $f(x) : \{0,1\}^n \rightarrow \{0,1\}$ with the promise that $f(x)$ is formed by taking the dot product under mod 2 with a *secret* n -bit string s , that is, $f(x) = x \cdot s = x_1 s_1 \oplus x_2 s_2 \oplus \dots \oplus x_n s_n$. The solution of this problem is to determine s by querying a unitary oracle encoding $f(x)$ as in the Deutsch-Jozsa problem. A restricted version of this problem, *Hamming weight detection*, is also solvable given a single heat exchange with a qubit when encoded in a thermal machine provided an energy gap promise, this time with linearly many qubits. Let the oracle construct an n thermal qubit machine at temperature T_M such that each qubit has an energetic gap $s_i \gamma$ and state $\tau_i = \frac{1}{(1 + e^{-\beta_M s_i \gamma})}(|0\rangle\langle 0| + e^{-\beta_M s_i \gamma}|1\rangle\langle 1|) : s_i \in \{0,1\}$, where s_i is the i th secret bit, γ is a fixed energy gap, and the global state is $\tau_f = \bigotimes_{i=1}^n \tau_i$. The 2^n energy levels of τ_f , $|x\rangle : x \in \{0,1\}^n$, have populations $e^{-\beta_M s x \bar{\gamma}}$, where $s \cdot x \cdot \bar{\gamma} = s_1 x_1 \gamma + s_2 x_2 \gamma + \dots + s_n x_n \gamma$ and $x_i \in \{0,1\}$ denotes whether the i th qubit is excited in $|x\rangle$. Thus, each population of τ_f features the output $f(x) = s \cdot x$ and should be queried using the probe via the thermal kickback introduced before, $V(x) = |0_S x\rangle\langle 1_S x \oplus 1| + |1_S x \oplus 1\rangle\langle 0_S x| + \mathbb{1}_{\text{Rest}}$. Notably, for $x = 1^n$ we have the postquery probe population $p'_0 = [1 + Z_f^{-1}(e^{-\beta_S \omega} - e^{-\beta_M \#(s) \gamma})]Z_S^{-1}$, where $\#(s)$ is the Hamming weight or number of 1's in s allowing an agent to

detect $\#(s)$ with knowledge of γ . This presents a challenge we did not encounter in the DJ problem; in this scenario the number of temperatures we need to distinguish to determine $\#(s)$ scales linearly with the size of s , posing a problem of distinguishability. This translates this computational problem into one of metrology [30] or multihypothesis testing, which we leave for future work. If the oracle used different energies γ_i for each τ_i , this exchange could also discriminate strings, but this would require a stricter promise condition where the agent knows each γ_i , allowing them to discriminate the output.

V. THERMODYNAMIC COST OF A QUERY

Given the nature of the model, we are able to investigate the energetics and thermodynamics of querying. (1) The change in ground-state population in S induced by $V(X)$ can be positive or negative, heating or cooling the probe. From Eq. (3) we see that for cooling $\Delta p_0 = p'_0 - p_0 > 0$ if $e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|} > 0$, and similarly, for heating $\Delta p_0 < 0$ if $e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|} < 0$. This provides two regimes where the probe is cooled or heated,

$$\frac{\omega}{T_S} < \frac{|\Gamma|}{T_M}, \quad \frac{\omega}{T_S} > \frac{|\Gamma|}{T_M}, \quad (5)$$

respectively, by the query depending on the ratios of energy and temperature of the probe ω/T_S and the ratio of sum of all energy gaps and temperature $|\Gamma|/T_M$ of the thermal machine oracle. (2) $e^{-\beta_M |\Gamma|}$ could become too small in comparison to $e^{-\beta_S \omega}$, making the probe energetically insensitive to this heat exchange and thus the properties of $f(x)$. For point 2 we would like $|\Delta p_0| > c$ where $0 < c < 1 - p_0$ is a sensitivity constant. This imposes the condition $c < |e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|}|/Z_f Z_S$, which leads to tightened versions of the conditions (5). Focusing on cooling, we require $e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|} > c Z_S Z_f$ so that rearranging and taking logarithms give the inequality $\omega/T_S < -\ln(c Z_S Z_f + e^{-\beta_M |\Gamma|})$, where the right-hand side is positive if $0 < c Z_S Z_f + e^{-\beta_M |\Gamma|} \leq 1$. Last, for the temperature of the probe after the query to be well-defined we examine Eq. (4), which can be simplified to $\beta'_S = \omega^{-1} \ln[(1 + Z_S \Delta p_0)/(e^{-\beta_S \omega} - Z_S \Delta p_0)]$. In the case of cooling we require $e^{-\beta_S \omega} > Z_S \Delta p_0$ for heating $1 > Z_S \Delta p_0$ for the term in the logarithm to be positive. The inequalities considered here together with their derivation are given in full in the Appendix.

The probe and machine qubits are embedded in an environment at temperature T_M where the probe is pushed out of equilibrium to T_S in preparation for the query. Resetting the machine after the query via thermalization would cost $E_{\text{Diss}} = \Delta p_0 |\Gamma|$ in dissipation, while setting the probe out of equilibrium again would cost $E_{\text{reset}} = \Delta p_0 \omega$ in work.

VI. READING OUT

Encoding the solution of a decision problem in the temperature of a thermal qubit presents two related difficulties. (1) The temperatures corresponding to different outcomes of the decision problem must be physically distinguishable. (2) The number of samples of the probe required to estimate its temperature, and thus learn the outcome of the problem, should be small in the size of the problem, ideally not exceeding the sample complexity of a classical probabilistic solution,

suggesting an advantage. We examine here these difficulties in the context of reading out the solution to a Deutsch-Jozsa problem via thermal kickback. Consider the GSP of the probe after thermal kickback for the different outcomes $p_0^{\text{Bal}}, p_0^{\text{Const}_1}$, and $p_0^{\text{Const}_2}$, and without loss of generality, let $p_0^{\text{Const}} = p_0^{\text{Const}_1}$, corresponding to $|\Gamma_{\text{Const}}| = NE_1$; then we are interested in $p_0^{\text{Const}} - p_0^{\text{Bal}} > t$, which is positive for $E_1 > E_2$ so that $NE_1 > N/2(E_1 + E_2)$. Since $p'_0 = (1 + Z_S \Delta p_0)/Z_S$, the inequality of interest becomes $|\Delta p_0^{\text{Const}} - \Delta p_0^{\text{Bal}}| > t$. We will consider the case where the query cools the probe so that $\Delta p_0^{\text{Const}}, \Delta p_0^{\text{Bal}} > 0$. We then have $\frac{e^{-\beta_S \omega} - e^{-\beta_M |\Gamma_{\text{Const}}|}}{Z_f^{\text{Const}} Z_S} - \frac{e^{-\beta_S \omega} - e^{-\beta_M |\Gamma_{\text{Bal}}|}}{Z_f^{\text{Bal}} Z_S} > t$ in general. The maximal cooling is possible when the probe is initially in the maximally mixed state where $e^{-\beta_S \omega} = 1$ and $Z_S = 2$. In this setting the above condition simplifies to a constraint on the difference between the largest population of the oracle's machine in the two cases

$$\frac{1}{Z_f^{\text{Const}}} - \frac{1}{Z_f^{\text{Bal}}} > 2t, \quad (6)$$

as detailed in the Appendix, where we also provide a plot showing values of t reachable for different E_1, E_2 , and input size n .

Determining whether $f(x)$ is balanced or constant reduces to a *classical* binary hypothesis testing scenario [31]. Here, with access to samples τ'_S of the postinteraction probe and energy basis measurements, an agent must determine whether S' is in temperature T_{Bal} (hypothesis A) or not in T_{Bal} (hypothesis B), i.e., it is in temperature T_{Const_1} or T_{Const_2} . This is a classical hypothesis test since $\tau'_S(T_{\text{Bal}})$ and T_{Const} are diagonal in the energy eigenbasis and so commute. As long as the type-A false-positive error δ can be controlled such that $0 < \delta < 1$, then the Chernoff-Stein lemma [31] states that the number of samples n^* of τ'_S required to determine the scenario is lower bounded by $\ln(1/\delta)/D[\tau'_S(T_{\text{Bal}})||\tau'_S(T_{\text{Const}})]$, where, without loss of generality, $\tau'_S(T_{\text{Const}})$ corresponds to the temperature T_{Const_1} or T_{Const_2} , which minimizes the relative entropy $D[\tau'_S(T_{\text{Bal}})||\tau'_S(T_{\text{Const}})]$. Using Pinsker's inequality [31] $D[\tau'_S(T_{\text{Bal}})||\tau'_S(T_{\text{Const}})] > 2d_{\text{TV}}(\tau'_S(T_{\text{Bal}}), \tau'_S(T_{\text{Const}}))^2$, where $d_{\text{TV}}(\cdot, \cdot)$ is the total variation distance, and because these are qubit thermal states and thus binary distributions, we have $d_{\text{TV}}(\tau'_S(T_{\text{Bal}}), \tau'_S(T_{\text{Const}})) = |p_0^{\text{Bal}} - p_0^{\text{Const}}|$. When the probe satisfies the distinguishability condition (6), we thus have the lower bound

$$n^* > \frac{\ln(1/\delta)}{2t^2}, \quad (7)$$

which is independent of the problem size! Given that this is a classical binary hypothesis test, this lower bound is asymptotically achievable using a likelihood ratio test [31].

VII. DISCUSSION AND CONCLUSION

As long as one can energetically ensure the distinguishability condition (6), the number of samples of S' required to solve the Deutsch-Jozsa problem is independent of n ; e.g., letting $\delta = t = 0.1$, one requires at least $50 \ln(10) \approx 116$ samples for any n using the bound (7). This suggests that while the quantum solution requires one query and one measurement and the classical solution requires $2^{n-1} + 1$ queries, quantum thermodynamics offers an intermediate regime

requiring one query and a constant number of samples $\Theta(\frac{\ln(1/\delta)}{2t^2})$. Although not advantageous over classical approaches for small n , this appears to offer an advantage for intermediate to large n . In particular, for the parameters $\delta = t = 0.1$ the sample complexity is lower than the classical query complexity for deterministic solutions at $n > 8$. If one considers probabilistic solutions to the Deutsch-Jozsa problem which allow for false-positive errors, there is no longer an exponential separation between classical and quantum approaches [32]. We examine how our sample complexity compares to that of a probabilistic classical approach in the Appendix and find a proof-of-principle advantage dependent on the size of t .

In this context, there are two ways to interpret having access to multiple samples of the postinteraction probe. First, since τ'_S is diagonal in the energy eigenbasis, a single sample may be used to unitarily reconstruct τ'_S in n^* pure ancillas (e.g., using a cnot conditioned on S acting on an ancilla), allowing for the desired repeated measurements. Second, the probe and machine can be thermodynamically reset as described earlier, allowing for repeated sampling.

One might wonder where the *quantumness* in our model is since no coherences were generated. Indeed, under which parameters autonomous thermal machines are genuinely quantum, i.e., their dynamics cannot be described by a purely stochastic master equation, is an active research question [33]. An interesting direction for future work would be to relate the distinguishability t to the quantumness of the model, thereby showing that sample complexity advantages over probabilistic classical models stem from genuinely quantum effects.

In this work we examined an agent's ability to learn the global properties of Boolean functions encoded in the energetic structure of a thermal machine via a heat exchange with a qubit. This serves as a fascinating model for the physics of quantum query complexity and a provocation to think of alternative models of quantum computation. In the Appendix we provide a sketch of an experimental implementation of the ideas discussed.

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DATA AVAILABILITY

No data were created or analyzed in this study.

APPENDIX A: DETAILED DERIVATIONS

1. Thermal kickback temperature

To obtain the probe qubit temperature (4) after the query, which determines whether $f(x)$ is constant or balanced, let

us consider the following. First, the ground-state population of a thermal qubit with Hamiltonian $H = 0|0\rangle\langle 0| + \omega|1\rangle\langle 1|$ at inverse temperature β can be expressed as $p_0 = \frac{1}{1+e^{-\beta\omega}}$. Then, solving for the inverse temperature β , we obtain $\beta = \frac{1}{\omega} \ln(p_0/1 - p_0)$. Next, we find the ground-state population of the probe after the query (3) to be $p'_0 = \frac{1 + \mathcal{Z}_f^{-1}(e^{-\beta_S\omega} - e^{-\beta_M|\Gamma|})}{\mathcal{Z}_S}$ and note that the change in ground-state population is therefore

$$\Delta p_0 = p'_0 - p_0 \quad (A1)$$

$$= \frac{1 + \mathcal{Z}_f^{-1}(e^{-\beta_S\omega} - e^{-\beta_M|\Gamma|})}{\mathcal{Z}_S} - \frac{1}{\mathcal{Z}_S} \quad (A2)$$

$$= \frac{e^{-\beta_S\omega} - e^{-\beta_M|\Gamma|}}{\mathcal{Z}_f \mathcal{Z}_S}, \quad (A3)$$

so that $\mathcal{Z}_f^{-1}(e^{-\beta_S\omega} - e^{-\beta_M|\Gamma|}) = \mathcal{Z}_S \Delta p_0$. Finally, to obtain β'_S , the inverse temperature of the probe qubit after the thermal machine oracle is queried, we substitute p'_0 into the logarithmic expression for β given earlier to obtain

$$\begin{aligned} \ln\left(\frac{p'_0}{1 - p'_0}\right) &= \ln(p'_0) - \ln(1 - p'_0) \\ &= \ln\left(\frac{1 + \mathcal{Z}_S \Delta p_0}{\mathcal{Z}_S}\right) \\ &\quad - \ln\left(\frac{\mathcal{Z}_S - 1 - \mathcal{Z}_S \Delta p_0}{\mathcal{Z}_S}\right) \end{aligned} \quad (A4)$$

$$= \ln(\mathcal{Z}_S) - \ln(\mathcal{Z}_S) + \ln(1 + \mathcal{Z}_S \Delta p_0) - \ln(\mathcal{Z}_S - 1 - \mathcal{Z}_S \Delta p_0) \quad (A5)$$

$$= \ln\left(\frac{1 + \mathcal{Z}_f^{-1}(e^{-\beta_S\omega} - e^{-\beta_M|\Gamma|})}{\mathcal{Z}_S - \mathcal{Z}_f^{-1}(e^{-\beta_S\omega} - e^{-\beta_M|\Gamma|}) - 1}\right), \quad (A6)$$

where we find

$$\beta'_S = \frac{1}{\omega} \ln\left(\frac{1 + \mathcal{Z}_f^{-1}(e^{-\beta_S\omega} - e^{-\beta_M|\Gamma|})}{\mathcal{Z}_S - \mathcal{Z}_f^{-1}(e^{-\beta_S\omega} - e^{-\beta_M|\Gamma|}) - 1}\right). \quad (A7)$$

For a general $V(X)$ as described in the main text, the change in temperature of the probe is instead

$$\begin{aligned} \beta'_S(X) &= \frac{1}{\omega} \ln\left(\frac{1 + \mathcal{Z}_f^{-1}(e^{-\beta_S\omega - \beta_M(|\Gamma| - X \cdot \Gamma)} - e^{-\beta_M X \cdot \Gamma})}{e^{-\beta_S\omega} - \mathcal{Z}_f^{-1}(e^{-\beta_S\omega - \beta_M(|\Gamma| - X \cdot \Gamma)} - e^{-\beta_M X \cdot \Gamma})}\right) \\ &= \frac{1}{\omega} \ln\left(\frac{1 + \mathcal{Z}_S \Delta p_0(X)}{e^{-\beta_S\omega} - \mathcal{Z}_S \Delta p_0(X)}\right). \end{aligned} \quad (A8)$$

In Fig. 4 we plot the thermal kickback temperature for a minimal two bit Deutsch-Jozsa problem.

2. Probe conditions for distinguishability

For distinguishability we require that the difference between the ground-state population of the probe after interaction for different cases be larger than a threshold value t . In the Deutsch-Jozsa case without loss of generality, let $p_0^{\text{Const}} = p_0^{\text{Const}_1}$, corresponding to $|\Gamma_{\text{Const}}| = NE_1$; then we are interested in

$$p_0^{\text{Const}} - p_0^{\text{Bal}} > t, \quad (A9)$$

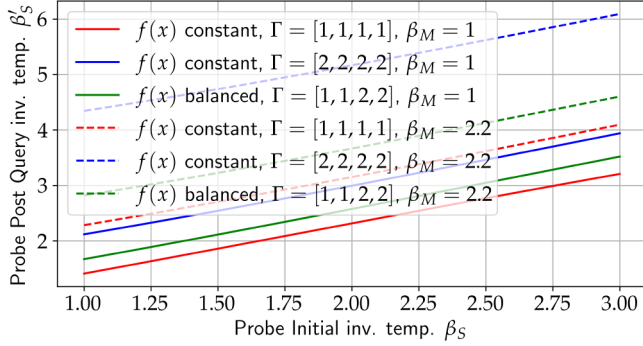


FIG. 4. Plot showing the inverse temperature of the probe with $\omega = 1$ after thermal query β'_S against the initial temperature β_S of the probe for the two-bit DJ problem. We see that the different outcomes are less distinguishable because the thermal-oracle qubits become warmer, i.e., smaller β_M (solid lines).

which is positive for $E_1 > E_2$ so that $NE_1 > N/2(E_1 + E_2)$. Substituting the form of the ground-state population after thermal kickback (3), we have

$$\frac{1 + \mathcal{Z}_S \Delta p_0^{\text{Const}}}{\mathcal{Z}_S} - \frac{1 + \mathcal{Z}_S \Delta p_0^{\text{Bal}}}{\mathcal{Z}_S} > t, \quad (A10)$$

$$\Delta p_0^{\text{Const}} - \Delta p_0^{\text{Bal}} > t.$$

We will consider the case where the query cools the probe so that $\Delta p_0 > 0$ for both scenarios. Expanding using Eq. (A3), we have

$$\frac{e^{-\beta_S \omega} - e^{-\beta_M |\Gamma_{\text{Const}}|}}{\mathcal{Z}_f^{\text{Const}} \mathcal{Z}_S} - \frac{e^{-\beta_S \omega} - e^{-\beta_M |\Gamma_{\text{Bal}}|}}{\mathcal{Z}_f^{\text{Bal}} \mathcal{Z}_S} > t, \quad (A11)$$

$$e^{-\beta_S \omega} (\mathcal{Z}_f^{\text{Bal}} - \mathcal{Z}_f^{\text{Const}}) - \chi > t \mathcal{Z}_S \mathcal{Z}_f^{\text{Bal}} \mathcal{Z}_f^{\text{Const}}, \quad (A12)$$

where we have cross multiplied to obtain a common denominator, then multiplied both sides by the common denominator $\mathcal{Z}_S \mathcal{Z}_f^{\text{Bal}} \mathcal{Z}_f^{\text{Const}}$, and adopted the notation $\chi = \mathcal{Z}_f^{\text{Const}} e^{-\beta_M |\Gamma_{\text{Bal}}|} - \mathcal{Z}_f^{\text{Bal}} e^{-\beta_M |\Gamma_{\text{Const}}|}$. The strongest scenario for cooling involves the probe initially in a maximally mixed state which sets $e^{-\beta_S \omega} = 1$ and $\mathcal{Z}_S = 2$ in this inequality. Also note that χ is exponentially suppressed and can be omitted up to additive error $O(\chi)$, giving the condition $\mathcal{Z}_f^{\text{Bal}} - \mathcal{Z}_f^{\text{Const}} > 2t \mathcal{Z}_f^{\text{Bal}} \mathcal{Z}_f^{\text{Const}}$, which leads to

$$\frac{\mathcal{Z}_f^{\text{Bal}} - \mathcal{Z}_f^{\text{Const}}}{\mathcal{Z}_f^{\text{Bal}} \mathcal{Z}_f^{\text{Const}}} > 2t \iff \frac{1}{\mathcal{Z}_f^{\text{Const}}} - \frac{1}{\mathcal{Z}_f^{\text{Bal}}} > 2t. \quad (A13)$$

Note that $1/\mathcal{Z}_f$ is the ground-state population of the thermal machine oracle; thus, this inequality communicates that a difference of t in ground-state populations of the probe for different cases is possible only if it corresponds to a difference of $2t$ in machine GSPs for these two cases. A plot showing which distinguishability values t are obtainable for a thermal machine oracle with $\beta_M = 1$ and various oracle qubit energies E_1 and E_2 is given in Fig. 5.

APPENDIX B: EXPERIMENTAL IMPLEMENTATION

As a stimulating example of the experimental feasibility of our ideas, consider that an implementation of the three-

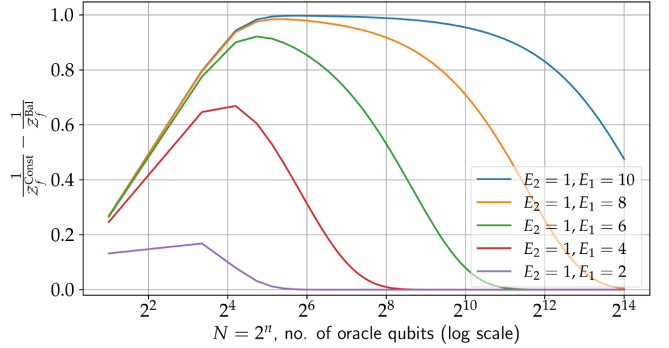


FIG. 5. Plot showing the left-hand side of inequality (6) $(1 + e^{-\beta_M E_1})^{-N} - (1 + e^{-\beta_M E_1})^{-N/2} (1 + e^{-\beta_M E_2})^{-N/2}$ for $\beta_M = 1$ and various oracle qubit energies E_1 and E_2 , showing that $t \in [0, 0.5]$ is achievable with different energies for different n .

bit Bernstein-Vazirani oracle presented can be implemented in a quantum thermodynamics experiment [35] using a quantum dot [36–38] or transmon qubit [39,40] setup. Here, three quantum dots or transmon qubits M_1, M_2 , and M_3 are each prepared in an individual thermal state at the same temperature T_M , with gate voltages or radio-frequency (rf) pulses, respectively, being used to modulate their energetic gaps $\Delta_1(s_1)$, $\Delta_2(s_2)$, and $\Delta_3(s_3)$ such that $\Delta_i(s_i) = s_i \gamma_i$, with γ_i being a fixed tunable energy parameter, e.g., gate voltage or rf pulse for qubit M_i , and $s_i \in \{\epsilon_1, \epsilon_2\}$ being an energetic bias which encodes the unknown string. A random number generator can then be used to shift the energetic gaps by s_1, s_2 , and s_3 . A fourth probe qubit is prepared in a thermal state at temperature $T_S > T_M$ with gap ω , with the goal of estimating s by exchanging heat with M . The machine gaps γ_1, γ_2 , and γ_3 would need to be engineered, and ϵ_1 and ϵ_2 would need to be small enough that $\omega \approx \Delta_1(s_1) + \Delta_2(s_2) + \Delta_3(s_3)$ for all possible strings $s_1 s_2 s_3 \in \{0, 1\}^3$, ensuring that an energy-selective coherent Rabi flip-flop interaction can occur between the $|0111\rangle$ and $|1000\rangle$ energy levels via the interaction Hamiltonian

$$H_{\text{query}} = g(\sigma_-^{(S)} \sigma_+^{(M_1)} \sigma_+^{(M_2)} \sigma_+^{(M_3)} + \text{H.c.}), \quad (B1)$$

which will effectively generate the desired U_{query} . The gap structure of the three oracle qubits is then estimated from a temperature reading of the probe qubit S to determine the secret string. This presents the challenge of such an experiment implementing a choice of parameters for which ϵ_1 and ϵ_2 are small enough to allow enough coherent heat exchange (minimizing detuning) but not so small that different strings are no longer thermodynamically distinguishable. In particular, we would like the energetic condition $\omega = s_1 \gamma_1 + s_2 \gamma_2 + s_3 \gamma_3$ to be satisfied for any secret string $s_1 s_2 s_3 \in \{\epsilon_1, \epsilon_2\}^3$. This is clearly not possible in general, so let us examine what happens if one approximately relaxes this condition, introducing some detuning. How would this imperfection influence the distinguishability of the temperatures corresponding to different strings? Let us assume that these bias energies take the form $\epsilon_1 = 1 - \epsilon$ and $\epsilon_2 = 1 + \epsilon$ for a small $\epsilon > 0$; then the total oracle energy deviates slightly from ω , producing a detuning for a given string s , $\delta(s) = s_1 \gamma_1 + s_2 \gamma_2 + s_3 \gamma_3 - \omega$. To keep the coherent Rabi oscillation strength large, the detuning must satisfy $\delta(s) \ll g$, where g is the interaction strength. In this

regime, the flip-flop probability is given by

$$P_{\text{flip}}(t) \approx \frac{g^2}{g^2 + \delta^2} \sin^2 \left(\frac{1}{2} \sqrt{g^2 + \delta^2} t \right), \quad (\text{B2})$$

indicating that population transfer remains high for small detuning. For short times t this simplifies to $\eta(\delta) = \frac{g^2}{g^2 + \delta^2}$, which acts as a suppression factor in the change in population inverse temperature of the probe system,

$$\beta'_S(\delta) = \frac{1}{\omega} \ln \left(\frac{1 + \eta(\delta) \mathcal{Z}_f^{-1}(e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|})}{\mathcal{Z}_f - \eta(\delta) \mathcal{Z}_f^{-1}(e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|}) - 1} \right). \quad (\text{B3})$$

We plot the inverse temperature of the probe after querying the oracle in this realization of the three-bit Bernstein-Vazirani example in Fig. 5.

APPENDIX C: PREPARING A THERMAL MACHINE ORACLE

In this section we will consider how an oracle with the ability to call a Boolean function $f(x)$ for a given input x could construct a thermal machine as described in the main text. Let us consider a channel $\mathcal{T}(\cdot)$ which takes as input an n -bit string x and a single-qubit pure state $|0\rangle$, where $x \in \{0, 1\}^{\times n}$ encodes the input for which the oracle is querying the function and $|0\rangle$ is qubit initialized in a pure state which will come to form part of the machine. The channel $\mathcal{T}(\cdot)$ will be formed of Kraus operators

$$K_x = |x\rangle\langle x| \otimes [(1 - p_x)\mathbb{1} + p_x \sigma_x], \quad (\text{C1})$$

which are conditional amplitude-damping or thermalization channels. In particular, for a given input x the oracle computes $E_x = f(x) + E$, where $E > 0$ is an arbitrary fixed positive constant, by querying $f(x)$ and obtains the probability $p_x = e^{-\beta_M E_x} / (1 + e^{-\beta_M E_x})$, which defines the parameter in the amplitude-damping channel. From here we see that the oracle will be able to obtain the thermal qubit

$$\tau_x = \frac{1}{1 + e^{-\beta_M E_x}} (|0\rangle\langle 0| + e^{-\beta_M E_x} |1\rangle\langle 1|) \quad (\text{C2})$$

for each input string $x \in \{0, 1\}^{\times n}$ by applying $\mathcal{T}(|x\rangle\langle x| \otimes |0\rangle\langle 0|)$, allowing them to construct the thermal machine described in the main text with access to an n -bit classical register and 2^n qubits, which will be used to generate the thermal machine.

Thermodynamic cost of a query

We present here for completeness the derivations of the inequalities discussed in Sec. V in full. Recall as presented in Appendix A 1 that

$$\Delta p_0 = p'_0 - p_0 \quad (\text{C3})$$

$$= \frac{e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|}}{\mathcal{Z}_S \mathcal{Z}_f}. \quad (\text{C4})$$

If the query cools the probe, then $\Delta p_0 > 0$, so $e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|} > 0$, which implies that $-\omega/T_S > -|\Gamma|/T_M$. Similarly, if the query heats the probe, then $\Delta p_0 < 0$, and $e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|} < 0$, which implies that $-\omega/T_S < |\Gamma|/T_M$. To

summarize we can state the conditions on the query as

$$\frac{\omega}{T_S} < \frac{|\Gamma|}{T_M}, \quad \frac{\omega}{T_S} > \frac{|\Gamma|}{T_M} \quad (\text{C5})$$

for cooling and heating, respectively. For the sensitivity condition we require $|\Delta p_0| > c$, with $0 < c < 1 - p_0$, giving the constraint

$$|e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|}| > c \mathcal{Z}_S \mathcal{Z}_f, \quad (\text{C6})$$

which leads to two cases, depending on whether the query heats or cools the probe. If the query cools the probe, then we have $e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|} > 0$, so that Eq. (C6) leads to $e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|} > c \mathcal{Z}_S \mathcal{Z}_f$, and rearranging and taking logarithms lead to $-\frac{\omega}{T_S} > \ln(c \mathcal{Z}_S \mathcal{Z}_f + e^{-\beta_M |\Gamma|})$, so finally, sensitive cooling

$$\frac{\omega}{T_S} < -\ln(c \mathcal{Z}_S \mathcal{Z}_f + e^{-\beta_M |\Gamma|}) \quad (\text{C7})$$

ensures the desired sensitivity if the probe is cooled. Note that this quantity is positive if $0 < c \mathcal{Z}_S \mathcal{Z}_f + e^{-\beta_M |\Gamma|} \leq 1$. In the case of heating we have $e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|} < 0$, so that Eq. (C6) takes the form $e^{-\beta_M |\Gamma|} - e^{-\beta_S \omega} > c \mathcal{Z}_S \mathcal{Z}_f$, and rearranging and taking logarithms lead to $\ln(e^{-\beta_M |\Gamma|} - c \mathcal{Z}_S \mathcal{Z}_f) > -\omega/T_S$, and finally, sensitive heating is

$$\frac{\omega}{T_S} > -\ln(e^{-\beta_M |\Gamma|} - c \mathcal{Z}_S \mathcal{Z}_f), \quad (\text{C8})$$

which is positive for $c \mathcal{Z}_S \mathcal{Z}_f < e^{-\beta_M |\Gamma|} \leq 1 + c \mathcal{Z}_S \mathcal{Z}_f$. These conditions that we have derived for the probe to satisfy the sensitivity we desire for a heat exchange $|\Delta p_0| > c$ can be thought of as tighter versions of the heating and cooling conditions.

Last, to make sure we have a well-defined temperature after heat exchange we require that the denominator and numerator of the fraction in Eq. (4) are both positive in the logarithm. Consider

$$\beta'_S = \frac{1}{\omega} \ln \left(\frac{1 + \mathcal{Z}_f^{-1}(e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|})}{e^{-\beta_S \omega} - \mathcal{Z}_f^{-1}(e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|})} \right) \quad (\text{C9})$$

$$= \frac{1}{\omega} \ln \left(\frac{1 + \mathcal{Z}_S \Delta p_0}{e^{-\beta_S \omega} - \mathcal{Z}_S \Delta p_0} \right). \quad (\text{C10})$$

In the case of cooling we have $\Delta p_0 > 0$, so the numerator is always positive, but for the denominator to be positive we require $e^{-\beta_S \omega} > \mathcal{Z}_S \Delta p_0$, that is, $e^{-\beta_S \omega} > \mathcal{Z}_f^{-1}(e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|})$. This leads to

$$-\frac{\omega}{T_S} > \ln \left(\frac{e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|}}{\mathcal{Z}_f} \right), \quad (\text{C11})$$

$$\frac{\omega}{T_S} < \ln(\mathcal{Z}_f) - \ln(e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|}), \quad (\text{C12})$$

so that up to $O(\ln(e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|}))$ for the temperature to be well-defined after a cooling interaction one requires $\omega/T_S < \ln(\mathcal{Z}_f)$, where the right-hand side is the free energy of the thermal machine oracle. In the case of heating, i.e., $\Delta p_0 < 0$, it is the numerator which can become negative, so we require $1 > \mathcal{Z}_S \Delta p_0$. This leads to $1 > \mathcal{Z}_f^{-1}(e^{-\beta_S \omega} - e^{-\beta_M |\Gamma|})$ and thus the condition

$$\frac{\omega}{T_S} > -\ln(\mathcal{Z}_f + e^{-\beta_M |\Gamma|}), \quad (\text{C13})$$

which is always true for $\omega/T_S > 0$.

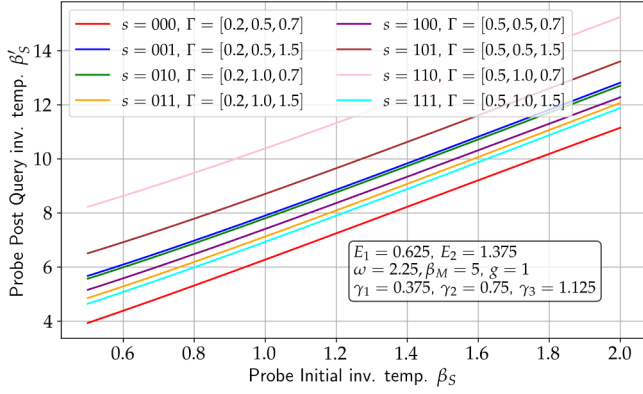


FIG. 6. Plot showing the inverse temperature of the probe after thermal query β'_S against the initial temperature β_S of the probe for the three-bit Bernstein-Vazirani problem, i.e., three-qubit oracle in the presented experimental implementation. In this plot we see that each of the eight potential secret strings is distinguishable in the temperature of the probe after the heat exchange despite the detuning condition $\delta(s)$.

APPENDIX D: COMPARING WITH CLASSICAL PROBABILISTIC APPROACHES TO THE DEUTSCH-JOZSA PROBLEM

An exponential query complexity separation exists for *deterministic* solutions of the Deutsch-Jozsa problem between classical and quantum approaches. A quantum algorithm can solve the problem in a single query, and a classical algorithm requires $2^{n-1} + 1$ queries to determine whether an n -bit function is balanced or constant. Similarly, we have shown that there is an exponential separation between solving this problem via a quantum thermodynamic heat exchange and the classical solution, but how does the sample complexity of the presented solution compare with a classical probabilistic approach?

Let us assume a classical agent is able to randomly sample from a uniform distribution of n -bit strings $\{0, 1\}^{\times n}$ and use this random sampling to determine whether the Boolean function f is balanced or constant. A strategy the agent might take is as follows: They make a hypothesis that $f(x)$ is constant and wish to verify this within an error of δ having sampled k random bit strings $x \in \{0, 1\}^{\times n}$ and examined each of their outputs $f(x)$. That is, δ is a false-positive error in which the agent has sampled $k < 2^{n-1} + 1$ and not yet found an output which alerts them the function is balanced and thus incorrectly thinks the function is balanced.

To obtain an expression for δ , assume that f is balanced but that the output of k samples has to be constant. With replacement, each sample x has a probability of $1/2$ of $f(x)$ being either 0 or 1 for a balanced f . Thus, the probability of getting an output of 0 or 1 k times is 2^k . Therefore, the false-positive probability after k samples is

$$\delta = P(k \text{ 0 outputs}) + P(k \text{ 1 outputs}) \quad (\text{D1})$$

$$= 2^{-k} + 2^{-k} \quad (\text{D2})$$

$$= \frac{1}{2^{k-1}}. \quad (\text{D3})$$

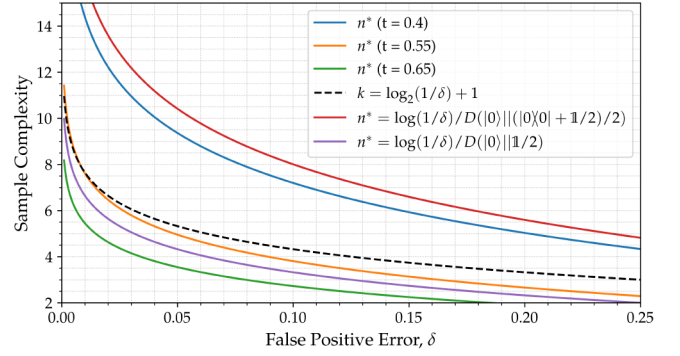


FIG. 7. Plot comparing the sample complexity k for the probabilistic classical approach to the DJ problem and the sample complexity of the approach presented in this paper n^* for different t values against different false-positive error values δ .

Without replacement, i.e., if a bit string x is sampled, it is then discarded from the set of n -bit strings; we instead have that there are $\binom{2^n}{k}$ ways to select k input bit strings. Since f is balanced, it will take half the domain to 0 and half the domain to 1, so that there are $\binom{2^{n-1}}{k}$ ways to obtain each constant outcome. This gives

$$\delta' = P(k \text{ 0 outputs}) + P(k \text{ 1 outputs}) \quad (\text{D4})$$

$$= \frac{\binom{2^{n-1}}{k} + \binom{2^{n-1}}{k}}{\binom{2^n}{k}} = \frac{2 \binom{2^{n-1}}{k}}{\binom{2^n}{k}}, \quad (\text{D5})$$

where $\lim_{k \rightarrow \infty} \delta' = \delta$; that is, the false-positive errors with and without replacement are asymptotically equivalent. Thus, asymptotically, we see that with this random sampling method we require

$$k = \Theta[\ln_2(1/\delta) + 1]. \quad (\text{D6})$$

Having obtained this expression, we are now in a position to make a comparison with the asymptotic sample complexity for the heat exchange approach presented in the main text, where we had $n^* = \Theta[\ln(1/\delta)/2t^2]$, where t is the distinguishability threshold. In Fig. 6 we show a comparison of the sample complexities for different values of t which suggests that a heat exchange approach could obtain a better sample complexity than the classical probabilistic approach if $t \gtrsim 0.55$. A numerical investigation exploring Fig. 5 for different t values suggests that it could be energetically challenging to obtain t values larger than 0.4 with a qubit probe, but it remains an open problem to show a limit on t for probes of arbitrary dimensions and arbitrary unitary heat exchanges beyond the proof-of-principle model presented in the main text.

We also plot the sample complexity (Fig. 7) for the probabilistic bipartite heat exchange presented early in the main text with sample complexity $n^* = \Theta\{\ln(1/\delta)/D[\tau_1|(\tau_1 + \tau_2)/2]\}$. In our model the oracle can at best prepare $\tau_1 = |0\rangle$, leading to the red line. But if we allowed the oracle to prepare $\tau_1 = |0\rangle$ and $\tau_2 = |1\rangle$, even this probabilistic bipartite heat exchange would be classical random sampling.

It is important to note that although the sample complexity in the classical probabilistic setting stems from sampling the function multiple times for different random inputs, the

sample complexity in our model is due to the need to estimate the temperature of the qubit encoding the solution. That is, the solution is already encoded in the state of the

probe after a single heat exchange, but reading this solution out requires a constant number of samples or copies of the probe.

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