

Forward-backward stochastic simulations: Q -based model for measurement and Bell nonlocality consistent with weak local realistic premises

M. D. Reid and P. D. Drummond *Centre for Quantum Science and Technology Theory, Swinburne University of Technology, Melbourne 3122, Australia*

(Received 8 December 2024; accepted 12 November 2025; published 2 January 2026)

While Bell nonlocality can be explained through mechanisms such as retrocausality and superluminal influences, it is not clear why such mechanisms do not manifest at an everyday macroscopic level. In this paper, we show how measurement and nonlocality can be explained consistently with macroscopic realism and no-signaling, and causal relations for macroscopic quantities. Considering the measurement of a field amplitude $\hat{x}$, we derive theorems that lead to an equivalence between a quantum phase-space probability distribution $Q(x, p, t)$ and stochastic trajectories for real amplitudes $x(t)$ and $p(t)$ propagating backward and forward in time, respectively. We present forward-backward stochastic simulations that motivate a Q -based objective-field model of reality. Amplification plays a key role in the measurement. With amplification, similar to decoherence, contributions due to interference become unobservable, leading to “branches” in x that correspond to distinct eigenvalues λ . This elucidates how the system evolves from a superposition to an eigenstate, from which Born’s rule follows. For superposition states, we find that the backward and forward trajectories are connected, leading to “hidden loops.” We deduce a hybrid causal structure involving causal deterministic relations for amplified variables, along with microscopic noise inputs and hidden loops for unobservable quantities. Causal consistency is confirmed, removing grandfather-type paradoxes. The simulations allow evaluation of a state inferred for the system, conditioned on a particular branch λ , from which we deduce a model for projection and the collapse of the wave function. The theory is extended to Einstein-Podolsky-Rosen correlations and Bell nonlocality. We demonstrate consistency with three weak local realistic premises: (1) the existence of real properties (defined for the system after operations that fix measurement settings); (2) a partial locality implying no-signaling; (3) “elements of reality” that apply to the predictions of a system by a meter, once meter settings are fixed. A Q -based hidden variable model and mechanism for Bell nonlocality is identified. Our work shows how forward-backward stochastic simulations lead to a hybrid causal structure, involving both deterministic causal relations and “hidden” stochastic loops, explaining measurement and entanglement, with paradoxes typically associated with retrocausality avoided.

DOI: [10.1103/qdyg-2nv4](https://doi.org/10.1103/qdyg-2nv4)

I. INTRODUCTION

All local, strictly causal hidden variable theories are falsifiable, since quantum predictions and experiments lead to a violation of Bell inequalities [1–4]. However, it is known that violations of Bell inequalities can indeed arise using the advanced solutions of classical electrodynamics [5–8]. Classical electrodynamics with future boundary conditions has been widely studied [9–14] and is essential to the theory of radiating fields. Following Bohr [15], who considered delayed-choice experiments [16], Wheeler speculated that retrocausality due to future boundary conditions may explain quantum paradoxes [17,18]. Bell’s work also motivated the question of potential superluminal disturbances, leading to no-signaling theorems [19]. These results have inspired analyses of causality in quantum physics [16,20–82].

Studies of causal structure are closely related to studies of realism in quantum mechanics. Classical models assume real properties that describe the system at a given time; retrocausality supposes these properties to be influenced by future events. Proposals to explain Bell violations and wave-function collapse include superluminal causal influences [83–88], retrocausality [5,8,28,29,47,89–94], cyclic causation [95,96], and unobservable loops [97–99], or superdeterminism [100]—but an open question is why then are such mechanisms not apparent at a macroscopic level? It has not been clear how to reconcile Bell nonlocality with the absence of superluminal signaling. Causal models based on superluminal or retrocausal mechanisms require a fine-tuning of parameters in order to explain no-signaling, which seems undesirable [53–55]. The fine-tuning may not always be a problem, however, and no-signaling does not rule out retrocausal or cyclic causation [29,92,95,96].

As well as no-signaling, we argue that macroscopic realism is a natural requirement for a model complying with causality at a macroscopic level. Macroscopic realism (MR) is the assumption that a system in a superposition of two macroscopically distinct states, “will at all times be in one

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI.

or other of these states” [101] (something that cannot be changed by a future event). Yet, Schrödinger argued that MR is inconsistent with the notion that quantum mechanics is a complete description of reality (“realdings”) [102]: Quantum causal models have been developed to address these problems [56,57,59–63,76]. It is not clear, however, whether one can present a unified framework combining causality and realism, and there remains the fundamental question of the level (from microscopic to macroscopic) at which causal concepts, or MR, will hold.

In this paper, we show how quantum stochastic phase-space theorems lead to stochastic simulations of real amplitudes x and p propagating in a backward and forward time-direction, respectively, that may contribute toward a resolution of these questions. The amplitudes are based on the well-known positive phase-space distribution, the Q function in quantum optics [103], and have been proposed as a way to address the “measurement problem” [104–109]. The measurement problem is to understand how a system, when being measured, evolves from a superposition to a single eigenstate (the “collapse of the wave function”) [4,110–113]. Questions arise, such as the timing of the “collapse” (when can the system be considered to be in the eigenstate), and Born’s question, of “what is the reality which our theory has been invented to describe?” [113]. These questions are closely related to the questions of cause and effect in Einstein-Podolsky-Rosen (EPR) and Bell experiments [53]. What hidden-variable theories are possible that describe the EPR and Bell correlations, and how would nonlocality be achieved, when there is no-signaling? Bohm’s pilot-wave hidden variable theory for quantum mechanics is deterministic [83], and the Pusey-Barrett-Rudolph theorem excludes most epistemic models [114].

In this paper, we seek to resolve these questions in the context of a model of reality motivated by the Q function. We analyze the measurement of $\hat{x}$ on a superposition of eigenstates $|x_j\rangle$ of $\hat{x}$, which we define as highly squeezed states in $\hat{x}$. Measurement of $\hat{x}$ is treated as amplification of $\hat{x}$, by a factor $G = e^{gt}$ (g is real) modeled by a Hamiltonian H_{amp} [104]. In the second part of the paper, we apply the model of reality to deduce how hidden variables satisfying weak forms of local realism put forward in Ref. [108,109] can explain EPR correlations and Bell nonlocality, consistently with no-signaling. This gives insight into a mechanism for Bell nonlocality, which is consistent with causality at a macroscopic level.

The resolution we propose involves an equivalence between the quantum state of the system at a fixed time t , which is given uniquely by the Q function $Q(x, p, t)$, and a set of amplitudes $\{x(t), p(t)\}$ that are solutions (“trajectories”) of a pair of forward-backward stochastic equations (Fig. 1). In this paper, we solve a pair of stochastic differential equations that model the measurement as H_{amp} : one (for p) is solved in the forward-time direction with a boundary condition at the initial time t_0 ; the other (for x) is solved in the backward-time direction with a boundary condition (referred to as a future boundary condition, FBC [12]) at the final time, t_f . Put simply, the initial-time boundary condition is determined by the initial state of the system. The final-time boundary condition is determined by the final state of the system. However, the use of an FBC to resolve problems about reality seems

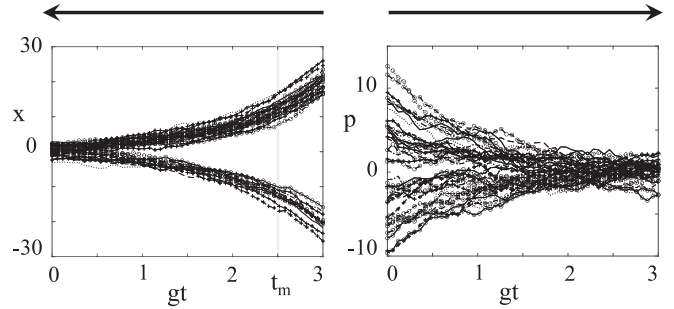


FIG. 1. Forward-backward stochastic simulation modeling quantum measurement. Individual trajectories for x and p model a measurement $\hat{x}$ on a system prepared in a superposition $|\psi_{\text{sup}}\rangle = \frac{1}{\sqrt{2}}(|x_1\rangle + i| -x_1\rangle)$ [Eq. (3.7) with $r = 2$] of eigenstates of $\hat{x}$. The system is amplified by H_{amp} [Eq. (2.2)], where $t_0 = 0$. The variable x is amplified (left) and p deamplified (right). The trajectories for x propagate in the negative-time direction from a future boundary condition at time $t_f = 3/g$; those for p propagate forward in time. Two branches for x are evident at time t_m when the system is macroscopic. Here, $x_1 = 0.8$, $G = e^{gt}$. (Refer to Secs. II and III.)

counterintuitive. The “paradox of the future boundary condition” is introduced—how can such a condition be consistent with our understanding of cause and effect? How can we determine what the FBC condition should be if we do not know the solutions in advance? A similar paradox applies to retrocausality, conventionally defined as the past affecting the future.

It is often also speculated that a phase-space interpretation of quantum mechanics is not possible, since the obvious candidate, the Wigner function $W(x, p)$, can be negative [115,116]. The Q function, being a Gaussian convolution of $W(x, p)$, has additional noise and is hence always positive—but then the x and p do not correspond to directly observable quantities. The Q function of an eigenstate $|x_j\rangle$ of $\hat{x}$ is a Gaussian distribution about a mean value of x_j . The fluctuations about the mean are at the level of the quantum vacuum, but the physical significance is unclear, since (for the eigenstate) these fluctuations must remain undetected in the measurement process. In this paper, we clarify the role of this “hidden noise” as part of the resolution of the FBC paradox.

The solutions we put forward resolve these paradoxes by avoiding traditional notions of retrocausality, instead leading to a different “hidden” type of causation, originating from microscopic inputs δx at the future boundary [109]. In the first part of this paper, we analyze the measurement problem by considering the measurement of $\hat{x}$ on a system prepared in a superposition $|\psi_{\text{sup}}\rangle = \sum_j c_j |x_j\rangle$ (c_j are probability amplitudes) of eigenstates $|x_j\rangle$ of $\hat{x}$. We find that the future boundary condition (FBC) for x has two parts: The first part originates from a deterministic causal relation $x_j \rightarrow Gx_j$, from the initial time t_0 to the final time t_f , that describes the amplification of $\hat{x}$. The first part of the FBC is hence of a classical nature, originating from the description of the state of the system at the initial time. Moreover, we show (Result III.1 of this paper) that the interference terms $\mathcal{I}nt$ that distinguish the superposition $|\psi_{\text{sup}}\rangle$ from a mixture of $|x_j\rangle$ vanish in the FBC. In other words, the amplification process plays a role similar to decoherence: This works, because the preparation

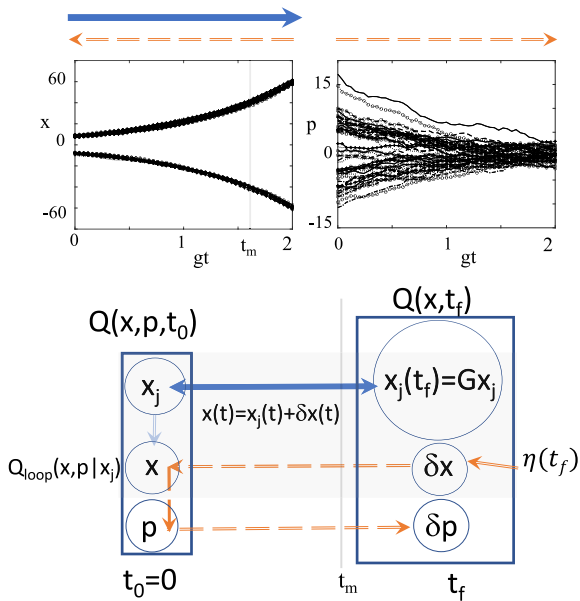


FIG. 2. Causal structure. (top) As for Fig. 1, with $x_1 = 8$ and $t_f = 2/g$. Here, the system starts in a superposition of macroscopically distinguishable states. Each branch follows the causal relation $x(0) \rightarrow Gx(0)$ (depicted by the solid blue line). (lower) Diagram of the causal relations applied to the simulation. The Q function $Q(x, p, t_0)$ describes the system at time t_0 . A future boundary condition is determined by the marginal $Q(x, t_f)$ in x , after amplification for a time t_f . A deterministic relation (solid two-way blue arrow) connects each x_j at time t_0 to the amplified value Gx_j at t_f . Gaussian noise $\delta x \equiv \eta(t_f)$ enters at the future boundary t_f . A connection between forward and backward trajectories $x(t)$ and $p(t)$ at time t_0 exists, which defines a distribution $Q_{\text{loop}}(x, p|x_j)$, giving rise to a “hidden loop” (orange dashed lines). (Refer to main text and Secs. III–V for details).

procedure has implemented the particular choice of measurement setting $\hat{x}$, and to fully describe the initial state, it is necessary that $|\psi_{\text{sup}}\rangle$ be expanded in terms of the eigenstates of $|x_j\rangle$, referred to as the measurement basis. This defines particular interference terms $\mathcal{Int}$ in $Q(x, p, t_0)$, which vanish in the FBC.

The second part of the future boundary condition for x constitutes a random microscopic input $\delta x \equiv \eta(t)$, the distribution of which is a Gaussian $\mathcal{G}(0, \sigma_{\text{vac}})$, with a mean of zero and a variance σ_{vac} at the level of quantum vacuum. The fluctuation δx has a constant average magnitude with time t , which hence corresponds to and explains the hidden noise level in the Q function of the eigenstate $|x_j\rangle$. The result is the emergence in the simulations of macroscopic “branches” $\mathcal{B}_j$ —distinct bundles of amplitudes $x(t)$, appearing as “lines” with a width $\approx \delta x$ that corresponds to a definite eigenvalue x_j (Figs. 1 and 2). Extending previous work [104–106, 109], this leads to a model of reality to explain quantum measurement, from which Born’s rule follows.

The *paradox of the future boundary condition* is hence addressed by the hybrid nature of the boundary condition (Fig. 2). There is no genuine backward causation, because the future inputs δx are randomly determined by the Gaussian $\mathcal{G}(0, \sigma_{\text{vac}})$, being independent of the time t_f at which the FBC

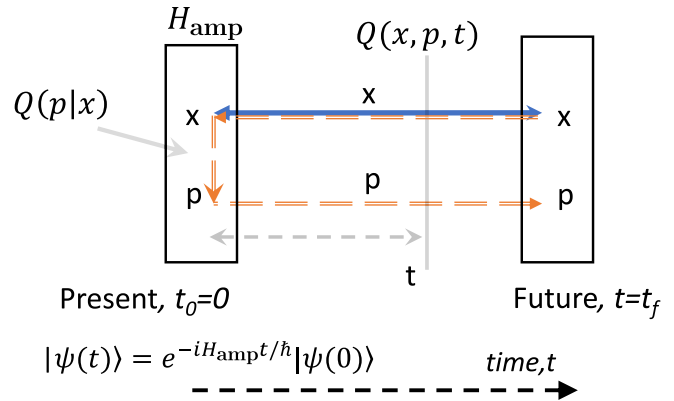


FIG. 3. Test of causal consistency. How can a future boundary condition be consistent with the causal behavior of the Q function? The diagram illustrates the equivalence of the probability density of the forward- and backward-propagating $p(t)$ and $x(t)$ to the Q function $Q(x, p, t)$, which defines the quantum state $|\psi(t)\rangle$. The system evolves under amplification H_{amp} . The equivalence is true for all t_f , even when t_f is far into the future. That the state at time t does not depend on the future inputs is explained by the causal structure of the simulation. An experimental test is feasible (refer to Sec. V).

is placed, and not determined by any choice of setting or outcome x_j at the final time t_f . As we show in the second part of the paper, this model can explain Bell nonlocality.

A main conclusion of this paper is that macroscopic realism (MR) as well as no-signaling follows naturally from the stochastic solutions and the proposed model of reality. The deterministic relation $x_j \rightarrow Gx_j$ that forms part of the future boundary condition (FBC) dominates the dynamics for the amplitudes $x(t)$, at the times t_m when the $x(t_m)$ are macroscopic. The branches have emerged at time t_m (Fig. 1), allowing definition of *macroscopic variables* $\tilde{\lambda}_x(t_m)$, the values of which represent the measurable property x of a system at a given time t_m , a property that can be detected. A main conclusion of this paper is that there is *no actual retrocausality*, because $\tilde{\lambda}(t_m)$ has a fixed value at the time t_m . A change to its value *requires a further local interaction*, which changes $\tilde{\lambda}(t)$ at a *future* time $t_{m2} > t_m$. Macroscopic realism and macroscopic causality [defined as causal relations for the variables $\tilde{\lambda}_x(t_m)$, refer to Sec. IB], hold (Fig. 2). This is despite the first-glance apparent “retrocausality” of the model in the form of the backward-in-time propagating solutions and the FBC.

Consistent with these results, we are able to prove *causal consistency*, as in absorber theory [8,13,14] and theories of closed timelike curves [117,118]. Causal consistency is defined as the requirement that the joint density of the amplitudes $\{x(t), p(t)\}$, when averaged over the “branches,” is given by the Q function $Q(x, p, t)$, at any time t during the evolution (Fig. 3). The proof is based on a conditional constraint determined by $Q(x, p, t_0)$ at the initial time. For a given branch $\mathcal{B}_j$ (associated with the eigenvalue x_j), the deterministic relation $x_j \rightarrow Gx_j$ when inverted implies a set of values around x_j at time t_0 . A correlation between the measured and complementary variables x and p at time t_0 allows a connection between backward and forward-propagating amplitudes, which we refer to as a “*hidden loop*.” The loop appears for

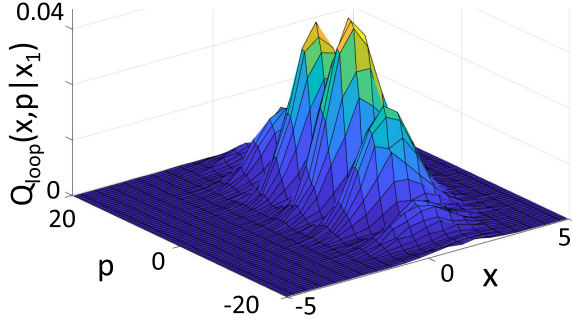


FIG. 4. Schrödinger’s cat paradox. How can cat states be consistent with macroscopic realism? A system is prepared in the superposition $|\psi_{\text{sup}}\rangle = \frac{1}{\sqrt{2}}(|x_1\rangle + i| -x_1\rangle)$ of eigenstates $|x_j\rangle$ of $\hat{x}$. The measurement of “which state the cat is in” proceeds by amplification of $\hat{x}$, via H_{amp} (Fig. 2). The simulation enables evaluation of $Q_{\text{loop}}(x, p|x_1)$, the postselected distribution inferred for the “cat” at the initial time, given that the final outcome is x_1 . It can be shown that $Q_{\text{loop}}(x, p|x_1)$ is not equivalent to a quantum state $|\psi\rangle$ (refer to Sec. IV), highlighting the argument for the incompleteness of quantum mechanics put forward by Schrödinger [102]. Here, $x_1 = -x_2 = 1$.

superpositions of $|x_j\rangle$, but not for mixtures, and is hence a distinguishing quantum feature.

Following from this, we show in this paper how a probability distribution that we call $Q_{\text{loop}}(x, p, t_0|x_j)$ for x and p at the initial time t_0 conditioned on a given branch B_j (associated with an outcome x_j for $\hat{x}$) can be defined and calculated. Perhaps surprisingly, (since the properties of the inputs δx at the FBC are unchanged with t) this *postselected distribution* $Q_{\text{loop}}(x, p, t_0|x_j)$ is inferred by *retrodiction* (not retrocausality), by reversing the deterministic relation $x_j \rightarrow Gx_j$, and using knowledge of the initial distribution $Q(x, p, t_0)$. Calculation shows that the amplitudes x and p of the distribution $Q_{\text{loop}}(x, p, t_0|x_j)$ are too precisely defined to correspond to a quantum state. This leads to a hidden-variable (HV) model for measurement on a superposition state, and an analysis of the Schrödinger cat paradox (Fig. 4).

Entanglement. In Part II of this paper, the results are extended to treat Einstein-Podolsky-Rosen (EPR) entanglement and Bell nonlocality. This enables the construction of a Q -based hidden-variable (HV) model for nonlocality, and also a full examination of quantum measurement. In particular, we provide a model for *projection* (“collapse of the wave function”) by analyzing the coupling of a system to a meter. We demonstrate forward-backward stochastic simulations of entangled systems, A and B , described by a Q function $Q(\lambda, t_0)$ where $\lambda \equiv (x_A, p_A, x_B, p_B)$ are phase-space amplitudes.

To examine entanglement, we allow for adjustable measurement settings, θ and ϕ , by considering measurement of quadrature phase amplitudes $\hat{x}_{\theta A} = \hat{x}_A \cos \theta + \hat{p}_A \sin \theta$ and $\hat{x}_{\phi B} = \hat{x}_B \cos \phi + \hat{p}_B \sin \phi$ at sites A and B , respectively. The $\hat{x}_K$ and $\hat{p}_K$ are quadrature amplitudes for the system $K \in \{A, B\}$, in analogy to position and momentum. The measurement settings are adjusted by independent local interactions H_{θ}^A and H_{ϕ}^B . We show that these interactions correspond to *deterministic causal transformations* of the local amplitudes x_K, p_K , which precede the independent amplifications H_{amp}^A

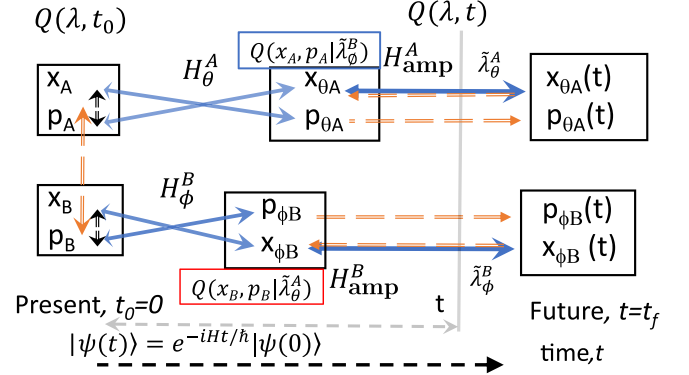


FIG. 5. Causal consistency for Bell-nonlocal systems A and B . The Q function $Q(\lambda, t)$ at time t is determined by $|\psi(t)\rangle$ which evolves causally in the positive-time direction, in accordance with Hamiltonians H_{θ}^A and H_{ϕ}^B that determine the measurement settings, and H_{amp}^A and H_{amp}^B that amplify $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$. Here, $\lambda = (x_A, p_A, x_B, p_B)$. The probability density $Q(\lambda, t)$ is equivalent to that of the amplitudes $x_K(t), p_K(t)$ [and $x_{\theta K}(t), p_{\theta K}(t)$], as defined by the forward-backward stochastic equations. The $x_{\theta K}, p_{\theta K}$ are deterministic functions of x_K, p_K for each $K \in \{A, B\}$ (depicted by two-way crossed blue arrows). The $Q(\lambda, t_0)$ exhibits local correlation between x_K and p_K (short vertical black dashed lines), as well as a correlation between A and B (vertical orange dashed line). Branches denoted by $\tilde{\lambda}_{\theta}^A$ and $\tilde{\lambda}_{\phi}^B$ emerge on amplification, where here $\tilde{\lambda}_{\theta}^A$ and $\tilde{\lambda}_{\phi}^B$ are variables that assume the values $x_{\theta j}^A$ and $x_{\phi k}^B$ representing the outcomes of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$, respectively. The distributions $Q(x_A, p_A|\tilde{\lambda}_{\theta}^A)$ and $Q(x_B, p_B|\tilde{\lambda}_{\phi}^B)$ determine the probabilities of outcomes at A given the outcome $\tilde{\lambda}_{\phi}^B$ at B , and the probabilities of outcomes at B given the outcome $\tilde{\lambda}_{\theta}^A$ at A . These are deduced from the backward trajectories, using knowledge of $Q(\lambda, t_0)$ and deterministic relations, and explain the Bell nonlocality. Being mutually consistent, Lorentz-invariance paradoxes are avoided (refer to Sec. IX for details).

and H_{amp}^B that complete the measurements of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ made on each system. For a Bell test, we consider binary outcomes $+1$ and -1 defined by the sign of the quadrature phase amplitude ($\hat{x}_{\theta A}$ or $\hat{x}_{\phi B}$), measured at each of site, with systems A and B spacelike separated. We prove the equivalence of the dynamics of the Q function $Q(\lambda, t)$ with the forward-backward dynamics of the amplitudes λ : Causal consistency is demonstrated in the bipartite case (refer to Fig. 5 and Result IX.7 of this paper).

Since the forward-backward simulations involve a future boundary condition (FBC), it might appear that the Bell violations are “put in by hand”: The put-in-by-hand argument is as follows: The final state at time t_f that determines the boundary condition of the backward stochastic differential equation is a bipartite amplified state that already possesses the correlations violating the Bell inequality. Hence, it would seem that nothing can be learned about the origin of the Bell correlations.

We explain in this paper that the violations (in the model of reality proposed here) are not put in by hand at the time t_f , but emerge at an earlier time t_{Bell} , through the dynamics of the interactions (Results VII.5, IX.4, and IX.8). Similar to the conclusions of Part I of the paper, the paradox of the future boundary condition is avoided (Fig. 5) and a genuine

retrocausality, where the future impacts the measurable past properties, does not occur. Moreover, there is no “spooky action” where the change of setting at one location (B) can change the existing property at the other A . For entangled states, the setting change at B can lead to correlations that violate a Bell inequality for *future* outcomes at A , if there is a *further change of setting* at A . Fine-tuning is explained consistently with a hidden-variable model, which includes hidden noise and interference.

These conclusions are arrived at from three different perspectives, which we summarize below for the reader’s convenience. First, a hidden variable model can be constructed based on the Q function (Result VII.9). The hidden variables (HV) are the phase-space amplitudes $\lambda \equiv (x_A, p_A, x_B, p_B)$; the Q function $Q(\lambda, t_0)$ is the associated probability distribution. Using Bell’s formalism, the probability of an outcome $+1$ at both sites is

$$P_{++}(\theta, \phi) = \int Q(\lambda, t_0) P(++|\theta, \phi, \lambda) d\lambda, \quad (1.1)$$

where $P(++|\theta, \phi, \lambda)$ is the probability of an outcome $+1$ at both sites conditioned on the values λ . Since we prove causal consistency, we can show that the distribution $Q(\lambda, t_0)$ is independent of the future settings, θ and ϕ (Fig. 5). This demonstrates there is no retrocausality, where the future affects the present “state” of the system, as defined by the Q function. In this Q -based HV model, we identify that the Bell violations arise from a breakdown of Bell’s locality assumption (Result IX.10).

The second perspective is to examine the macroscopic properties $\tilde{\lambda}$ associated with the branches defined from the solutions (trajectories) of the forward-backward stochastic equations. We conclude that the Q -based HV model satisfies a *partial locality*, which applies to the systems after the settings are established. Such weak forms of local realism were proposed in Refs. [108,109,119] and shown to be not contradicted by Bell’s theorem. This leads to a main result of this paper, that the Q -based model of reality is consistent with *weak local realism*. We first demonstrate consistency with the three premises of weak macroscopic realism (wMR) [108,109,119–122]. We consider both systems A and B to be at rest in a frame F .

(1) The first premise specifies macroscopic realism: Macroscopic properties $\tilde{\lambda}_x^A(t)$ for a system A (say) exist *once the local settings are fixed* and after amplification, at a time $t = t_m$ (Fig. 1).

(2) The second premise specifies a partial locality: The macroscopic value $\tilde{\lambda}_x^A(t)$ cannot be changed by a spacelike-separated event occurring after t_m , as in a subsequent change of setting ϕ at B . Once the setting is fixed at A , only a further change of setting at A can change the value of $\tilde{\lambda}_x^A(t)$, which requires further dynamics.

(3) The third premise concerns projection: Once the value $\tilde{\lambda}_x^A(t_{mA})$ of the outcome [as in (1)] for system A is determined, at a time t_{mA} , consistency with the initial state at time t_0 may imply restrictions on outcomes at system B . Details are given below in Sec. IB

Examination of the simulations reveals that the weak local realistic premises can be generalized to apply to systems after the measurement settings are implemented, but that are

not necessarily macroscopic. Hidden variables $\{\lambda_K\}$ can be identified that define probabilities for outcomes at site $K \in \{A, B\}$. In this paper, we demonstrate the consistency of the simulations with wLR [108], defined below in Sec. IB.

The weak local realistic premises are useful for understanding EPR correlations. We deduce a consequence for the EPR paradox: there can be predetermined values (weak “elements of reality”) for measurements at B , prior to the choice of setting ϕ at B , provided the settings are fixed at A . In this paper, we demonstrate these elements of reality in a forward-backward stochastic simulation of EPR correlations, which illustrates the emergence of macroscopic correlated EPR “branches.” The analysis may explain recent experiments, which detect EPR correlations for macroscopic BEC systems [123].

The third perspective given in this paper is to provide a model for *projection*. This allows us to understand the workings of a meter. Extending the analysis of the postselected distribution defined in Part I of the paper, we evaluate the probability distribution $Q(x_B, p_B|x_j^A)$ inferred for system B conditioned on a branch B_j (with outcome x_j^A) of a second system A , correlated with B . The $Q(x_B, p_B|x_j^A)$ describes the “projected state” for system B , modeling the collapse of the wave function at B . We show that the collapse arises as information is lost about complementary variables of the meter A . Limits on the timing of projection are deduced.

We ask the following question about a meter. If the measurement at A allows inference of a state, or outcome, at B , then does the measurement at A *cause* the outcome at B , through some “spooky action-at-a-distance,” or is the predicted outcome at B due to correlations present at an earlier time? We show, by constructing a local HV theory consistent with the simulations, that, provided the settings of the meter A are fixed, the latter is correct.

How then can a violation of a Bell inequality occur? The mechanism relates to *hidden interference* that arises from the entangled nature of the initial state. If the hidden-variable amplitudes λ given in (1.1) were directly measurable, there would be no Bell violation, since the transformations H_θ^B and H_{amp}^K are local. However, the amplitudes are not directly measurable: This is evident from the Q function $Q(\lambda, t_0)$, which possesses hidden interference terms $\mathcal{I}nt_{AB}$. These terms are defined with respect to the initial measurement basis, determined by the preparation of the state $Q(\lambda, t_0)$. We choose the initial basis so that $\theta = \phi = 0$ at time t_0 . Extending Result III.1 from Part I of the paper, we prove (Result VII.2) that the $\mathcal{I}nt_{AB}$ terms are not amplified by H_{amp} and are hence not detected by measurement of $\hat{x}_A$ and $\hat{x}_B$. Where changes of settings take place however, the Q function changes: The $\mathcal{I}nt_{AB}$ terms change, and can contribute to measurable probabilities, in a way that violates the Bell inequalities (Result VII.5). We show that this can only happen if certain conditions are satisfied: The settings θ and ϕ must change at both locations A and B , mixing $\hat{x}_K$ with $\hat{p}_K$ ($K \in \{A, B\}$) locally (Result VII.7), so that there is a “loop” between the backward and forward trajectories of the same system K (refer to definitions below).

More detail is evident from the stochastic solutions (trajectories), where we find that the Bell nonlocality can be

explained by the projection mechanism. A hidden interference effect comes into play if we change the setting ϕ at B , to measure $\hat{x}_{\phi B}$, keeping the setting at A fixed. The projected distribution $Q(x_A, p_A | x_{\phi k}^B)$ at A conditioned on a branch with outcome $x_{\phi k}^B$ at B can be inferred, by *retrodiction*, at B , based solely on a deterministic relation $x_{\phi k}^B \leftrightarrow Gx_{\phi k}^B$, and a further set of deterministic relations $\mathcal{D}$ that specify correlations between $\hat{x}_{\phi B}$ and $\hat{x}_A$, as well as knowledge of the setting ϕ at B and of the initial state $Q(\lambda, t_0)$. This is surprising because the projected state for A is sufficient to explain the Bell violations (Result IX.4). The $Q(x_A, p_A | x_{\phi k}^B)$ defines the state at A , consistent with any future adjustment of setting θ at A . For entangled states, the projected state $Q(x_A, p_A | x_{\phi k}^B)$ contains extra hidden interference terms $\mathcal{I}_A$, as in a quantum superposition, if ϕ satisfies $\phi \neq n\pi$ (n is an integer)—but these terms are undetectable, with the setting at A is fixed ($\theta = 0$). Hence, the predictions for outcomes of $\hat{x}_A$ and $\hat{x}_{\phi B}$ are consistent with a local HV model. We show there is consistency with *no-signaling*: With the setting fixed at A , a change of setting at B does not influence the outcome at A . The interference $\mathcal{I}_A$ of system A can be detected, however, if there is a *further change* of setting to $\theta \neq n\pi$ at A , and this leads to correlations that can violate a Bell inequality. The conclusion is that Bell nonlocality emerges over the time span t_{Bell} , as settings are changed at both sites, consistent with the premises of weak local realism.

A final question examined in this paper is how to resolve an apparent inconsistency with Lorentz invariance, since for spacelike-separated systems, the time order of setting changes may be reversed, in a different frame. In one frame, the outcomes for A for any setting are constrained conditionally on the outcomes of B ; in another frame, the outcomes for B are constrained by those of A . We suggest this is resolved by the symmetry of the setting changes, which must occur at both sites, for Bell violations to occur. There are two projected states, one for system A and one for system B , which are each defined based on one setting-change, but are unchanged by the time-order of the setting-changes. The *projected states are mutually consistent* with each other and with the initial distribution $Q(\lambda, t_0)$ (Fig. 5).

To conclude, we explain how experimental tests of the compatibility of the forward-backward simulations with quantum predictions are possible, elucidating paradoxes (Figs. 1–5). Below is a summary of the layout of the paper, followed by a list of definitions associated with the main concepts.

A. Layout of paper

In Part I of the paper (Secs. II–V), we treat *quantum measurement*. In Sec. II, we review the measurement Hamiltonian H_{amp} , the Q function equations of motion, and the equivalent forward-backward stochastic dynamics. The path-integral and stochastic equivalence theorems on which the paper is based are derived in Appendix A, along with the numerical verification of the computed distribution in Appendix B. In Sec. III, the equations modeling a measurement $\hat{x}$ on a system in a superposition $|\psi_{\text{sup}}\rangle$ of eigenstates of $\hat{x}$ are solved. The Q -based model of reality is explained in Sec. IV. Born's rule is derived, and the role of the Wigner function is explained. Justification of macroscopic realism and macroscopic causality

is presented in Sec. IV C. The derivation of the postselected distribution based on an outcome of a measurement, and explanation of a hidden loop is given in Secs. IV D and IV E. Hidden-variable models $\mathcal{P}_{\text{sup}}$ and $\mathcal{P}_{\text{det}}$ for measurement on a superposition state and the justification of weak local realism are presented in Secs. IV F and IV G. A causal model for measurement and a proof of causal consistency are given in Sec. V.

In Part II of the paper, we treat *nonlocality and entanglement*. In Sec. VI, we present the simulation for continuous-variable EPR correlations. The simulation of Bell nonlocality is explained in Sec. VII A, along with an analysis of the mechanism for Bell nonlocality based on the evolution of the Q function in Sec. VII B. The different sorts of hidden loops that can occur for entangled states, and explanation of the further hidden variable (HV) models $\mathcal{P}_{\text{asym},A}$, $\mathcal{P}_B$, $\mathcal{P}_{\text{det}B}$ and $\mathcal{P}_{\text{Bell}}$ that can be deduced from the simulation are given in Secs. VII C and VII D. A model for the projection and collapse of the wave function is presented in Sec. VIII, with causal relations for projection and Bell nonlocality given in Sec. IX. It is shown through a series of results in Secs. VII, VIII, VII, and IX that the simulations are consistent with three premises referred to as *weak macroscopic realism* (wMR). Based on the HV models, we extend (in Secs. IV, VII, and IX) to explain consistency with *weak local realism* (wLR). Proofs of the mutual consistency of the projected states and of the causal consistency of the forward-backward stochastic equations for the entangled system (as in Fig. 5) are given in Sec. IX B, Sec. IX C and Appendix H. The distinction between the hidden variables of the Q -based model and those of Bell's local hidden variables is analyzed in Sec. IX, along with an explanation of how Lorentz-invariance paradoxes may be resolved. Fine-tuning is explained in Appendix J. In the Conclusion (Sec. X), potential experiments are outlined.

B. Definitions

In Part I of the paper, we consider measurement on a single-mode field, and define in a rotating frame the quadrature phase amplitude $\hat{x} = \hat{a} + \hat{a}^\dagger$ where $\hat{a}^\dagger$ and $\hat{a}$ are the standard boson creation and destruction operators [124]. The complementary observable is $\hat{p} = (\hat{a} - \hat{a}^\dagger)/i$. Measurement of $\hat{x}$ is treated as amplification of $\hat{x}$, by a factor G , modeled by the interaction Hamiltonian $H_{\text{amp}} = \frac{i\hbar g}{2}[\hat{a}^{\dagger 2} - \hat{a}^2]$ [104].

In Part II of the paper, we treat entangled systems A and B described by a two-mode Q function. We consider quadrature phase amplitudes $\hat{x}_{\theta A} = \hat{x}_A \cos \theta + \hat{p}_A \sin \theta$ and $\hat{x}_{\phi B} = \hat{x}_B \cos \phi + \hat{p}_B \sin \phi$ measured at A and B , respectively. Here, $\hat{x}_K$ and $\hat{p}_K$ (where $K = A, B$) are the $\hat{x}$ and $\hat{p}$ quadrature phase amplitudes of the system K .

Definition (1): Q function. The Q function of a single-mode field in a quantum state $|\psi\rangle$ is defined as $Q(\alpha) = \frac{1}{\pi} |\langle \alpha | \psi \rangle|^2$ where $|\alpha\rangle$ is a coherent state [103]. The Q function uniquely defines the quantum state. The phase-space coordinates λ are the real coordinates x and p , given by $\alpha = (x + ip)/2$. In this paper, we denote the Q function of the quantum state at a time t as $Q(x, p, t)$.

The Q function of a two-mode field (modes A and B) is defined as $Q(\alpha, \beta) = \frac{1}{\pi^2} |\langle \beta | \langle \alpha | \psi \rangle|^2$ where $|\alpha\rangle$, $|\beta\rangle$ are coherent states for A and B , respectively. The phase-space

coordinates λ are the real coordinates x_K and p_K , given by $\alpha = (x_A + ip_A)/2$ and $\beta = (x_B + ip_B)/2$. We denote the Q function of the quantum state at a time t as $Q(\lambda, t)$ where $\lambda = (x_A, p_A, x_B, p_B)$.

Definition (2): Forward-backward stochastic simulation. Consider the equations $\frac{dp}{dt} = -gp + \xi_p$ and $\frac{dx}{dt} = -gx + \xi_x$ where $t_- = -t$, and $\xi_x(t)$, $\xi_p(t)$ are Gaussian noise terms and g is real [104,107]. The boundary conditions are determined by distributions given by Q functions that are sampled. The solutions $x(t)$ and $p(t)$ differ for each run. We refer to the $x(t)$ and $p(t)$ as *trajectories*, and to the set of runs as a forward-backward stochastic equation (FBSE) simulation.

Definition (3): Eigenstate of $\hat{x}$. The eigenstate of $\hat{x}$ is denoted $|x_j\rangle$ and is defined as the squeezed state $|\frac{x_j}{2}, r\rangle_{sq} = D(\frac{x_j}{2})S(r)|0\rangle$ in the limit where $r \rightarrow \infty$, where x_j and r are real. Here, $|0\rangle$ is the vacuum state of the field and $D(\beta_j) = e^{\beta_j \hat{a}^\dagger - \beta_j^* \hat{a}}$ and $S(z) = e^{\frac{1}{2}(z^* \hat{a}^2 - z \hat{a}^{\dagger 2})}$.

Definition (4): Hidden interference. Consider the expansion $|\psi_{\text{sup}}\rangle = \sum_j c_j |x_j\rangle$ of a quantum state in terms of the measurement basis of $\hat{x}$, i.e., the eigenstates of $\hat{x}$. Here, c_j are probability amplitudes. The corresponding Q function $Q_{\text{sup}}(x, p, t)$ is a sum of type $\sum_j |c_j|^2 Q_j(x, p) + \mathcal{I}nt$ where $Q_j(x, p)$ is the Q function of $|x_j\rangle$. We refer to the extra terms $\mathcal{I}nt$ as “hidden interference” terms. These terms vanish in the marginal distribution defined as $Q_{\text{sup}}(x, t) = \int dp Q_{\text{sup}}(x, p, t)$, in the limit of r large where $|x_j\rangle$ are eigenstates of $\hat{x}$, and are not amplified by H_{amp} .

Definition (5): Branches and the time t_m . We consider the forward-backward stochastic simulation of the amplification H_{amp} of $\hat{x}$ on the system prepared in the superposition $|\psi_{\text{sup}}\rangle$ defined in Definition (4). With sufficient time of evolution when the system has become macroscopic, at a time we denote by t_m , distinct groups of amplitudes $x(t_m)$ emerge. Each group can be identified with a distinct eigenvalue x_j . We refer to such a group of amplitudes as the *branch* $\mathcal{B}_j$ associated with the eigenvalue x_j . The group of amplitudes $\mathcal{B}_j$ of a single branch will be associated with a single eigenvalue x_j . In this paper, the branches are labeled by a variable $\tilde{\lambda}_x$, the value of which indicates the eigenvalue associated with the branch. The value represents a measurable property of the system at the time t_m .

Definition (6a): Weak macroscopic realism: First premise [wMR(1)]. Consider a system A that has N (where $N \geq 2$) macroscopically distinct states available to it, e.g., a system prepared at a time t_m in a superposition of N macroscopically distinct states. Suppose a measurement $\hat{O}$ with N discrete outcomes can distinguish between the macroscopic states. For example, where $N = 2$, we consider two outcomes, $+1$ and -1 , which identify and distinguish the two states. The measurement $\hat{O}$ can be macroscopically coarse-grained, as the requirement is only to distinguish between the states.

Macroscopic realism (MR) posits that the system, prior to the measurement $\hat{O}$, is in *one and only one* of the macroscopic states [101]. To be more specific, MR posits that the outcome of the measurement $\hat{O}$ is determined at the time t_m . The system can be assigned a variable $\tilde{\lambda}$, the value of which indicates the outcome of $\hat{O}$ if the measurement $\hat{O}$ takes place. Where $N = 2$, the value of $\tilde{\lambda}$ can be $+1$ or -1 .

To define *weak macroscopic realism* (wMR) [108,109,119–122,125], we further qualify the definition of MR, to specify

that at the time t_m , any operations U needed to fix measurement settings (e.g., θ in the measurement where $\hat{O} \equiv \hat{x}_{\theta A} = \hat{x}_{\theta} \cos \theta + \hat{p} \sin \theta$) have been carried out. In this paper, we consider the system prepared in a superposition $(|x_1\rangle + |-x_1\rangle)/\sqrt{2}$ of eigenstates of $\hat{x}$, where x_1 is real and $|x_1| \rightarrow \infty$. Here, $\hat{O}$ is defined as the sign of the outcome of $\hat{x}$. To apply wMR, we assume that the operations U have been implemented, so that $\theta = 0$. The amplification H_{amp} of the system will complete the measurement of $\hat{x}$.

Comment. The terminology “weak” is used to distinguish alternative definitions based on stronger assumptions, such as in classical mechanics, where MR is defined for the system as it exists *prior* to the operations U . This stronger assumption of MR is falsifiable [122,126,127]. The definition is also “weaker” than that of macro-realism, defined by Leggett and Garg in Ref. [101], which makes the additional assumption of noninvasive measurability.

Definition (6b): Weak macroscopic realism: Second premise [wMR(2)]. Consider two spacelike-separated systems A and B both stationary in a frame F . The two systems may be entangled. Consider the macroscopic system A at the time t_m described in Definition (6a) of wMR(1), which has N macroscopically distinct states available to it. According to the premise wMR(1), the outcome of the measurement $\hat{O}$ is determined at the time t_m . We denote the value of the outcome by the symbol $\tilde{\lambda}^A$. We assume all settings are fixed for system A . The value $\tilde{\lambda}^A$ can be considered a property of the system A at the time t_m , and denoted $\tilde{\lambda}^A(t_m)$. The premise wMR(2) of weak macroscopic realism posits that the value of $\tilde{\lambda}^A(t_m)$ is not changed by any interactions or measurements that might take place at system B at a time *after* t_m [108]. The property $\tilde{\lambda}^A$ can only be changed if a local interaction takes place at A , in which case the property is denoted by $\tilde{\lambda}^A(t)$, and is defined at a future time $t > t_m$.

Definition (6c): Weak macroscopic realism: Third premise [wMR(3)]. Consider the system A described in Definitions (6a) and (6b) at a time t_{mA} . For example, in this paper, we take $\hat{O} \equiv \hat{x}_A$ and assume that at a time t_{mA} all operations that determine the measurement setting for system A (e.g. $\theta = 0$) have been carried out. We may consider that the system A was initially prepared in the superposition $\sum_j c_j |x_j^A\rangle$ given in Definition (4), and has been amplified according to H_{amp} , so that the system at time t_{mA} is in the superposition $\sum_j c_j e^{-iH_{\text{amp}}t/\hbar} |x_j^A\rangle$ of macroscopically distinct states $e^{-iH_{\text{amp}}t/\hbar} |x_j^A\rangle$, each with a definite outcome for $\hat{x}_A$.

According to wMR(1), the outcome for $\hat{x}_A$ is determined at the time t_{mA} . The value of this outcome is denoted by a variable $\tilde{\lambda}_x^A$. The outcomes correspond to eigenvalues of the measurement, in this case, the eigenvalues denoted x_j^A [refer to Definition (3)]. Quantum mechanics predicts that the outcomes of measurements at B conditioned on an outcome x_j^A at A are determined by a *projected* quantum state that we denote by $|\psi_B\rangle_j$.

The third premise wMR(3) of weak macroscopic realism, introduced in Refs. [108,119,125], posits that the projected state $|\psi_B\rangle_j$ (as defined by its predictions for B) is determined for the local system B at (or by) the time t_{mA} . For example, in the EPR paradox, the outcome of $\hat{x}_B$ for system B can be inferred from the outcome of $\hat{x}_A$ at A . The premise wMR(3)

posits that the outcome at B is determined at (or by) the time t_{mA} when the system A has [according to wMR(1)] a predetermined value for measurement $\hat{x}_A$. Hence, there is an “element of reality” for the measurement at B defined at time t_{mA} . There is no contradiction with Bell’s theorem because the time t_{mA} is *after* the settings at A have been fixed [108].

Definition (7): Macroscopic causality. As a consequence of wMR [Definitions (6a) and (6b)], a variable $\tilde{\lambda}$ is defined at a time t_m for a macroscopic system A in the frame F . This variable signifies the predetermination of the outcome of the measurement, i.e., $\hat{O}$, that distinguishes the macroscopic states of the system A . Macroscopic causality is the assertion that such variables satisfy causality, i.e., the value of $\tilde{\lambda}$ is not changed retrocausally by any future event and is not changed by any spacelike-separated event at system B occurring after time t_m .

Definition (8a): Postselected state. Consider the system described by a Q function $Q(x, p, t)$ at time t_0 . A correlation can exist between the values of x and p , as indicated by the conditional distribution $Q(p|x) = Q(x, p, t_0)/Q(x, t_0)$ where $Q(x, t_0) = \int dp Q(x, p, t_0)$ is the marginal of x . Hence, if we consider the solutions $x(t)$ that arise from the backward-propagating trajectories of a given branch $\mathcal{B}_j$ (corresponding to an outcome x_j , refer above), then there is a joint distribution of variables $\{x(t), p(t)\}$, corresponding to the branch, defined at the initial time t_0 . The values $x(t_0)$ are consistent with the deterministic relation $x_j \rightarrow Gx_j$ associated with the amplification H_{amp} , in the evolution of the Q function. The joint distribution is denoted $Q_{\text{loop}}(x, p, t_0|\mathcal{B}_j)$. We refer to this distribution as the postselected distribution, or else as the postselected state. The notations $Q_{\text{loop}}(x, p, t_0|x_j)$ and $Q_0(x, p|x_j)$ are also used to denote $Q_{\text{loop}}(x, p, t_0|\mathcal{B}_j)$. The notation $Q_{\text{loop}}(x, p, t_0|\tilde{\lambda}_x)$ is also used, where the variable $\tilde{\lambda}_x$ denotes a branch associated with a particular outcome $\tilde{\lambda}_x \in \{x_j\}$.

Definition (8b): Postselected state for system B conditioned on an outcome x_j^A for A : the projected distribution. Consider the postselected state defined in Definition (8a). For a given branch $\mathcal{B}_j^A$ that defines the outcome x_j^A at A , the postselected state as defined by the Q function of the bipartite system is $Q_{\text{loop}}(\lambda, t_0|\mathcal{B}_j^A)$ where we abbreviate $\lambda = (x_A, p_A, x_B, p_A)$. To obtain the distribution for system B alone, we define

$$Q(x_B, p_B|\mathcal{B}_j) = \int dp_A dx_A Q_{\text{loop}}(\lambda, t_0|\mathcal{B}_j),$$

which corresponds to the *projected* distribution (or state) for B , given an outcome x_j^A at A . The bipartite distribution $Q_{\text{loop}}(\lambda, t_0|\mathcal{B}_j^A)$ is conditioned on the outcome x_j^A at A , and must be consistent with the deterministic relations that account for amplification ($x_j^A \rightarrow Gx_j^A, x_k^B \rightarrow Gx_k^B$), and also the deterministic relations $\mathcal{D}_1$ that account for the correlations between the outcomes x_k^B at B and x_j^A at A . Here, $\{x_k^B\}$ denotes the set of outcomes at B for the measurement of $\hat{x}_B$. According to the causal model developed in this paper, the deterministic relations exist for the system prepared at time t_0 , since the measurement settings are fixed, as consistent with the assumptions of weak local realism defined below. Where the meaning is clear, we denote $Q_{\text{loop}}(\lambda, t_0|\mathcal{B}_j^A)$ as $Q_{\text{loop}}(\lambda, t_0|x_j^A)$.

Definition (9a): Hidden loop. Continuing from Definition (8a), there is hence a connection between the backward-propagating trajectories $x(t)$ defining a branch $\mathcal{B}_j$, and a *correlated* set of forward-propagating trajectories $p(t)$ that are determined by the $Q_{\text{loop}}(x, p, t_0|\mathcal{B}_j)$. This connection we refer to as a loop. The loop is referred to as *hidden*, if the associated distribution $Q_{\text{loop}}(x, p, t_0|\mathcal{B}_j)$ cannot be equivalent to a quantum state, or else if variables involved in the loop, such as p , are not detectable in the model of measurement.

Definition (9b): Local and nonlocal loops. The local loop is a loop [refer to Definition (9a)] that connects backward and forward amplitudes of the same system, A or B . A nonlocal loop connects backward and forward trajectories of different systems A and B .

Definition (10a): Weak local realism: First premise [wLR(1)]. We consider a system A at a time t_1 , *after* operations U needed to fix measurement settings have been carried out, so that the system is prepared for a measurement $\hat{O}$, i.e., in this paper $\hat{O} \equiv \hat{x}_{\theta A} = \hat{x}_A \cos \theta + \hat{p}_A \sin \theta$. In a spin-Bell experiment, the time t_0 corresponds to the time after a system (i.e., a photon or electron) has passed through an analyzer (a polarizer or Stern-Gerlach apparatus), so that a final detection will register the outcome of the polarization or spin along a direction θ , determined by the analyzer. In the examples of this paper, the measurement is of $\hat{O} \equiv \hat{x}_{\theta A}$ and the “analyzer” corresponds to a phase-shift device that determines θ . Unlike the definition of wMR, the system need not be macroscopic.

For such a system, two versions of weak local realism (wLR) can be defined: *Deterministic* wLR(1) and *Probabilistic* wLR(1). In the deterministic version, it is posited that the value for the outcome of $\hat{O}$ is determined by the time t_0 [108,120]. We denote the value for the outcome by λ^A . In the probabilistic version, it is posited that a *set* of hidden variables $\{\lambda^A\}$ exists to describe the system, such that there is a definite probability $P(x|\{\lambda^A\})$ for the outcome x of $\hat{O}$, given a set of values for $\{\lambda^A\}$.

Definition (10b): Weak local realism: Second premise [wLR(2)]. Consider two spacelike-separated systems A and B both stationary in a frame F . Consider the system A at the time t_{1A} after the measurement settings at A are fixed, as described in Definition (10a) of wLR(1).

According to the deterministic version of premise wLR(1), the value for the outcome $x_{\theta A}$ of the measurement $\hat{O} \equiv \hat{x}_{\theta A}$ on system A is determined at time t_{1A} . We denote the value by $\lambda_{x_{\theta}}^A$. Premise wLR(2) posits that this value is not changed by any interactions or measurements that might take place at system B at a time after t_{1A} [108]. Similar to premise wMR(2), the value can only be changed if there is a local interaction at A .

According to the probabilistic version of wLR(2), one can define a probability $P(x|\{\lambda^A\})$ for an outcome $x_{\theta A}$ of $\hat{O}$, given a set of values for the hidden variables of $\{\lambda^A\}$ that describe system A . The premise wLR(2) posits that these probabilities are unchanged by the interactions at B that might take place at a time $t > t_{1A}$.

Definition (10c): Weak local realism: Third premise [wLR(3)]. Consider the system A described in Definitions (10a) and (10b) of wLR(1) and wLR(2). We consider the measurement of $\hat{x}_{\theta A}$. Unlike the Definition (5) for wMR, we do not need to assume the system A is macroscopic.

Consider the deterministic version of wLR, defined by premise wLR(1) [Definition (10a)]. This premise posits that the value for the outcome of $\hat{x}_{\theta A}$ is determined at (or by) the time t_{1A} , after the operations that fix the measurement settings for system A have been carried out. Consider the projected state $|\psi_B\rangle_j$ defined in the Definition (6c) wMR(3) above. Premise of wLR(3) posits that this state (as defined by its predictions for B) is determined at the time t_{1A} [108]. In the EPR example given for Definition (6c), this implies that the outcome for $\hat{x}_B$ is determined at (or by) the time t_{1A} , after the setting θ is fixed at A . In other words, an element of reality exists for B based on the prediction at A , at time t_{1A} .

Alternatively, we consider the probabilistic version of wLR, defined by premise wLR(1) in Definition (10a). This premise posits that probabilities $P(x_j^A|\{\lambda^A\})$ for outcomes at A are fixed at the time t_{1A} . For any one of these outcomes x_j^A , a projected state $|\psi_B\rangle_j$ for system B is defined by quantum mechanics, as in Definition (6c). Premise wLR(3) posits that at the time t_{1A} , the system B is described by the projected state $|\psi_B\rangle_j$ (as defined by its predictions of outcomes of future measurements at B) with probability $P(x_j^A|\{\lambda^A\})$.

Definition (11): No-signaling. No-signaling is defined as the conditional independence of the outcome at A and the setting at B , given the setting at A [53].

II. MEASUREMENT MODEL

In this paper, the system dynamics is described by a unified stochastic model motivated by the Q function [103]. The examples of this paper are restricted to bosonic fields. Our approach requires the measurement process to be included in the dynamics.

A. Measurement model: Parametric amplification

First, restricting our analysis to a single mode of the field, we define the complementary quadrature phase amplitude observables

$$\begin{aligned}\hat{x} &= \hat{a} + \hat{a}^\dagger, \\ \hat{p} &= (\hat{a} - \hat{a}^\dagger)/i\end{aligned}\quad (2.1)$$

in a rotating frame, where $\hat{a}$ is the boson destruction operator [124]. Hence $\hat{a} = (\hat{x} + i\hat{p})/2$. This implies the uncertainty relation $\Delta\hat{x}\Delta\hat{p} \geq 1$ for any state $|\psi\rangle$ of the single-mode field, where here we denote $(\Delta\hat{O})^2 = \langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2$ as the variance of the observable $\hat{O}$.

We analyze the simplest measurement procedure that can take place—that of direct amplification to a macroscopic signal. Measurement is modeled by a measuring device (a meter) that produces macroscopic outputs, namely, the parametric amplifier [104]. To measure $\hat{x}$, the system prepared in a state $|\psi\rangle$ is amplified according to the Hamiltonian [128]

$$H_{\text{amp}} = \frac{i\hbar g}{2}[\hat{a}^{\dagger 2} - \hat{a}^2], \quad (2.2)$$

where $g > 0$ is real. The system evolves under H_{amp} for a time t_f , to give the final measurement output. The measurement is completed simply by the fields being amplified to become macroscopic, and hence detectable to macroscopic devices or observers. The importance of amplification is explained by Bohr, who regarded measurement as amplification [15].

For $g > 0$, the dynamics of H_{amp} gives solutions that amplify the “position” $\hat{x}$ but attenuate the orthogonal “momentum” quadrature $\hat{p}$. This is clear from the standard operator Heisenberg equations, which give solutions

$$\begin{aligned}\hat{x}(t) &= \hat{x}(0)e^{gt}, \\ \hat{p}(t) &= \hat{p}(0)e^{-gt}.\end{aligned}\quad (2.3)$$

H_{amp} induces squeezing in $\hat{p}$ [128]. The interaction H_{amp} has been realized experimentally [129,130]. A related model of measurement based on amplification was presented by Glauber for a two-slit experiment [131,132].

B. Equation of motion for the Q function: Generalized Fokker-Planck equation

To solve for single-mode evolution, we use the Q function [103]

$$Q(\alpha) = \frac{1}{\pi} |\langle\alpha|\psi\rangle|^2 \quad (2.4)$$

defined for a quantum state $|\psi\rangle$ with respect to the nonorthogonal basis of coherent states $|\alpha\rangle$. The coherent state $|\alpha\rangle$ satisfies $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ where α is a complex number. The phase-space coordinates λ are the real coordinates x and p , given by $\alpha = (x + ip)/2$. In this paper, we denote the Q function of the quantum state at a time t as $Q(x, p, t)$. The single-mode Q function defines the quantum state $|\psi\rangle$ uniquely as a positive probability distribution. The moments evaluated from the Q function distribution are antinormally ordered operator moments [133]. The antinormally ordered variances $\sigma_x^2(t)$ and $\sigma_p^2(t)$ are precisely the variances of x and p as defined by the Q function $Q(x, p, t)$.

The dynamics for real variables x and p can be deduced by deriving a generalized Fokker-Planck equation, which gives an equation of motion of Q for a given Hamiltonian H . In general terms, the Q function probability density $Q(\lambda, t)$ for a phase-space coordinate λ is defined with respect to a nonorthogonal operator basis $\hat{\Lambda}(\lambda)$ as

$$Q(\lambda, t) \equiv \text{Tr}(\hat{\Lambda}(\lambda)\hat{\rho}(t)), \quad (2.5)$$

where $\hat{\rho}$ is the density operator of the system. As $Q(\lambda, t)$ is normalized to unity, it is necessary to normalize the basis so it integrates to unity, and the normalization condition is that $\int \hat{\Lambda}(\lambda)d\lambda = \hat{1}$. The basis satisfies $\hat{\Lambda}^2(\lambda) = \mathcal{N}(\lambda)\hat{\Lambda}(\lambda)$, which is different from the condition for projectors that $\hat{P}^2 = \hat{P}$, because it is a continuous nonorthogonal basis.

From the Schrödinger equation, the dynamics of the probability distribution is obtained from the usual equation $i\hbar d\rho/dt = [H, \rho]$. As a result, one obtains an equation for the Q function time-evolution:

$$\frac{dQ(\lambda, t)}{dt} = \frac{i}{\hbar} \text{Tr}\{[H, \hat{\Lambda}(\lambda)]\rho(t)\}. \quad (2.6)$$

This is equivalent to a zero-trace diffusion equation for the variables λ , of the form $\dot{Q}(\lambda, t) = \mathcal{L}(\lambda)Q(\lambda, t)$, where $\mathcal{L}(\lambda)$ is the differential operator for Q function dynamics. For the single-mode, the phase-space coordinates λ are the real coordinates x and p .

For the system evolving according to the Hamiltonian H_{amp} given by (2.2), a dynamical equation for the Q function can be

derived [104]. Applying the correspondence rules to transform operators into differential operators, one obtains a generalized Fokker-Planck type equation in terms of complex coherent-state variables α :

$$\frac{dQ}{dt} = - \left[g \frac{\partial}{\partial \alpha} \alpha^* + g \frac{\partial}{\partial \alpha^*} \alpha + \frac{g}{2} \frac{\partial^2}{\partial \alpha^2} + \frac{g}{2} \frac{\partial^2}{\partial \alpha^{*2}} \right] Q. \quad (2.7)$$

Here the differential operators act on all terms to the right [134–136]. Defining $\partial_p \equiv \partial/\partial p$ and $\partial_x \equiv \partial/\partial x$, we obtain in real coordinates

$$\frac{dQ}{dt} = [\partial_p(gp) - \partial_x(gx) + g(\partial_p^2 - \partial_x^2)]Q. \quad (2.8)$$

The above equation demonstrates a diffusion matrix which is traceless and equally divided into positive and negative definite parts, and a drift matrix:

$$A = \begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} gx \\ -gp \end{bmatrix}. \quad (2.9)$$

C. Forward-backward stochastic equations

Next, we derive from the equation of motion (2.8) for $Q(x, p)$ the forward-backward stochastic equations that determine the dynamics $x(t)$ and $p(t)$ of the Q function amplitudes $\lambda = (x, p)$. There is an equivalent time-symmetric stochastic action principle for $Q(\lambda, t)$, leading to probabilistic path integrals [107]. These have sample trajectories λ that define the realistic path for all times.

In this paper, we derive two theorems generalizing these results to the case of an initial superposition state, where a conditional relation implies a feedback between the forward-backward stochastic solutions. The theorems have a general applicability and are hence presented as a distinct result in Appendix A. We prove the equivalence between the averages calculated from the trajectories $\lambda(t)$ (x and p) and those calculated from the Q function $Q(x, p, t)$ (refer also to Secs. VB, IX C, and Appendix H). This result is independently confirmed by statistical comparisons of the numerical simulations with the Q function (Appendix B).

Using the equivalence theorems in Appendix A, we solve the measurement dynamics for H_{amp} . Here, the x and p dynamics decouple. However, to obtain a mathematically tractable equation for the traceless noise matrix, we follow Ref. [104] and the sign of t is reversed in the amplified dynamics of x . The corresponding integrated stochastic equations for $g > 0$ are

$$\begin{aligned} p(t) &= p(t_0|x_0) - \int_{t_0}^t g p dt' + \int_{t_0}^t dw_p, \\ x(t) &= x(t_f) - \int_t^{t_f} g x dt' + \int_t^{t_f} dw_x, \end{aligned} \quad (2.10)$$

where the terms dw_p and dw_x are infinitesimal Gaussian noise increments such that $\langle dw_\mu dw_\nu \rangle = 2g\delta_{\mu\nu}dt$ [134–137]. Both $x(t_f)$ and $p(t_0|x_0)$ represent random boundary conditions with a specified marginal and conditional probability, at the final time t_f and the initial time t_0 of the interaction, respectively. Here, $x_0 \equiv x(t_0)$.

Alternatively, in differential form, we obtain for p the equation [104,105]

$$\frac{dp}{dt} = -gp + \xi_p \quad (2.11)$$

propagating *forward* in time, which has a boundary condition determined by the Q function at the initial time $t_0 = 0$. The equation for x is

$$\frac{dx}{dt_-} = -gx + \xi_x, \quad (2.12)$$

where $t_- = -t$. The equation is solved *backward* in time, and therefore has a boundary condition in the future, defined as the time t_f when the amplification is completed. Defining $\xi = (\xi_p, \xi_x)$, the Gaussian random noises $\xi_\mu(t)$ satisfy: $\langle \xi_\mu(t) \xi_\nu(t') \rangle = 2g\delta_{\mu\nu}\delta(t - t')$, where the noise terms are defined as $\xi_\mu(t) = \lim_{dt \rightarrow 0} dw_\mu/dt$ [135].

Thus, there is a *forward-backward stochastic differential equation* (FBSDE) for individual trajectories. This describes two individual stochastic trajectories, such that the average of the dynamical trajectories equals the Q -function averages. The trajectories are decoupled dynamically, with decay and stochastic noise occurring in each of the time directions. One propagates forward, and one backward, in time. As there are random noise inputs, and the boundary condition is a distribution given by the Q function that is sampled, a series of runs is required. The solutions $x(t)$ and $p(t)$ differ for each run, and averages are evaluated to determine moments at a given time t . The $x(t)$ and $p(t)$ are *trajectories* of the forward-backward stochastic equation (FBSE) simulation [Definition (1) in Sec. IB].

In examples treated here, the trajectory in x depends on a future marginal, $Q(x, t_f)$, while the trajectory in p depends on a conditional distribution, $Q(p|x)$ at the initial time t_0 , which depends on x in the future (Figs. 1 and 2), giving a type of cyclic causal behavior in the form of a “loop”. It is possible to exchange these by time-reversal symmetry. There is an apparent grandfather-type paradox arising from the present seeming to depend on the future. This is resolved in Sec. V by nature of the causal model.

Causal consistency, defined in the Introduction, follows from the proofs and results in Sec. VB, Appendices A and H (refer Fig. 3 for a depiction and proposed experiment): The quantum state $|\psi\rangle$ as determined uniquely by the Q function $Q(x, p, t)$ depends on the time t of evolution, starting from an initial time $t = 0$. The consistency ensures that the trajectories at large time $t_f \rightarrow \infty$ satisfy the conditional relations connecting $x(t_f)$ and $p(t_f)$ for the amplified state, which quantifies the end part of the loop (refer to Figs. 2 and 3). For the measurement interaction H_{amp} , we find that at the final time $t_f \rightarrow \infty$, the p decouples completely from x , and there is no further feedback in the simulation from p to x : the loop is hence incomplete, disconnecting at the end-time t_f .

III. FORWARD-BACKWARD STOCHASTIC SIMULATIONS

A. Measurement on an eigenstate of $\hat{x}$

We first consider the measurement of $\hat{x}$ on an eigenstate $|x_j\rangle$ of $\hat{x}$, where x_j is the eigenvalue [Definition (3) in Sec. IB]. We resolve problems with normalization by defining the

eigenstates in $\hat{x}$ as highly squeezed states. The squeezed state is defined as [128]

$$|\beta_j, z\rangle_{sq} = D(\beta_j)S(z)|0\rangle. \quad (3.1)$$

Here, $|0\rangle$ is the vacuum state satisfying $\hat{a}|0\rangle = 0$, and $D(\beta_j) = e^{\beta_j \hat{a}^\dagger - \beta_j^* \hat{a}}$ and $S(z) = e^{\frac{1}{2}(z^* \hat{a}^2 - z \hat{a}^{\dagger 2})}$ are the displacement and squeezing operators, where z and β_j are complex numbers. For the state with squeezed fluctuations in $\hat{x}$, we note that $z = r$ where r is a real, positive number, referred to as the *squeeze parameter*. Hence $(\Delta\hat{x})^2 = e^{-2r}$ and $(\Delta\hat{p})^2 = e^{2r}$. The parameter r determines the variances and hence the amount of squeezing in $\hat{x}$. The mean values are given by $\langle\hat{a}\rangle = \beta_j$ where $\beta_j = (x_j + ip_j)/2$ and x_j and p_j are real. The eigenstate denoted $|x_j\rangle$ is thus defined as the limiting squeezed state

$$|x_j\rangle \equiv \lim_{r \rightarrow \infty} \left| \frac{x_j}{2}, r \right\rangle_{sq}, \quad (3.2)$$

with x_j real and $p_j = 0$.

The Q function of the eigenstate $|x_j\rangle$ is the bivariate Gaussian (refer to Appendix C)

$$Q_{x_j}(x, p) = \frac{e^{-p^2/2\sigma_p^2}}{2\pi\sigma_x\sigma_p} e^{-(x-x_j)^2/2\sigma_x^2}, \quad (3.3)$$

with variances $\sigma_x^2 = 1 + e^{-2r}$ and $\sigma_p^2 = 1 + e^{2r}$, where $r \rightarrow \infty$. In this limit, $\sigma_x^2 = 1$ and σ_p^2 is large. The mean value of x is the eigenvalue x_j . The variance in x is $\sigma_x^2 = 1$, which is the level of the vacuum noise determined by the Heisenberg uncertainty relation $\Delta\hat{x}\Delta\hat{p} \geq 1$. The *measured* variance of $\hat{x}$ is zero (since for an eigenstate $\Delta\hat{x} = 0$), indicating that the fluctuations given by σ_x are not detectable in the measured output. We refer to these fluctuations where $\sigma_x^2 = 1$ as hidden noise, symbolized by $\eta(t)$.

To solve the forward-backward stochastic equations, we evaluate the Q function for the system at the final time t_f , after the amplification H_{amp} . The state of the amplified system is given by

$$|\psi(t)\rangle = e^{-iH_{\text{amp}}t/\hbar} \left| \frac{x_j}{2}, r \right\rangle_{sq} = \left| G(t) \frac{x_j}{2}, r' \right\rangle_{sq}, \quad (3.4)$$

(refer to Appendix C) where $r' = -gt + r$ and $G(t) = e^{gt}$ is the amplification factor [Eq. (2.3)]. The Q function for the amplified system at a time t is hence

$$Q(x, p, t) = \frac{e^{-p^2/2\sigma_p^2(t)}}{2\pi\sigma_x(t)\sigma_p(t)} e^{-(x-G(t)x_j)^2/2\sigma_x^2(t)}, \quad (3.5)$$

with variances

$$\begin{aligned} \sigma_x^2(t) &= 1 + e^{-2r+2gt}, \\ \sigma_p^2(t) &= 1 + e^{2r-2gt}. \end{aligned} \quad (3.6)$$

It is convenient to write $\sigma_x^2(t) = 1 + G(t)^2[\sigma_x^2(0) - 1]$ and $\sigma_p^2(t) = 1 + [\sigma_p^2(0) - 1]/G(t)^2$ where $G(t) = e^{gt}$. We then see that as $G(t) \rightarrow \infty$, the hidden noise $\eta(t)$ (with variance $\sigma_x^2(0) = 1$) associated with the eigenstate will *not* be amplified (but any noise above this level will be). The noise in p decays to the hidden noise level ($\sigma_p^2(t) \rightarrow 1$).

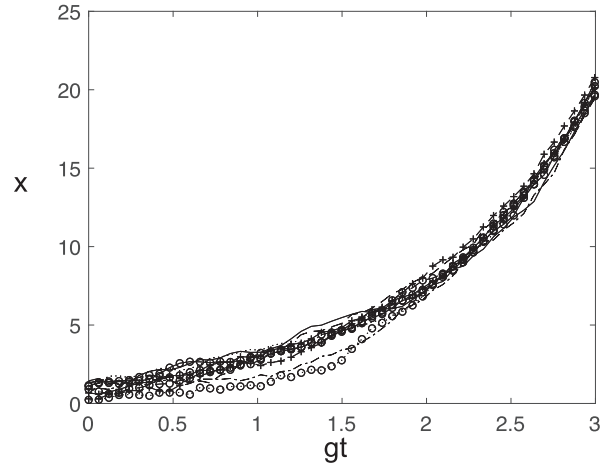


FIG. 6. Stochastic solutions $x(t)$ of the backward equation (2.12). Measurement of $\hat{x}$ on a system prepared in an eigenstate $|x_j\rangle$ of $\hat{x}$. The mean of $x(t)$ is amplified to Gx_j but the variance σ_x is unchanged. $r = 2$, $x_1 = 1$, and $gt_f = 3$.

The forward-backward equations (2.11) and (2.12) can now be solved [107]. The boundary condition for the forward equation is the marginal of $Q(x, p, t_0)$ in p , which is $Q(p, t_0) = e^{-p^2/2\sigma_p^2}/(\sigma_p\sqrt{2\pi})$. The boundary condition for the backward equation is the marginal $Q(x, t_f)$ of $Q(x, p, t_f)$ in x , which is $Q(x, t_f) = e^{-(x-Gx_j)^2/2\sigma_x^2(t_f)}/(\sigma_x(t_f)\sqrt{2\pi})$. Solutions are shown in Figs. 6 and 7.

B. Measurement on a superposition of eigenstates

To address the measurement problem, we need to consider measurements on a system prepared in a superposition of eigenstates. Here, we consider the superposition

$$\begin{aligned} |\psi_{\text{sup}}\rangle &= c_1|x_1\rangle + c_2|x_2\rangle \\ &\rightarrow Nc_1 \left| \frac{x_1}{2}, r \right\rangle_{sq} + c_2 \left| \frac{x_2}{2}, r \right\rangle_{sq} \end{aligned} \quad (3.7)$$

of eigenstates of $\hat{x}$, approximated as a superposition of squeezed states with $r \rightarrow \infty$. The c_j are probability amplitudes. Without loss of generality, we take c_1 to be real and

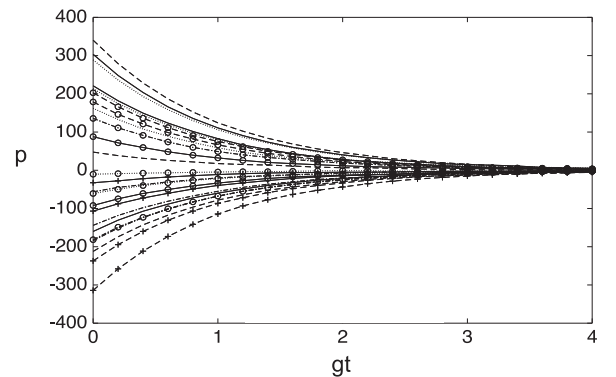


FIG. 7. Stochastic solutions $p(t)$ of the forward equation (2.11). Measurement of $\hat{x}$ as in Fig. 6. There is attenuation of the complementary variable p to the undetected (hidden) vacuum level, where $\sigma_p^2 = 1$. Similar results are obtained for all values of x_j .

positive, and write $c_2 = |c_2|e^{i\varphi}$ where φ is a phase factor. The Q function is

$$Q_{\text{sup}}(x, p, t_0) = N \frac{e^{-p^2/2\sigma_p^2}}{2\pi\sigma_x\sigma_p} (|c_1|^2 e^{-(x-x_1)^2/2\sigma_x^2} + |c_2|^2 e^{-(x-x_2)^2/2\sigma_x^2}) + \mathcal{I}nt, \quad (3.8)$$

where $\mathcal{I}nt = N \frac{e^{-p^2/2\sigma_p^2}}{2\pi\sigma_x\sigma_p} 2|c_1c_2|\mathcal{I}$. The variances are $\sigma_x^2 = 1 + e^{-2r}$ and $\sigma_p^2 = 1 + e^{2r}$. N is the normalization factor. We find

$$\mathcal{I} = e^{-[(x-x_1)^2 + (x-x_2)^2]/4\sigma_x^2} \mathcal{F}, \quad (3.9)$$

where

$$\mathcal{F} = (\cos \varphi) \cos \left[\frac{p}{2\sigma_x^2} (x_1 - x_2) \right] - (\sin \varphi) \sin \left[\frac{p}{2\sigma_x^2} (x_1 - x_2) \right], \quad (3.10)$$

and $N = (1 + 2c_1|c_2|(\cos \varphi)e^{-\frac{(x_1-x_2)^2}{8\sigma_x^2}\{1+\frac{\sigma_p^2}{\sigma_x^2}\}})^{-1}$. NB: $N \rightarrow 1$ as $\sigma_p^2 \rightarrow \infty$, in the limit of large r when the squeezed states become eigenstates of $\hat{x}$.

Taking the limit $r \rightarrow \infty$, we see that the Q function has two Gaussian peaks with fixed variance $\sigma_x^2 = 1$, centered at the eigenvalues x_1 and x_2 . There is an additional interference term, which we symbolize by $\mathcal{I}nt$, corresponding to a central Gaussian peak at $x = 0$ modulated by a fringe pattern $\mathcal{F}$. The Q function of the superposition differs from the Q function $Q_{\text{mix}}(x, p)$ of the mixture of eigenstates

$$\rho_{\text{mix}} = \sum_j |c_j|^2 |x_j\rangle \langle x_j| \quad (3.11)$$

only by the addition of the third term $\mathcal{I}nt$ [138].

Forward solutions for p . The boundary condition for the forward equation (2.11) for p is given by the marginal $Q_{\text{sup}}(p, t_0) = \int Q_{\text{sup}}(x, p, t_0) dx$ for p at $t_0 = 0$. We find

$$Q_{\text{sup}}(p, t_0) = N \frac{e^{-p^2/2\sigma_p^2}}{\sqrt{2\pi}\sigma_p} \left[1 + 2|c_1c_2|e^{-\frac{(x_1-x_2)^2}{8\sigma_x^2}} \mathcal{F} \right]. \quad (3.12)$$

Taking $\cos \varphi = 0$ (c_2 is imaginary) and $x_1 = -x_2$, this reduces to

$$\frac{e^{-p^2/2\sigma_p^2}}{\sqrt{2\pi}\sigma_p} \left\{ 1 - e^{-x_1^2/2\sigma_x^2} \sin(p x_1 / \sigma_x^2) \right\}.$$

Forward trajectories for the attenuated variable p are shown in Figs. 8 and 9. From the solution (3.12), we see that with an increasing separation of the two eigenstates $|x_1\rangle$ and $|-x_1\rangle$ in the superposition $|\psi_{\text{sup}}\rangle$ (so that $|x_1|$ increases), the fringes become less prominent. The evolution of $Q_{\text{sup}}(p, t)$ is shown in Fig. 10 (top panel). The trajectories of p decay to the hidden noise level $\sigma_p^2(t_f) = 1$ (Fig. 10, lower panel).

Backward solutions for x . The boundary condition for the backward equation (2.12) determines the dynamics of the amplified variable x , corresponding to the measured quantity $\hat{x}$, which is of most interest. The boundary condition is determined by the state $|\psi_{\text{sup}}(t)\rangle = e^{-iH_{\text{amp}}t/\hbar} |\psi_{\text{sup}}\rangle$ after amplification H_{amp} . We find

$$|\psi_{\text{sup}}(t)\rangle = N(c_1|Gx_1, r'\rangle_{sq} + c_2|Gx_2, r'\rangle_{sq}), \quad (3.13)$$

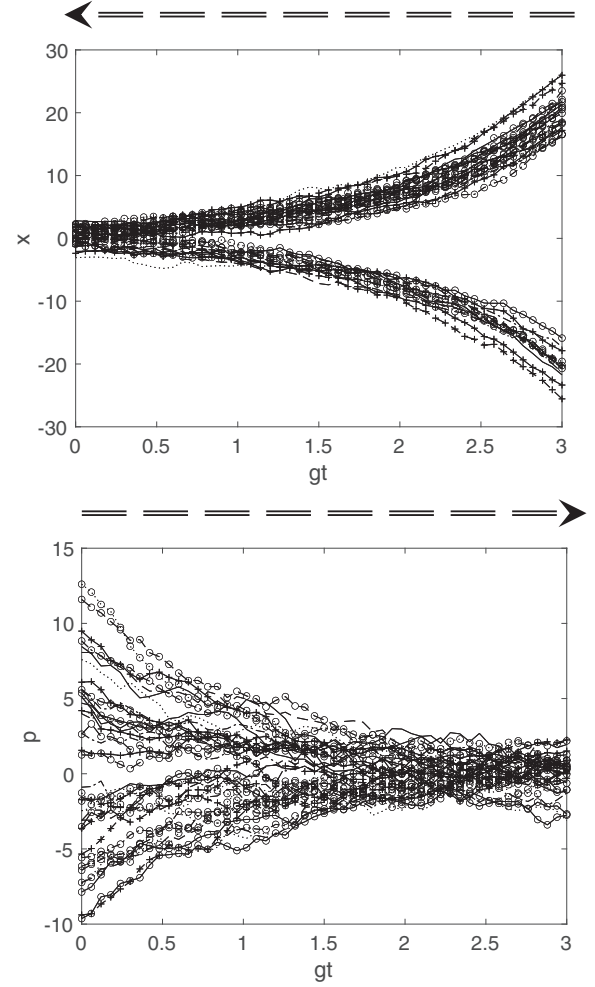


FIG. 8. Emergence of branches with amplification: Measurement of $\hat{x}$ on a system prepared in a superposition given by $|\psi_{\text{sup}}\rangle$ of eigenstates of $\hat{x}$ [Eq. (3.7)]. We take $c_1 = -ic_2 = 1/\sqrt{2}$, $x_1 = -x_2 = 1$ and $gt_f = 3$. The top figure gives individual stochastic solutions (trajectories) of the backward equation (2.12), showing the emergence of branches associated with each eigenvalue x_j . Here, the measurement is made on a microscopic superposition. We note the “width” of each branch is constant. The lower plot shows individual solutions of the forward equation (2.11) for the variable p . The values are attenuated to an undetected (hidden) vacuum noise level, $\sigma_p^2 = 1$. The superposition of eigenstates is approximated by (3.7) with $r = 2$.

where $G = e^{gt}$ is the amplification factor, $r' = -gt + r$ and t is the time of evolution (refer to Appendix C). The Q function of the amplified system is

$$Q_{\text{sup}}(x, p, t) = N_t \frac{e^{-p^2/2\sigma_p^2(t)}}{2\pi\sigma_x(t)\sigma_p(t)} (|c_1|^2 e^{-(x-G(t)x_1)^2/2\sigma_x^2(t)} + |c_2|^2 e^{-(x-G(t)x_2)^2/2\sigma_x^2(t)} + 2|c_1c_2|\mathcal{I}(t)), \quad (3.14)$$

where $G(t) = e^{gt}$, N_t is a normalization constant and $\mathcal{I}(t)$ is given by $\mathcal{I}$ and $\mathcal{F}$ of Eq. (3.9), but substituting $x_j \rightarrow G(t)x_j$, and $\sigma_x \rightarrow \sigma_x(t)$, $\sigma_p \rightarrow \sigma_p(t)$. The variances $\sigma_x(t)$ and $\sigma_p(t)$ of the amplified state are given by Eq. (3.6). Importantly, we note that the means x_1 and x_2 of the Gaussian functions

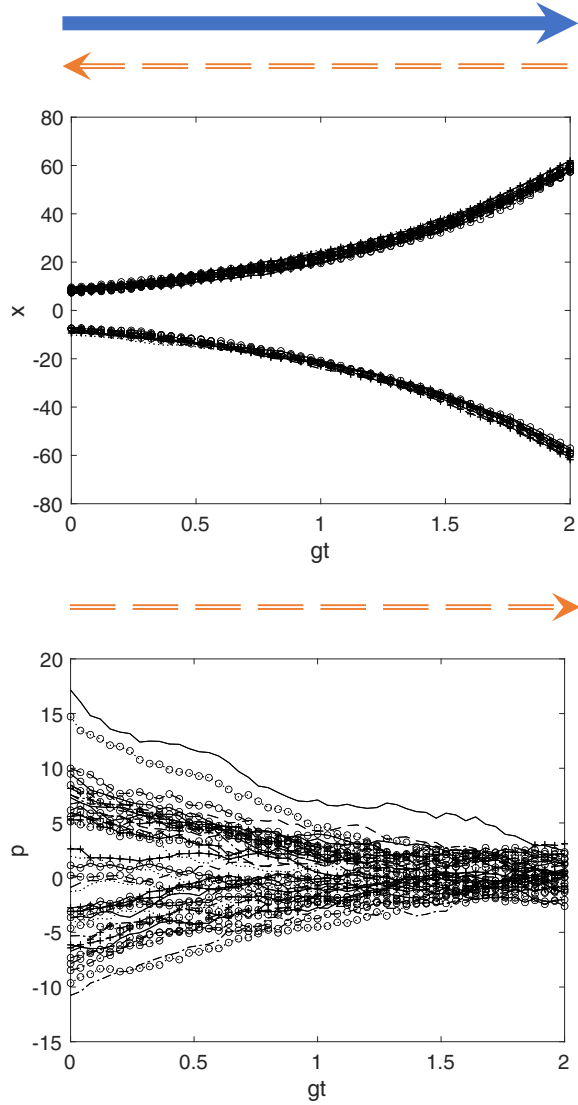


FIG. 9. Measurement on a macroscopic superposition. As in Fig. 8, but with $gt_f = 2$ and $x_1 = 8$. The top and lower figures show solutions of the backward and forward equations (2.12) and (2.11), respectively. The solid blue line denotes the causal relation $x_j \rightarrow Gx_j$ that amplifies the mean x_j of the Gaussian terms, as evident in the Q function [Eq. (3.14)] of $|\psi_{\text{sup}}(t)\rangle$. The individual trajectories propagate in the backward-time direction, and are subject to input noises, $\delta x(t_f) \equiv \eta(t_f)$ and $\xi_x(t)$, as defined by Eqs. (3.23) and (2.12). The noise $\delta x(t)$ gives the width of each branch a time t , and is constant in average magnitude throughout the amplification, as determined by the variance σ_x . The orange dashed lines represent backward-propagating (right to left) and forward-propagating (left to right) trajectories for x and p . In the Q model of reality (Sec. IV), the value given by $x(t_f)$ is the detected outcome. The inferred outcome for $\hat{x}$ is $x(t)/G$ giving either x_1 or $-x_1$ ($G = e^{gt_f}$). The p and $\delta x(t_f)$ are not amplified and are hence not measured.

amplify according to

$$x_j \rightarrow X_j = Gx_j, \quad (3.15)$$

where $G \equiv G(t) = e^{gt}$, which being a deterministic relation can be reversed, so that $Gx_j \rightarrow x_j$. By contrast, the variances in x of the Gaussian distributions do not amplify. The vari-

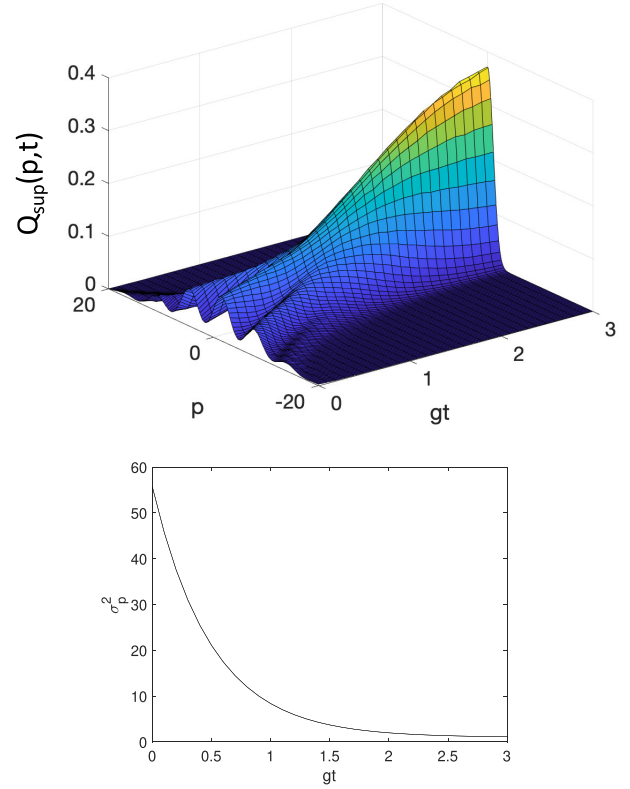


FIG. 10. Decay of the complementary variable. Solutions for the complementary variable p , for measurement of $\hat{x}$ on the system in the superposition $|\psi_{\text{sup}}\rangle$ of two eigenstates of $\hat{x}$. Parameters are as for Fig. 8, with $x_1 = 1$. The top figure shows the distribution $Q_{\text{sup}}(p, t)$ of the forward-propagating trajectories of p , as evaluated from 10^6 trajectories, a sample of which is given in Fig. 8. Fringes are evident at $t = 0$. The lower plot confirms the reduction in the variance $\sigma_p^2(t)$ to the hidden unobservable value of 1, as $t_f \rightarrow \infty$.

ances are

$$\sigma_x^2(t) = 1 \quad (3.16)$$

throughout the evolution, the fluctuations hence remain constant at the “hidden level.” This is evident in Figs. 8 and 9, where the width of each band of trajectories, as associated with each eigenvalue, is constant with t . By contrast, σ_p^2 is initially large as for an eigenstate $|x_j\rangle$, but reduces to the hidden noise level $\sigma_p^2(t) = 1$ as $t \rightarrow \infty$. We see that the Gaussian functions become increasingly sharp as the system amplifies. We must also examine how the interference term $\mathcal{I}(t)$ in the Q function (3.14) behaves on amplification.

The future boundary condition for the backward equation is given by the marginal

$$Q_{\text{sup}}(x, t) = \int Q_{\text{sup}}(x, p, t) dp. \quad (3.17)$$

at the final time $t = t_f$. This requires integration over the variable p . To calculate the marginal, we evaluate $\mathcal{I}(x, t) = \int dp \mathcal{I}$, where $\mathcal{I}$ is given by (3.9), which requires the

integration

$$\int_{-\infty}^{\infty} dp \frac{e^{-p^2/2\sigma_p^2(t)}}{2\pi\sigma_x(t)\sigma_p(t)} \left\{ (\cos\varphi) \cos \left[\frac{p}{2\sigma_x^2(t)} G(t)(x_1 - x_2) \right] - (\sin\varphi) \sin \left[\frac{p}{2\sigma_x^2(t)} G(t)(x_1 - x_2) \right] \right\}. \quad (3.18)$$

The second integral proportional to $\sin\varphi$ is zero by symmetry. The first term simplifies to

$$\begin{aligned} & \cos\varphi \int_{-\infty}^{\infty} dp \frac{e^{-p^2/2\sigma_p^2(t)}}{\sqrt{2\pi}\sigma_p} \cos \left[\frac{p}{2\sigma_x^2(t)} (x_1 - x_2) \right] \\ &= (\cos\varphi) e^{-G(t)^2(x_1-x_2)^2\sigma_p^2(t)/8\sigma_x^4(t)}. \end{aligned} \quad (3.19)$$

This term gives the contribution of the interference to the marginal, and becomes important for the mechanism of Bell violations in Secs. VII and IX. It depends on the variance σ_p of p , which tends to ∞ at the initial time t_0 , since we have considered a superposition $|\psi_{\text{sup}}\rangle = \sum_j c_j |x_j\rangle$ of eigenstates of $\hat{x}$. The solution for the marginal is

$$\begin{aligned} Q_{\text{sup}}(x, t) &= N_t \frac{1}{\sqrt{2\pi}\sigma_x(t)} (|c_1|^2 e^{-(x-G(t)x_1)^2/2\sigma_x^2(t)} \\ &\quad + |c_2|^2 e^{-(x-G(t)x_2)^2/2\sigma_x^2(t)} + 2|c_1 c_2| \mathcal{I}(x, t)), \end{aligned} \quad (3.20)$$

where

$$\begin{aligned} \mathcal{I}(x, t) &= \cos\varphi e^{-[(x-G(t)x_1)^2 + (x-G(t)x_2)^2]/4\sigma_x^2(t)} \\ &\quad \times e^{-G(t)^2(x_1-x_2)^2\sigma_p^2(t)/8\sigma_x^4(t)}, \end{aligned} \quad (3.21)$$

Here, $\sigma_x^2(t) = 1 + G(t)^2[\sigma_x^2(0) - 1]$ and $\sigma_p^2(t) = 1 + [\sigma_p^2(0) - 1]/G(t)^2$ [Eq. (3.6)]. We see that $\sigma_p(0) \rightarrow \infty$ and $\sigma_x^2(t) = 1$. Immediately, we see that the interference term $\mathcal{I}$ will vanish at the initial time $t = 0$ in the marginal for x . Hence $\mathcal{I}(x, 0) \rightarrow 0$ in the limit $r \rightarrow \infty$, which corresponds to the expansion of $|\psi_{\text{sup}}\rangle$ in terms of the measurement basis, as in Eq. (3.7).

The vanishing of the interference term $\mathcal{I}(x, 0)$ in the marginal is an important result that is due mathematically to the integral (3.19), which vanishes when $\sigma_p(t)$ is large. Regardless, on substituting $\sigma_p(t_f) = \sigma_x(t_f) = 1$, we see that $\mathcal{I}(x, t_f)$ rapidly de-amplifies with amplification G , with $\mathcal{I}(x, t_f) \rightarrow (\cos\varphi) e^{-x^2/2-G^2x_1^2} \rightarrow 0$ for $x_1 = -x_2$. In this way, we see that *amplification plays a similar role to decoherence in attenuating the interference terms* that distinguish the superposition $|\psi_{\text{sup}}\rangle$ from the mixture ρ_{mix} .

The generalization to expansions $|\psi_{\text{sup}}\rangle = \sum_j c_j |x_j\rangle$ is straightforward (refer to Appendix C). This leads to the following Definition [given as Definition (4) in Sec. IB] and result:

Definition: Hidden interference terms. Consider measurement of $\hat{x}$. The interference terms $\mathcal{I}$ appearing in the Q function $Q(x, p, t)$ when the Q function is expanded in terms of the Q functions of the eigenstates of $\hat{x}$ are referred to as hidden interference terms. These terms are calculated by writing $|\psi\rangle$ in terms of the eigenstates $|x_j\rangle$ (i.e., expanding $|\psi\rangle$ in the measurement basis) and calculating the Q function directly from Eq. (2.4).

Result III.1: Interference terms and the future boundary condition. The hidden interference terms arising in the Q function of a state $|\psi\rangle$ do not contribute to the future boundary condition, given by Eq. (3.17) (proved above). Consider the measurement of $\hat{x}$ on the system $|\psi_{\text{sup}}\rangle = \sum_j c_j |x_j\rangle$ [Eq. (3.7)] at time $t_0 = 0$. From Eq. (3.20) with $\mathcal{I}(x, t) \rightarrow 0$ and $\sigma_x(t_f) = 1$, the future boundary condition for the backward equation (2.12) reduces to ($G = e^{gt_f}$)

$$Q_{\text{sup}}(x, t_f) = \frac{1}{\sqrt{2\pi}\sigma_x(t_f)} \sum_j |c_j|^2 e^{-(x-Gx_j)^2/2}. \quad (3.22)$$

Solutions showing individual trajectories for the amplified variable x are given in Figs. 8 and 9.

Comment. The interference terms $\mathcal{I}(x, t)$ of the marginal $Q(x)$ vanish in the limit of large r , because the integral (3.19) vanishes. However, the interference $\mathcal{I}(t)$ does not vanish in the initial joint distribution $Q(x, p, 0)$ [Eq. (3.14)]. Moreover, the integral will not vanish where there is a change of basis so that $\sigma_p(0)$ is no longer large, which is relevant to Bell violations (refer to Sec. VII). ■

Comment. The future boundary condition (3.22) is the same as that for the system prepared in the mixed state ρ_{mix} [Eq. (3.11)].

Procedure of simulation. Result III.1 defines the procedure of the simulation. The interference terms in $Q(x, t)$ are justified to be ignored *provided we write in the measurement basis* as in Result III.1, where $r \rightarrow \infty$ (or else for all r , if $\text{Re}\{c_2\} = 0$ [refer to Eq. (3.7)]). Then, the future boundary condition for simulation of (2.12) is determined by the marginal (3.22), which comprises a set of Gaussian peaks (one for each outcome x_j) with equal variance $\sigma_x^2(t_f) = 1$. To model measurement, strong amplification is needed, and gt_f is large.

We consider the superposition of two eigenstates $|x_1\rangle$ and $|x_2\rangle$ as in Eq. (3.7). The initial condition for a given run of the simulation of Eq. (2.12) is

$$x(t_f) = Gx_j + \delta x(t_f), \quad (3.23)$$

where the eigenvalue x_j is selected to be x_1 or x_2 with probability $|c_j|^2$. The $\delta x(t_f)$ is a noise term, which we also label $\eta(t_f)$, sampled from the Gaussian distribution $\mathcal{G}(0, 1)$ of mean 0 and variance $\sigma^2 = 1$. In this paper, we define the Gaussian distribution using the notation $\mathcal{G}(\bar{x}, \sigma^2) = \frac{e^{-(x-\bar{x})^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$, where $\bar{x}$ is the mean and σ^2 is the variance of x .

Comment. Importantly, we see that value $\delta x(t_f)$ [denoted by $\eta(t)$ in Fig. 2] is hence *independent* of the choice of eigenvalue x_j and independent of t_f (Fig. 11). This is relevant to understanding how grandfather paradoxes are avoided. The sampling is determined by (3.22), which is identical to that for the system prepared in a mixture of eigenstates, ρ_{mix} [Eq. (3.11)]. We will see that the procedure of the simulation defines the causal model depicted in Fig. 2, and explained in Sec. V.

The simulations reveal *branches*, as $gt_f \rightarrow \infty$ (Figs. 8 and 9). This leads to the following definition [given as Definition (5) in Sec. IB].

Definition: Branches $\mathcal{B}_j$ are defined as groups of amplitudes $x(t)$, each associated with a distinct eigenstate $|x_j\rangle$ in the superposition $|\psi_{\text{sup}}\rangle$ (Figs. 8 and 9).

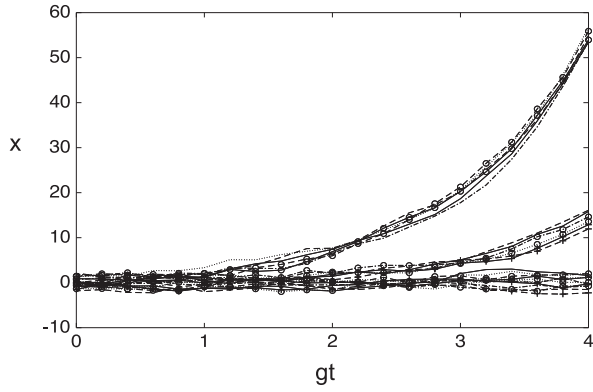


FIG. 11. Measurement on eigenstates showing branches have constant width independent of eigenvalue. The solutions show the amplification of $\hat{x}$ for the eigenstate $|x_j\rangle$, for different values of x_j : The curves correspond to $x_j = 1$ (top curve), $x_j = 0.25$ (center top), $x_j = 0.01$ (center lower) and $x_j = 0$ (lowest curve), with $gt_f = 4$. The curves $x = 0.01$ and $x_j = 0$ cannot be distinguished with the level of amplification given by $gt_f = 4$. The curve for $x_j = 0$ is the solution δx of Eq. (3.25), validating the expansion $x(t) = x_j(t) + \delta x(t)$ of Eq. (3.24) in which δx is a hidden noise term of constant average magnitude independent of x_j .

Result III.2: Branches emerge with amplification.

Proof. The fluctuations with variance σ_x about the mean x_j , as associated with each Gaussian in the Q function $Q_{\text{sup}}(x, p, t_0)$ [Eq. (3.8)] of the initial state, are not amplified [Eq. (3.14)] with the evolution. However, the mean values x_j are amplified according to $x_j \rightarrow X_j = Gx_j$, where $G = e^{gt_f}$ [Eq. (3.15)]. Hence, as $gt \rightarrow \infty$, we see distinct peaks centered at Gx_j with relative probabilities $|c_j|^2$. The eigenvalues x_j are amplified, but the hidden noise δx about these values is not. Hence, branches emerge for $x(t)$ as t increases. The hidden noise represents the width $w \sim \delta x$ of the lines that form the branches, and is constant, independent of x_j , as in Fig. 8. ■

Result III.3: The hidden noise δx (the width of the branch) is independent of the eigenvalue x_j .

Proof. The width w of each branch $\mathcal{B}_j$ becomes indiscernible relative to the mean value Gx_j , as $gt \rightarrow \infty$. The hidden vacuum fluctuation $\delta x(t)$ does not lead to a macroscopic value. The amplitudes can be written

$$x(t) = x_j(t) + \delta x(t), \quad (3.24)$$

where $x_j(t) = e^{gt} x_j$, the solutions for $\delta x(t)$ being independent of x_j , satisfying the stochastic equation

$$\frac{d[\delta x(t)]}{dt_-} = -g\delta x(t) + \xi_x(t), \quad (3.25)$$

with input $\delta x(t_f) \equiv \eta(t_f)$ given by sampling the Gaussian $\mathcal{G}(0, 1)$. This is verified in Fig. 11, where the $\delta x(t)$ can be taken as the solution of (2.12) with $x_j = 0$. The noise $\delta x(t) = \eta(t)$ adds equally to the exponential lines given for the different eigenvalues x_j in Fig. 11. ■

IV. Q MODEL OF REALITY

A. Assumptions and basic results

The solutions for $x(t_f)$ in Figs. 8 and 9 show two branches, associated with each eigenstate of the superposition $|\psi_{\text{sup}}\rangle$. The stochastic equations hence motivate a “hidden-variable” model for quantum measurement, referred to as the Q -based reality model (Q model of reality) or objective-field model [104–106]. We regard the amplitudes λ (i.e., x and p) of the Q function $Q(x, p)$ as representing a realization of the quantum state, immediately prior to and during the measurement. The physical assumptions made for the meter H_{amp} that defines the measurement of $\hat{x}$ are as follows:

(1) Measurement is identified as an amplification process (after measurement settings are fixed).

(2) The amplitudes x and p are phase-space variables, with distribution $Q(x, p, t)$ at time t . We denote the values of x and p at time t as $x(t)$ and $p(t)$. Here, we denote $x(t_f)$ as the amplified variable.

(3) The detected value is $x(t_f)$, when macroscopic. The amplitudes at the hidden vacuum level (i.e., where the variances σ_x^2 or σ_p^2 are 1) are not observable. The inferred (or “measured”) outcome is $x_0 = x(t_f)/G$ where G is the amplification factor.

(4) The probability density function for the detection of $x(t_f)$ is $Q(x, t_f) = \int Q(x, p, t_f) dp$ where $t_f \rightarrow \infty$ is the time of interaction with the amplifier.

(5) Amplified variables satisfy future boundary conditions and attenuated variables satisfy past boundary conditions. The amplifier does not couple amplified and attenuated quadratures.

The last condition applies to the model of measurement given by H_{amp} [Eq. (2.2)] but may be adjusted when the measurement model includes a coupling to a separate system that is a meter (refer to Ref. [109]). The condition is motivated by the stochastic method (Appendices A and B) and by the causal consistency proved in Secs. VB, IX C, and Appendix H. In the model, the system evolves under H_{amp} for a time t_f to give the final measurement output. The measurement is completed simply by the fields being amplified to become macroscopic, and hence detectable to macroscopic devices or observers. No special collapse is required, only standard physical processes. The analysis is hence compatible with Bell’s requirement [111] that there should be no “shifty split” between system and measurement apparatus.

We present a series of results applying to the Q -based model of reality.

Result IV.1: The inferred outcomes are the eigenvalues. Consider measurement of $\hat{x}$. After sufficient amplification, the $x(t_f)$ separate into branches each associated with a single eigenstate $|x_j\rangle$ of $\hat{x}$ [Definition (5) of Sec. IB and Result III.2, Fig. 8]. The detected value according to the model of reality is $x(t_f)$, where $t_f \rightarrow \infty$ so that $x(t_f)$ is macroscopic, and the inferred outcome for $\hat{x}$ is $x(t_f)/G$. The inferred outcome will always be one of the eigenvalues x_j .

Proof. Since from Eq. (3.24) [based on the solutions of (2.12)], we write $x(t) = e^{gt} x_j + \delta x(t)$, we see that with sufficient amplification, the inferred outcome is one of the eigenvalues x_j . The vacuum fluctuation $\delta x(t)$ does not lead to a macroscopic value and is undetectable, becoming $\delta x(t)/G$.

As t_f increases, each branch becomes identifiable as a thin line, which represents a distinct value x_j for $\hat{x}$. ■

Result IV.2: Incompleteness. The amplitudes x and p of $Q(x, p, t)$ at a time t [along with the forward-backward stochastic equations, (2.12) and (2.11)] are not sufficient to completely define the state and future dynamics of the system.

Proof. Consider the measurement $\hat{x}$ on an eigenstate $|x_j\rangle$ of $\hat{x}$ where $x_j \rightarrow 0$. The measurement always yields the outcome x_j , which is the eigenvalue, but which is not identifiable from the amplitudes x and p alone (refer Fig. 11). The value x_j is masked by the hidden Gaussian noise when x_j is small (Fig. 8). The future boundary condition for the backward equation is required for the solutions $x(t)$. This requires knowledge of the Q function $Q(x, p, t_0)$, or at least (in a hidden-variable model, refer to Secs. IV G and V) *knowledge of the Gaussian-mean value x_j* . ■

Comment. Our interpretation of the Q model of reality and associated causal model for measurement (Sec. V) includes that the system at time t_0 is defined as prepared with respect to a measurement basis, such as $|x_j\rangle$ of $\hat{x}$ [39]. The Q function uniquely describes the quantum state $|\psi\rangle$ which can be expanded as $|\psi\rangle = \sum_j c_j |x_j\rangle$, from which the Q function will be of the form (3.8), a sum of Gaussian functions in x and p with mean x_j , as well as interference terms of type *Int*, as generalized for an arbitrary number of states $|x_j\rangle$. The means x_j and c_j are hence in principle identifiable as part of the description of the state at time t_0 (Appendix C). We show in Sec. V A that the dynamics of $x(t)$ as implied by H_{amp} can be deduced from the backward stochastic equation (2.12), if the x_j and c_j are known, based on the deterministic relation $x_j \leftrightarrow Gx_j$ (which assumes the system at time t_0 once H_{amp} is implemented to be associated probabilistically with a particular x_j , refer Secs. IV G and V A).

B. Derivation of Born's rule

In the Q model of reality, the density of the amplitudes $x(t_f)$ as $gt_f \rightarrow \infty$ gives the probability for detection, which leads to Born's rule. The rule follows because the measurement of $\hat{x}$ amplifies the variable x , but attenuates the complementary variable p and interference terms $\mathcal{I}$ (Result III.1). We give a derivation below.

Result IV.3a: Born's rule. Consider a general superposition $|\psi_{\text{sup}}\rangle = \sum_j c_j |x_j\rangle$ of eigenstates $|x_j\rangle$ of $\hat{x}$, where c_j are probability amplitudes. Here, we define the eigenstates as the squeezed states $|\frac{x_j}{2}, r\rangle_{sq}$ [Eq. (3.2)] in the limit where $r \rightarrow \infty$. The final density $Q_{\text{sup}}(x, t_f)$ given by the Q function of the amplified state $e^{-iH_{\text{amp}}t/\hbar}|\psi_{\text{sup}}\rangle$ leads to Born's rule, that the probability of an outcome x_j is $|c_j|^2$.

Proof. In the limit of $gt \rightarrow \infty$, the marginal distribution for the amplified state is

$$Q_{\text{sup}}(x, t) = \frac{1}{\sqrt{2\pi}\sigma_x(t)} \sum_j |c_j|^2 e^{-(x-G(t)x_j)^2/2\sigma_x^2(t)},$$

where $G = e^{gt}$ ($g > 0$) and $\sigma_x^2(t) = 1 + e^{2gt-2r}$ [refer Eq. (3.3)]. Considering the measured inferred variable $x_0 = x/G$ where $G = e^{gt}$, the rescaled distribution

becomes

$$\begin{aligned} Q_{sc}(x_0, t) &= \frac{1}{\sqrt{2\pi}\sigma_{x_0}(t)} \sum_j |c_j|^2 e^{-(x_0-x_j)^2/2(e^{-2gt}+e^{-2r})} \\ &\rightarrow \frac{1}{\sqrt{2\pi}\sigma_{x_0}(t)} \sum_j |c_j|^2 e^{-(x_0-x_j)^2/(2e^{-2r})} \end{aligned} \quad (4.1)$$

on taking gt large. The eigenstates of $\hat{x}$ correspond to $r \rightarrow \infty$ where the Gaussian peaks become infinitely narrow. The separation between peaks is much larger than the width of the Gaussian distributions. Hence, in the Q model, the probability of observing x_j is given by the weighting $|c_j|^2$, in accordance with Born's rule. ■

Result IV.3b: We demonstrate Born's rule for measurement of the rotated quadrature

$$\hat{x}_\theta = \hat{x} \cos \theta + \hat{p} \sin \theta \equiv \hat{a} e^{-i\theta} + \hat{a}^\dagger e^{i\theta}. \quad (4.2)$$

The complementary observable is $\hat{p}_\theta = -\hat{x} \sin \theta + \hat{p} \cos \theta$.

Proof. We expand in eigenstates $|x_{\theta j}\rangle$ of $\hat{x}_\theta$ as $|\psi_{\text{sup},\theta}\rangle = \sum_j d_j |x_{\theta j}\rangle$, where d_j are probability amplitudes. The eigenstates are defined similarly to $|x_j\rangle$ [Eq. (3.2)] as highly squeezed states in $\hat{x}_\theta$ (Appendix C). We define real coordinates x_θ and p_θ as $\alpha e^{-i\theta} = (x_\theta + ip_\theta)/2$. Extending the derivation of the Q function of a squeezed state as in Appendix C, the Q function $Q_{\text{sup},\theta}(x_\theta, p_\theta)$ can be derived for $|\psi_{\text{sup},\theta}\rangle$. Modeling the amplification of $|\psi_{\text{sup},\theta}\rangle$ by applying

$$H_{\text{amp},\theta} = i\hbar g(\hat{a}^{\dagger 2} e^{2i\theta} - \hat{a}^2 e^{-2i\theta})/2 \quad (4.3)$$

[which gives $\hat{x}_\theta(t) = e^{gt} \hat{x}_\theta(0)$] the result is proved by evaluating the marginal $Q_{\text{sup},\theta}(x_\theta, t)$ of the Q function of the amplified state [following Eqs. (3.14)–(3.20)], and applying the same analysis as for Result IV.3a. ■

Summarizing, the probability density for an outcome x_j , given $gt_f \rightarrow \infty$, is

$$P(x_j) \equiv Q_{sc}(x_j, t_f) = |\langle x_j | \psi_{\text{sup}} \rangle|^2, \quad (4.4)$$

in agreement with the quantum prediction (Born's rule). Here, $Q_{sc}(x_j, t_f)$ is the rescaled distribution, found by taking $Q_{\text{sup}}(x, t_f)$, rewriting with respect to the scaled variable $x_0 = x/G$, and taking the limit where $G = e^{gt} \rightarrow \infty$.

Role of the Wigner function. The realization of Born's rule for the *continuous-valued* measurements $\hat{x}_\theta$ can be best demonstrated by the connection with the Wigner function, $W(x, p)$. We find this elucidates the nature of the backward stochastic equation (2.12). It is known that the marginal

$$W(x) = \int W(x, p) dp \quad (4.5)$$

of the Wigner function gives the distribution for observed outcomes x of $\hat{x}$.

To establish the connection with the Q distribution in the model of reality, we write the Q function $Q(\lambda, t_0)$ for $|\psi\rangle$ at time t_0 with respect to the measurement basis. We consider measurement of $\hat{x}$ and expand $Q(\lambda, t_0)$ in terms of the Q functions of the $\hat{x}$ eigenstates [Definition (3), Sec. IB]. We find the required form is obtained by writing the Q function in terms of the Wigner function. The Q function of a state $|\psi\rangle$ is the convolution of the Wigner function $W(\beta)$: $Q(\alpha) = \frac{2}{\pi} \int W(\beta) e^{-2|\alpha-\beta|^2} d^2\beta$ [115,116]. Using $\alpha = (x + ip)/2$, we

find

$$Q(\lambda, t_0) = \frac{1}{2\pi} \int W(\lambda_0) e^{-(x-x_0)^2/2} e^{-(p-p_0)^2/2} d\lambda_0, \quad (4.6)$$

where $\lambda \equiv (x, p)$, $\lambda_0 \equiv (x_0, p_0)$, $d\lambda_0 = dx_0 dp_0$ and $W(\lambda_0)$ is the Wigner function of the state $|\psi\rangle$ at time t_0 . The convolution ensures the positivity of Q , given W can be negative. The marginal $Q(x, t_0) = \int dp Q(\lambda, t_0)$ is

$$Q(x, t_0) = \frac{1}{\sqrt{2\pi}} \int dx_0 W(x_0) e^{-(x-x_0)^2/2}, \quad (4.7)$$

which we can rewrite as

$$Q(x, t_0) = \int dx_0 W(x_0) Q_{x_0}(x), \quad (4.8)$$

where we note that $Q_{x_j}(x) = \frac{e^{-(x-x_j)^2/2}}{\sqrt{2\pi}}$ is the marginal in x of the Q function $Q_{x_j}(x, p)$ of the x eigenstate $|x_j\rangle$ with eigenvalue x_j , given by Eq. (3.3) on putting $\sigma_x^2 = 1$ as valid for eigenstates of $\hat{x}$ where $r \rightarrow \infty$. Comparing $Q(x)$ with the marginal $Q_{\text{sup}}(x, 0)$ of Eq. (3.20), and using Result III.1 where we show interference terms $\mathcal{I}$ do not contribute to final probabilities for $\hat{x}$, we identify $W(x_0)$ as the probabilities $|c_j|^2$ (taking the limit of a continuum of states $|x_j\rangle$). Hence, $W(x_0)$ is the probability density for an outcome x_0 of $\hat{x}$. This gives consistency with the earlier Result III.1, which ignores interference in the future boundary. In summary, we arrive at the following two results:

Result IV.4: *The future boundary condition is determined by the marginal of the Wigner function of the state defined at the time t_0 . The Q function $Q(x, p, t_0)$ represents the quantum state $|\psi\rangle$ of the system at time t_0 . The marginal $Q(x, t_f) = \int dp Q(x, p, t_f)$ of the Q function $Q(x, p, t_f)$ applies to the state of the amplified system after interaction with H_{amp} , at a time t_f . Rewriting $Q(x, t_f)$ in terms of the scaled variable $x_0 = x/G$ that represents the measured outcome, we establish from Result IV.3a that the resulting function $Q_{sc}(x_0, t_f)$ gives the probability distribution for the outcomes x_0 . Hence,*

$$Q_{sc}(x_0, t_f) \rightarrow W(x_0) \quad (4.9)$$

as $G = e^{gt_f} \rightarrow \infty$, where $W(x_0)$ is the marginal of the Wigner function defined by (4.5). Hence, the marginal $W(x)$ of the Wigner function defined at the time t_0 determines the future boundary condition for x .

Result IV.5: *Physical interpretation of the Wigner function:* The simulation gives a physical interpretation of the Wigner function. The quantum-optics relation (4.6) between the Wigner and Q function can be derived from the simulation.

Proof. The Q function at time t_0 is $Q(x, p, t_0)$ represents the quantum state at time t_0 . The marginal $W(x)$ of the Wigner function corresponds to the marginal $Q_{sc}(x_0, t_f)$ of the Q function of the amplified state, when rescaled so that the hidden Gaussian noise is treated as indiscernible. The Wigner function determines the future boundary condition. The simulation of the backward equation (2.12) proceeds by rewriting $W(x_0)$ in terms of $x' = Gx_0$, where $G = G(t_f) = e^{gt_f}$, to obtain the rescaled distribution $W_{sc}(x')$, and convoluting with the hidden Gaussian noise of variance $\sigma_x = 1$ that describes the input δx

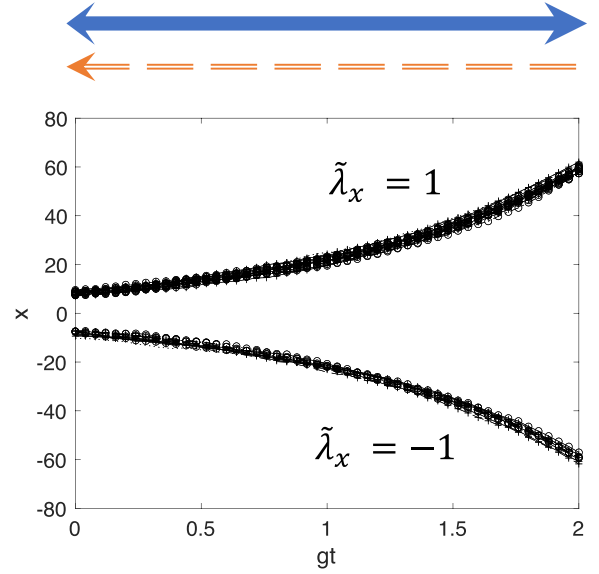


FIG. 12. Macroscopic realism. Trajectories modeling the measurement of $\hat{x}$ on a macroscopic superposition $|\psi_{\text{sup}}\rangle$ of two eigenstates of $\hat{x}$, as in Fig. 9. The solid blue arrow pointing forward in time denotes the causal relation $x_j \rightarrow Gx_j$ that amplifies the mean of the Gaussian peaks in the Q function of $|\psi_{\text{sup}}\rangle$. This relation is deterministic, and hence can be reversed, giving a retrodiction $Gx_j \rightarrow x_j$, indicated by the backward-pointing arrow. For the macroscopic superposition depicted, the amplitudes $x(0)$ and $x(t_f)$ of x at the initial and final times are on the same branch, $\tilde{\lambda}_x$. The value of $x(0)$ predetermines the outcome of $\hat{x}$ (Results IV.6 and IV.7). The amplification, as in $x_j \rightarrow Gx_j$ represents a macroscopic causal relation (forward-pointing arrow).

[refer to Eq. (3.23)]. This is given by

$$Q(x, t_f) = \frac{1}{\sqrt{2\pi}} \int dx' W_{sc}(x') e^{-(x-x')^2/2} \quad (4.10)$$

We have seen from the simulation of the backward equation (Figs. 8 and 9) that $Q(x, t_0)$ retains the same distribution as $Q(x, t_f)$ over the Gaussian terms $e^{-(x-Gx_j)^2/2}/\sqrt{2\pi}$, except that the means decay ($x' = Gx_0 \rightarrow x_0$) in the negative time direction. Hence [since $G(t) = 1$ at t_0 implying $W_{sc}(x') = W(x_0)$], we find from the simulation that

$$Q(x, t_0) = \frac{1}{\sqrt{2\pi}} \int dx_0 W(x_0) e^{-(x-x_0)^2/2}. \quad (4.11)$$

This explains the fundamental relation (4.6) used in quantum optics, that gives the relationship between the Wigner and the Q -function distributions. ■

C. (Weak) macroscopic realism

Weak macroscopic realism is defined in Definition (6) of Sec. IB. We summarize below, for application to the system of Figs. 9 and 12, where there is a measurement of $\hat{x}$.

Definition: *Weak macroscopic realism (wMR).* Consider a system in a superposition of two macroscopically distinct states φ_1 and φ_2 that are distinguished by a binary measurement $\hat{O}$. We assume that at a time t_0 , any operations U needed to fix the measurement setting have been carried out. Weak

macroscopic realism posits that the outcome of $\hat{O}$ is determined, at the time t_0 [108,119].

In the simulation shown in Fig. 9, the macroscopically distinct states are the eigenstates $|x_1\rangle$ and $| -x_1\rangle$, where $|x_1| \gg 1$. The binary-valued measurement $\hat{O}$ given by the sign of the outcome of $\hat{x}$ distinguishes these states. We refer to the outcomes of $\hat{O}$ as $+1$ and -1 . The premise of wMR posits that the system at time t_0 has a definite value for the outcome (denoted by $\tilde{\lambda}_x$) of the measurement $\hat{O}$. That is, each realization of the system can be regarded as *either* in a state with $\tilde{\lambda}_x = 1$ *or* in a state with $\tilde{\lambda}_x = -1$.

Result IV.6: Weak macroscopic realism holds in the Q model of reality: Premise wMR(1).

Proof. We consider the system described above and depicted in Fig. 12, which is prepared at time t_0 in the superposition $|\psi_{\text{sup}}\rangle$. There are two ways to carry out the measurement of $\hat{x}$ (and hence of $\hat{O}$) on the system at time t_0 .

(1) The first is to directly detect the value $x(t_0)$ of x at time t_0 , since this value is macroscopic by definition of the macroscopic superposition state, i.e., we take $t_0 \equiv t_f$. According to the assumptions of the Q model of reality, the value of $x(t_f)$ determines the measurement outcome. For the macroscopic superposition, the $x(t_0)$ is associated with one or other branch, which implies the outcome of $\hat{x}$ is determined by the value $x(t_0)$ at the time t_0 , prior to detection. Hence, wMR holds for the macroscopic superposition state at time t_0 .

(2) The second way to measure $\hat{x}$ is to further amplify the system, by applying H_{amp} , as in Fig. 12, and to then detect $x(t_f)$ where $t_f > t_0$. According to the Q model of reality, the macroscopic system is in one or other branch at time t_0 , as indicated by $x(t_0)$. From the Figure, we note that the trajectories propagating (backward) from a given branch $\tilde{\lambda}_x$ at time t_f are at time t_0 on the same branch. This follows from the deterministic relation $x_j \rightarrow Gx_j$ of the simulation [Eq. (3.15)], which, being deterministic, can be reversed, so that $Gx_j \rightarrow x_j$, and is hence indicated by the blue two-way arrow. Now, for wMR to fail, we would require some instance where the value $x(t_0)$ would not imply the outcome for $\hat{x}$ as measured by the detection of $x(t_f)$. Since the branches are distinct at the time t_0 , this can only happen if the branch of $x(t_0)$ is different to that $x(t_f)$, which appears to contradict the solutions of Fig. 12 which for macroscopic states are consistent with the relation $x_j \leftrightarrow Gx_j$. We conclude consistency with wMR but the following Result IV.7 is needed for clarity. ■

Result IV.7: Assumption of branch continuity. Consider the system described by the macroscopic superposition $|\psi_{\text{sup}}\rangle$ at a time t_m (Fig. 12). According to the Q model of reality, the system is, at time t_m , in one or other of the branches $\tilde{\lambda}_x$, which we denote by $\lambda_x(t_m)$. Suppose a unitary operation $U = e^{-iH_{\text{amp}}t/\hbar}$ then takes place, with Hamiltonian H_{amp} . In quantum mechanics, the initial state for the operation U is $|\psi_{\text{sup}}\rangle$ (both branches). In the Q model, we define the *assumption of branch continuity* in the macroscopic limit, that the initial state for the operation U is given by the set of amplitudes of the same branch, $\tilde{\lambda}_x(t_m)$. This assumption allows consistency with wMR in Result IV.6, part (2).

Proof. We prove that the assumption allows consistency with wMR. Suppose for a system in a macroscopic superposition state $|\psi_{\text{sup}}\rangle$ at time $t_0 \equiv t_{m1}$, as in Fig. 12, we consider

successive $N - 1$ operations U that give further amplification, to a time $t_f = Nt_m$. In the Q model of reality, the amplitude $x(t_0)$ would give the outcome of $\hat{x}$ if measured at time t_0 . For the macroscopic superposition we see from Fig. 12 that an individual system is on one or other branch. If we assume branch continuity, then the individual macroscopic system will remain on the same branch throughout the successive applications. If there is a detection of $x(t_f)$ indicating the branch $\tilde{\lambda}_x(t_m)$ corresponding to an eigenvalue x_j at time t_f , then this implies the system was on the same branch at the earlier times $(N - 1)t_m, \dots, t_m$, which gives consistency with wMR. ■

Comment. In the assumption of branch continuity, the predictions for the probability of an outcome of $\hat{x}$ are the same as for the quantum state $|\psi_{\text{sup}}\rangle$. Consider the state at time t_m . We see from the Results IV.3a that in the Q model, the probability for outcome x_j is given by the probability density of amplitudes $x(t_f)$ at time t_f , which gives $|c_j|^2$ as correct for $|\psi_{\text{sup}}\rangle$. The prediction is based on the marginal $Q_{\text{sup}}(x, t_f)$ (Result III.1) which is not changed over the ensemble if we apply the assumption.

For an analysis of EPR correlations (Secs. VI–IX), we extend the assumption Result IV.7.

Result IV.8: Branch continuity for spacelike separated systems A and B stationary in a frame F . Suppose system A has been amplified so that branches $\tilde{\lambda}_x^A$ of amplitudes $x_A(t_{mA})$ emerge at time t_{mA} . An operation U or interaction H then takes place at B at time $t_{m2} > t_{m1}$. An individual system A at time t_{mA} is associated with one or other branch, as determined by the value $x_A(t_{mA})$. It is assumed that the branch for A is unchanged by the operations or interactions at B . We give a justification of this Result in Part II of the paper, based on the properties of the bipartite Q function (refer to Results IX.2 and IX.3).

D. Postselected state

As part of analyzing the measurement problem, we seek to understand the collapse of the wave function. With this motivation, we evaluate the postselected state that is inferred at the initial time t_0 , given a particular branch (with outcome x_j) at the later time t_f [Fig. 13 and Definition (8a) of Sec. IB].

Definition: The distribution postselected on a given outcome x_j . The trajectories $x(t)$ for a given branch $\tilde{\lambda}_x$ (associated with eigenvalue x_j) can be tracked from time t_f to time t_0 . This defines a set $\{x(t)\}$ of values of x at time t_0 , which we symbolize by $\mathcal{B}_j$. Associated with this set for x is a distribution for $p(t_0)$, as determined by the conditional distribution

$$Q_0(p|x) = Q(x, p, t_0)/Q(x, t_0) \quad (4.12)$$

evaluated from the Q function $Q(x, p, t_0)$ at $t_0 = 0$. Here, $Q(x, t_0) = \int dp Q(x, p, t_0)$. This defines a set of coordinates $\{x(t_0), p(t_0)\}$ and an associated distribution

$$\begin{aligned} Q_{\text{loop}}(x, p, t_0|\mathcal{B}_j) &\equiv Q(x(t_0), p(t_0)|x(t_0) \in \mathcal{B}_j) \\ &\equiv Q_{\text{loop}}(x, p, t_0|x_j). \end{aligned} \quad (4.13)$$

We define $Q_{\text{loop}}(x, p, t_0|\mathcal{B}_j)$ as the *post-selected state* or *post-selected distribution*, inferred on the outcome x_j .

Notation. Where it is clear that we restrict to the initial time t_0 , and refer to the branch associated with eigenvalue

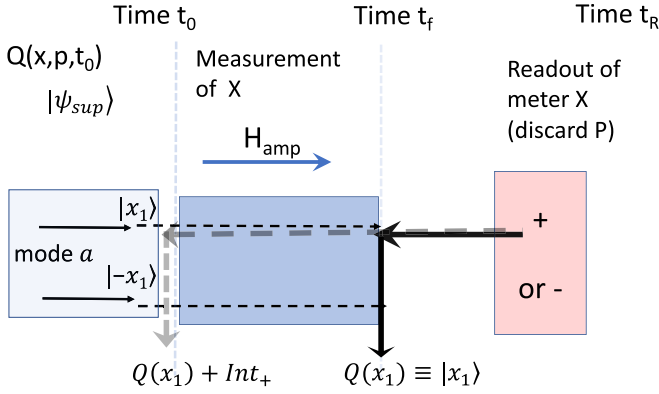


FIG. 13. Measurement by amplification. Diagram of the measurement of $\hat{x}$ on a superposition of eigenstates $|x_1\rangle$ and $|-x_1\rangle$ of $\hat{x}$. The measurement proceeds at a time t_0 with the interaction H_{amp} , which amplifies $\hat{x}$ by the factor G . The final stage of measurement is the irreversible readout at time t_R , which gives the detected value $x(t_f)$ as either Gx_1 or $-Gx_1$, discarding information about $\hat{p}$. The measured outcome for $\hat{x}$ is $x(t_f)/G$, implying outcomes of either x_1 or $-x_1$. The solid black arrows show the conditioning on the outcome x_1 , to infer $|x_1\rangle$ for the macroscopic superposition at time t_f . However, tracing the trajectories back to the initial time t_0 (reverse dashed arrows) the postselected state carries a contribution due to interference (denoted Int_+), as evident by the fringes in Fig. 14.

x_j , we abbreviate to write $Q_{\text{loop}}(x, p, t_0 | \mathcal{B}_j) \equiv Q_{\text{loop}}(x, p | x_j)$, or else $Q_0(x, p | x_j)$. Since the branch is symbolized by λ_x , with values $\tilde{\lambda}_x \in \{x_j\}$, we sometimes write $Q_{\text{loop}}(x, p | x_j) \equiv Q_{\text{loop}}(x, p | \tilde{\lambda}_x)$.

Result IV.9: *The postselected state is not equivalent to the eigenstate associated with the outcome.* The postselected distribution $Q_{\text{loop}}(x, p, t_0 | x_j)$ for a superposition $|\psi_{\text{sup}}\rangle = \sum_k c_k |x_k\rangle$ of eigenstates $|x_k\rangle$ of $\hat{x}$ (where $c_j \neq 0$) contains a sinusoidal term. Hence, the postselected state is not the eigenstate $|x_j\rangle$.

Proof. We prove by evaluating the postselected distribution for $|\psi_{\text{sup}}\rangle = c_1 |x_1\rangle + c_2 |x_2\rangle$ of Eq. (3.7), where we take c_1 to be real and positive, and write $c_2 = |c_2|e^{i\varphi}$. We trace the trajectories of $x(t)$ from time t_f to t_0 , for each $x(t_f)$ that is on the branch with eigenvalue x_j . The superposition has just two branches (Fig. 8). We take $x_2 = -x_1$ and $x_1 > 0$. For sufficiently large gt , each $x(t_f)$ is either positive or negative, associated with the outcome x_1 or $-x_1$, which we denote by $+$ or $-$. We first consider just one branch, $\mathcal{B}_j$ (of eigenvalue x_j). The simulation starts at the future boundary. At time t_f , the distribution of the values $x(t)$ of the branch $\mathcal{B}_j$ is given as $\frac{e^{-(x - e^{gt} x_j)^2/2}}{\sqrt{2\pi}}$ ($t_0 < t < t_f$). The distribution of $\mathcal{B}_j$ at time t_0 is $Q_j(x) = \frac{e^{-(x - x_j)^2/2}}{\sqrt{2\pi}}$. It is convenient to denote $\mathcal{B}_1$ and $\mathcal{B}_2$ by $\mathcal{B}_+$ and $\mathcal{B}_-$, respectively.

At time t_0 , there is a connection of the values $\mathcal{B}_+$ with a set of values for p determined by $Q_0(p|x)$ [Eq. (4.12)]. Denoting the distribution of the branch $\mathcal{B}_{\pm}$ at t_0 by $Q_{\pm}(x) = \frac{e^{-(x \mp x_1)^2/2}}{\sqrt{2\pi}}$, we find [using Eq. (4.13)]

$$Q_{\text{loop}}(x, p, t_0 | x \in \mathcal{B}_{\pm}) = Q_{\pm}(x) Q_0(p|x). \quad (4.14)$$

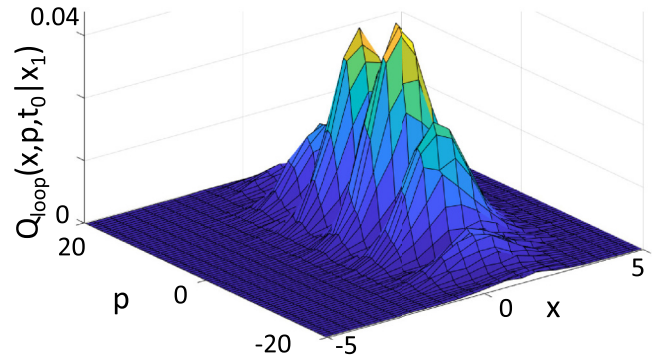


FIG. 14. Postselected state. The postselected distribution $Q_{\text{loop}}(x, p, t_0 | x_1)$ at time $t_0 = 0$, given the branch x_1 at time t_f , for the state $|\psi_{\text{sup}}\rangle$ of Eq. (3.7) considered in Fig. 8, where $x_1 = 1$. In practice, we trace back the trajectories $x(t)$ with a positive value for $x(t_f)$.

Calculation gives

$$\begin{aligned} Q_0(p|x) &= N \frac{e^{-p^2/2\sigma_p^2}}{\sqrt{2\pi}\sigma_p} \left(1 + 2|c_1 c_2| e^{-x^2/2} e^{-x_1^2/2} \frac{\mathcal{F}}{S_G} \right) \\ &\rightarrow N \frac{e^{-p^2/2\sigma_p^2}}{\sqrt{2\pi}\sigma_p} \left(1 + \frac{\mathcal{F}}{\cosh(x x_1)} \right), \end{aligned} \quad (4.15)$$

where $Q(x, p, t_0)$ is given by Eq. (3.8), $Q(x, 0)$ is given by Eq. (3.20), and

$$\mathcal{F} = (\cos \varphi) \cos(x_1 p) - (\sin \varphi) \sin(x_1 p) \quad (4.16)$$

is given by Eq. (3.10). The sinusoidal function $\mathcal{F}$ depends on p but not x , and is independent of the branch. Also, $S_G = \sqrt{2\pi} [|c_1|^2 Q_+(x) + |c_2|^2 Q_-(x)]$ and N is a normalization factor; NB: $N \rightarrow 1$ as $r \rightarrow \infty$ for (3.8). Fringes are evident, becoming finer for large x_1 (when the superposition is macroscopic) and also increasingly damped, provided $x \neq 0$. The last line of Eq. (4.15) gives the special case where $|c_1| = |c_2|$.

From (4.14), the postselected distribution is

$$Q_{\text{loop}}(x, p, t_0 | \pm x_1) = N [Q(\pm x_1) + \text{Int}_{\pm}], \quad (4.17)$$

where we take $|c_1| = |c_2|$ for simplicity. Here $Q(\pm x_1) = \frac{e^{-p^2/2\sigma_p^2}}{2\pi\sigma_x\sigma_p} e^{-(x \mp x_1)^2/2}$ is the Q function of the eigenstates $|\pm x_1\rangle$, and

$$\begin{aligned} \text{Int}_{\pm} &= \frac{e^{-p^2/2\sigma_p^2}}{2\pi\sigma_x\sigma_p} 2e^{-x^2/2\sigma_x^2} e^{-x_1^2/2\sigma_x^2} \frac{\mathcal{F}}{[1 + Q_{\mp}(x)/Q_{\pm}(x)]} \\ &= \frac{e^{-p^2/2\sigma_p^2}}{2\pi\sigma_x\sigma_p} 2e^{-x^2/2\sigma_x^2} e^{-x_1^2/2\sigma_x^2} \frac{\mathcal{F}}{(1 + e^{\mp 2x_1 x})}. \end{aligned} \quad (4.18)$$

We confirm the Result IV.9: For example, $Q_{\text{loop}}(x, p, t_0 | x_1)$, which is conditioned on the positive branch, includes a contribution from the bivariate Gaussian $Q(x_1)$ with positive mean, as well as a sinusoidal term Int_+ (Figs. 13 and 14). ■

A plot of the distribution as evaluated numerically for the state $|\psi_{\text{sup}}\rangle$ of Eq. (3.7) is presented in Fig. 14. The fringe pattern arising from $\mathcal{F}$ is evident. The fringe distribution is independent of the outcome (whether x_1 or $-x_1$) because the conditional $Q_0(p|x)$ is independent of sign of x . The marginal

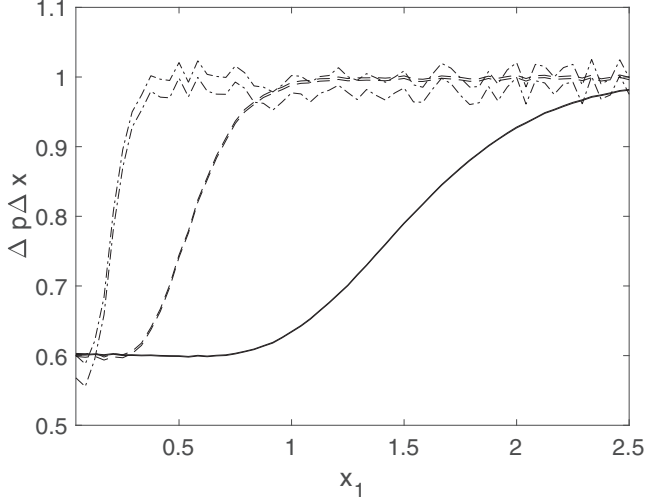


FIG. 15. Uncertainty product for the postselected distribution. We plot the uncertainty product $\Delta\hat{x}\Delta\hat{p}$ associated with the postselected distribution $Q_{\text{loop}}(x, p, t_0|x_1)$ conditioned on the outcome x_1 for $\hat{x}$, for the state $|\psi_{\text{sup}}\rangle$ of Fig. 14. Here, $t_0 = 0$. Here, Δx and Δp correspond to the observed uncertainties if $Q_{\text{loop}}(x, p, t_0|x_1)$ were the Q function of a quantum state (we drop the operator hats in $\Delta\hat{x}$ and $\Delta\hat{p}$ and write $\Delta\hat{x}\Delta\hat{p} = \Delta x\Delta p$ for convenience). The upper dashed-dotted line is for the superposition $|\psi_{\text{sup}}\rangle$ of two eigenstates of $\hat{x}$, with $r = 2$. The dashed line is for $|\psi_{\text{sup}}(r)\rangle$ with $r = 1$. The solid line is for $r = 0$. In each case, $gt = 4$. The two parallel lines indicate the upper and lower error bounds from sampling errors, with 1.2×10^7 trajectories.

of the distribution hence corresponds to that given by Fig. 10 at time $t = 0$.

Comment. The fringes decay as the separation between the eigenstates is larger, so that $|x_1| \gg 1$. Hence, $Q_{\text{loop}}(x, p, t_0|x_1)$ becomes the Q function of the eigenstate $|x_j\rangle$ in the limit of a macroscopic superposition.

Result IV.10: *The postselected state does not (generally) correspond to a quantum state.* The distribution $Q_{\text{loop}}(x, p|x_j)$ at the time t_0 , associated with the branch x_j , is not equivalent to the Q function of any quantum state. Hence, the postselected state is not a quantum state.

Proof. Let us assume $Q_{\text{loop}}(x, p, t_0|x_1)$ to be the Q function of a quantum state $|\psi\rangle$. The measured variances for $\hat{x}$ and $\hat{p}$ are $(\Delta\hat{x})^2 = \sigma_{x,+}^2(0) - 1$ and $(\Delta\hat{p})^2 = \sigma_{p,+}^2(0) - 1$, where $\sigma_{x,+}^2 \equiv \sigma_{x,+}^2(0)$ and $\sigma_{p,+}^2 \equiv \sigma_{p,+}^2(0)$ are the variances of x and p as given by $Q_{\text{loop}}(x, p, t_0|x_1)$ (refer to Sec. II). The variance product $\Delta\hat{x}\Delta\hat{p}$ is given in Fig. 15 versus x_1 (the separation between the states of the superposition) for large gt . From Fig. 15, see that $\Delta\hat{x}\Delta\hat{p} < 1$ for all x_1 (and α_0) (although $\Delta\hat{x}\Delta\hat{p} \rightarrow 1$ as $|x_1| \rightarrow \infty$). A contradiction is obtained with the Heisenberg uncertainty relation $\Delta\hat{x}\Delta\hat{p} \geq 1$. Hence, the distribution $Q_{\text{loop}}(x, p, t_0|x_1)$ is not a Q function and cannot represent a quantum state. ■

In summary, the postselected distribution $Q_{\text{loop}}(x, p, t_0|x_1)$ does not reflect a quantum state $|\psi\rangle$. The variables x and p are hence “hidden variables.” This is depicted in Fig. 4, with reference to the Schrödinger-cat paradox. The simulation shows how macroscopic realism can be consistent with quantum mechanics for the system in a macroscopic superposition.

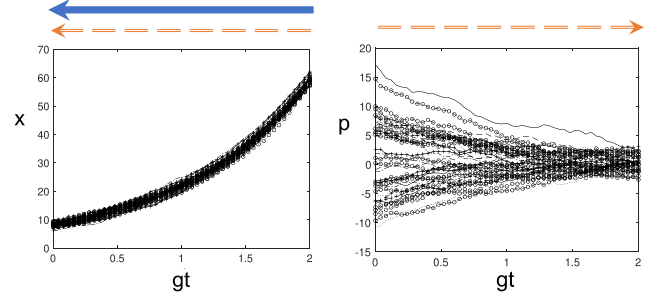


FIG. 16. Connected trajectories forming a hidden loop. The trajectories $x(t)$ of one branch for $\hat{x}$ can be traced backward in time to the initial time $t_0 = 0$ (left figure), where they couple to certain forward-going trajectories for p (right figure). Plots are for the system prepared in a superposition $|\psi_{\text{sup}}\rangle$ of eigenstates $|x_1\rangle$ and $|-x_1\rangle$ of $\hat{x}$ as in Fig. 9, conditioned on an outcome x_1 for measurement of $\hat{x}$, with $x_1 = 8$ and $gt_f = 2$. The resulting distribution for p is independent of the branch-value, whether x_1 or $-x_1$.

Yet, the postselected state for the macroscopic superposition is not a quantum state, which highlights the argument for the incompleteness of quantum mechanics, as put forward by Schrödinger [102]. However, the uncertainty product *approaches* 1 with increasing separation $|x_1|$ of the eigenstates (Fig. 15), where one has a true macroscopic superposition—a “cat state.”

E. Hidden loop

At the initial time t_0 , there is a link between the backward trajectories $x(t)$ of one branch and those going forward in time for the complementary observable p . The link is determined by the conditional distribution $Q_0(p|x)$ [Eq. (4.12)]. This enables definition of the joint distribution $Q_{\text{loop}}(x, p, t|x_j)$ throughout the evolution, which is used to confirm causal consistency (refer to Secs. VB and IX C and Appendices B and H).

Definition: Hidden loop. For the set of values of $x(t_f)$ at the time t_f on a given branch with outcome x_j , we have defined the set $\{x(t_0)\}$ at the initial time t_0 , found by tracing the backward trajectories $x(t)$. The set $\{x(t_0)\}$ is denoted $\mathcal{B}_j$. The $\mathcal{B}_j$ defines a set of *connected forward-going p trajectories*, found by propagating each $p(t_0)$ at time $t_0 = 0$ from the sample generated by $Q_0(p|x)$, where $x \in \{x(t_0)\}$, back to the future time t_f . We define the sets of variables $\{x(t_f), x(t_0), p(t_0), p(t_f)\}$ and all intermediate values on the trajectories. We refer to such a set of trajectories as a *loop*. The distribution of the amplitudes $\{x(t), p(t)\}$ at a given time t ($t_0 \leq t \leq t_f$) is denoted

$$Q_{\text{loop}}(x, p, t|x \in \mathcal{B}_j). \quad (4.19)$$

Such a set of trajectories is plotted in Fig. 16. The definition is summarized as Definition 9(a) in Sec. IB.

Comment. The term hidden is used because the loop arises from a distribution such as $Q_{\text{loop}}(x, p, t_0|x_j)$ which cannot correspond to a quantum state $|\psi\rangle$ (Result IV.10, Fig. 4). The loop involves the unobserved variables δx and the deamplified p . We note the term “hidden causal loop” might also be used [97]. However, we avoid this term, because there is no backward information transfer of the value x_j . The input at

time t_f is the stochastic noise, while the observed value $e^{gt} x_j$ is governed by the (causal) deterministic relation $x_j \rightarrow e^{gt} x_j$.

Result IV.11: The coupling of the p trajectories with those for x is determined by $Q(p|x)$, which is independent of the sign of x and is not sensitive to the outcome x_j . This is evident in Fig. 16 which plots trajectories of p conditioned on the outcome x_1 . The distribution of trajectories for p is the same for both branches x_1 and $-x_1$, and is hence given by Fig. 9 (lower panel).

Result IV.12: *Hidden loops occur for superpositions, not classical mixtures.* The conditional distribution $Q_0(p|x)$ for the mixture ρ_{mix} (3.11) of eigenstates of $|x_j\rangle$ is given by

$$Q_0(p|x) = \frac{e^{-p^2/2\sigma_p^2}}{\sigma_p\sqrt{2\pi}} = Q(p), \quad (4.20)$$

which implies complete independence of the trajectories for x and p . There is no interference term $\mathcal{I}$, and the $Q_0(p|x)$ factorizes. Hence, the hidden loop occurs only for the superposition (not mixtures) of eigenstates $|x_j\rangle$.

F. Hidden variable models

We have seen how the amplitudes x and p themselves do not give a complete description (Result IV.2). Hidden variable (HV) models can be deduced from the simulation, however, which is useful in developing a causal model for measurement (Sec. V) and for justifying weak local realism [Definition (10a)]. Consider the system in the superposition $|\psi_{\text{sup}}\rangle$ of eigenstates $|x_j\rangle$ of $\hat{x}$, as in Eq. (3.7). The system is prepared with respect to the measurement basis $\hat{x}$ at the time t_0 .

Result IV.13a: *Hidden variable model $\mathcal{P}_{\text{sup}}$.* A hidden variable model $\mathcal{P}_{\text{sup}}$ exists such that for any given amplitudes x and p at time t_0 , there is a definite probability $P(x_j|x, p)$ of an outcome x_j of the measurement $\hat{x}$.

Proof. The Q function of $|\psi_{\text{sup}}\rangle$ is $Q_{\text{sup}}(x, p, t_0)$ of Eq. (3.8). We take $x_1 = -x_2 > 0$. From postselection, we can determine the distributions $Q_{\text{loop}}(x, p, t_0 | \pm x_1)$, since there is a well-defined set of x and p at the time t_0 , given the branch x_1 or $-x_1$. Using that for events A and B , the conditional probabilities satisfy

$$P(A|B) = \frac{P(A \cup B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}, \quad (4.21)$$

it follows that

$$P(\pm x_1|x, p) = \frac{Q_{\text{loop}}(x, p, t_0 | \pm x_1)P(\pm)}{Q_{\text{sup}}(x, p, t_0)}, \quad (4.22)$$

which can be evaluated from the simulation, where $P(+)$ and $P(-)$ are the respective probabilities of the positive and negative outcomes. Using (4.14) and that $Q(p|x) = Q_{\text{sup}}(x, p, t_0)/Q_{\text{sup}}(x, t_0)$, where from (3.20) we see that because we consider superpositions of eigenstates of $\hat{x}$ $Q_{\text{sup}}(x, t_0) = [Q_+(x) + Q_-(x)]/2$, we find

$$P(\pm x_1|x, p) = \frac{Q_{\pm}(x)}{Q_+(x) + Q_-(x)}, \quad (4.23)$$

where $Q_{\pm}(x) = \frac{e^{-(x \mp x_1)^2/2}}{\sqrt{2\pi}}$. Hence, $P(x_1|x, p) + P(-x_1|x, p) = 1$, as required. We note the distribution is derived accounting

for interference, but the final result is the same as that obtained for the mixed state ρ_{mix} . ■

Result IV.13b: *(Deterministic) Hidden variable model $\mathcal{P}_{\text{det}}$.* There is another interpretation possible from the postselected states (4.17) as derived for the superposition state $|\psi_{\text{sup}}\rangle = c_1|x_1\rangle + c_2|x_2\rangle$ of Eq. (3.7). Here, we take for simplicity c_1 to be real and positive, $c_2 = |c_2|e^{i\varphi}$, $|c_1| = |c_2|$ and $x_2 = -x_1$.

It can be posited that the system *prior to measurement* is in a *probabilistic mixture* of the postselected states, i.e., is *either* in the state with distribution $Q_{\text{loop}}(x, p, t_0 | +x_1)$ with probability 1/2, *or* in the state with distribution $Q_{\text{loop}}(x, p, t_0 | -x_1)$ with probability 1/2. Defining $\mathcal{I}nt = \mathcal{I}nt_+ + \mathcal{I}nt_-$, we see that $\mathcal{I}nt = N \frac{e^{-p^2/2\sigma_p^2}}{2\pi\sigma_x\sigma_p} 2e^{-x^2/2\sigma_x^2} e^{-x_1^2/2\sigma_x^2} \mathcal{F}$ where $\mathcal{F}$ is given by (4.16). Hence, since

$$Q_{\text{sup}}(x, p, t_0) = \frac{Q(x_1)}{2} + \frac{Q(-x_1)}{2} + \frac{\mathcal{I}nt}{2}, \quad (4.24)$$

the probability distribution for the mixture of postselected states is indeed given by the Q function $Q_{\text{sup}}(x, p, t_0)$, as required for the proposed model.

In the model, we note that for a system that could be described as being in the state $Q_{\text{loop}}(x, p, t_0 | +x_1)$, the marginal $Q_{\text{loop}}(x, t_0 | x_1) = \int dp Q_{\text{loop}}(x, t_0 | x_1)$ in x is

$$N \int dp [Q(x_1) + \mathcal{I}nt_+].$$

Applying the techniques of Eqs. (3.18) and (3.19) in Sec. III, we see that the contribution due to $\mathcal{I}nt_+$ will vanish. Hence, the marginal $Q_{\text{loop}}(x, t_0 | x_1)$ is as for the eigenstate $|x_1\rangle$. Similarly, the marginal in x for the state $Q_{\text{loop}}(x, p, t_0 | -x_1)$ is as for the eigenstate $|-x_1\rangle$. In the proposed model, we make the assumption that the backward stochastic equation (2.12) applies to describe the measurement of $\hat{x}$ on the system, which is in the state given by *either* $Q_{\text{loop}}(x, t_0 | x_1)$ *or* $Q_{\text{loop}}(x, t_0 | -x_1)$ at the time t_0 . Then, the conclusion is that the state $Q_{\text{loop}}(x, p, t_0 | x_1)$ will always give the outcome x_1 ; and the state $Q_{\text{loop}}(x, p, t_0 | -x_1)$ will always give the outcome $-x_1$. Hence, the system is at time t_0 in a state with a predetermined outcome for $\hat{x}$, meaning either definitely x_1 or definitely $-x_1$. For this reason, we refer to the model as “deterministic.”

G. Weak local realism (wLR)

The concept of weak macroscopic realism (wMR) introduced in Result IV.6 can be generalized to *weak local realism*, as given by Definition (10a) in Sec. IB. Consider measurement of $\hat{x}$ on a system at time t_0 (Fig. 13). The system is *prepared with respect to the measurement basis $\hat{x}$* at t_0 , meaning that the operations U that fix the measurement settings [i.e., the sign of g in H_{amp} ; Eq. (2.2)] have been performed.

Result IV.14: *Weak local realism: Premise wLR(1).* The Q -based model of reality is consistent with the assumption of weak local realism (wLR). There are two versions of wLR, *deterministic* and *probabilistic* [refer to Definition (10) in Sec. IB]. The first assumes a deterministic hidden variable (HV) model: there is a predetermination of the outcome of $\hat{x}$ at the time t_0 (after settings are fixed). The second assumes a probabilistic HV model: hidden variables exist for the system

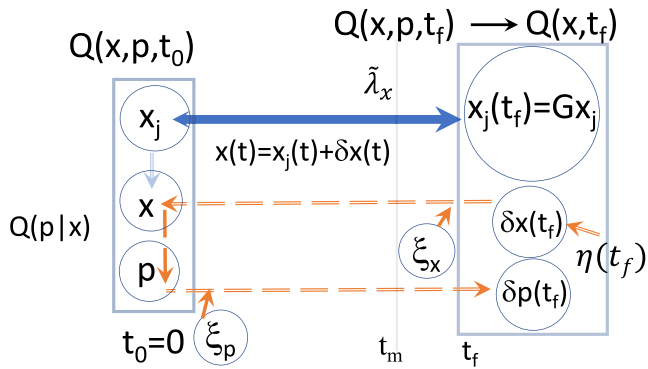


FIG. 17. Causal model for quantum measurement. Diagram of the model inferred from the simulation of measurement $\hat{x}$ on a superposition $|\psi_{\text{sup}}\rangle$ of eigenstates $|x_j\rangle$ of $\hat{x}$. The relation $x_j \rightarrow Gx_j$ (blue line) is deterministic and causal. Branches emerge after sufficient amplification when the system is macroscopic, at a time t_m , as depicted by the value $\tilde{\lambda}_x$ that determines the outcome of the measurement $\hat{x}$ (Figs. 8 and 9). The retrocausality present in the model is embodied in the noise input $\delta x(t_f) \equiv \eta(t_f)$, generated at the future boundary t_f . The properties of the input $\delta x(t_f) \equiv \eta(t_f)$ at time t_f are independent of the value t_f and of the eigenvalue x_j , and similarly for ξ_x , which is determined by H_{amp} . Furthermore, the properties of the stochastic term $\delta x(t)$ remain unchanged with time t . A hidden loop (orange dashed lines) is present for superpositions $|\psi_{\text{sup}}\rangle$, but not mixtures, of $|x_j\rangle$. The value x_j can be inferred from Gx_j (depicted by the two-way blue arrow).

at the time t_0 (after settings are fixed), which imply definite probabilities for outcomes of $\hat{x}$.

Justification. In the Q -based simulation and model, the means x_j of the Gaussian functions in $Q(x, p, t)$ are amplified to Gx_j [refer to Eqs. (3.8) and (3.14)]. The interference terms and directly implies the premise wLR(1). Hence, the probability of an outcome x_j of $\hat{x}$ for the system, when prepared with respect to the measurement basis, is the same as for the mixture, ρ_{mix} , defined by Eq. (3.11). For the mixture, the system *can* be viewed as being (with probability $|c_j|^2$) in a state with a definite outcome x_j . Hence, a deterministic HV model for the observed probabilities exists. Similarly, the HV model given by Result IV.13b is a deterministic HV model deduced from the simulation, which accounts for the interference terms and directly implies the premise wLR(1).

Alternatively, the probabilistic HV model $\mathcal{P}_{\text{sup}}$ can be developed from the simulation (Result IV.13a). At time t_0 , the system is in an HV state given by variables $\lambda \equiv (x, p)$, with a probability density function $Q(\lambda, t_0)$. For a given set of values λ , there is defined a probability $P(x_j|x, p)$ for an outcome x_j , as given by Eqs. (4.22) and (4.23). ■

V. CAUSAL MODEL FOR MEASUREMENT, CAUSAL CONSISTENCY, AND THE MEASUREMENT PROBLEM

The simulations involve classical-like amplitudes x and p and individual realizations, and hence we can identify “cause and effect” relations for quantum measurement, referred to as a “causal model.” A causal model is constructed based on the procedure of the simulation (Sec. III), as depicted in Fig. 17. This elucidates why retrocausality, defined as the future

affecting the past (as in a transfer of information backward in time), is not present despite the use of future boundary conditions. The causal model also explains how causal consistency is achieved, and why there are no grandfather paradoxes.

A. Causal model

A causal model is deduced to explain quantum measurement. We denote the measurement by $\hat{X}$, where $\hat{X}$ is either $\hat{x}$ or $\hat{p}$ or $\hat{x}_\theta = \hat{x} \cos \theta + \hat{p} \sin \theta$. The *model parameters* are the measurement setting θ , and the variables that describe the state of the system, at the initial time $t = 0$ and at the final time $t = t_f$, as well as the noise inputs $\xi_1(t)$ and $\xi_2(t)$ defined by (2.12) and (2.11), and the noise input $\delta x(t_f) \equiv \eta(t)$ defined by the future boundary condition (3.23) at time t_f . Where we measure $\hat{x}$, the notation implies $\xi_1(t) \equiv \xi_x(t)$ and $\xi_2(t) \equiv \xi_p(t)$.

In the simulation, the setting θ is in principle adjustable, as in the Hamiltonian (4.3). Experimentally, this corresponds to a unitary operation $U_\theta = e^{-iH_\theta t/\hbar}$ where H_θ induces a phase shift. In the analysis below (Fig. 17), we assume that at time t_0 , the operation U_θ has been carried out so that $g > 0$, and the measurement is to be of $\hat{x}$.

Prepare with respect to the measurement basis. The system variables at time $t_0 = 0$ are x and p , with a distribution given by the Q function $Q(x, p, t_0)$. The Q function for a state of the system is unique but can be written as a superposition of the eigenstates $|\chi_{\theta j}\rangle$ of $\hat{x}_\theta$, for any choice of θ . In an experiment, at the time t_0 after the operation U_θ where we choose $\theta = 0$, we assume that the system is prepared for the amplification H_{amp} (with $g > 0$) to finalize a measurement of $\hat{x}$.

Assumption (1). It is assumed as part of the Q model that the physical state for the system at the time t_0 , after the settings are determined, is given by the Q function written *with respect to the coordinates of the measurement basis*, meaning in this case that Q is expanded in terms of the Q function of the eigenstates $|x_j\rangle$ of $\hat{x}$ [as in Eq (3.8)]. Denoting the Gaussian with mean $\bar{x}$ and variance σ by $\mathcal{G}(\bar{x}, \sigma) = \frac{e^{-(x-\bar{x})^2/2\sigma^2}}{\sqrt{2\pi}\sigma}$, the Q function $Q(x, p, t_0)$ for the superposition $|\psi_{\text{sup}}\rangle = \sum_j c_j |x_j\rangle$ is the sum of terms involving the Gaussians $\mathcal{G}(x_j, \sigma_x)$ associated with each eigenstate $|x_j\rangle$, where $\sigma_x = 1$, plus hidden interference terms Int .

Assumption (2). The simplest causal model is based on the procedure of the simulation (Sec. III) and the models $\mathcal{P}_{\text{sup}}$ and $\mathcal{P}_{\text{det}}$ (Result IV.13). The state of the system at the time t_0 is identified by x and p , along with *one* of the mean values, x_j . Model $\mathcal{P}_{\text{sup}}$ identifies a distribution $P(x_j|x, p)$ for each x and p .

Future boundary condition. The simulation begins with the future boundary condition of the backward equation (2.12). The boundary condition is a probabilistic mixture of Gaussians $\mathcal{G}(Gx_j, \sigma_x)$ with $\sigma_x = 1$ and an amplified mean Gx_j , with relative probabilities $|c_j|^2$ (Result III.1), which is identical to that of a mixed state

$$\rho_{\text{mix}} = |c_1|^2 |x_1\rangle\langle x_1| + |c_2|^2 |-x_1\rangle\langle -x_1|. \quad (5.1)$$

Each individual trajectory (or run) is specified by a value of x_j , the value being one of the set of eigenvalues $\{x_j\}$ of the measurement $\hat{x}$. The relative weighting for x_j in the simula-

tions is determined by the probability amplitudes, as in (5.1). The future boundary condition at time t_f includes the input fluctuation $\delta x(t_f)$, which we depict by the stochastic symbol $\eta(t_f)$ in the figures, and is not measured the amplification H_{amp} . The value of $\delta x(t_f)$ is sampled using a random Gaussian function $\mathcal{G}(0, 1)$ with mean 0 and variance $\sigma_x^2 = 1$ at time t_f (refer to the procedure of the simulation, Sec. III). The backward equation (2.12) involves the noise inputs $\xi_x(t)$, as well as the decay. Remarkably, there is no change to the properties of the fluctuations $\delta x(t_f)$ as the amplitudes evolve according to Eq. (2.12).

Causal deterministic relation. The future boundary condition is hence determined by an initial condition at time t_0 , which probabilistically specifies the value x_j [according to (5.1)]. More generally, where the eigenvalues x_j form a continuous set, we denote the set of outcomes by the continuous variable x_0 . The future boundary condition is then determined by the marginal $W(x_0)$ of the Wigner function $W(x_0, p_0)$ (Results IV.4 and IV.5). $W(x_0)$ is the probability distribution for possible outcomes forming a continuous set $\{x_0\}$, which in Fig. 17 is symbolized by the set $\{x_j\}$. The amplification of the value x_j is given as a deterministic causal relation

$$x_j \rightarrow Gx_j, \quad (5.2)$$

where by causal we mean forward in time. This causal deterministic relation is represented by the blue forward-going solid arrow in Fig. 9. This is also justified, based on the models $\mathcal{P}_{\text{sup}}$ and $\mathcal{P}_{\text{det}}$ (Results IV.13a and IV.13b): for any x, p , a value x_j can be selected for the initial state at time t_0 , with probability $P(x_j|x, p)$. We have noted that being deterministic, the relation $x_j \rightarrow Gx_j$ gives a retrodictive, backward relation, $Gx_j \rightarrow x_j$. This is depicted by the solid blue two-way arrows in Fig. 17.

Stochastic retrocausality and hidden loops. We express the amplitude as $x(t) = x_j(t) + \delta x(t)$, where $x_j(t)$ satisfies the causal deterministic equation

$$\frac{dx_j(t)}{dt} = gx_j(t), \quad (5.3)$$

with boundary condition $x_j(0) = x_j$, and solution $x_j(t) = x_j(0)e^{gt}$ (solid blue line in Fig. 17). The exponential amplification with respect to t is evident in the trajectories for x plotted in Figs. 6–9.

The retrocausality of the process is embodied stochastically, in the noise term $\delta x(t)$ [Eq. (3.24)]. The noise δx satisfies the stochastic equation

$$\frac{d[\delta x(t)]}{dt_-} = -g\delta x(t) + \xi_x(t), \quad (5.4)$$

with input $\delta x(t_f)$ given as a Gaussian $\mathcal{G}(0, 1)$, at the vacuum level. This is depicted by the orange reversed dashed arrows in Fig. 17. We have seen in Sec. III that $\delta x(t_f)$ is added to the mean macroscopic value $x_j(t)$ in a way that is independent of the final time t_f , and independent of the branch value x_j . Solutions for δx alone are given in Fig. 11. Similarly, the properties of the noise ξ_x do not depend on t_f or x_j .

The stochastic behavior is also independent of the setting θ , i.e., whether $\hat{x}$ or $\hat{p}$ is measured. In considering the measurement of $\hat{p}$ where $\theta = -\pi/2$ [Eq. (4.3)], we would solve

for trajectories $p(t) = p_j(t) + \delta p(t)$, where p_j satisfies

$$\frac{dp_j(t)}{dt} = gp_j(t), \quad (5.5)$$

and the $\delta p(t)$ satisfies

$$\frac{d[\delta p(t)]}{dt_-} = -g\delta p(t) + \xi_p(t), \quad (5.6)$$

with input $\delta p(t_f)$ given as a Gaussian $\mathcal{G}(0, 1)$, at the vacuum level. The $\delta p(t)$ has the same initial condition $\delta p(t_f)$ and trajectory equations as $\delta x(t)$.

The hidden loop for measurement of $\hat{x}$ is depicted in Fig. 17 and is based on the conditional distribution $Q_0(p|x)$ [Eq. (4.12)], which is determined from the initial Q function $Q(x, p, t_0)$. The value of $x(t_0)$ is given as $x_j + \delta x(t_0)$. The joint distribution for $\{x(t_0), p(t_0)\}$ is given by $Q_{\text{loop}}(x, p, t_0|x_j)$ [Eq. (4.13)] as in Fig. 2. The values $p(t_0)$ propagate forward in time according to (2.11), depicted by the orange dashed arrows in the forward-time direction. Since they are at the hidden vacuum level at time t_f , both p and δx are unobservable.

In summary, although retrocausality is present in terms of Eqs. (2.12) for $x(t)$, the causal model for the measurement shows that this reflects only in the way the backward equation is solved, at a microscopic stochastic level. There is no genuine retrocausality in the sense of a backward-in-time transfer of information about the outcome x_j or the setting θ , whether $\hat{x}$ or $\hat{p}$. The trajectory for δx has a stochastic solution involving the noise ξ_x , but the noise inputs $\delta x(t_f)$ and ξ_x do not depend on $x_j(t)$ or the setting θ .

B. Causal consistency and causality at a macroscopic level

The Q function, which represents the quantum state $|\psi(t)\rangle = e^{-iH_{\text{amp}}t/\hbar}$, evolves causally, forward in time. The causal model must explain how consistency is achieved between the causal evolution of the Q function and the intrinsic retrocausality implicit in the forward-backward stochastic dynamics.

Definition: Causal consistency. We demonstrate the causal consistency of the measurement model by showing that the joint distribution for $x(t)$ and $p(t)$ at a given time t ($t_0 < t < t_f$), as provided by the hidden loops averaged over all branches $\mathcal{B}_j$, is equivalent to the Q function $Q(x, p, t)$ at the time t . This must be true for all choices of t_f (the time of the future boundary). The test of causal consistency is depicted in Fig. 3.

Result V.1: Causal consistency of the Q model. The Q -based model of reality is causally consistent.

Proof. Verification that the numerical solutions from the simulations are causally consistent is given in Appendix B. Here, we prove this analytically.

The simulation for the state $\sum_j c_j |x_j\rangle$ begins at the future boundary at time t_f , with a value $x(t)$ emanating from a particular branch $\mathcal{B}_j$ (associated with the eigenvalue x_j), with probability $|c_j|^2$. For each branch, the distribution of $x(t_f)$ is given by the Gaussian $\frac{e^{-(x-x_j)^2/2}}{\sqrt{2\pi}}$ where $G = e^{gt_f}$. We first consider just one branch, x_j . At time t , the distribution is $\frac{e^{-(x-G(t)x_j)^2/2}}{\sqrt{2\pi}}$ ($t_0 < t < t_f$) where $G(t) = e^{gt}$. The distribution at time $t_0 = 0$ is $Q_j(x) = \frac{e^{-(x-x_j)^2/2}}{\sqrt{2\pi}}$, a set of values for x that we

denote by $\mathcal{B}_j$. At time t_0 , there is a connection with a set of values for p determined by $Q_0(p|x)$ [Eq. (4.12)]. For simplicity, in this paper, we consider the superposition of $|x_1\rangle$ and $|x_2\rangle$ where there are just two branches, and take $|c_1| = |c_2|$, $c_2 = |c_2|e^{i\varphi}$, and $x_2 = -x_1$ with $x_1 > 0$, and denote the two branches by $\mathcal{B}_\pm$. Following on from Eq. (4.17) of the proof of Result IV.9, denoting the distribution of the branch $\mathcal{B}_\pm$ at t_0 by $Q_\pm(x) = \frac{e^{-(x \mp x_1)^2/2}}{\sqrt{2\pi}}$, we consider the branch with eigenvalue $x_j = \pm x_1$, and find

$$Q_{\text{loop}}(x, p, t_0|x \in \mathcal{B}_\pm) = Q_\pm(x)Q_0(p|x), \quad (5.7)$$

where $Q_0(p|x)$ is given by Eq. (4.15).

Continuing, we examine the *forward evolution* of the distribution of the p amplitudes. The forward and backward equations are separable, so that trajectories for p propagate independently of those for x . The marginal of p is $Q_{\text{sup}}(p, t) = \int dx Q_{\text{sup}}(x, p, t)$ where $Q_{\text{sup}}(x, p, t)$ is given by Eq. (3.14). We find

$$Q_{\text{sup}}(p, t) = N_0 \frac{e^{-p^2/2\sigma_p^2(t)}}{\sqrt{2\pi}\sigma_p(t)} (1 + e^{-G(t)x_1^2} \mathcal{F}(t)), \quad (5.8)$$

where

$$\mathcal{F}(t) = (\cos \varphi) \cos [pG(t)x_1] - (\sin \varphi) \sin [pG(t)x_1],$$

$G(t) = e^{gt}$, $\sigma_p^2(t) = 1 + [\sigma_p^2(0) - 1]/G(t)^2$, and N_0 is a normalization factor. The distribution is evaluated from the trajectories of p as shown in Figs. 8 and 9, and, notably, is independent of the branch, whether $\mathcal{B}_+$ or $\mathcal{B}_-$.

The joint densities for $x(t)$ and $p(t)$ at a given time $t_0 \leq t \leq t_f$ can now be evaluated for each branch, by examining the distribution $Q_{\text{loop}}(x, p, t|x \in \mathcal{B}_\pm)$, as in Eq. (4.19), which is the evolution of (5.7) (refer to Fig. 3). The value of x_1 is deterministically amplified to $G(t)x_1$ as the system evolves from the time t_0 , so that

$$Q_\pm(x) \rightarrow Q_{G\pm}(x) = \frac{e^{-(x \mp G(t)x_1)^2/2}}{\sqrt{2\pi}}. \quad (5.9)$$

Hence, denoting $Q_{\text{loop}}(x, p, t|x \in \mathcal{B}_\pm)$ by $Q_{\text{loop}\pm}(x, p, t)$ for simplicity of notation, we find $Q_{\text{loop}\pm}(x, p, t) = N_t Q_{G\pm}(x) Q_t(p|x)$ where $Q_t(p|x) = Q(x, p, t)/Q(x, t)$ is the conditional of the evolved Q function at time t , and [noting Eq. (3.14)] is obtained from $Q_0(p|x)$ of Eq. (4.15) by replacing $\mathcal{F} \rightarrow \mathcal{F}(t)$, $x_1 \rightarrow Gx_1$, $\sigma_p \rightarrow \sigma_p(t)$. Hence,

$$Q_{\text{loop}\pm}(x, p, t) = N_t Q_{G\pm}(x) \frac{e^{-p^2/2\sigma_p^2(t)}}{\sqrt{2\pi}\sigma_p(t)} \times \left(1 + \frac{\mathcal{F}(t)}{\cosh(xG(t)x_1)} \right), \quad (5.10)$$

where N_t is a normalization factor. The Q function $Q_{\text{sup}}(x, p, t)$ for the state $|\psi_{\text{sup}}(t)\rangle = e^{-iH_{\text{amp}}t/\hbar}$ at time t corresponds to the joint density given as the average of that for the branches. Hence

$$Q_{\text{sup}}(x, p, t) = N_t \frac{e^{-p^2/2\sigma_p^2(t)}}{2\sqrt{2\pi}\sigma_p(t)} \left(Q_{G+}(x) + Q_{G-}(x) + \{Q_{G+}(x) + Q_{G-}(x)\} \frac{\mathcal{F}(t)}{\cosh(xG(t)x_1)} \right), \quad (5.11)$$

which [on noting that $\sigma_x(t) = 1$], reduces to the Q function $Q_{\text{sup}}(x, p, t)$ given by Eq. (3.14), as required. We note the proof follows because of the independence of the forward-distribution for p on the branch, and because of the simple deterministic relation $x_j \rightarrow Gx_j$ for amplification. The proof for a superposition with more branches follows along the same lines. ■

Several related Results for the Q model of reality apply.

Result V.2a: Causality for macroscopic variables: No retrocausality at a macroscopic level. As the system amplifies, the superposition $\sum_j c_j |x_j\rangle$ becomes a macroscopic superposition $\sum_j e^{-iH_{\text{amp}}t/\hbar} |x_j\rangle$ of amplified states [Eq. (3.13) and Fig. 8]. The $x(t)$ becomes macroscopic and associated with one of the branches $\tilde{\lambda}_x$, for which the outcome of the measurement $\hat{x}$ is specified, as one of the eigenvalues x_j , consistent with the premise of weak macroscopic realism (Result IV.6). The causal relation is that the value $\tilde{\lambda}_x$ determines the measurement outcome *subsequently*, if there is to be a final detection, assuming there is no change of settings (Result IV.6). The branch-value $\tilde{\lambda}_x$ is not changed retrocausally by the future boundary condition.

For such a macroscopic system, the deterministic relation $x_j \rightarrow Gx_j$ applies fully to macroscopic variables $x(t)$, since for the macroscopic superposition, the x_j are macroscopically distinct at the initial time t_0 . Hence, we obtain causal relations at the level of macroscopic amplitudes x_j . This is illustrated in Fig. 12, where the system is in a macroscopic superposition at time t_0 . Each branch (symbolized by $\tilde{\lambda}_x$) is distinct throughout the evolution, and shows exponential amplification forward in time (solid blue arrow in Fig. 9). ■

We extend this Result in Secs. VII and IX to include that the model is consistent with no-signaling.

Result V.2b: No grandfather paradoxes. The causal model explains why there is no grandfather-type paradox arising from the solutions. The grandfather paradox is where the state of a system at time t is apparently affected by a future event. If we define the state of the system as being determined by $Q(x, p, t)$, then we have verified causal consistency (Result V.1), that $Q(x, p, t)$ is not changed by the future boundary condition. If we define the state of the system by the variable $\tilde{\lambda}_x$ that denotes the branch for a physical observable $\hat{x}$, and hence defines a measurable physical property for the system, then we have verified (Result V.2a) that this value is not changed retrocausally.

C. Measurement problem

The solutions of this paper give a model for measurement, contributing an explanation of the measurement postulates. In the Q model, amplification plays the key role in the quantum-to-classical transition. From Result III.1 (Sec. III), we see that amplification attenuates the interference terms that distinguish the superposition from a mixed state. The macroscopic properties $\tilde{\lambda}_x$ (given by the branches) emerge as the system is amplified and the inferred outcome $x(t_f)/G$ is one of the eigenvalues x_j . Born's rule is derived in Sec. IV B.

Amplification also contributes to an explanation of the collapse of the wave-function. As the system is amplified, the superposition $|\psi_{\text{sup}}\rangle$ evolves into a macroscopic superposition (Fig. 8). The results from Sec. IV D (Fig. 15) show that for a

macroscopic superposition, the postselected state conditioned on the branch with outcome x_j approaches the eigenstate $|x_j\rangle$.

However, the amplification H_{amp} that creates the macroscopic superposition state is a *reversible* process. The complete collapse to the eigenstate must occur when the process is *not reversible*. We see that this corresponds to when information about the complementary variable p is removed at the final detection (Fig. 13). In the calculations (Sec. IV D), this corresponds to averaging the distribution $Q_{\text{loop}}(x, p, t_0|x_j)$ for the inferred state (given an outcome x_j) over p . A better model of the collapse allows for a coupling to a meter [109]. A model for projection (and the collapse of the wave function) when a measurement is made on a system B by a meter A is given in Sec. IX.

VI. CONTINUOUS-VARIABLE EPR ENTANGLEMENT

The stochastic equations can be applied to treat bipartite systems. In this section, we examine EPR entanglement as detected using measurements of $\hat{x}$ and $\hat{p}$.

Einstein, Podolsky, and Rosen (EPR) presented an argument for the completion of quantum mechanics in 1935 [139]. The argument can be realized for two separated field modes A and B [140,141]. Two modes prepared in a two-mode squeezed state

$$|\psi_{\text{epr}}\rangle = (1 - \eta^2)^{1/2} \sum_{n=0}^{\infty} \tanh^n r_2 |n\rangle_A |n\rangle_B \quad (6.1)$$

at time $t_0 = 0$ possess EPR correlations. Here $\eta = \tanh r_2$ where r_2 is the two-mode squeeze parameter, and $|n\rangle_{A/B}$ are number states. boson operators $\hat{a}$, $\hat{b}$ and quadrature phase amplitudes $\hat{x}_A$, $\hat{p}_A$, $\hat{x}_B$ and $\hat{p}_B$ are defined for each mode, where $\hat{x}_A = \hat{a} + \hat{a}^\dagger$, $\hat{x}_B = \hat{b} + \hat{b}^\dagger$, $\hat{p}_A = (\hat{a} - \hat{a}^\dagger)/i$, $\hat{p}_B = (\hat{b} - \hat{b}^\dagger)/i$.

The Q function for the two-mode quantum state $|\psi\rangle$ is defined as $Q(\alpha, \beta) = \frac{1}{\pi^2} |\langle \alpha | \langle \beta | \psi \rangle|^2$ where $|\alpha\rangle$ and $|\beta\rangle$ are coherent states of modes A and B , respectively. Introducing real coordinates $x_A = \alpha + \alpha^*$, $p_A = (\alpha - \alpha^*)/i$, $x_B = \beta + \beta^*$, $p_B = (\beta - \beta^*)/i$, the Q function of $|\psi_{\text{epr}}\rangle$ is

$$Q_{\text{epr}}(\lambda, t_0) = \frac{(1 - \eta^2)}{16\pi^2} e^{-\frac{1}{8}(x_A - x_B)^2(1+\eta)} e^{-\frac{1}{8}(p_A + p_B)^2(1+\eta)} \times e^{-\frac{1}{8}(x_A + x_B)^2(1-\eta)} e^{-\frac{1}{8}(p_A - p_B)^2(1-\eta)}, \quad (6.2)$$

where $\lambda = (x_A, x_B, p_A, p_B)$. As $r_2 \rightarrow \infty$, $|\psi_{\text{epr}}\rangle$ is an eigenstate of $\hat{x}_- = \hat{x}_A - \hat{x}_B$ and $\hat{p}_+ = \hat{p}_A + \hat{p}_B$, giving a realization of the EPR paradox [141].

Following Sec. II, a measurement of $\hat{x}_A$ or $\hat{p}_A$ is modeled as a direct amplification, given by the Hamiltonian

$$H_{\text{amp}}^A = i\hbar g(\hat{a}^{\dagger 2} e^{2i\theta} - \hat{a}^2 e^{-2i\theta}), \quad (6.3)$$

where g is real and positive. The choice $\theta = 0$ or $\pi/2$ allows measurement of $\hat{x}_A$ or $\hat{p}_A$, respectively. Similarly, $H_{\text{amp}}^B = i\hbar g(\hat{b}^{\dagger 2} e^{2i\phi} - \hat{b}^2 e^{-2i\phi})/2$ with $\phi = 0$ or $\pi/2$ allows measurement of $\hat{x}_B$ or $\hat{p}_B$.

First, we consider joint measurements of $\hat{x}_A$ and $\hat{x}_B$. Following Sec. III, the stochastic equations for the measurements

$\hat{x}_A$ and $\hat{x}_B$ are ($K \in \{A, B\}$)

$$\frac{dx_K}{dt_-} = -gx_K + \xi_{K1}(t), \quad (6.4)$$

with a boundary condition at the time t_f and $\frac{dp_K}{dt} = -gp_K + \xi_{K2}(t)$ with a boundary condition at time t_0 . We find $\langle \xi_{K\mu}(t) \xi_{K\nu}(t') \rangle = 2g\delta_{\mu\nu}\delta(t - t')$. The noise terms for A and B are independent.

The equations can be solved by transforming to $x_{\pm} = x_A \pm x_B$, $p_{\pm} = p_A \pm p_B$. The equations are

$$\frac{dx_{\pm}}{dt_-} = -gx_{\pm} + \xi_{1\pm}(t), \quad (6.5)$$

with boundary conditions at the future time t_f . Those for p_+ and p_- are $\frac{dp_{\pm}}{dt} = -gp_{\pm} + \xi_{2\pm}(t)$ with boundary conditions at the initial time $t_0 = 0$. Here $\langle \xi_{\mu+}(t) \xi_{\nu+}(t') \rangle = 4g\delta_{\mu\nu}\delta(t - t')$ and $\langle \xi_{\mu-}(t) \xi_{\nu-}(t') \rangle = 4g\delta_{\mu\nu}\delta(t - t')$ and cross-terms are zero. We examine the solutions for $x_{\pm}$, since these are of most interest to us, representing the amplified, or measured, quantities. As in previous examples, the dynamics for the x and p separate, so that the amplification of x is determined by the marginals for x_A and x_B . The original distribution (6.2) is a function of x_+ , x_- , p_+ , and p_- . Transforming $Q_{\text{epr}}(\lambda, t_0)$ to these variables to obtain the function $Q_{\text{epr},\pm}(x_{\pm}, p_{\pm}, t_0)$, the marginal for x_+ and x_- at time t [defined as $\int Q_{\text{epr},\pm}(x_{\pm}, p_{\pm}, t) dp_+ dp_-$] is

$$Q_{\text{epr},\pm}(x_+, x_-, t) = \frac{e^{-x_-^2/2\sigma_-^2(t)} e^{-x_+^2/2\sigma_+^2(t)}}{2\pi\sigma_+(t)\sigma_-(t)}. \quad (6.6)$$

Here the gain is $G(t) = e^{gt}$ and the variances are $\sigma_{\pm}^2(t) = 2 + G^2(t)[\sigma_{\pm}^2(0) - 2]$, which we rewrite as

$$\sigma_{\pm}^2(t) = 2(1 + e^{2gt} e^{\pm 2r_2}). \quad (6.7)$$

The function $Q_{\text{epr},\pm}(x_+, x_-, t)$ is separable with respect to x_- and x_+ . The value $\sigma_{\pm}^2(t) = 2$ is the hidden vacuum noise level associated with the Q function. The boundary condition for the backward trajectories of $x_{\pm}$ is determined by the marginal $Q_{\text{epr},\pm}(x_+, x_-, t_f)$ of the system at time t_f .

The results of the simulations for $x_{\pm}$ are shown in Figs. 18 and 19. The solutions for x_- reveal the correlation for the EPR state between x_A and x_B . The variances $\sigma_{\pm}^2(t)$ increase with gt due to the gain associated with the measurement. However, the trajectories when scaled according to the amplification factor $G = e^{gt}$ reveal a reduced variance $\sigma_{\pm}^2(t)/G^2 = 2(e^{-2gt} + e^{\pm 2r_2})$. We note that $\sigma_-(t) \rightarrow 2(e^{-2gt} + e^{-2r_2}) \rightarrow 0$, as consistent with the prediction of zero variance in the measured quantity $\hat{x}_- = \hat{x}_A - \hat{x}_B$, when r_2 is large. On the other hand, there is an enhanced variance in x_+ beyond that of the vacuum state (as the x_A and x_B are uncorrelated), which becomes larger with the measurement, which amplifies the field.

The agreement of the predictions from the simulations with quantum mechanics can be demonstrated by introducing the scaled variables $\tilde{x}_{\pm} = x_{\pm}/e^{gt}$ and $\tilde{x}_{A/B} = x_{A/B}/e^{gt}$. These scaled variables correspond to the measured outcomes in the Q model of reality (Result IV.1, Sec. IV), which we denoted by x_0 in the single-mode case. In the large-amplification limit,

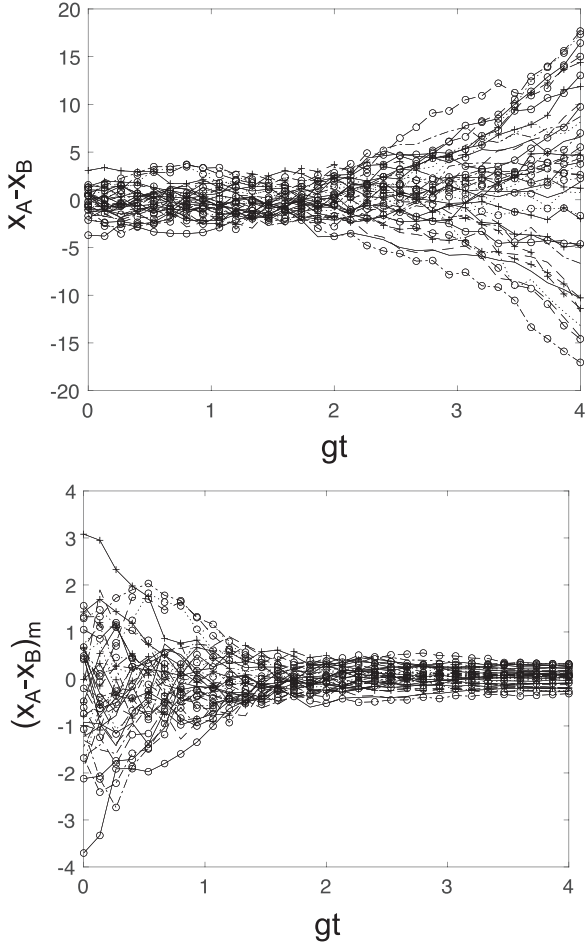


FIG. 18. EPR simulation. Simulations for the joint measurement of $\hat{x}_A$ and $\hat{x}_B$ on the EPR state $|\psi_{\text{ep}}\rangle$. The top figure shows trajectories of x_- . The lower figure shows trajectories scaled according to $(x_A - x_B)_m \equiv \tilde{x}_- = x_-/e^{gt}$ which is the inferred (measured) outcome of the measurement $\hat{x}_- = \hat{x}_A - \hat{x}_B$, as $gt \rightarrow \infty$. Here $r_2 = 2$.

$\sigma_+^2(t) \rightarrow 2e^{2gt+2r_2}$. The marginal (6.6) at time t_f becomes

$$Q_{sc}(\tilde{x}_+, \tilde{x}_-, t_f) = \frac{e^{-\tilde{x}_-^2/(4(e^{-2gt} + e^{-2r_2}))} e^{-\tilde{x}_+^2/4e^{2r_2}}}{2\pi\tilde{\sigma}_+(t_f)\tilde{\sigma}_-(t_f)} \rightarrow \frac{1}{4\pi} e^{-\tilde{x}_-^2/(4e^{-2r_2})} e^{-\tilde{x}_+^2/4e^{2r_2}}, \quad (6.8)$$

where we take $gt \rightarrow \infty$ and $\tilde{\sigma}_{\pm}^2(t_f) = 2e^{\pm 2r_2}$. Once the amplification factor G is accounted for in the measurement, the final distributions for x_- and x_+ are

$$Q_{sc}(\tilde{x}_{\mp}, t_f) = \frac{1}{2e^{\mp r_2}\sqrt{\pi}} e^{-\tilde{x}_{\mp}^2/(4e^{\mp 2r_2})}. \quad (6.9)$$

These distributions have variances of $2e^{\mp 2r_2}$. The hidden vacuum noise terms associated with the Q function, as given by $\sigma_{\pm}^2(t) = 2$ in the expression (6.7) for $\sigma_{\pm}^2(t)$, have become indiscernible. The limiting distributions give precisely the distributions predicted from quantum mechanics. The two-mode squeezed state $|\psi_{\text{ep}}\rangle$ predicts $[\Delta(\hat{x}_{\pm})]^2 = 2e^{\pm 2r_2}$, where $\hat{x}_{\pm} = \hat{x}_A \pm \hat{x}_B$, consistent with the result given by Eq. (6.9).

The EPR argument is based not only on the perfect correlation between $\hat{x}_A$ and $\hat{x}_B$, but also on the perfect anticorrelation

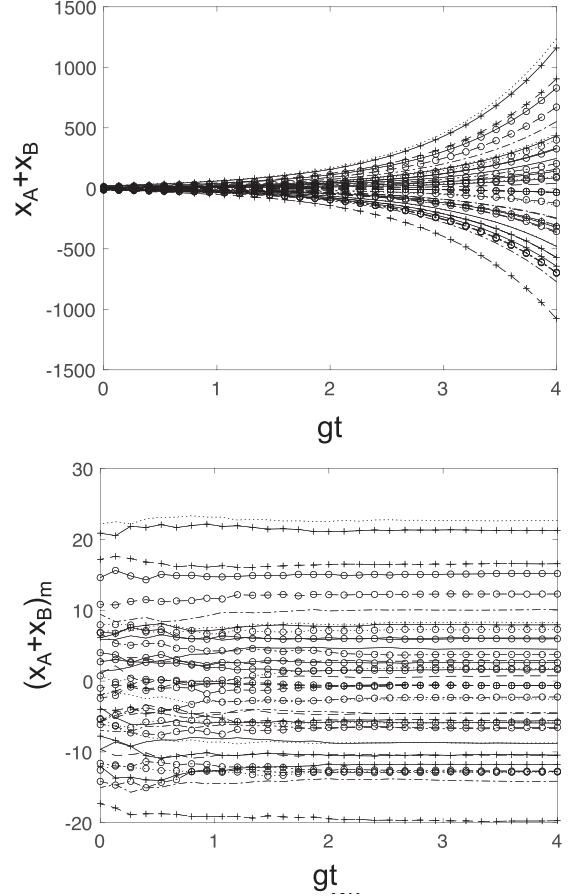


FIG. 19. EPR simulation. Measurement of $\hat{x}_A$ and $\hat{x}_B$ on the EPR state $|\psi_{\text{ep}}\rangle$. The top figure shows trajectories of x_+ . The lower figure plots trajectories of the scaled value $(x_A + x_B)_m \equiv \tilde{x}_+ = x_+/e^{gt}$, which becomes the inferred (measured) outcome of the measurement $\hat{x}_+ = \hat{x}_A + \hat{x}_B$, as $gt \rightarrow \infty$. Here, $r_2 = 2$.

between $\hat{p}_A$ and $\hat{p}_B$ [139]. The stochastic equations for joint measurements of $\hat{p}_A$ and $\hat{p}_B$ are

$$\frac{dp_K}{dt_-} = -gp_K + \xi_{K1}(t), \quad (6.10)$$

and $\frac{dx_K}{dt} = -gx_K + \xi_{K2}(t)$. The equations of interest are those for p_K since these give solutions for the amplified variables that are measured. The marginal for $p_{\pm}$ for the initial state (6.2) is

$$Q_{\text{ep},\pm}(p_-, p_+, 0) = \frac{e^{-p_-^2/2\sigma_+^2(0)} e^{-p_+^2/2\sigma_-^2(0)}}{2\pi\sigma_+(0)\sigma_-(0)}, \quad (6.11)$$

where $\sigma_{\mp}^2(0) = 2(1 + e^{\mp 2r_2})$. The fluctuations are decreased in p_+ . The marginal used for the backward equation (6.10) is

$$Q_{\text{ep},\pm}(p_+, p_-, t_f) = \frac{e^{-p_-^2/2\sigma_+^2(t_f)} e^{-p_+^2/2\sigma_-^2(t_f)}}{2\pi\sigma_+(t_f)\sigma_-(t_f)}, \quad (6.12)$$

where the gain is $G(t) = e^{gt}$ and the variances are $\sigma_{\pm}^2(t) = 2(1 + e^{2gt}e^{\pm 2r_2})$. The $Q_{\text{ep},\pm}(p_-, p_+, t_f)$ is obtained from $Q_{\text{ep},\pm}(x_+, x_-, t_f)$ by replacing x_+ with p_- , and x_- with p_+ . The stochastic solutions for $p_{\pm}$ are hence given by those of $x_{\pm}$, but with $p_{\pm}$ replacing $x_{\mp}$. The results are in agreement with those of quantum mechanics, showing an anticorrelation

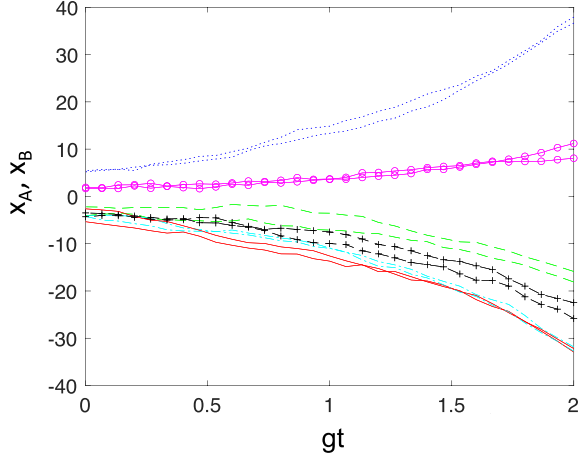


FIG. 20. Model of EPR correlations and weak elements of reality. We consider joint measurement of $\hat{x}_A$ and $\hat{x}_B$ on the EPR state $|\psi_{\text{EPR}}\rangle$, with $r_2 = 2$. The figure shows trajectories for x_A and x_B . Distinct branches emerge as the system is amplified. The pair (x_A, x_B) corresponding to the same run is shown with the same color and symbol. The outcome of $\hat{x}_A$ is the value $x_A(t)/e^{gt}$. Similarly, the outcome for $\hat{x}_B$ is $x_B(t)/e^{gt}$. For large r and $G = e^{gt}$, the correlation between x_A and x_B is perfect so that the outcome for $\hat{x}_A$ can be inferred from $\hat{x}_B$. The trajectories p_A and $-p_B$ for the joint measurement of $\hat{p}_A$ and $-\hat{p}_B$ show the same behavior as for x_A and x_B , satisfying the same equations.

between the outcomes of $\hat{p}_A$ and $\hat{p}_B$, as r_2 becomes large. The simulation leads to the following conclusions.

Result VI.1: Branches show EPR correlations. Branches emerge as both systems are amplified, consistent with the prediction of EPR correlations.

Proof. First, we note that the interactions H_{amp}^A and H_{amp}^B [Eq. (6.3)] that induce the choice of settings and amplification at A and B are local, acting only on the sets of variables $\{x_A, p_A\}$ and $\{x_B, p_B\}$, respectively. In the Q model, when the fields A and B are spatially separated, we identify amplitudes $\{x_A, p_A\}$ with field A , and amplitudes $\{x_B, p_B\}$ with field B .

The solutions for x_A and x_B are found as $x_A = (x_+ + x_-)/2$ and $x_B = (x_+ - x_-)/2$. For large amplification gt , the solutions are dominated by those of x_+ . In Fig. 20, we plot x_A and x_B , using the same symbol (and color) for the same run of the simulation. As the amplification due to measurement increases, the branches (denoted $\tilde{\lambda}_x^A$ and $\tilde{\lambda}_x^B$) emerge, given by the lines in the Fig. 20 (Secs. III B and IV A). With amplification, at a time t_m when the values $x_A(t)$ and $x_B(t)$ are macroscopic, the premise of weak macroscopic realism applies (Result IV.6, Sec. IV C) and the result of the measurements $\hat{x}_A$ and $\hat{x}_B$ are assumed to be determined. The value $x_A(t)/G$ or $x_B(t)/G$ given by the “line” as $gt \rightarrow \infty$ represents the outcome, if a detection is to be made. We see that $x_A(t) - x_B(t) \rightarrow 0$ as r_2 becomes large, implying perfect correlation between the outcomes of $\hat{x}_A$ and $\hat{x}_B$, in the Q model of reality. Similarly, for measurements of $\hat{p}_A$ and $\hat{p}_B$, we find branches (which we label $\tilde{\lambda}_p^A$ and $\tilde{\lambda}_p^B$) emerging for amplitudes $p_A(t)$ and $p_B(t)$ (Fig. 20). The amplitudes satisfy $p_A(t) + p_B(t) \rightarrow 0$ as $r_2 \rightarrow \infty$, implying perfect anticorrelation between outcomes $\hat{p}_A$ and $\hat{p}_B$. ■

Result VI.2: EPR correlations are consistent with (weak) macroscopic realism: Weak elements of reality.

Proof. This follows from the proof of Result IV.1 and from the definition of weak macroscopic realism [refer to Definition 6(a) of Sec. I B]. If we consider measurement of $\hat{x}_A$ and $\hat{x}_B$, the trajectories separate and show distinct branches as $gt \rightarrow \infty$, at a time we refer to as t_m , consistent with the emergence of macroscopic superposition states as the system is amplified (Result V.2a). Applying Result IV.6, weak macroscopic realism holds: the outcomes are determined at the time t_m , denoted by the value indicated by the lines in the Fig. 20. We refer to these values as “weak elements of reality.” ■

The existence of the weak elements of reality does not conflict with the known failure of the EPR elements of reality as shown by Bell’s theorem. This is because the weak elements of reality are defined for the system after the local settings are fixed, at the time t_m . Results VI.1 and VI.2 are hence consistent with proposals that elements of reality exist for the EPR experiment, after the settings are fixed, in a way that does not conflict with Bell nonlocality [108, 119, 142].

VII. CV BELL NONLOCALITY

The demonstration of Bell nonlocality involves a choice of measurement settings θ and ϕ for systems A and B , respectively. In this paper, we examine measurements of $\hat{x}$ and $\hat{p}$, for which the outcomes are continuous variables (CVs). It is known that certain states will violate a Bell inequality for such measurements [143–146]. We define the rotated quadrature phase amplitudes

$$\begin{aligned}\hat{x}_{\theta A} &= \hat{x}_A \cos \theta + \hat{p}_A \sin \theta, \\ \hat{p}_{\theta A} &= -\hat{x}_A \sin \theta + \hat{p}_A \cos \theta\end{aligned}\quad (7.1)$$

for system A and

$$\begin{aligned}\hat{x}_{\phi B} &= \hat{x}_B \cos \phi + \hat{p}_B \sin \phi, \\ \hat{p}_{\phi B} &= -\hat{x}_B \sin \phi + \hat{p}_B \cos \phi\end{aligned}\quad (7.2)$$

for system B . In the proposed Bell test, the amplitudes $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ are measured, and the outcomes are binned to be $+1$ or -1 , according to the sign S of the outcome. Binary outcomes $S = \pm 1$ apply at each site, for given settings θ and ϕ . The joint probability $P_{++}^{AB}(\theta, \phi)$ for positive outcomes at A and B , given θ and ϕ , can be determined experimentally. Bell’s local hidden variables (LHVs) assumption is that this probability is given by

$$P_{++}^{AB}(\theta, \phi) = \int d\lambda \rho(\lambda) P_A(+|\lambda, \theta) P_B(+|\lambda, \phi), \quad (7.3)$$

where $\rho(\lambda)$ is the distribution for a set of hidden variables λ that describe the hidden-variable state of the system, and $P_A(+|\lambda, \theta)$ and $P_B(+|\lambda, \phi)$ are the respective probabilities for an outcome $+1$ at A and B , given the values λ , θ , and ϕ [1–3, 147]. The locality assumption is the independence of $P_A(+|\lambda, \theta)$ on ϕ , and of $P_B(+|\lambda, \phi)$ on θ . The condition (7.3) allows a Bell inequality to be derived as for spins [143, 144]. The Clauser-Horne (CH) Bell inequality $S \leq 1$, where

$$S = \frac{P_{++}^{AB}(\theta, \phi) - P_{++}^{AB}(\theta, \phi') + P_{++}^{AB}(\theta', \phi) + P_{++}^{AB}(\theta', \phi')}{P_+^A(\theta') + P_+^B(\phi)} \quad (7.4)$$

follows, where $P_+^A(\theta)$ and $P_+^B(\phi)$ are the marginal probabilities of obtaining +1 at site A and B , with the settings fixed at θ and ϕ , respectively. The Clauser-Horne-Shimony-Holt (CHSH) Bell inequality

$$|E(\theta, \phi) - E(\theta, \phi') + E(\theta', \phi) + E(\theta', \phi')| \leq 2 \quad (7.5)$$

can also be derived, where $E(\theta, \phi)$ is the expected value of the product of the outcome S at each site.

An example of a state that will violate the Bell inequalities is the two-mode-squeezed cat state [145]

$$|\psi_{\text{Bell}}\rangle = N_0^2 e^{-iH_I t_I/\hbar} (|\alpha_0\rangle_A |\alpha_0\rangle_B + |-\alpha_0\rangle_A |-\alpha_0\rangle_B), \quad (7.6)$$

$$\begin{aligned} Q_{\text{Bell}}(\alpha, \beta) = & \frac{N_0^2 e^{-p_A^2/4 - p_B^2/4 - \tanh r_0 p_A p_B/2}}{16\pi^2 \cosh^2 r_0} \left\{ e^{-(x_A - 2\alpha_0 e^{r_0})^2/4} e^{-(x_B - 2\alpha_0 e^{r_0})^2/4} e^{2(\tanh r_0)[(x_A - 2\alpha_0 e^{r_0})(x_B - 2\alpha_0 e^{r_0})/4]} \right. \\ & + e^{-(x_A + 2\alpha_0 e^{r_0})^2/4} e^{-(x_B + 2\alpha_0 e^{r_0})^2/4} e^{2(\tanh r_0)[(x_A + 2\alpha_0 e^{r_0})(x_B + 2\alpha_0 e^{r_0})/4]} \\ & \left. + 2e^{-2(1 - \tanh r_0)\alpha_0^2 e^{2r_0}} e^{-x_A^2/4} e^{-x_B^2/4} e^{\frac{1}{2}(\tanh r_0)[x_A x_B]} \cos[(1 - \tanh r_0)\alpha_0 e^{r_0}(p_A + p_B)] \right\}. \end{aligned} \quad (7.7)$$

This function has the form

$$Q_{\text{Bell}}(\lambda, t_0) \sim G(p_A, p_B) \{G_1(x_A, x_B) + G_2(x_A, x_B) + I\}, \quad (7.8)$$

where we denote the real variables $x_A = \alpha + \alpha^*$, $p_A = (\alpha - \alpha^*)/i$, $x_B = \beta + \beta^*$, $p_B = (\beta - \beta^*)/i$ by $\lambda = (x_A, p_A, x_B, p_B)$ as in Eq. (6.2). Here, G denotes a bivariate Gaussian function and I is a sinusoidal interference term arising due to entanglement.

Measurement settings. After preparation of the state at time t_0 , the field modes A and B are spatially separated. Each field then interacts locally with a device that determines the measurement setting. The amplitudes $\{x_A, p_A\}$ represent field A , and amplitudes $\{x_B, p_B\}$ represent field B . For measurements $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ given by (7.1) and (7.2), the device that determines the measurement settings θ and ϕ is a phase-shifter, modeled by the interaction Hamiltonians

$$H_\theta^A = \tilde{g}\hat{n}_A, \quad H_\phi^B = \tilde{g}\hat{n}_B \quad (7.9)$$

for systems A and B , respectively ($\tilde{g} > 0$). Here, $\hat{n}_A$ and $\hat{n}_B$ are the number operators for modes A and B . Straightforward solutions of the operator equations of motion yield for H_θ^A that $\hat{x}_A = \tilde{g}\hat{p}_A/\hbar$ and $\hat{p}_A = -\tilde{g}\hat{x}_A/\hbar$, and similarly for H_ϕ^B . The solutions are

$$\begin{aligned} \hat{x}_A(t) &= \hat{x}_A(0) \cos \theta + \hat{p}_A(0) \sin \theta, \\ \hat{x}_B(t) &= \hat{x}_B(0) \cos \phi + \hat{p}_B(0) \sin \phi, \end{aligned} \quad (7.10)$$

where $\theta = \tilde{g}t_A/\hbar$ and $\phi = \tilde{g}t_B/\hbar$, the t_A and t_B being the times of interaction. This gives the required transformations (7.1) and (7.2). The interactions H_θ^A and H_ϕ^B correspond to the unitary operations $U_A(\theta) = e^{-iH_\theta^A t_A/\hbar} = e^{-i\theta \hat{n}_A}$ and $U_B(\phi) = e^{-iH_\phi^B t_B/\hbar} = e^{-i\phi \hat{n}_B}$. We may consider that the operations act simultaneously, or consecutively, in which case one interaction only may be completed at a time t_1 . We denote the time at which both interactions are completed as the time $t_2 > t_1$.

where $H_I = i\kappa\hbar(\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b})$, $N_0^2 = 1/[2(1 + e^{-4|\alpha_0|^2})]$ and α_0 is real. Violation of the CH-Bell inequality (7.4) with $S = 1.008$ for $\alpha_0 \sim 0.9$ and $\kappa t_I = 0.6$ has been predicted, for certain choices of θ and ϕ . For example, letting $\delta_1 = \theta - \theta'$ and $\delta_2 = \phi' - \phi$, the authors of Ref. [145] report violations for $\theta = \phi = 0.42\pi$ and $\delta_1 = \delta_2 = \delta = 0.7\pi$.

A. CV Bell simulation

We next outline the simulation of the CV Bell violations. The system is prepared in a bipartite state, such as $|\psi_{\text{Bell}}\rangle$, at a time t_0 . The Q function $Q_{\text{Bell}}(\alpha, \beta) = \frac{1}{\pi^2} |\langle \alpha | \langle \beta | \psi_{\text{Bell}} \rangle|^2$ for the state $|\psi_{\text{Bell}}\rangle$ is ($r_0 = \kappa t_I$)

After interaction with both the phase-shifters, at a time t_2 , the state of the system initially in state $|\psi\rangle$ is

$$|\psi(t_2)\rangle = e^{-iH_\theta^A t_A/\hbar} e^{-iH_\phi^B t_B/\hbar} |\psi\rangle. \quad (7.11)$$

The corresponding Q function at time t_2 is

$$\begin{aligned} Q(\lambda_{\text{rot}}, t_2) &= \frac{1}{\pi^2} |\langle \alpha | \langle \beta | e^{-i\theta \hat{n}_A} e^{-i\phi \hat{n}_B} |\psi\rangle|^2 \\ &= \frac{1}{\pi^2} |\langle \alpha e^{i\theta} | \langle \beta e^{i\phi} | \psi \rangle|^2, \end{aligned} \quad (7.12)$$

where $\lambda_{\text{rot}} = (x_{\theta A}, p_{\theta A}, x_{\phi B}, p_{\phi B})$ represents the rotated coordinates: $x_{\theta A} = \alpha e^{i\theta} + \alpha^* e^{-i\theta}$, $p_{\theta A} = (\alpha e^{i\theta} - \alpha^* e^{-i\theta})/i$, $x_{\phi B} = \beta e^{i\phi} + \beta^* e^{-i\phi}$, and $p_{\phi B} = (\beta e^{i\phi} - \beta^* e^{-i\phi})/i$. The last line follows since $e^{i\theta \hat{n}} |\alpha\rangle = |\alpha e^{i\theta}\rangle$. From this result, it is shown in the Appendix D that the Q function $Q(\lambda_{\text{rot}}, t_2)$ of the new state at time t_2 is that found by substituting the coordinates $\lambda = (x_A, p_A, x_B, p_B)$ in the original Q function $Q_{\text{Bell}}(\lambda, t_0)$ for the new coordinates λ_{rot} , given by the rotation transformation

$$\begin{aligned} x_{\theta A} &= x_A \cos \theta + p_A \sin \theta, \\ p_{\theta A} &= -x_A \sin \theta + p_A \cos \theta, \end{aligned} \quad (7.13)$$

and

$$\begin{aligned} x_{\phi B} &= x_B \cos \phi + p_B \sin \phi, \\ p_{\phi B} &= -x_B \sin \phi + p_B \cos \phi. \end{aligned} \quad (7.14)$$

This is a general result, valid for arbitrary two-mode states. Alternatively, we see that the Fokker-Planck equation of type (2.8) corresponding to the interactions H_θ^A and H_ϕ^B contains drift terms only, without diffusion terms. The equivalent equations for the amplitudes are hence deterministic (without stochastic terms) [135]. Consistent with the above analysis, the evolution of the amplitudes λ associated with the dynamics of the Q function is hence given by

$$\begin{aligned} x_A(t_A) &= x_A(0) \cos \theta + p_A(0) \sin \theta, \\ p_A(t_A) &= -x_A(0) \sin \theta + p_A(0) \cos \theta, \end{aligned} \quad (7.15)$$

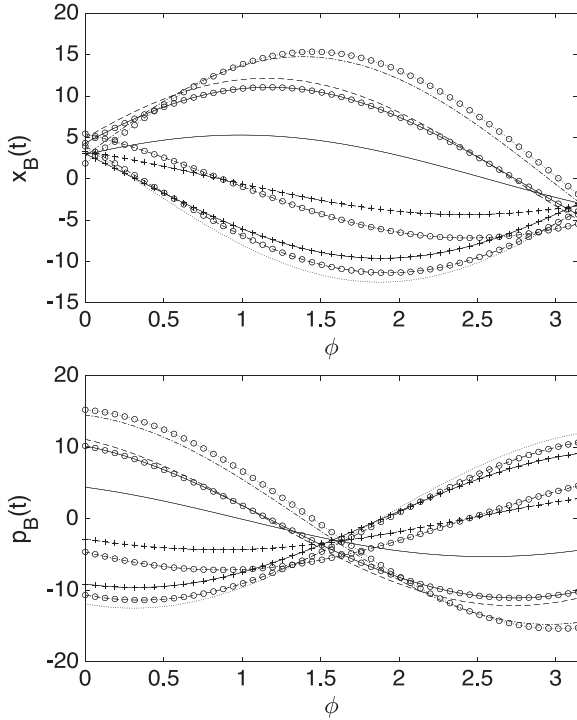


FIG. 21. Evolution of amplitudes $x_B(t)$ and $p_B(t)$ as the system B interacts according to H_ϕ^B [Eq. (7.9)], which induces a change of measurement setting $\phi = \tilde{g}t/\hbar$ at B . The amplitudes evolve as (7.16). Here, the initial distribution of x_B and p_B is given by the Q function (3.3) of the squeezed state with $x_k = 4$, $r = 2$ (which approximates an eigenstate $|x_k\rangle$ of $\hat{x}_B$). For this approximate eigenstate, the variance of p_B is large, that of x_B is small. After a time $t = \hbar\phi/\tilde{g}$, the system is prepared for measurement of $\hat{x}_B$. Note that at the intermediate times such as $\phi \approx 0.7$, where $\phi \neq n\pi/2$ (n is an integer), neither the variance of x_B nor p_B is large.

and

$$\begin{aligned} x_B(t_B) &= x_B(0) \cos \phi + p_B(0) \sin \phi, \\ p_B(t_B) &= -x_B(0) \sin \phi + p_B(0) \cos \phi, \end{aligned} \quad (7.16)$$

where $\theta = \tilde{g}t_A/\hbar$ and $\phi = \tilde{g}t_B/\hbar$. The final amplitudes are denoted $x_{\theta A} \equiv x_A(t_A)$, $p_{\theta A} \equiv p_B(t_B)$, $x_{\phi B} \equiv x_B(t_B)$, and $p_{\phi B} \equiv p_B(t_B)$. We depict the dynamics in Fig. 21. There is a change of setting at B , for a time $t_B = t$. If the distribution of the amplitudes at time $t_0 = 0$ is given by $Q(x_A, p_A, x_B, p_B, t_0)$, then the evolved amplitudes at time t have a distribution $Q(x_A, p_A, x_B, p_B, t)$ obtained by substituting $x_B \rightarrow x_B \cos \phi - p_B \sin \phi$, $p_B \rightarrow x_B \sin \phi + p_B \cos \phi$ in accordance with Eq. (7.16) (refer Appendix E).

The transformations (7.15) and (7.16) corresponding to the setting changes are *local and deterministic*. They give a *causal relation*, from time t_0 to time $t_2 > t_0$, but are reversible. Being deterministic, *there is a two-way relation* (depicted by thin two-way arrows in Fig. 22), since the values λ_{rot} of coordinates at time t_2 allow a retrodictive inference of the values λ at t_0 . Here, time t_2 is the time at which both settings have been adjusted.

Amplification. After both settings are fixed, at a time t_2 , the measurements of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ proceed by amplifying $x_{\theta A}$ and $x_{\phi B}$ according to H_{amp}^A and H_{amp}^B [Eq. (6.3)] followed by a final

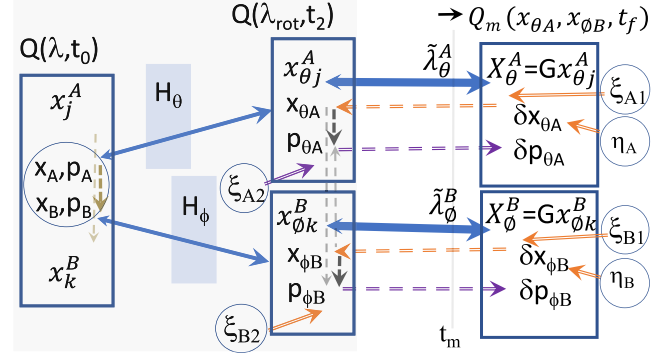


FIG. 22. Steps in the simulation of the measurement of CV EPR or Bell nonlocality. The entangled state given by $Q(\lambda, t_0)$ is generated at time t_0 . The settings are changed according to H_θ^A and H_ϕ^B , which give causal local reversible deterministic relations (thin blue two-way arrows). The state at time t_2 is given by $Q(\lambda_{\text{rot}}, t_2)$. Note that while the amplitudes transform locally, the means $x_j^A \rightarrow x_{\theta j}^A$ and $x_k^B \rightarrow x_{\phi k}^B$ do not. After the settings are fixed, the measurements $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ are carried out by amplification, according to H_{amp}^A and H_{amp}^B , giving stochastic forward-backward dynamics, with branches $\tilde{\lambda}_\theta^A$ and $\tilde{\lambda}_\phi^B$ emerging at time t_m . There is a discontinuity at this stage for the forward-time dynamics. The solid thick blue two-way arrows denote the deterministic relations ($x_{\theta j}^A \rightarrow Gx_{\theta j}^A$, $x_{\phi k}^B \rightarrow Gx_{\phi k}^B$), in which the means $x_{\theta j}^A$ and $x_{\phi k}^B$ of the Gaussians in $Q(\lambda_{\text{rot}}, t_2)$ are amplified to $Gx_{\theta j}^A$ and $Gx_{\phi k}^B$. The microscopic noise inputs $\eta_A(t_f)$ and $\eta_B(t_f)$ at the boundary at time t_f give the reverse-time trajectories $x_{\theta A}(t) = x_{\theta j}^A + \delta x_{\theta A}$ and $x_{\phi B}(t) = x_{\phi k}^B + \delta x_{\phi B}$ (orange dashed lines). $Q(\lambda_{\text{rot}}, t_2)$ and $Q(\lambda, t_0)$ determine the correlations giving links between trajectories (indicated by the faint brown lines at t_0). Local loops between $x_{\theta A}$ and $p_{\theta A}$ and between $x_{\phi B}$ and $p_{\phi B}$, are depicted by dashed black vertical lines.

detection. The equations for the amplitudes $x_{\theta A}$ and $x_{\phi B}$ are *local, stochastic, and retrocausal*, given by

$$\begin{aligned} \frac{dx_{\theta A}}{dt_-} &= -gx_{\theta A} + \xi_{A1}(t), \\ \frac{dx_{\phi B}}{dt_-} &= -gx_{\phi B} + \xi_{B1}(t), \end{aligned} \quad (7.17)$$

with a boundary condition at the future time t_f , where $t_- = -t$. Here, we use the term retrocausal only to imply the propagation in the negative time direction t_- . The equations for p are $\frac{dp_{\theta A}}{dt} = -gp_{\theta A} + \xi_{A2}(t)$ and $\frac{dp_{\phi B}}{dt} = -gp_{\phi B} + \xi_{B2}(t)$, which are causal with a boundary condition at the time t_2 . The Gaussian random noises satisfy $\langle \xi_{A\mu}(t) \xi_{A\nu}(t') \rangle = 2g\delta_{\mu\nu}\delta(t - t')$ and $\langle \xi_{B\mu}(t) \xi_{B\nu}(t') \rangle = 2g\delta_{\mu\nu}\delta(t - t')$ with cross terms zero. Here, we are imposing a new initial boundary for Eq. (7.17) at time t_2 , where a *discontinuity* is introduced. The backward nature of the propagation of the stochastic amplitudes clearly implies a discontinuity at the time t_2 in the forward-time direction. The amplitudes do not propagate continuously in the forward-time direction.

Procedure of simulation. There is hence a two-step process in the forward-time direction. At the time t_0 , the system is specified with respect to a particular measurement basis, in this case $\hat{x}_A$ and $\hat{x}_B$ ($\theta = \phi = 0$), as determined by the preparation of the system. This implies that the Q function $Q(\lambda, t_0)$ is expanded with respect to this basis, in terms of the Q function

of $|x_j^A\rangle$ and $|x_k^B\rangle$, the eigenstates of $\hat{x}_A$ and $\hat{x}_B$, as explained in the causal model of Sec. V. The deterministic transformations associated with H_θ^A and H_ϕ^B rotate the amplitudes, so that the Q function $Q(\lambda_{\text{rot}}, t_2)$ at time t_2 is expressed in the coordinates $\lambda_{\text{rot}} = (x_{\theta A}, p_{\theta A}, x_{\phi B}, p_{\phi B})$ of the new measurement basis, $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$.

$Q(\lambda_{\text{rot}}, t_2)$ is unique for the quantum state and is equivalently expressible as an expansion in terms of the Q functions of the eigenstates $|x_{\theta j}^A\rangle$ and $|x_{\phi k}^B\rangle$ of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ (Appendix C). This expansion $Q(\lambda_{\text{rot}}, t_2)$ involves bivariate Gaussian functions $\mathcal{G}$ with means $x_{\theta j}^A$ and $x_{\phi k}^B$, and interference terms $\mathcal{Int}_{AB}$, similar to Eq. (3.8). The future boundary condition for Eq. (7.17) is determined by the marginal

$$Q_m(x_{\theta A}, x_{\phi B}, t) = \int dp_{\theta A} dp_{\phi B} Q(\lambda_{\text{rot}}, t) \quad (7.18)$$

of the Q function $Q(\lambda_{\text{rot}}, t)$ of the amplified state, where $t \geq t_2$. NB: This function is defined for the bipartite system and hence differs from $Q(x, p, t)$ defined for a single system, in Part I of the paper. We use the subscript m to avoid confusion.

We confirm below in Result VII.3 that the interference terms of the marginal (7.18), as for (3.20) in Part I of the paper, vanish, and, as for Result III.1, the future boundary condition for Eq. (7.17) is the sum of Gaussian terms. Extending the analysis of Results IV.3 and IV.4 of Sec. IV B to two modes, the final distribution $P(x_{\theta A}, x_{\phi B})$ for the probability of outcomes of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ is given by the rescaled marginal

$$P(x_{\theta A}, x_{\phi B}) \rightarrow Q_{\text{sc}}(x_{\theta A}, x_{\phi B}, t_f) \quad (7.19)$$

as $t_f \rightarrow \infty$, which in turn is given by the marginal $W(x_{\theta A}, x_{\phi B})$ of the Wigner function of the state $|\psi_{\text{Bell}}\rangle$. The Wigner function for $|\psi_{\text{Bell}}\rangle$ is derived in Ref. [145], revealing a violation of the Bell inequality for certain choices of θ and ϕ .

Paradox and projection. Questions and paradoxes arise. As in Result III.1, the amplified quantities that appear in the scaled distribution $Q_{\text{sc}}(x_{\theta A}, x_{\phi B}, t_f)$ at a time t_f are the means (denoted $x_{\theta j}^A$ and $x_{\phi k}^B$ in Fig. 22) of the bivariate Gaussian functions. These means are amplified by each H_{amp} , and correspond to the final detected outcomes, but are *not* directly determinable from the values of the amplitudes $x_{\theta A}(t)$ and $x_{\phi B}(t)$ at time t_2 (Result IV.2).

Moreover, we show below that the transformation of the Gaussian means $x_j^A \rightarrow x_{\theta j}^A$ and $x_k^B \rightarrow x_{\phi k}^B$ from time t_0 to time t_2 is nonlocal. In Part I of this paper, we show consistency with an HV model where there is a probability for the $x_{\theta j}^A$ given values for the amplitudes x_A and p_A , so that the mean $x_{\theta j}^A$ can be considered part of the description of the state at the time t_1 , after the setting is fixed locally at A. The interaction H_{amp}^A is assumed to “extract” and amplify the mean. But the H_{amp}^A is local, as is H_θ^A . How then can the change of mean, $x_j^A \rightarrow x_{\theta j}^A$, be determined, if knowledge of ϕ is required? It seems that there is no mechanism provided for the Bell nonlocality by the simulation, other than that the violations are put in by hand as a future boundary condition (FBC) (Sec. I).

In the following two Secs. VIII and IX, we counter this argument and elucidate the paradox by considering *projection*. While there is a discontinuity in the forward-time direction, the amplitudes propagate backward in a continuous way. This puts a constraint on the possible states for one system (for

future measurements), given the setting of the other. This also leads to a demonstration of the consistency of the Bell nonlocality with the premises of weak macroscopic realism and weak local realism, as defined by Definitions (6) and (10) in Sec. IB. These premises posit that real properties exist, prior to the time t_f at which the future boundary condition is applied.

B. Role of hidden interference: Mechanism for Bell nonlocality

How does the simulation produce the Bell nonlocality? We examine this in two parts. First, in this section, we examine the dynamics of the Q function, as it evolves through each step of the measurement in the simulation, showing the role of the hidden interference. In Sec. IX, we examine the trajectories of the forward-backward amplitudes, emphasizing the role of projection.

The term I that appears for $|\psi_{\text{Bell}}\rangle$ [Eq. (7.6)] includes a detectable type of interference which will contribute a measurable central “spot” in the final detected joint probability $P(x_{\theta A}, x_{\phi B})$ for outcomes $x_{\theta A}$ and $x_{\phi B}$ of measurement $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$. We hence distinguish between the measurable interference I present for a superposition of coherent states, and the hidden interference $\mathcal{Int}$ present in the superposition of eigenstates $|x_j\rangle$ of $\hat{x}$, as in Eq. (3.8) with $r \rightarrow \infty$: The hidden interference is not detectable when the system undergoes amplification H_{amp} (Result III.1). The role of the hidden interference terms is clarified by the simulation, from which we deduce a mechanism for Bell violations.

Definition: Hidden interference terms. We consider measurements of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$. The Q function is expressed in terms of the “measurement basis,” i.e., as a summation of the Q functions of the eigenfunctions $|x_{\theta A}\rangle$ and $|x_{\phi B}\rangle$ of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$. These are defined as for the squeezed states (3.3) but with rotated axes θ and ϕ (Appendix C). The interference terms appearing when the Q function for the state $|\psi\rangle$ is written with respect to the *measurement* basis are referred to as hidden interference terms. These are denoted by $\mathcal{Int}_{AB}$ in the bipartite case.

Result VII.1: Mechanism for Bell violations in the simulation. While the transformations $x_A, p_A \rightarrow x_{\theta A}, p_{\theta A}$ and $x_B, p_B \rightarrow x_{\phi B}, p_{\phi B}$ of the amplitudes are local [refer to (7.13) and (7.14)], the transformation of the means $x_j^A \rightarrow x_{\theta j}^A$ and $x_k^B \rightarrow x_{\phi k}^B$ of the Gaussian functions is nonlocal (Fig. 22). This is explained by the dynamics of the amplitudes in the simulation, as we show below.

Proof. The transformations $x_A, p_A \rightarrow x_{\theta A}, p_{\theta A}$ and $x_B, p_B \rightarrow x_{\phi B}, p_{\phi B}$ given by (7.13) and (7.14) that arise due to the adjustment of measurement settings are local and deterministic. Hence, if the amplitudes $x_{\theta A}$ and $x_{\phi B}$ were directly detectable in the final measurement, a violation of the Bell inequality would not be possible. A Bell LHV theory would hold, where x_A, p_A, x_B, p_B are the hidden variables given by λ in the expression (7.3). In fact, the detected outcomes of the measurements (based on the amplifications H_{amp}^A and H_{amp}^B) are given by the means $x_{\theta j}^A$ and $x_{\phi k}^B$ of the Gaussians comprising the Q function $Q(\lambda, t_2)$, when this Q function is written with respect to the measurement basis. Since the means $x_{\theta j}^A$ and $x_{\phi k}^B$ lead to the outcomes via the deterministic relations $x_{\theta j}^A \rightarrow Gx_{\theta j}^A$ and $x_{\phi k}^B \rightarrow Gx_{\phi k}^B$, it is this

stage that leads to any violation of the Bell model (and which is hence “nonlocal”). ■

Result VII.2: Bipartite hidden interference and the future boundary condition. The hidden interference terms $\mathcal{I}nt_{AB}$ do not contribute to the measured probabilities (as modeled by the measurements H_{amp}^A and H_{amp}^B). This is a bipartite extension of Result III.1.

Proof. To clarify the role of the hidden interference $\mathcal{I}nt_{AB}$, we express the Q function in terms of the measurement basis. We take the case where the system is prepared for measurements $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ (as at time t_2 in Fig. 22), and expand the state for the system as

$$|\psi\rangle = \sum_{jk} c_{jk} |x_{\theta j}^A\rangle |x_{\phi k}^B\rangle, \quad (7.20)$$

where $|x_{\theta j}^A\rangle$ and $|x_{\phi k}^B\rangle$ are the eigenstates of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ (refer to Appendix C for definitions). The Q function $Q(\alpha, \beta) = \frac{1}{\pi^2} \langle \alpha | \langle \beta | \rho | \beta \rangle | \alpha \rangle$ can be written as

$$Q(\alpha, \beta) = \frac{1}{\pi^2} \langle \alpha | \langle \beta | (\rho_D + \rho_{OD}) | \beta \rangle | \alpha \rangle, \quad (7.21)$$

where the density operator $\rho = |\psi\rangle\langle\psi| = \rho_D + \rho_{OD}$ is divided into a sum of diagonal and off-diagonal terms,

$$\rho_D = \sum_{jk} |c_{jk}|^2 |x_{\theta j}^A\rangle \langle x_{\phi k}^B| \langle x_{\phi k}^B| \langle x_{\theta j}^A|, \quad (7.22)$$

and

$$\rho_{OD} = \sum_{j \neq j', k \neq k'} c_{jk} c_{j'k'}^* |x_{\theta j}^A\rangle \langle x_{\phi k}^B| \langle x_{\phi k'}^B| \langle x_{\theta j'}^A|. \quad (7.23)$$

The Q function is hence

$$\begin{aligned} Q &= \frac{1}{\pi^2} \sum_{jk} |c_{jk}|^2 \langle \alpha | \langle \beta | x_{\theta j}^A \rangle | x_{\phi k}^B \rangle|^2 + \mathcal{I}nt_{AB} \\ &= N \left(\sum_{jk} |c_{jk}|^2 \{ \mathcal{G}_{0, \sigma_{\rho_{\theta A}}} (p_{\theta A}) \mathcal{G}_{x_{\theta j}^A, 1} (x_{\theta A}) \right. \\ &\quad \left. \times \mathcal{G}_{0, \sigma_{\rho_{\phi B}}} (p_{\phi B}) \mathcal{G}_{x_{\phi k}^B, 1} (x_{\phi B}) \} + \mathcal{I}nt_{AB} \right). \end{aligned} \quad (7.24)$$

Here we denote the Gaussian function of x with mean $\bar{x}$ and variance σ_x by

$$\mathcal{G}_{\bar{x}, \sigma_x}(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-(x-\bar{x})^2/2\sigma_x}. \quad (7.25)$$

We have taken the variance of x_{θ} for the eigenstate $|x_{\theta}\rangle$ of $\hat{x}_{\theta}$ to be 1. Equation (7.24) is a two-mode generalization of Eq. (3.8). The summation involving the Gaussian functions $\mathcal{G}$ arises from the diagonal part of the density matrix ρ_D . $\mathcal{I}nt_{AB}$ is the hidden interference term arising from ρ_{OD} .

In the simulation, the joint measurement of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ is modeled by H_{amp}^A and H_{amp}^B acting independently on each system. As in the single-mode case, the means of the Gaussian functions of $x_{\theta A}$ and $x_{\phi B}$ are amplified, but their variances remain at 1, while amplitudes $p_{\theta A}$ and $p_{\phi B}$ deamplify. It is shown in Result III.1 (Sec. III) that hidden interference terms are not detected in the amplified field, a result that can be straightforwardly generalized to apply to (7.24). The probabilities $P(x_{\theta j}^A, x_{\phi k}^B)$ for joint outcomes $x_{\theta j}^A$ and $x_{\phi k}^B$ of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ are known (in quantum mechanics) to be given

by the coefficients $|c_{jk}|^2$ in the diagonal expansion of ρ [Eq. (7.22)]. This is in agreement with the simulation, where the detected probabilities correspond to the coefficients of the Gaussians in the expansion (7.24), which are given by $|c_{jk}|^2$ (Sec. IV B). ■

To summarize, the observed probabilities are given by the mixed state ρ_D defined by the diagonal matrix (7.22). The hidden interference terms $\mathcal{I}nt_{AB}$ correspond to off-diagonal terms that are not detected with amplification.

Result VII.3: Changing the setting at one location. No observable nonlocality. Changing the setting at one location, say B (leaving the other setting fixed) can change the hidden interference, but not the observed probabilities, relative to a nonentangled state. Hence, Bell nonlocality is not observed with a change of setting at just one site. There are, however, changes to the hidden interference terms.

Proof. The analysis in Appendix F shows what happens to the system prepared in $|\psi\rangle$ [Eq. (7.20)] when we change the measurement setting at one location, say B . The transformed state of $|\psi\rangle$ (i.e., of $\rho = |\psi\rangle\langle\psi|$) is compared with the transformed state of the *partially mixed* state $\rho_{\text{mix}, A}$ given as

$$\begin{aligned} \rho_{\text{mix}, A} &= \sum_l \{ \langle x_{\theta l}^A | \rho | x_{\theta l}^A \rangle \} |x_{\theta l}^A\rangle \langle x_{\theta l}^A| \\ &= \sum_l |f_l|^2 |\psi_l^B\rangle \langle x_{\theta l}^A| \langle x_{\theta l}^A| \langle \psi_l^B|. \end{aligned} \quad (7.26)$$

To understand the meaning of $\rho_{\text{mix}, A}$, we express the state $|\psi\rangle$ of the system as $|\psi\rangle = \sum_l f_l |x_{\theta l}^A\rangle |\psi_l^B\rangle$, where $\sum_l |f_l|^2 = 1$ and $|\psi_l^B\rangle$ are normalized states for B . From the last line of (7.26), we see that $\rho_{\text{mix}, A}$ represents a system whereby a collapse to the eigenstates at A has occurred. The $\rho_{\text{mix}, A}$ is hence not entangled, although local superposition states remain possible for B .

To model the change of setting at B in the simulation, we transform to the new basis of B . This changes the expansion of ρ (and of the Q function of the state $|\psi\rangle$), since in the Q model we write with respect to the measurement basis (Secs. V A and VII A). However, the changes to the diagonal part of ρ [ρ_D of Eq. (7.21)] are the same for $|\psi\rangle$ and $\rho_{\text{mix}, A}$ (refer to Appendix F). The new expansions of ρ as given for $|\psi\rangle$ and $\rho_{\text{mix}, A}$ are different, but the difference is only in the off-diagonal terms ρ_{OD} . This difference will affect the hidden interference terms $\mathcal{I}nt_{AB}$ in the Q function but do not contribute to the observable probabilities (Result VII.2). Hence, the observable joint probabilities for $|\psi\rangle$ are the same as given for the nonentangled state $\rho_{\text{mix}, A}$. ■

Result VII.4: Bell nonlocality. After a change of setting at *both* locations A and B , the hidden interference terms can contribute to measurable probabilities. This can lead to Bell nonlocality.

Proof. That changes of settings at both A and B can lead to Bell nonlocality is illustrated by the spin Bell state (Appendix G). We explicitly demonstrate how this occurs for the CV-Bell example below (Result VII.5). ■

Result VII.5: Dynamics of the mechanism for Bell nonlocality. We trace the dynamics of the amplitudes in the simulation depicted in Fig. 22 to demonstrate how Bell nonlocality arises in accordance with Result VII.4.

Proof. We consider first for simplicity that the state at time t_0 is prepared, for measurements $\hat{x}_A$ and $\hat{x}_B$, in the entangled state

$$|x_j^A\rangle|x_k^B\rangle + |x_l^A\rangle|x_m^B\rangle. \quad (7.27)$$

Here, $|x_j^A\rangle$ and $|x_k^B\rangle$ are eigenstates of $\hat{x}_A$ and $\hat{x}_B$, respectively, which we define as highly squeezed states in $\hat{x}_A$ and $\hat{x}_B$ [Eq. (3.2)]. The Q function is

$$Q(\lambda, t_0) = N\{\mathcal{G}_{0,\sigma_{p_A}}(p_A)\mathcal{G}_{0,\sigma_{p_B}}(p_B)\{\mathcal{G}_{x_j^A,1}(x_A)\mathcal{G}_{x_k^B,1}(x_B) + \mathcal{G}_{x_l^A,1}(x_A)\mathcal{G}_{x_m^B,1}(x_B)\} + \mathcal{I}nt_{AB}\}, \quad (7.28)$$

where

$$\mathcal{I}nt_{AB} = 2e^{-\frac{(\Delta_A)^2}{2}}e^{-\frac{(\Delta_B)^2}{2}}\mathcal{G}_{\frac{(x_j^A+x_l^A)}{2},1}(x_A)\mathcal{G}_{\frac{(x_k^B+x_m^B)}{2},1}(x_B) \times \mathcal{G}_{0,\sigma_{p_A}}(p_A)\mathcal{G}_{0,\sigma_{p_B}}(p_B)\mathcal{F}_{AB}, \quad (7.29)$$

with the sinusoidal component

$$\mathcal{F}_{AB} = \cos[p_A\Delta_A/\sigma_{x_A} + p_B\Delta_B/\sigma_{x_B}]. \quad (7.30)$$

The $\mathcal{G}$ are the Gaussian functions defined by (7.25). Here $\Delta_A = (x_j^A - x_l^A)/2\sigma_{x_A}$ and $\Delta_B = (x_k^B - x_m^B)/2\sigma_{x_B}$ are the separations between the states of the superposition. The $\sigma_{x_A}^2$ and $\sigma_{p_A}^2$ are the variances of x_A and p_A in the Q functions of the squeezed states that approximate the eigenstates $|x_j^A\rangle$ and $|x_l^A\rangle$. The $\sigma_{x_B}^2$ and $\sigma_{p_B}^2$ are defined similarly for eigenstates $|x_k^B\rangle$ and $|x_m^B\rangle$ of system B . Hence, $\sigma_{x_A}^2 = \sigma_{x_B}^2 = 1$ and $\sigma_{p_A}^2 \rightarrow \infty, \sigma_{p_B}^2 \rightarrow \infty$. The fringe term is modulated by Gaussian functions with peaks at the average position of the means x_j^A and x_l^A (x_k^B and x_m^B) and is damped by the separation Δ_A (Δ_B) between them.

From Results III.1 and VII.2, we know that the hidden interference term $\mathcal{I}nt_{AB}$ does not contribute to the marginal defined as the integral $Q_m(x_A, x_B, t_0) = \int dp_A dp_B Q(\lambda, t_0)$. This is evident here as follows: When the cosine function in $\mathcal{F}_{AB}$ is expanded, the integral reduces to a combination of integrals over p_A or p_B individually. Those involving a sine function are zero due to symmetry. Those involving the cosine function reduce to integrals of the type

$$\frac{1}{\sqrt{a\pi}} \int_{-\infty}^{\infty} e^{-p^2/a} \cos(kp) dp = e^{-ak^2/4}, \quad (7.31)$$

as used in (3.19), which vanish since we take $a \equiv \sigma_{p_A}^2$ (or $\sigma_{p_B}^2$) to be large for the eigenstate. The vanishing of the $\mathcal{I}nt_{AB}$ terms in the marginal $Q_m(x_A, x_B, t_0)$ is consistent with Result VII.2, that the hidden terms $\mathcal{I}nt_{AB}$ do not contribute to the probabilities measured directly under H_{amp}^A and H_{amp}^B .

If we change the setting at B to ϕ , then (following Sec. VII A) we substitute $x_B \rightarrow x_{\phi B} \cos \phi - p_{\phi B} \sin \phi$ and $p_B \rightarrow x_{\phi B} \sin \phi + p_{\phi B} \cos \phi$ in the expression for $Q(\lambda, t_0)$. The $Q(\lambda, t_0)$ is transformed to give a new function (refer to Appendix E):

$$Q(\lambda_{\text{rot}}, t_1) = N_1 \mathcal{G}_{0,\sigma_{p_A}}(p_A) \{\mathcal{G}_{x_j^A,1}(x_A) f_{x_k^B}(x_{\phi B}, p_{\phi B}) + \mathcal{G}_{x_l^A,1}(x_A) f_{x_m^B}(x_{\phi B}, p_{\phi B})\} + \mathcal{I}nt_{AB}(\phi), \quad (7.32)$$

where N_1 is a normalization term. The bivariate distributions $\mathcal{G}_{0,\sigma_{p_B}}(p_B)\mathcal{G}_{x_k^B,1}(x_B)$ and $\mathcal{G}_{0,\sigma_{p_B}}(p_B)\mathcal{G}_{x_m^B,1}(x_B)$ in (7.28) are

transformed to the new coordinates, as depicted in Fig. 21. These transformed distributions are denoted by $f_{x_k^B}(x_{\phi B}, p_{\phi B})$ and $f_{x_m^B}(x_{\phi B}, p_{\phi B})$, respectively. The transformed interference is

$$\mathcal{I}nt_{AB}(\phi) = 2e^{-\frac{(\Delta_A)^2}{2}}\mathcal{G}_{\frac{(x_j^A+x_l^A)}{2},1}(x_A)\mathcal{G}_{0,\sigma_{p_A}}(p_A) \times e^{-\frac{(\Delta_B)^2}{2}}f_{\frac{(x_k^B+x_m^B)}{2}}(x_{\phi B}, p_{\phi B})\mathcal{F}_{AB}(\phi), \quad (7.33)$$

where the fringe term is

$$\mathcal{F}_{AB}(\phi) = \cos\left[\frac{p_A}{2}(x_j^A - x_l^A) + \frac{p_B(\phi)}{2}(x_k^B - x_m^B)\right]. \quad (7.34)$$

The fringe term $\mathcal{F}_{AB}(\phi)$ involves the rotated coordinate $p_B(\phi) = x_{\phi B} \sin \phi + p_{\phi B} \cos \phi$ for B . However, the p_A is unchanged. We evaluate the new marginal defined as $Q_m(x_A, x_{\phi B}, t_1) = \int dp_A dp_{\phi B} Q(\lambda_{\text{rot}}, t_1)$. We can expand both terms in $\mathcal{F}_{AB}(\phi)$ as the sum of two terms, one involving $\sin[p_A(x_j^A - x_l^A)/2]$ and the other $\cos[p_A(x_j^A - x_l^A)/2]$. The sine integration vanishes due to symmetry. The cosine integration vanishes in the limit of $\sigma_{p_A}^2 \rightarrow \infty$ [as for Eq. (3.19)], justified for the eigenstates $|x_j^A\rangle$ and $|x_l^A\rangle$. Now, we see that the integration over p_A ensures that $\mathcal{I}nt_{AB}$ vanishes from the marginal $Q_m(x_A, x_{\phi B}, t_1)$: Hence, the change of basis in B changes the hidden interference term $\mathcal{I}nt_{AB}$, but this change *remains hidden*, since it does not contribute to the marginal $Q_m(x_A, x_{\phi B}, t_1)$ (consistent with Result VII.3).

However, if there is a change of setting at *both* sites A and B , then *both* p_A and p_B transform as in (7.13) and (7.14), meaning that both p_A and p_B are rotated. The final transformed function $Q(\lambda_{\text{rot}}, t_2)$ is given by

$$Q(\lambda_{\text{rot}}, t_2) = N_2 \{f_{x_j^A}(x_{\theta A}, p_{\theta A})f_{x_k^B}(x_{\phi B}, p_{\phi B}) + f_{x_l^A}(x_{\theta A}, p_{\theta A})f_{x_m^B}(x_{\phi B}, p_{\phi B}) + \mathcal{I}nt_{AB}(\theta, \phi)\}, \quad (7.35)$$

where N_2 is the normalization constant. The interference term is

$$\mathcal{I}nt_{AB}(\theta, \phi) = f_{\frac{(x_j^A+x_l^A)}{2}}(x_{\theta A}, p_{\theta A})f_{\frac{(x_k^B+x_m^B)}{2}}(x_{\phi B}, p_{\phi B}) \times e^{-\frac{(\Delta_A)^2}{2}}e^{-\frac{(\Delta_B)^2}{2}}\mathcal{F}_{AB}(\theta, \phi), \quad (7.36)$$

where

$$\mathcal{F}_{AB}(\theta, \phi) = \cos\left[\frac{p_A(\theta)}{2}(x_j^A - x_l^A) + \frac{p_B(\phi)}{2}(x_k^B - x_m^B)\right] \quad (7.37)$$

Here, $p_A(\theta) = x_{\theta A} \sin \theta + p_{\theta A} \cos \theta$ and $p_B(\phi) = x_{\phi B} \sin \phi + p_{\phi B} \cos \phi$. To determine the marginal $Q_m(x_{\theta A}, x_{\phi B}, t_2) = \int dp_{\theta A} dp_{\phi B} Q(\lambda_{\text{rot}}, t_2)$ at time t_2 , it is necessary to integrate over $p_{\theta A}$ and $p_{\phi B}$. With both transformations, the integrals over $p_{\theta A}$ and $p_{\phi B}$ involving the cosine function *no longer necessarily vanish*, since with $\theta, \phi \neq n\pi$ (where n is an integer) the variances $\sigma_{p_{\theta A}}^2$ and $\sigma_{p_{\phi B}}^2$ are *no longer restricted to be very large*. This is evident in Fig. 21 and by Eq. (7.31), where a need not be large (refer to Appendix E). Hence, we require $\theta, \phi \neq n\pi$ to observe nonlocality. This implies that the rotated state after a change of setting is a superposition of the original

eigenstates. In conclusion, with such a selection of settings, the interference term $\mathcal{Int}_{AB}(\theta, \phi)$ is no longer hidden and can contribute nonzero terms to the marginal $Q_m(x_{\theta A}, x_{\phi B}, t_2)$. We conclude that the interference terms may contribute to the measurable probability distribution $P(x_{\theta A}, x_{\phi B})$ for outcomes $x_{\theta A}$ and $x_{\phi B}$ of joint measurements $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$.

This result can be generalized to an arbitrary state prepared at time t_0 by considering $|\psi\rangle = \sum_{jk} c_{jk} |x_j^A\rangle |x_k^B\rangle$ (refer to Appendix C).

Comment. It is the interference term $\mathcal{Int}_{AB}$ present in the original Q function $Q(\lambda, t_0)$ that leads to the Bell nonlocality.

Proof. The marginal $Q_m(x_{\theta A}, x_{\phi B}, t_2)$, defined by (7.18) and used in the above proof, can be expressed as the sum of three terms, Q_1 , Q_2 , and Q_3 . These are defined as the integration over $p_{\theta A}$ and $p_{\phi B}$ of the three terms appearing in $Q(\lambda_{\text{rot}}, t_2)$ in (7.35). The integration of the first two terms in $Q(\lambda_{\text{rot}}, t_2)$ gives Q_1 and Q_2 , which lead to peaks in the distribution $Q(x_{\theta A}, x_{\phi B}, t)$ corresponding to the states $|x_j^A\rangle |x_k^B\rangle$ and $|x_l^A\rangle |x_m^B\rangle$, respectively, as in a mixture of the $|x_j^A\rangle |x_k^B\rangle$ and $|x_l^A\rangle |x_m^B\rangle$. The expressions Q_1 and Q_2 are expressible as a product of a function of $x_{\theta A}$ and a function of $x_{\phi B}$. The $x_{\theta A}$ is a function only of local variables x_A , p_A , and θ ; $x_{\phi B}$ is a function only of local variables x_B , p_B , and ϕ . We see that any mixture of the $|x_j^A\rangle |x_k^B\rangle$ and $|x_l^A\rangle |x_m^B\rangle$ would satisfy a Bell LHV model, where x_A , p_A , x_B and p_B play the role of hidden variables. The third term given by $Q_3 = \int dp_{\theta A} dp_{\phi B} \mathcal{Int}_{AB}(\theta, \phi)$ does not factorize in this way, and hence leaves open the possibility to violate a Bell inequality.

Result VII.6: Nonlocality. Consistent with Results VII.1 and VII.4, the effect due to the interference term $\mathcal{Int}_{AB}(\theta, \phi)$ on the observed probabilities is *nonlocal*, implying it can lead to a violation of a Bell inequality.

Proof. The proof follows from the above comment. We note that the impact of the change of setting θ at A on any observed probability that is due to $\mathcal{Int}_{AB}$ is only present if there is also a change of setting ϕ at B. This is evident by the mathematical form of $\mathcal{Int}_{AB}$ in Eq. (7.36). If $\phi = 0$, there is no change to observed probabilities due to any change of setting θ at A that can be attributed to $\mathcal{Int}_{AB}$; if $\phi \neq 0$, such a change becomes possible (Result VII.5). Hence, the effect due to $\mathcal{Int}_{AB}$ at A depends on ϕ at B.

Result VII.7: The CV Bell nonlocality requires the settings to mix x and p locally. The constraint $\theta, \phi \neq n\pi$ (where n is an integer) is derived above in Result VII.5 as a requirement for the Bell nonlocality. This implies the settings need to be changed (relative to the initial settings, denoted x_A , p_A and x_B , p_B) so that *both* the measured quadratures $\hat{x}_\theta^A$ and $\hat{x}_\phi^B$ are rotated to combine the local amplitudes x_A , p_A and x_B , p_B , respectively (so that the rotated states are superpositions of the original eigenstates in $\hat{x}_A$ and $\hat{x}_B$). This is consistent with quantum predictions.

C. Local loops, nonlocal loops, and correlation

We see from Sec. IV E that the hidden interference term $\mathcal{Int}$ can contribute to a hidden loop, which distinguishes the superposition from the mixed state (Figs. 16 and 17). The correlations are more complex in the two-mode case. However, we see that the term $\mathcal{Int}_{AB}$ present in the Q function (7.28) of the superposition $|x_j^A\rangle |x_k^B\rangle + |x_l^A\rangle |x_m^B\rangle$ implies that the

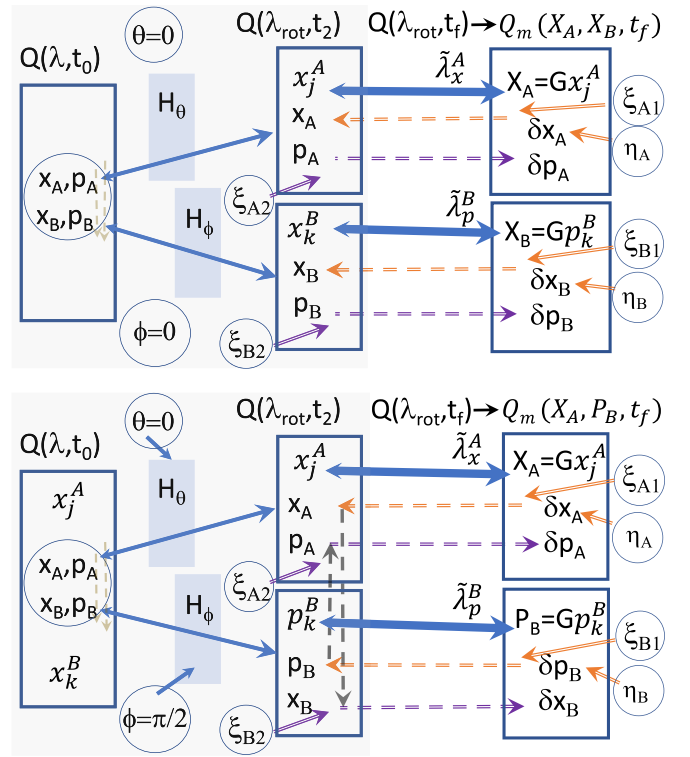


FIG. 23. Nonlocal loops and EPR states. Diagram as in Fig. 22, depicting the special case of the simulation of the measurement of the CV EPR entanglement for $|\psi_{\text{EPR}}\rangle$. The system is prepared for measurements $\hat{x}_A$ and $\hat{x}_B$ at time t_0 . The top figure depicts measurement of $\hat{x}_A$ and $\hat{x}_B$, where x and p trajectories are not connected. The lower figure depicts measurement of $\hat{x}_A$ and $\hat{p}_B$, showing nonlocal loops. The local loops evident in Fig. 22 for the Bell state are absent for the EPR state $|\psi_{\text{EPR}}\rangle$.

conditional distribution $Q(p_A|x_A)$ (unlike that of a mixture of $|x_j^A\rangle |x_k^B\rangle$ and $|x_l^A\rangle |x_m^B\rangle$) does not factorize to give $Q(p_A|x_A) = Q(p_A)$. Hence, a local loop connecting variables x_A and p_A will be present.

Definition: Local loop. We define a loop as local, if it connects backward and forward-propagating variables for a single system (A or B).

Result VII.8: Local loops are associated with the Bell violation. We see from Result VII.5 (Comment) that the Bell nonlocality arises from hidden interference terms $\mathcal{Int}_{AB}$. These terms [as for $\mathcal{Int}$ in Eq. (3.8)] lead to a correlation between the local coordinates. We see that $Q(p_A|x_A) \neq Q(p_A)$ and $Q(p_B|x_B) \neq Q(p_B)$. There are hence local loops for each system A and B. The local loops are depicted in Fig. 23. Being a consequence of $\mathcal{Int}_{AB}$, they are present for the Bell nonlocal system.

Comment. This is consistent with the observation that a local loop does not exist for the CV EPR entangled state $|\psi_{\text{EPR}}\rangle$ given by Eq. (6.2) (Fig. 23), for which a local hidden variable theory (LHV) exists to explain measurements of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ [148].

For the two-mode case, it is possible to consider other correlations, i.e., we see that $Q(p_B|x_A) \neq Q(p_B)$ but also $Q(p_A|x_B) \neq Q(p_A)$ and $Q(x_A|x_B) \neq Q(x_A)$. The latter

correlations, between x_A and x_B (say), imply a correlation between $\hat{x}_A$ and $\hat{x}_B$ that can clearly also have a classical nature.

Definition: Nonlocal loop. We define a loop as nonlocal if it connects backward and forward-propagating variables for systems at separated sites.

Correlation. There are constraints that exist between the trajectories of different systems, due to the correlation between the two systems, as defined by $Q(\lambda, t_0)$. This occurs in the EPR case $|\psi_{\text{epr}}\rangle$ [Eq. (6.1)], when both $\hat{x}_A$ and $\hat{x}_B$ are measured, since $Q(x_A|x_B) \neq Q(x_A)$ and x_A and x_B are correlated (Fig. 23, top panel). These constraints ensure certain backward trajectories are correlated between the two systems in a given run, as in Fig. 20, but can create nonlocal loops, for example, in the EPR case where $\hat{x}_A$ and $\hat{p}_B$ are measured. This is depicted in Fig. 23 (lower panel) for $|\psi_{\text{epr}}\rangle$, where the existence of the perfect correlation between $\hat{x}_A$ and $\hat{x}_B$, and perfect anticorrelation between $\hat{p}_A$ and $\hat{p}_B$, allows for two nonlocal loops, and is also a signature of entanglement [149].

The above correlations and connections between the trajectories of the simulation are determined by the $Q(\lambda_{\text{rot}}, t_2)$, which are in turn determined by $Q(\lambda, t_0)$, at the initial time, since the transformations (7.13) and (7.14) that change the measurement settings are deterministic (refer to Appendix H). This is depicted in Figs. 22 and 23 by the vertical dashed lines at both times, t_1 and t_0 .

D. Hidden-variable (HV) models $\mathcal{P}_B$, $\mathcal{P}_{\text{det}B}$, and $\mathcal{P}_{\text{Bell}}$

In Sec. IV F, hidden-variable model $\mathcal{P}_{\text{sup}}$ was developed from the simulation involving $|\psi_{\text{sup}}\rangle$. A similar approach can be applied to the Bell simulation. In the proposed Bell test, the amplitudes $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ are measured, and the outcomes binned to be ± 1 or -1 , according to the sign S of the outcome. Binary outcomes $S_A = \pm 1$ and $S_B = \pm 1$ apply at each site, for given settings θ and ϕ .

Result VII.9a: Hidden variable model $\mathcal{P}_B$. Unlike the forward trajectories, the backward-propagating trajectories are continuous. Hence, for any given joint “spin” outcomes $S_A = \pm 1$ and $S_B = \pm 1$ at A and B , respectively, we can deduce a distribution $P(\lambda_{\text{rot}}|S_A, S_B)$ for the amplitudes $\lambda_{\text{rot}} = (x_{\theta A}, p_{\theta A}, x_{\phi B}, p_{\phi B})$ at time t_2 . Here, t_2 is the time by which both the settings θ and ϕ are fixed (Fig. 22). The distribution can be found by tracing the backward trajectories from t_f to the time t_2 . Using that for events A and B , the conditional probabilities satisfy $P(A|B) = P(B|A)P(A)/P(B)$, it follows (refer to Result IV.13, Sec. IV F) that for a given set of values for the variables $\lambda_{\text{rot}} = (x_{\theta A}, p_{\theta A}, x_{\phi B}, p_{\phi B})$, there exists a probability distribution

$$P_B(S_A, S_B|\lambda_{\text{rot}}) \quad (7.38)$$

for joint outcomes at A and B . Hence, a probabilistic HV model $\mathcal{P}_B$ exists, in which the system given by amplitudes λ_{rot} can be viewed as being in a state with a certain probability for outcomes S_A and S_B at A and B .

Proof. Following Sec. IV F, we identify A with outcomes $x_{\theta j}^A$, $x_{\phi k}^B$, and B with variables λ_{rot} . Following Sec. IV D, we define the distribution $Q_{\text{loop}}(\lambda_{\text{rot}}, t_2|x_{\theta j}^A, x_{\phi k}^B)$ for the variables λ_{rot} defined at time t_2 , given the amplitudes $x_{\theta A}(t)$, $x_{\phi B}(t)$ that are traced back from the branches $\mathcal{B}_{Aj}$ and $\mathcal{B}_{Bk}$ associated with outcomes $x_{\theta j}^A$ and $x_{\phi k}^B$. The probability distribution for

outcomes $x_{\theta j}^A$, $x_{\phi k}^B$ given λ_{rot} is

$$P_B(x_{\theta j}^A, x_{\phi k}^B|\lambda_{\text{rot}}) = \frac{Q_{\text{loop}}(\lambda_{\text{rot}}, t_2|x_{\theta j}^A, x_{\phi k}^B)P(x_{\theta j}^A, x_{\phi k}^B)}{Q(\lambda_{\text{rot}}, t_2)}.$$

Extending Eq. (4.14) in Sec. IV D to bipartite case, we write

$$Q_{\text{loop}}(\lambda_{\text{rot}}|x_{\theta j}^A, x_{\phi k}^B) = Q_{jk}(x_{\theta A}, x_{\phi B})Q(\lambda_{\text{rot}}|x_{\theta A}, x_{\phi B}),$$

where $Q_{jk}(x_{\theta A}, x_{\phi B})$ is the distribution at the time t_2 associated with the branches $\mathcal{B}_{Aj}$ and $\mathcal{B}_{Bk}$, and

$$Q(\lambda_{\text{rot}}|x_{\theta A}, x_{\phi B}) = Q(\lambda_{\text{rot}}, t_2)/Q_m(x_{\theta A}, x_{\phi B}, t_2)$$

is the conditional distribution [similar to $Q_0(p|x)$ in Eq. (4.12)], defined from the initial state $Q(\lambda_{\text{rot}}, t_2)$. Here, we define the marginal $Q_m(x_{\theta A}, x_{\phi B}, t_2) = \int dp_{\theta A} dp_{\phi B} Q(\lambda_{\text{rot}}, t_2)$, which (extending Result III.1) is the summation of the $P(x_{\theta l}^A, x_{\phi m}^B)Q_{lm}(x_{\theta A}, x_{\phi B})$ over all l and m . Hence,

$$P_B(x_{\theta j}^A, x_{\phi k}^B|\lambda_{\text{rot}}) = \frac{Q_{jk}(x_{\theta A}, x_{\phi B})P(x_{\theta j}^A, x_{\phi k}^B)}{Q_m(x_{\theta A}, x_{\phi B}, t_2)}. \quad (7.39)$$

Following Sec. IV F, we find $Q_{jk}(x_{\theta A}, x_{\phi B}) = \frac{e^{-(x_{\theta A} - x_{\theta j}^A)^2/2}}{\sqrt{2\pi}} \frac{e^{-(x_{\phi B} - x_{\phi k}^B)^2/2}}{\sqrt{2\pi}}$. The probability of outcomes S_A and S_B given the set of hidden variables λ_{rot} can be found by summing the $P_B(x_{\theta j}^A, x_{\phi k}^B|\lambda_{\text{rot}})$ where outcomes $x_{\theta j}^A$, $x_{\phi k}^B$ have the appropriate sign. ■

Comment. In Sec. IV F (Result IV.13b), the deterministic model $\mathcal{P}_{\text{det}}$ was also put forward, based on the postselected state. A similar model $\mathcal{P}_{\text{det}B}$ can be constructed in the bipartite case, to describe the predetermination of the outcomes $x_{\theta j}^A$ and $x_{\phi k}^B$ of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ as given by the system defined at the time t_2 in Fig. 22, which is the time by which the measurement settings have been fixed. The postselected states are derived in Appendix H.

Result VII.9b: Hidden variable model $\mathcal{P}_{\text{Bell}}$. A second model $\mathcal{P}_{\text{Bell}}$ traces back from branches at time t_f to the time t_0 , prior to the unitary operations that determine the setting choices θ and ϕ . This is possible because the relations (7.13) and (7.14) are deterministic, allowing an algebraic transformation of variables (Appendix H). Hence, we define the HV model

$$P_{\text{Bell}}(S_A, S_B|\lambda, \theta, \phi), \quad (7.40)$$

where $\lambda = (x_A, p_A, x_B, p_B)$ are the variables defined for the state of the system at the time t_0 , prior to the interactions that determine θ and ϕ . Hence, a probabilistic HV model $\mathcal{P}_{\text{Bell}}$ exists, in which the system given by amplitudes $\lambda = (x_A, p_A, x_B, p_B)$ can be viewed as being in a state with a certain probability for outcomes S_A and S_B at A and B .

Proof. We define

$$P(x_{\theta j}^A, x_{\phi k}^B|\lambda) = \frac{Q_{\text{loop}}(\lambda, t_0|x_{\theta j}^A, x_{\phi k}^B)P(x_{\theta j}^A, x_{\phi k}^B)}{Q(\lambda, t_0)} \quad (7.41)$$

and evaluate the transformed probability density function from $Q_{\text{loop}}(\lambda_{\text{rot}}, t_2|x_{\theta j}^A, x_{\phi k}^B)$. ■

Result VII.9c: Asymmetric local hidden variable model $\mathcal{P}_{\text{asym}, A}$. A third HV model is deduced from Result VII.3. Suppose we fix the setting at system A but allow a change of setting ϕ at B . We consider measurements $\hat{x}_A$ and $\hat{x}_{\phi B}$. We see

from Result VII.3 that an LHV model exists based on $\rho_{\text{mix},B}$ to correctly describe the outcomes of $\hat{x}_A$ and $\hat{x}_{\phi B}$. Details are in Appendix F.

E. Weak macroscopic realism and weak local realism for bipartite systems

We now extend the premises wMR(1) and wLR(1) of weak macroscopic realism (wMR) and weak local realism (wLR), given by Definitions (6) and (10) in Sec. IB, and by Results IV.6 and IV.14, so that they apply to bipartite systems. Consider the system of Fig. 22 prepared generally at time t_2 for measurement of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$, so that an independent amplification H_{amp} for each system will finalize the measurement. The possible outcomes at A are denoted by the set $\{x_{\theta j}^A\}$. The possible outcomes at B are denoted by the set $\{x_{\phi k}^B\}$.

Result VII.10: Weak macroscopic realism: Premise wMR(1). The simulation (and Q model) of the bipartite system is consistent with the premise wMR(1) of wMR [Definition (6a) in Sec. IB], which posits that *both* the values x_j^A and x_k^B for the outcomes at A and B are determined at a time t_m , after the settings are fixed, and after sufficient amplification, H_{amp}^A and H_{amp}^B .

Proof. Where there is a joint measurement as in the amplifications H_{amp}^A and H_{amp}^B , distinct branches emerge at a time t_m (as in Fig. 20, at time $t_m = 2/g$ where $G = e^{|g|t_f} \rightarrow \infty$). In the Q model, the measured outcomes (if detection takes place) are given by $\tilde{x}_{\theta A} = x_{\theta A}(t)/G$ and $\tilde{x}_{\phi B} = x_{\phi B}(t)/G$ (Result IV.6 in Sec. IV C). We symbolize the predetermined values by variables $\tilde{\lambda}_{\theta j}^A$ and $\tilde{\lambda}_{\phi k}^B$. ■

Result VII.11: Weak local realism: Premise wLR(1). At the time t_2 , the system of Fig. 22 has undergone the unitary transformations U_{θ}^A and U_{ϕ}^B that fix the settings to be θ and ϕ . There are two versions of wLR(1), *deterministic* and *probabilistic*, defined in Definition (10a) (refer also to Result IV.14, Sec. IV G). The bipartite simulation is consistent with the premise wMR(1) of both versions of wLR, which posit that *both* the outcomes at A and B are determined (at least with a certain probability) at time t_2 .

Proof. The deterministic version of wLR posits consistency with a hidden variable (HV) model in which the outcomes are determined at the time t_2 . In the Q model as applied to Fig. 22, the means $x_{\theta j}^A$ and $x_{\phi k}^B$ of the Gaussian functions in $Q(\lambda, t_2)$ are amplified to $Gx_{\theta j}^A$ and $Gx_{\phi k}^B$, and the correlation between the outcomes is hence determined by $Q(\lambda_{\text{rot}}, t_2)$. The interference terms Int_{AB} in $Q(\lambda, t_2)$ are not amplified (Result VII.2, Sec. VIIB). Hence, the prediction for joint probabilities of outcomes $\hat{x}_A$ and $\hat{x}_B$ is the same as for the *mixed* state, ρ_D defined by Eq. (7.22), for which the outcomes $x_{\theta j}^A$ and $x_{\phi k}^B$ can be viewed as determined for an individual realization of the system. Hence, a deterministic HV model consistent with wLR(1) exists. The model $\mathcal{P}_{\text{det}B}$ outlined above in Sec. VIID provides a more detailed deterministic HV model consistent with wLR(1), and also accounts for the interference terms in the Q function.

Alternatively, for the probabilistic version of wLR, a hidden variable (HV) model $\mathcal{P}_B$ [Eq. (7.38)] can be developed from the simulation. At time t_2 , we consider the system to be in an HV state $\lambda_{\text{rot}} = (x_{\theta A}, p_{\theta A}, x_{\phi B}, p_{\phi B})$ with probability

$Q(\lambda_{\text{rot}}, t_2)$. It is possible to construct a joint probability for the outcomes $x_{\theta j}^A$ and $x_{\phi k}^B$ (and hence of $\mathcal{S}_A$ and $\mathcal{S}_B$) at the sites, for any given values of λ_{rot} [Result VII.9a, Eq. (7.39)]. Hence, a probabilistic HV model satisfying the probabilistic version of wLR(1) exists. ■

Result VII.12: Deterministic relations $\mathcal{D}$ for correlated outcomes. The results specify the outcomes $x_{\theta j}^A$ and $x_{\phi k}^B$ to be determined at the time t_m , or else, for Result VII.11 (at least probabilistically) at time t_2 . Hence, the results imply certain deterministic relations $\mathcal{D}$ that specify the correlations between the outcomes $x_{\theta j}^A$ and $x_{\phi k}^B$, *at these times*. For Result VII.1, the deterministic relations $\mathcal{D}$ will apply to the system at time t_2 , because the settings at A and B are fixed at θ and ϕ at this time. The system has been prepared by the appropriate unitary operations with respect to the measurement basis $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$. Specifically, examining state $|x_j^A\rangle|x_k^B\rangle + |x_l^A\rangle|x_m^B\rangle$ of Eq. (7.27), the Q function (7.28) shows that the outcomes x_j^A and x_k^B are correlated, as are those of x_l^A and x_m^B .

Result VII.13: Deterministic relations $\mathcal{D}$ for correlated outcomes after a single setting change. Suppose we fix the setting at system A but allow a change of setting ϕ at B. We consider measurements $\hat{x}_A$ and $\hat{x}_{\phi B}$. We see from Result VII.3 that a local hidden variable (LHV) model $\mathcal{P}_{\text{asym},B}$ exists based on $\rho_{\text{mix},B}$ to correctly describe the outcomes of $\hat{x}_A$ and $\hat{x}_{\phi B}$ (Result VII.9c and Appendix F). We hence deduce a set $\mathcal{D}$ of deterministic relations implied by $\mathcal{P}_{\text{asym},B}$. In the weak local realistic model (which is implemented by the Q -model of reality), the values for the outcomes for $\hat{x}_A$ and $\hat{x}_{\phi B}$ are determined (at least probabilistically) as x_j^A and x_k^B , at a time t_1 once the settings are fixed at $\theta = 0$ and ϕ . The purpose of the relations $\mathcal{D}$ is to give the correlation between the (hidden) variables, x_j^A and x_k^B .

Proof. The $Q(\lambda, t_0)$ denoting an initial state $|\psi(t_0)\rangle$ prepared in the measurement basis with $\theta = \phi = 0$ is transformed to $Q(\lambda_{\text{rot}B}, t_{1B})$ at the time t_{1B} , after the setting change ϕ at B, where $\lambda_{\text{rot}B} = (x_A, p_A, x_{\phi B}, p_{\phi B})$. The $Q(\lambda_{\text{rot}B}, t_{1B})$ can be written in the new measurement basis (Appendix E) using (7.14). We can reexpress in the form (7.24): a sum of Gaussian functions and interference terms $\text{Int}_{AB}(\phi)$. This is done by noting that the initial quantum state $|\psi(t_0)\rangle = \sum_{i,l} c_{il} |x_i^A\rangle |x_l^B\rangle$ is determined by the c_{il} . The $|\psi(t_0)\rangle$ is defined uniquely by $Q(\lambda, t_0)$ and can in principle be determined by $Q(\lambda, t_0)$ (Appendix C). Hence, we can write in the new basis, using $|x_j^B\rangle = \sum_k d_{lk} |x_{\phi k}^B\rangle$ where d_{lk} are transformation coefficients given in Appendix E, as functions of ϕ . Hence $|\psi(t_{1B})\rangle = \sum_{j,k} e_{jk} |x_j^A\rangle |x_{\phi k}^B\rangle$, where $e_{jk} = \sum_l c_{jl} d_{lk}$ can be determined solely from the c_{il} and ϕ , which defines $Q(\lambda_{\text{rot}B}, t_{1B})$ in the form (7.32).

This also defines the joint probabilities of outcomes x_j^A and $x_{\phi k}^B$, as $|e_{jk}|^2$. The deterministic relations $\mathcal{D}_1$ that determine the correlations between outcomes x_j^A and $x_{\phi k}^B$ can hence be established from knowledge of the initial state $Q(\lambda, t_0)$ and ϕ at B. These are $x_j^A \leftrightarrow x_{\phi k}^B$ with probability $|e_{jk}|^2$. ■

Result VII.13b: Compatibility between wMR and wLR, and the deterministic relations. It is remarked that weak macroscopic realism (wMR) is compatible with weak local realism (wLR), and vice versa. In particular, the Q model of reality supports both wMR and wLR, as evident from the simu-

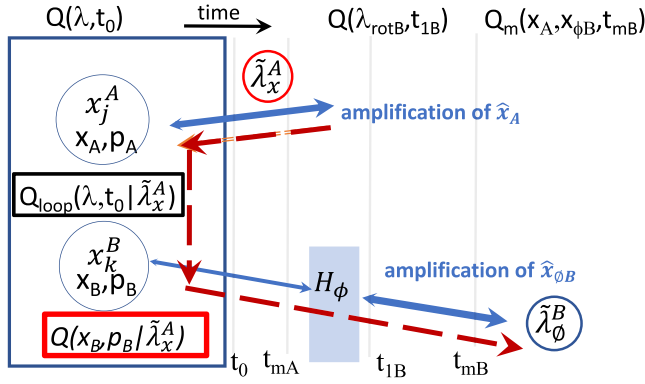


FIG. 24. Projection in the Q model. The system is prepared at time t_0 with respect to the measurement basis of $\hat{x}_A$ and $\hat{x}_B$, so that $\theta = \phi = 0$. The field A is amplified (at time t_{mA}) with the setting fixed at $\theta = 0$. The trajectories $x_A(t)$ conditioned on the branch $\tilde{\lambda}_x^A$ corresponding to an outcome x_j^A trace back to time t_0 (orange dashed and solid blue line), where they correlate with certain amplitudes, to determine the postselected distribution $Q_{\text{loop}}(\lambda, t_0 | \tilde{\lambda}_x^A) \equiv Q_{\text{loop}}(\lambda, t_0 | x_j^A)$ (accounting for the deterministic relations $\mathcal{D}$ that constrain correlations with outcomes x_k^B). The distribution $Q(x_B, p_B | \tilde{\lambda}_x^A) \equiv Q(x_B, p_B | x_j^A)$ for system B alone [Eq. (8.2)] corresponds to the *projected state* of B given the outcome x_j^A at A . The projected state is determined at (or by) the time t_{mA} , which can be prior to a setting change H_ϕ and final detection at B . The projected state is consistent with the marginal $Q_m(x_A, x_{\phi B}, t_{mB})$, which gives the joint probability density of amplitudes $x_A(t_{mB})$ and $x_{\phi B}(t_{mB})$.

lations, which show consistency with wLR as well as the emergence of branches. Hence, the causal model and the above deterministic relations $\mathcal{D}$ are compatible with wMR.

VIII. MEASUREMENT WITH A METER: STATE PROJECTION

For the EPR system (6.1), it is well known that the Wigner function provides a local hidden variable (LHV) theory which explains the correlation between spacelike-separated systems A and B [148]. However, the LHV model cannot explain other correlated systems, such as $|\psi_{\text{Bell}}\rangle$ of Eq. (7.6).

In this section, we use the Q -based simulation to analyze measurement by a meter, hence analyzing *projection*, where a measurement on one of two entangled systems “collapses” the state of the second system. We show consistency with the premises wMR(3) and wLR(3) of weak local and macroscopic realism [Definitions (6c) and (10c) in Sec. IB] which leads to conclusions about the mechanism and the timing of the collapse of the wave function.

A. Model for projection

Figure 24 shows a two-mode system prepared at time t_0 for measurements $\hat{x}_A$ and $\hat{x}_B$ (where $\theta = \phi = 0$). The system is described by a two-mode Q function $Q(\lambda, t_0)$, where $\lambda = (x_A, p_A, x_B, p_B)$. We use the symbols λ and λ interchangeably to represent the phase-space coordinates, which can also symbolize hidden variables in the Q -based model. The system might be in an EPR state such as given by Eq. (6.1), or in a two-mode cat state similar to $|\psi_{\text{Bell}}\rangle$ of Eq. (7.6). We consider

quadrature phase amplitude measurements $\hat{x}_{\theta A} = \hat{x}_A \cos \theta + \hat{p}_A \sin \theta$ and $\hat{x}_{\phi B} = \hat{x}_B \cos \phi + \hat{p}_B \sin \phi$.

If there is a change of setting ϕ at B , then there are *constraints imposed* on system B by the configuration at A . This gives a model for *projection*.

In Fig. 24, the setting at A has been fixed at $\theta = 0$, ready for a measurement of $\hat{x}_A$ (Fig. 24). Let us suppose that system A at the time t_{mA} has been amplified according to H_{amp}^A [Eq. (6.3)]. The *bipartite postselected distribution* at the time t_0 conditioned on a branch symbolized by $\tilde{\lambda}_x^A$ with outcome x_j^A for $\hat{x}_A$ at A is denoted by [refer Eq. (4.13) and Definition (8) in Sec. IB]

$$Q_{\text{loop}}(\lambda, t_0 | x_j^A). \quad (8.1)$$

The bipartite distribution $Q_{\text{loop}}(\lambda, t_0 | x_j^A)$ originates from the backward-propagating trajectories that trace back to the initial time t_0 , from a given branch x_j^A for A . The initial state $Q(\lambda, t_0)$ implies a restriction (red dashed lines) on the state of system B , as depicted in Fig. 24.

The Q_{loop} is an explicit function of $\lambda = (x_A, p_A, x_B, p_B)$ but is derived within the framework of the Q model of reality, and Results VII.10–VII.12. Hence, its derivation must take into account the possible outcomes x_k^B at B , as given by the deterministic relations $\mathcal{D}$ that specify the correlations between the outcomes at A and B for the specified measurement basis. The bipartite postselected distribution *must be consistent with any known deterministic relations* $\mathcal{D}$ that determine the correlation between any of the possible outcomes x_k^B at B , and x_j^A at A (Results VII.10–VII.12).

The state for system B alone conditioned on the branch x_j^A of A is

$$Q(x_B, p_B | x_j^A) = \int dx_A dp_A Q_{\text{loop}}(\lambda, t_0 | x_j^A), \quad (8.2)$$

as calculated by integrating $Q_{\text{loop}}(\lambda, t_0 | x_j^A)$ over x_A and p_A . The distribution (8.2) constrains the outcomes for B , including for any future changes of settings at B . Figure 24 depicts a future change of setting at B to ϕ , as induced by the Hamiltonian H_ϕ .

Result VIII.1a: Retrodiction not retrocausality. The distribution function $Q(x_B, p_B | x_j^A)$ for the projected state for system A is derived based on knowledge of the setting and outcome x_j^A at A and of the initial distribution $Q(\lambda, t_0)$. No knowledge is required about the setting ϕ at B . Hence, we claim the distribution is deduced by *retrodiction* (not *retrocausality*), based on the relation $x_j^A \rightarrow Gx_j^A$ [Eq. (5.2)], and the deterministic relations $\mathcal{D}$ that specify the correlations between outcomes x_j^A and x_k^B (assuming weak local or macroscopic realism, as in Results VII.10–VII.12). Here, we assume the setting is fixed at A .

Proof. The proof follows since $Q(x_B, p_B | x_j^A)$ is derived directly from $Q_{\text{loop}}(\lambda, t_0 | x_j^A)$. The method of derivation of $Q_{\text{loop}}(\lambda, t_0 | x_j^A)$ involves tracking the trajectories $x_A(t)$ for the branch x_j^A back to the original time t_0 , but the nature of the backward simulation means the distribution $Q_{\text{loop}}(\lambda, t_0 | x_j^A)$ can be deduced by retrodiction, by reversing the deterministic relation $x_j^A \rightarrow Gx_j^A$ and similar deterministic relations $\mathcal{D}$ between x_j^A and x_k^B (refer below). This is because *there is no change to the properties of the noise* $\delta x_A \equiv \eta_A(t_f)$, which is an *input at the future boundary*, as the system propagates in the

backward-time direction, from time t_f (which determines the future boundary) to time t_0 (refer to Secs. III B and V A). ■

The projected distribution at B , however, gives more information than just the outcome for the current measurement $\hat{x}_B$ at B , given an outcome at A . This is determined by the relations $\mathcal{D}$ as fixed by the current setting at B (Result VII.12). The projected distribution gives information about the *state* for B , and hence information about future measurements at B after further setting-changes. This is because the projected distribution is derived not only from x_j^A , but from the Q function $Q(\lambda, t_0)$, which includes noise and interference terms. We clarify with the following result.

Result VIII.1b: Future settings at B . The distribution $Q(x_B, p_B | x_j^A)$ determines the probabilities of outcomes at B , conditioned on the outcome x_j^A of $\hat{x}_A$ at A , for any future choice of setting ϕ . The predictions are consistent with the correlations between outcomes at A and B as predicted by the Q function $Q(x_A, x_{\phi B}, t_{mB})$ of the system at time t_{mB} , after the setting-change and amplification at B (refer Fig. 24).

Proof. The proof is given in Appendix H (Result AH.3) and follows because the change of setting at B is a local transformation (D3) corresponding mathematically to changing coordinates in the distribution $Q(x_B, p_B | x_j^A)$. ■

Hence, as expected, the distribution $Q(x_B, p_B | x_j^A)$ will correspond to the projected state, given quantum mechanically. This models the “collapse to the projected state” for bipartite systems.

Result VIII.1c: Projected state. The probability distribution $Q(x_B, p_B | x_j^A)$ corresponds to the Q function of the projected state for B in quantum mechanics, given the outcome x_j^A for system A .

Proof. Consider the general state $\sum_{jk} c_{jk} |x_j^A\rangle |x_k^B\rangle$ as in Eq. (7.20). We consider that the systems A and B are prepared with respect to the measurement bases of $\hat{x}_A$ and $\hat{x}_B$. We first take the simplest case of

$$|x_j^A\rangle |x_k^B\rangle + |x_l^A\rangle |x_m^B\rangle \quad (8.3)$$

as in (7.27) where $j \neq l, k \neq m$, for which according to quantum mechanics the projected state at B given the outcome x_j^A at A is $|x_k^B\rangle$. (This follows because the eigenstates are orthogonal, proved in Appendix C). Applying Results VII.11 and 12, the correlations between the outcomes at A and B define a set of deterministic relations $\mathcal{D}$ in the Q model, in this case

$$x_j^A \leftrightarrow x_k^B, \quad x_l^A \leftrightarrow x_m^B. \quad (8.4)$$

We consider the outcome x_j^A . Following the procedure of Sec. IV D and Appendix H, we trace back the branch $\mathcal{B}_j^A$, to obtain the distribution for x_A at time t_0 , as $Q_{x_j^A}(x_A) = \frac{e^{-(x_A - x_j^A)^2/2}}{\sqrt{2\pi}}$. This gives for the inferred distribution at t_0 ,

$$\begin{aligned} Q_{\text{loop}}(\lambda, t_0 | x_j^A) &= Q_{x_j^A}(x_A) Q(p_A, x_B, p_B | x_A) \\ &= \frac{Q_{x_j^A}(x_A) Q(\lambda, t_0)}{Q_m(x_A, t_0)}, \end{aligned} \quad (8.5)$$

where $Q(\lambda, t_0)$ is the Q function of the system at time t_0 [Eq. (7.28)] and $Q_m(x_A, t_0) = \int dp_A dx_B dp_B Q(\lambda, t_0)$ denotes

the marginal distribution for x_A . However, a direct evaluation does not fully define the projected state, due to the overlap of the distributions of the eigenfunctions $|x_j^A\rangle$ and $|x_l^A\rangle$ for small separations $|x_j^A - x_l^A| \lesssim 1$. The correct procedure also takes into account the deterministic relation $\mathcal{D}$ that defines the correlation that x_j^A implies x_k^B for B , which is embedded in the causal model, based on the fixed settings at t_0 . This relation rules out the bivariate Gaussian in $Q(\lambda, t_0)$ with means x_j^A and x_m^B . The interference term $\mathcal{Int}_{AB}$ as given by Eq. (7.29) is shared between the two states of the superposition (8.3). This term is a function of p_A and cannot be inferred from the outcome being x_j^A or x_l^A : this term is not ruled out. However, the projected state for B is deduced by integrating over p_A . We see from Result VII.5 that the integration over p_A leads to a vanishing of the term involving $\mathcal{Int}_{AB}$. The projected state in the Q model is hence

$$Q_{x_j^A}(x_B, p_B) = \mathcal{G}_{0, \sigma_{p_B}}(p_B) \mathcal{G}_{x_k^B, 1}(x_B), \quad (8.6)$$

corresponding to the Q function of state $|x_k^B\rangle$ as required. Here $\mathcal{G}$ are the Gaussian functions defined by Eqs. (7.28) and (7.25).

Generalizing, the projected state of $|\psi\rangle = \sum_{jk} c_{jk} |x_j^A\rangle |x_k^B\rangle$ given x_j^A is found by rewriting $|\psi\rangle = \sum_j f_j |x_j^A\rangle |\psi_j^B\rangle$ where $|\psi_j^B\rangle = \sum_k d_{jk} |x_k^B\rangle$ is normalized as in Eq. (7.26). The projected state is $|\psi_j^B\rangle$. Hence, we consider a superposition of type

$$|x_j^A\rangle (d_{j1} |x_{k1}^B\rangle + d_{j2} |x_{k2}^B\rangle + \dots) + |x_l^A\rangle |x_m^B\rangle. \quad (8.7)$$

The Q function $Q(\lambda, t_0)$ is of the form

$$\begin{aligned} Q(\lambda, t_0) &= N(\mathcal{G}_{0, \sigma_{p_A}}(p_A) \mathcal{G}_{x_j^A, 1}(x_A) Q_{Bj}(x_B, p_B) \\ &\quad + \mathcal{G}_{0, \sigma_{p_A}}(p_A) \mathcal{G}_{x_l^A, 1}(x_A) \mathcal{G}_{0, \sigma_{p_B}}(p_B) \mathcal{G}_{x_m^B, 1}(x_B) \\ &\quad + \mathcal{Int}_{AB}), \end{aligned} \quad (8.8)$$

where $Q_{Bj}(x_B, p_B)$ is the Q function of the superposition $d_{j1} |x_{k1}^B\rangle + d_{j2} |x_{k2}^B\rangle + \dots$, so that

$$\begin{aligned} Q_{Bj}(x_B, p_B) &= \mathcal{G}_{0, \sigma_{p_B}}(p_B) \{ |d_{j1}|^2 \mathcal{G}_{x_{k1}^B, 1}(x_B, p_B) \\ &\quad + |d_{j2}|^2 \mathcal{G}_{x_{k2}^B, 1}(x_B, p_B) + \dots + \mathcal{I}_B \}, \end{aligned} \quad (8.9)$$

where here the interference term $\mathcal{I}_B$ is a sum of terms such as $\mathcal{Int}$ given for a single-mode two-state superposition, in Eq. (3.8). The projected state conditioned on x_j^A rules out the second Gaussian in (8.8), and the bipartite interference $\mathcal{Int}_{AB}$ vanishes over the integration with respect to p_A . Integrating over the coordinates of A , the projected distribution given the outcome x_j^A at A is hence

$$Q_{x_j^A}(x_B, p_B) = Q_{Bj}(x_B, p_B), \quad (8.10)$$

in agreement with the Q function of the projected state $|\psi_j^B\rangle$. The projected distribution $Q_{x_j^A}(x_B, p_B)$ describes the projected state at time t_0 . The predictions for the outcomes if there is a change of setting is given by the transformation (D3) at B , which corresponds to the change of coordinates in $Q_{x_j^A}(x_B, p_B)$ (refer to Appendix H). ■

The projection models the collapse to the projected state for bipartite systems. What does this tell us about the process of the collapse of the wave function in the Q model of reality?

Result VIII.1d: Wave-function collapse. The conditioned state (8.2) for B alone has reduced information. For example, fringe terms involving p_A vanish in $Q(x_B, p_B|x_j^A)$ (refer above). Hence, in the Q -based simulation and model, the final collapse to the projected state occurs when an inference about the state of the system B is made, deduced from the detected outcome x_j^A at A . There is a loss of information about the complementary variable p_A of system A . The projected state does not fully reflect the state of the system B prior to the measurement at A (Fig. 13).

B. Measurement by a meter

We illustrate projection as applied to the measurement problem with the example of the entangled Schrödinger-cat state

$$|\psi_{\text{ent}}\rangle_M = \frac{1}{\sqrt{2}}\{|\alpha_0\rangle|x_1\rangle + i|-\alpha_0\rangle|-x_1\rangle\}. \quad (8.11)$$

Here, $|\pm\alpha_0\rangle$ are macroscopic coherent states for the field mode A with $\alpha_0 > 0$ and large. $|\pm x_1\rangle$ are eigenstates of $\hat{x}_B$ with eigenvalues $\pm x_1$. As in Part 1 of the paper, we model the eigenstates $|x_1\rangle$ as highly squeezed states in $\hat{x}_B$. The state (8.11) becomes

$$|\psi_{\text{ent}}\rangle_M = \frac{1}{\sqrt{2}}\left\{|\alpha_0\rangle\left|\frac{x_1}{2}, r\right\rangle_{sq} + i|-\alpha_0\rangle\left|-\frac{x_1}{2}, r\right\rangle_{sq}\right\}, \quad (8.12)$$

where $|\frac{x_1}{2}, r\rangle_{sq}$ is the squeezed state for mode B defined by Eqs. (3.1)–(3.3). For r large, the squeezed state $|\frac{x_1}{2}, r\rangle_{sq}$ becomes the eigenstate $|x_1\rangle$ of $\hat{x}_B$.

When α_0 is large, the entangled state (8.12) models the state created when a *macroscopic meter* (field A) has coupled to a system B . The outcome of a measurement $\hat{x}_A$ on system A gives (for α_0 real and large) the outcome of the sign of $\hat{x}_B$ for system B . According to the measurement postulate, if the outcome for $\hat{x}_A$ at A is positive (+) then, for this superposition, the system B collapses to the eigenstate $|x_1\rangle$ (for r large).

We evaluate numerically from the simulation the distribution for system B conditioned on the branch corresponding to the positive outcome (where $x_A(t_f) > 0$, denoted by the symbol +) for $\hat{x}_A$ of system A i.e., $Q(x_B, p_B|+)$. Here, because the system A is macroscopic, given by macroscopically distinct states $|\alpha_0\rangle$ and $|-\alpha_0\rangle$ that become orthogonal as $\alpha_0 \rightarrow \infty$, the projected distribution $Q(x_B, p_B|+)$ for B given the outcome + at A is deduced directly by integration of $Q_{\text{loop}}(x_B, p_B, p_A|+)$ over p_A , without the need for direct implementation of the deterministic relations $\mathcal{D}$. The function $Q(x_B, p_B|+)$ is plotted in Fig. 25, verifying equivalence to the Q function of the squeezed state (modeling the eigenstate $|x_1\rangle$), as expected from the meter-system state (8.11) and consistent with the measurement postulates of quantum mechanics.

C. Projection after a change of setting

The projection procedure for one system, say B , conditioned on an outcome at the second system A , can be applied even when there has been a change of setting at the system A after the time t_0 at which the bipartite states are prepared. We

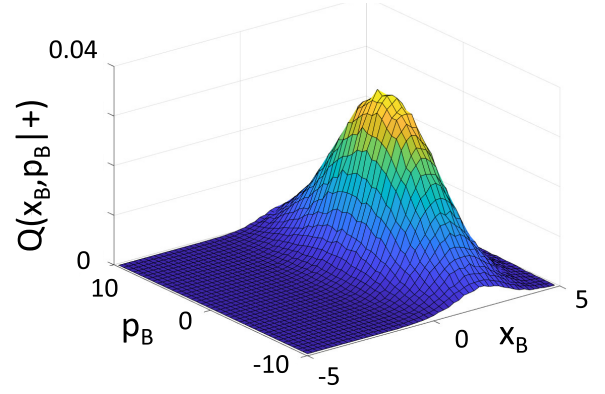


FIG. 25. Projected distribution and collapse of the wave function. The projected state for system B entangled with a meter A , as in Eq. (8.12). We plot $Q(x_B, p_B|x_1) \equiv Q(x_B, p_B|+)$ for B conditioned on the positive outcome x_1 of $\hat{x}_A$ on the meter A i.e., we trace back trajectories with $x_A(t) > 0$ at the time t_f , where t_f is large (refer to Fig. 24). Here, we take $\alpha_0 = 2, x_1 = 2$ with $r = 1.5$ and 1.2×10^6 trajectories. The Q function $Q(x_B, p_B|+)$ corresponds to that of the squeezed state $|x_1/2, r\rangle$, which approximates the eigenstate $|x_1\rangle$, modeling the collapse of system B to eigenstate $|x_1\rangle$ after measurement.

examine this case, in order to understand the violation of Bell inequalities in the next section.

Result VIII.1e: Projected state at B conditioned on A after a change of setting at A . A projected state and distribution $Q(x_B, p_B|x_{\theta_j}^A)$ can be defined for the system B at a time t_{mA} , given an outcome $x_{\theta_j}^A$ for a measurement $\hat{x}_{\theta}^A$ (Fig. 26). Here, we are allowing that there has been a change of setting to θ at A after the initial preparation time t_0 . The time t_{mA} is the time after sufficient amplification at A , so that the outcome for $\hat{x}_{\theta}^A$

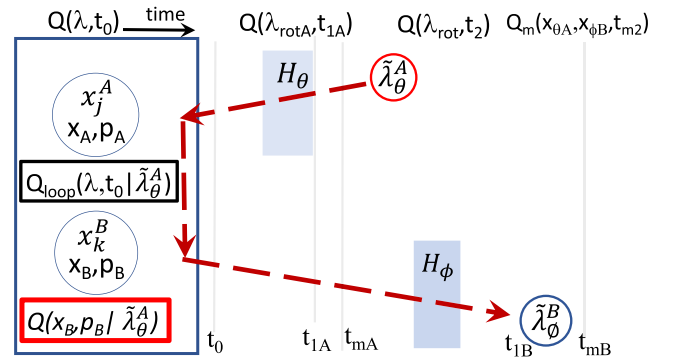


FIG. 26. Depiction of projection where there has been a setting change θ at A . The projected state $Q(x_B, p_B|\tilde{\lambda}_{\theta}^A)$ is defined to describe the system at time t_0 , based on the setting change θ at A and the Q function $Q(\lambda, t_0)$. The distribution $Q(x_B, p_B|\tilde{\lambda}_{\theta}^A)$ defines predictions for the system B for further changes of setting ϕ that might take place, or have taken place, after the time t_0 . The derivation is explained in Result VIII.1e and depicted in Fig. 27, involving the derivation of $Q_{\text{loop}}(\lambda_{\text{rotA}}, t_{1A}|x_{\theta_j}^A)$, followed by a change of coordinates, equivalent to tracing trajectories back through the interaction due to the setting-change H_{θ} (red dashed line). The projected state $Q(x_B, p_B|\tilde{\lambda}_{\theta}^A)$ is consistent with the marginal distribution $Q_m(x_{\theta_j}^A, x_{\theta_j}^B, t_{mB})$ that defines the systems after both setting changes.

is defined according to the assumptions of weak macroscopic realism (wMR).

Proof. We allow for a setting change θ at A , but consider system B prior to any setting change at B . The $Q(\lambda, t_0)$ is transformed to $Q(\lambda_{\text{rot}A}, t_{1A})$ at the time t_{1A} , after the setting change at A , where $\lambda_{\text{rot}A} = (x_{\theta A}, p_{\theta A}, x_B, p_B)$. The $Q(\lambda_{\text{rot}A}, t_{1A})$ can be written in the new measurement basis as explained in Result VII.13 (exchanging the roles of A and B). From Result VII.13, the deterministic relations $\mathcal{D}_1$ that determine the correlations between outcomes $x_{\theta j}^A$ and x_k^B can hence be established from knowledge of the initial state $Q(\lambda, t_0)$ and θ at A . This is justified because the local hidden variable model $\mathcal{P}_{\text{asym},B}$ given by the partially mixed state, in this case denoted $\rho_{\text{mix},B}$, as in Result VII.3 and Appendix F, applies to correctly describe the outcomes of $\hat{x}_{\theta A}$ and $\hat{x}_B$.

Applying the technique given in proof of Result VIII.1c, the inferred distribution $Q_{\text{loop}}(\lambda_{\text{rot}A}, t_{1A} | x_{\theta j}^A)$ conditioned on the branch $\mathcal{B}_{\theta j}$ with outcome $x_{\theta j}^A$ will comprise the relevant Gaussian functions that are consistent with deterministic relations $\mathcal{D}_1$, as well as relevant interference terms $\mathcal{I}nt_{AB}$ (which will vanish over integration of $p_{\theta A}$). This allows evaluation of the projected state $Q(x_B, p_B | x_{\theta j}^A)$. The outcomes $x_{\theta j}^A$ and x_k^B may not be maximally correlated, and the projected state can be a superposition of eigenstates of $\hat{x}_B$, as in Result VIII.1c.

There is consistency with the projected state that can be evaluated by tracking the trajectories of the branch $\mathcal{B}_{\theta j}$ back to the initial time t_0 (Fig. 27). To obtain the inferred distribution $Q_{\text{loop}}(\lambda, t_0 | x_{\theta j}^A)$ at time t_0 , the amplitudes $x_{\theta A}(t_1)$, $p_{\theta A}(t_1)$ of $\mathcal{B}_{\theta j}$ are transformed according to (7.15) to $x_A(t)$ and $p_A(t)$, corresponding to a change of setting, but which is equivalent to changing coordinates in $Q_{\text{loop}}(\lambda_{\text{rot}A}, t_{1A} | x_{\theta j}^A)$. Mathematically, the distribution for B conditioned on $x_{\theta j}^A$ does not change under a coordinate transformation, and hence the projected distribution is $Q(x_B, p_B | x_{\theta j}^A)$ (refer to Appendix H). ■

We illustrate the proof below by calculating the projected state at B where the initial state of the system at time t_0 is the entangled state $|x_i^A\rangle |x_k^B\rangle + |x_l^A\rangle |x_m^B\rangle$ given by Eq. (8.3). This leads to the following comment:

Comment (1). Entangled states give projected states that are superpositions. Where there is entanglement, the projected state at B given an outcome at A is a superposition, at least for one of the outcomes and for particular choices of setting θ . For an entangled state, this is also true for the projected state at A given an outcome at B . The result would not hold for a nonentangled state, i.e., consider $|x_i^A\rangle (|x_k^B\rangle + |x_m^B\rangle)$ for which the projected state at B given an outcome $x_{\theta j}^A$ is $|x_k^B\rangle + |x_m^B\rangle$ but the projected state at A given an outcome $x_{\phi q}^B$ after a setting change ϕ at B is $|x_i^A\rangle$.

We demonstrate by considering the initial state $|\psi\rangle$ at time t_0 to be $|x_i^A\rangle |x_k^B\rangle + |x_l^A\rangle |x_m^B\rangle$. Changing the basis at A to θ gives $|x_i^A\rangle = \sum_r d_{ir} |x_{\theta r}^A\rangle$, $|x_l^A\rangle = \sum_r d_{lr} |x_{\theta r}^A\rangle$ so that the expression for the state becomes

$$|\psi\rangle \rightarrow \sum_r |x_{\theta r}^A\rangle (d_{ir} |x_k^B\rangle + d_{lr} |x_m^B\rangle), \quad (8.13)$$

where d_{kr} are defined in Appendix E. This state is given by the Q function $Q(\lambda_{\text{rot}A}, t_{1A})$ in Figs. 26 and 27, which is derived from the initial state $Q(\lambda, t_0)$ using knowledge of the coefficients c_i and θ , as explained in the proof above. The

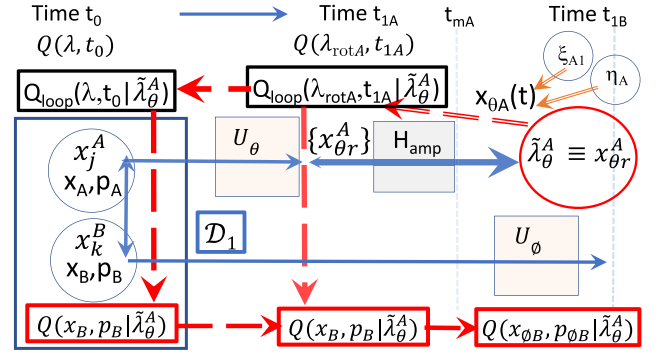


FIG. 27. Derivation of the projected state with a setting change θ at A . The diagram depicts the inference of the projected distribution $Q(x_B, p_B | x_{\theta j}^A)$ for system B , conditioned on an outcome $x_{\theta r}^A$, symbolized by a branch $\tilde{\lambda}_{\theta}^A$, defined after the rotation U_{θ} and amplification H_{amp} . The system is prepared at time t_0 for measurements $\hat{x}_A$ and $\hat{x}_B$ in the state $Q(\lambda, t_0)$. The derivation requires only knowledge of θ and the initial state, $Q(\lambda, t_0)$ (given by the c_{jk} defined in Result VII.13) and is based on deterministic relations (blue arrows). The change of setting θ as given by the deterministic transformation (D2) at A is consistent with deterministic relations $\mathcal{D}_1$ that account for the correlations between x_k^B and $x_{\theta r}^A$ in $Q(\lambda_{\text{rot}A}, t_{1A})$. Result VIII.1e and the associated Comment (1) explain the calculation of $Q(x_B, p_B | x_{\theta j}^A)$, based on tracking trajectories $x_{\theta A}(t)$ associated with the branch $\tilde{\lambda}_{\theta}^A \equiv x_{\theta j}^A$ to deduce $Q_{\text{loop}}(\lambda_{\text{rot}A}, t_{1A} | x_{\theta j}^A)$, where deterministic relations are accounted for. The projected distribution $Q(x_B, p_B | x_{\theta j}^A)$ is evaluated by integrating over coordinates of A , from either $Q_{\text{loop}}(\lambda_{\text{rot}A}, t_{1A} | x_{\theta j}^A)$, or by converting to the distribution $Q_{\text{loop}}(\lambda, t_0 | x_{\theta j}^A)$ at time t_0 . The probabilities of outcomes $x_{\phi k}^B$ conditioned on the outcome $\tilde{\lambda}_{\theta}^A = x_{\theta r}^A$ at A , for a setting-change ϕ at B are given by $Q(x_{\phi B}, p_{\phi B} | x_{\theta j}^A)$, consistent with the coordinate transformation (D3) for B .

deterministic relations $\mathcal{D}_1$ are (Result VII.13)

$$x_k^A \leftrightarrow x_{\theta r}^A, \quad x_m^B \leftrightarrow x_{\theta r}^A, \quad (8.14)$$

where $x_k^B \leftrightarrow x_{\theta r}^A$ occurs with probability $|d_{ir}|^2$, and $x_m^B \leftrightarrow x_{\theta r}^A$ occurs with probability $|d_{lr}|^2$. From (8.13), the projected state given an outcome $x_{\theta j}^A$ at A for $\tilde{\lambda}_{\theta}^A$ is the superposition

$$|\psi_B\rangle = d_{ij} |x_k^B\rangle + d_{lj} |x_m^B\rangle$$

provided $d_{ij}, d_{lj} \neq 0$. This corresponds to the projected distribution $Q(x_B, p_B | x_{\theta j}^A)$ derived from the distribution $Q_{\text{loop}}(\lambda_{\text{rot}A}, t_{1A} | x_{\theta j}^A)$, which is deduced from the branch $\tilde{\lambda}_{\theta}^A \equiv x_{\theta j}^A$ associated with outcome $x_{\theta j}^A$ (red double dashed lines in Fig. 27), as explained in the techniques of Result VIII.1c. The projected distribution $Q(x_B, p_B | x_{\theta j}^A)$ is derived on integrating $Q_{\text{loop}}(\lambda_{\text{rot}A}, t_{1A} | x_{\theta j}^A)$ over coordinates $x_{\theta A}$ and $p_{\theta A}$ (vertical dashed lines in Fig. 27). The same projected distribution $Q(x_B, p_B | x_{\theta j}^A)$ is derived, if we transform the coordinates of A to x_A and p_A (top horizontal red dashed lines in Fig. 27). This corresponds to the tracing of the trajectories of A through the reversible transformation (D2) given by the measurement-setting adjustment H_{θ} .

Considering the projected state of A given an outcome $x_{\phi B}^B$ at B , we find the projected state at A is the superposition of type $|\psi_A\rangle = d_{ks} |x_i^A\rangle + d_{ms} |x_l^A\rangle$.

Comment (2). The inference of the projected state at A requires knowledge of the setting θ at A , and the determin-

istic relations. The inference is based on *retrodiction*. The postselected state $Q_{\text{loop}}(\lambda_{\text{rotA}}, t_{1A}|x_{\theta j}^A)$ is derived solely from knowledge of $x_{\theta j}^A$ and the relation $x_{\theta j}^A \leftrightarrow Gx_{\theta j}^A$, where G is the amplification factor from H_{amp} , as well as the deterministic relations $\mathcal{D}_1$, and the setting transformation (D2). The specific value of the noise inputs η or ξ is not required. This is because there is no change to the noise properties as the system propagates in the backward-time direction.

Comment (3). The appearance of the superposition $|\psi_B\rangle$ in Comment (1) above is consistent with Result VII.5. A change of setting at A changes the interference term $\mathcal{I}_{AB}$ in the Q function of $|x_j^A\rangle|x_k^B\rangle + |x_j^A\rangle|x_m^B\rangle$ at time t_0 , so that the integration over $x_{\theta A}$ and $p_{\theta A}$ leads to a contribution that need not be zero. This means that the interference term can contribute to the projected state calculated from $Q_{\text{loop}}(\lambda_{\text{rotA}}, t_{1A}|x_{\theta j}^A)$, and the resulting Q function $Q(x_B, p_B|x_{\theta j}^A)$ can contain an interference term $\mathcal{I}_B$, thus allowing for the superposition $|\psi_B\rangle$. Similarly, in reverse, going from time t_{1A} to t_0 , the projected distribution $Q(x_B, p_B|x_{\theta j}^A)$ calculated by integrating $Q_{\text{loop}}(\lambda, t_0|x_{\theta j}^A)$ over x_A and p_A can contain interference terms $\mathcal{I}_B$ in B .

Comment (4). Analogy for spin systems. The procedure can be illustrated by an analogy with basis changes for the system prepared at time t_0 in the spin Bell state

$$|\psi_{\text{Bell}}, t_0\rangle = \frac{1}{\sqrt{2}}(|+\rangle_{zA}|-\rangle_{zB} - |-\rangle_{zA}|+\rangle_{zB}), \quad (8.15)$$

which corresponds in the analogy to $Q(\lambda, t_0)$. Here $|\pm\rangle_{zK}$ are the eigenstates of $\hat{\sigma}_z$ for system $K \in \{A, B\}$. We take $\hat{\sigma}_z^K = \hat{\sigma}_z^K \cos \theta + \hat{\sigma}_x^K \sin \theta$. At A , a change of setting from spin z to spin $\hat{\sigma}_\theta^A$ changes the basis: the new basis states are $|\pm\rangle_{\theta A} = \cos \frac{\theta}{2} |\pm\rangle_{zA} \pm \sin \frac{\theta}{2} |\mp\rangle_{zA}$. The state at time t_{1A} is

$$|\psi_{\text{Bell}}, t_{1A}\rangle = \frac{1}{\sqrt{2}} \left\{ |+\rangle_{\theta A} \left(\cos \frac{\theta}{2} |-\rangle_{zB} - \sin \frac{\theta}{2} |+\rangle_{zB} \right) - |-\rangle_{\theta A} \left(\cos \frac{\theta}{2} |+\rangle_{zB} + \sin \frac{\theta}{2} |-\rangle_{zB} \right) \right\}, \quad (8.16)$$

corresponding in the analogy to $Q(\lambda_{\text{rotA}}, t_{1A})$. This gives the expression in terms of the measurement basis for spin $\hat{\sigma}_\theta^A$ at A and spin z at B . The state conditioned on the $+$ outcome at A is $|+\rangle_{\theta A} |\psi_{B\text{sup}}\rangle$ where $|\psi_{B\text{sup}}\rangle = \cos \frac{\theta}{2} |-\rangle_{zB} - \sin \frac{\theta}{2} |+\rangle_{zB}$. The projected state at B given a $+$ outcome at A is the superposition $|\psi_{B\text{sup}}\rangle$, in analogy with the Q function $Q(x_B, p_B|x_{\theta j}^A)$. There is no direct analogy with $Q_{\text{loop}}(\lambda_{\text{rotA}}, t_{1A}|x_{\theta j}^A)$. The change of basis back to spin σ_z^A at A gives

$$\left(\cos \frac{\theta}{2} |+\rangle_{zA} + \sin \frac{\theta}{2} |-\rangle_{zA} \right) |\psi_{B\text{sup}}\rangle, \quad (8.17)$$

which corresponds to $Q_{\text{loop}}(\lambda, t_0|x_{\theta j}^A)$. The projected state for B is unchanged, given by $|\psi_{B\text{sup}}\rangle$ and hence $Q(x_B, p_B|x_{\theta j}^A)$. The projected state describes the predictions at B , for future setting changes, i.e., if there is a change of setting to ϕ at B , then the state at B is

$$|\psi_{B\text{sup}, \phi}\rangle = \cos \left(\frac{\theta - \phi}{2} \right) |-\rangle_{\phi B} + \sin \left(\frac{\theta - \phi}{2} \right) |+\rangle_{\phi B},$$

where $|\pm\rangle_{\phi B} = \cos \frac{\phi}{2} |\pm\rangle_{zB} \pm \sin \frac{\phi}{2} |\mp\rangle_{zB}$. The expected value of the spin product given $+$ at A is hence $E_+(\theta, \phi) = -\cos(\theta - \phi)$. The expected value conditioned on a negative spin $\hat{\sigma}_\theta^A$ at A can be similarly calculated, to give agreement with the overall expected value, $E(\theta, \phi) = -\cos(\theta - \phi)$.

We consider the case where A and B are reversed, so that a setting change ϕ takes place first at B . The projected state for A given a positive spin outcome for $\hat{\sigma}_\phi^B$ is the superposition $|\psi_{A\text{sup}}\rangle = (\cos \frac{\phi}{2} |-\rangle_{zA} - \sin \frac{\phi}{2} |+\rangle_{zA})$. There is hence a *mutual consistency* of the projected states with respect to the initial state $|\psi_{\text{Bell}}\rangle$ (refer to Result IX.3).

We compare with the mixed state ρ_{mix}^{AB} defined as the 50-50 mixture of $|+\rangle_{zA}|-\rangle_{zB}$ and $|-\rangle_{zA}|+\rangle_{zB}$. In the analogy, the initial state $Q(\lambda, t_0)$ has no interference term. At A , a change of setting from spin z to spin $\hat{\sigma}_\theta^A$ changes the basis to $|\pm\rangle_{\theta A} = \cos \frac{\theta}{2} |\pm\rangle_{zA} \pm \sin \frac{\theta}{2} |\mp\rangle_{zA}$, but the projected state for B given a positive spin outcome at A is a mixture of up- and down-spin states at B .

D. Timing of the projection

In the Introduction, we raised the question as to the timing of the collapse of the wave function. From the results, we see there is consistency with premises (3) of *weak macroscopic realism* (wMR) and *weak local realism* (wLR), defined in Definitions (6c) and (10c) in Sec. IB, which imply a constraint on the time of projection.

Result VIII.2: Weak macroscopic realism: Premise wMR(3). Consider the systems A and B prepared at time t_0 with respect to a particular measurement basis which, without loss of generality, we assume to be $\hat{x}_A$ and $\hat{x}_B$ as in Fig. 24. Suppose the measurement setting at A has been fixed and the system A amplified, so that distinct branches symbolized by λ_x^A have emerged at a time t_{mA} (Fig. 24). The first premise of weak macroscopic realism asserts that the outcome x_j^A for $\hat{x}_A$ at A is specified at time t_{mA} (Result IV.6). The Q model of reality and simulations are consistent with the third premise of wMR, premise wMR(3), defined below.

The premise wMR(3) of *weak macroscopic realism* posits that the constraints given by the projected distributions $Q(x_B, p_B|\lambda_x^A) \equiv Q(x_B, p_B|x_j^A)$ on the outcomes for B describe the system B at (or by) the time t_{mA} , after sufficient amplification at A has taken place so that weak macroscopic realism [premise wMR(1)] can be applied. $Q(x_B, p_B|x_j^A)$ implies predictions for B , for any future setting change ϕ at B that might take place.

Proof. This follows from Result VIII.1a. The trajectories for $x_A(t_{mA})$ associated with the branch λ_x^A corresponding to outcome x_j^A are defined at time t_{mA} . NB: The noise properties $\delta x(t)$ are unchanged with the reverse propagation and the branch has fixed width due to the noise $\delta x(t)$ (Results III.2 and III.3). The Q function $Q(\lambda, t_0)$ for the entangled system is defined at the earlier time t_0 . The distribution $Q_{\text{loop}}(\lambda|x_j^A)$ can be deduced from $Q(\lambda, t_0)$, and hence so can $Q(x_B, p_B|x_j^A)$. Hence, $Q(x_B, p_B|x_j^A)$ is determined at time t_{mA} . The second statement of premise wMR(3) follows from Result VIII.1b.

If a setting change θ has taken place at A at a time t_2 between the times t_0 and t_{mA} , then the distribution $Q(x_B, p_B|x_{\theta j}^A)$ can be deduced from the $Q_{\text{loop}}(\lambda_{\text{rotA}}, t_2|x_{\theta j}^A)$ where $\lambda_{\text{rotA}} = (x_{\theta A}, p_{\theta A}, x_B, p_B)$ or else equivalently by tracing the

amplitudes $x_A(t)$ back to the time t_0 , to deduce $Q_{\text{loop}}(\lambda, t_0|x_{\theta_j}^A)$, as explained in Result VIII.1e. ■

Comment. The projected state $Q(x_B, p_B|x_j^A)$ is defined at the time t_{mA} , and it has been assumed that no setting change has taken place at B , as in Fig. 24. Hence, the last statement refers to predictions for future changes at B . In fact, the projected distribution $Q(x_B, p_B|x_j^A)$ is defined to describe the system B at the time t_0 . We see from Result VIII.1b (and Appendix H) that this distribution $Q(x_B, p_B|x_j^A)$ then defines the predictions at B conditioned on outcome x_j^A at A for any time $t > t_0$, which may be prior to t_{mA} .

We generalize the result to apply to *weak local realism* (wLR), as introduced in Definition 10 in Sec. IB and in Sec. IV G (Result IV.14).

Result VIII.3: Weak local realism: Premise wLR(3). Suppose the measurement setting at A has been fixed at a time t_{1A} , meaning that the unitary operations U_{θ}^A that fix the setting as θ have been carried out. In Fig. 24, the time t_{1A} corresponds to time t_0 . There are two versions of weak local realism, deterministic and probabilistic, as explained in Result IV.14 (Sec. IV G), for which the third premises are defined below. The Q model of reality and simulations is consistent with premises wLR(3).

(i) Premise wLR(3) of the *deterministic* version of weak local realism is based on premise wLR(1), that the value $x_{\theta_j}^A$ for the outcome of $\hat{x}_{\theta A}$ is determined for system A at time t_{1A} . Premise wLR(3) asserts that the system B is accordingly constrained according to $Q(x_B, p_B|x_{\theta_j}^A)$ at (or by) the time t_{1A} , when the setting at A is specified (provided the setting at A remains fixed).

(ii) Premise wLR(3) of the *probabilistic* version of weak local realism assumes that hidden variables $\{\lambda_{\theta}^A\}$ exist to describe the system A at time t_{1A} , after the settings at A are fixed. There exists a projected distribution $Q(x_B, p_B|\{\lambda_{\theta}^A\})$ defined for system B given that system A is specified by $\{\lambda_{\theta}^A\}$. The probabilistic version of premise wLR(1) [Result IV.14, Definition (10a)] asserts that there exists a probability $P(x_{\theta_j}^A|\{\lambda_{\theta}^A\})$ of an outcome $x_{\theta_j}^A$ for $\hat{x}_{\theta A}$ at A , given the system A is described by $\{\lambda_{\theta}^A\}$. The projected distribution $Q(x_B, p_B|\{\lambda_{\theta}^A\})$ is the mixture of projected distributions $Q(x_B, p_B|x_{\theta_j}^A)$ weighted by $P(x_{\theta_j}^A|\{\lambda_{\theta}^A\})$.

Proof. (i) First, we consider the deterministic version of wLR(3), where the outcome $x_{\theta_j}^A$ at A is assumed to be determined at time t_{1A} [Result IV.14 for premise wLR(1)], after the setting at A is fixed. Provided the setting at A is fixed, we know from Result VII.3 and Appendix F that the quantum predictions (and hence those of the Q model and simulation) are consistent with an LHV model $\mathcal{P}_{\text{asym},A}$, based on the partially mixed state $\rho_{\text{mix},A}$ given by (7.26). In the model $\mathcal{P}_{\text{asym},A}$, each realization of the system A is in a state with a definite predetermined outcome $x_{\theta_j}^A$ for $\hat{x}_{\theta A}$ at A at time t_{1A} . The state of system B is the projected state for B , as correlated with a given outcome $x_{\theta_j}^A$: Hence, in the model $\mathcal{P}_{\text{asym},A}$, the predictions for B are defined at the time t_{1A} , consistent with premise wLR(3). The result follows, since the predictions of the Q model (and of quantum mechanics) do not contradict those of the model $\mathcal{P}_{\text{asym},A}$ when the setting at A is fixed. The HV model $\mathcal{P}_{\text{det}B}$ suggested as a Comment in Result VII.9 directly supports the validity of the premise wLR(3), and motivates further study.

(ii) Second, we assume the probabilistic version of wLR. This version is directly verified by the Q model. The probabilities $P(x_{\theta_j}^A|\{\lambda_{\theta}^A\})$ for system A at time t_{1A} , where the variables λ_{θ}^A correspond to $x_{\theta A}$, $p_{\theta A}$, can be deduced (by integration over variables for B) from Result VII.9a, Eq. (7.39), where an HV distribution $P(x_{\theta_j}^A, x_{\theta_k}^B|\lambda_{\text{rot}})$ is derived consistently with the Q model and the predictions of quantum mechanics. Here, $\lambda_{\text{rot}} = (x_{\theta A}, p_{\theta A}, x_B, p_B)$, as we take $\phi = 0$. Hence, considering the system A at time t_{1A} as defined by the distribution $Q(\lambda_{\text{rot}A}, t_{1A})$, the probability of an outcome x_j^A at A is known from the values of the local hidden variable, $x_{\theta A}$ and $p_{\theta A}$. For each x_j^A , the projected distribution $Q(x_B, p_B|x_j^A)$ can be deduced at A based on knowledge of the initial state $Q(\lambda, t_0)$ and any local setting change θ at A (Result VIII.1a). Hence, the overall projected distribution for system B can be deduced by Alice at A given $\{\lambda_{\theta}^A\}$ at time t_{1A} (within the framework of wLR) as the weighted mixture given by $\sum_{x_j^A} P(x_j^A|\{\lambda_{\theta}^A\})Q(x_B, p_B|x_j^A)$. The total probability distribution $P(x_{\theta_j}^A)$ for an outcome x_j^A at A is $\int dx_{\theta A} dp_{\theta A} P(\{\lambda_{\theta}^A\})P(x_{\theta_j}^A|\{\lambda_{\theta}^A\})$ where $P(\{\lambda_{\theta}^A\})$ is the probability of variables $\{\lambda_{\theta}^A\} = \{x_{\theta A}, p_{\theta A}\}$ as given by the marginal $Q_m(x_{\theta A}, p_{\theta A}) = \int dx_B dp_B Q(\lambda_{\text{rot}}, t_{1A})$. This correctly describes the projected state at B given outcome x_j^A at A , according to quantum mechanics. Hence, the premise wLR(3) applies. ■

Result VIII.4: Distinction between cause and correlation in projection. We consider correlated systems, where the state of system B can be determined by a measurement $\hat{x}_A$ at A . The measurement of $\hat{x}_A$ at A involves, after the setting is fixed at a time t_0 , an amplification given by H_{amp}^A , followed by, at a time after t_{mA} (as depicted in Fig. 24), a final detection. If we assume weak local or macroscopic realism (wLR), as in the Q model, it is not the case that either the amplification H_{amp} or final detection at A *cause* the outcome, or the projected state $Q(x_B, p_B|\{\lambda_{\theta}^A\})$, at B .

Proof. From Results VII.12 and VII.13, we see that the deterministic relations $\mathcal{D}$ that give the correlations between the outcomes at A and B are determined at, or by, the time t_0 , which is before H_{amp}^A and the final detection at A . According to wLR (Result VIII.3), the projected state is specified for system B , at (or by) the time t_0 , when the settings at A are fixed. ■

Comment. No nonlocality of the meter. A meter is defined as a system, such as A in the Schrödinger-cat example [Eq. (8.11)], which is correlated with B so that the outcome at A gives information about the outcome of a measurement at B . The meter may be spatially separated from B . The meter is configured so that *its settings are fixed*. Prior interaction with the system B has generated a state such as (8.11), where the measurement basis for A is fixed at $\hat{x}_A$. The meter A is also macroscopic. Hence, the systems A and B are analogous to those described in Fig. 24 at the time t_{mA} . Result VIII.4 implies that the final part of the measurement to detect the outcome at A does not cause the outcome at B .

IX. CAUSAL MODEL FOR BELL NONLOCALITY

In the Q model of reality, the x_A, x_B, p_A, p_B are hidden variables. The simulation can be examined to identify a causal (cause-and-effect) model consistent with Bell nonlocality.

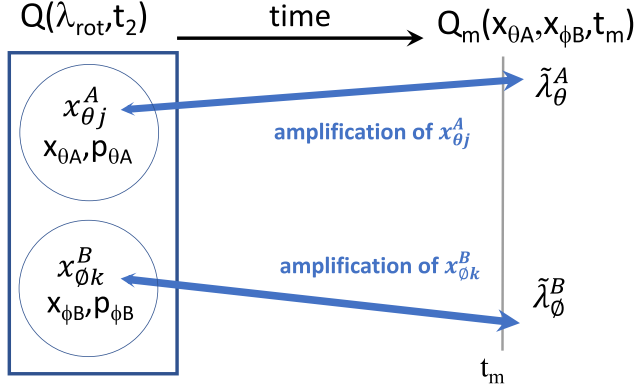


FIG. 28. Weak macroscopic realism (wMR) and weak local realism (wLR). At time t_2 , the system is prepared for measurement of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$. The distribution $Q_m(x_{\theta A}, x_{\phi B}, t_m)$ gives the probability density of amplitudes $x_{\theta A}(t)$ and $x_{\phi B}(t)$ at time t_m after amplification H_{amp} at each site. The blue two-way arrows denote the causal deterministic relations in which the means $x_{\theta j}^A$ and $x_{\phi k}^B$ of the Gaussians in $Q(\lambda_{\text{rot}}, t_2)$ are amplified to $Gx_{\theta j}^A$ and $Gx_{\phi k}^B$. The interference terms $\mathcal{I}nt_{AB}$ in $Q(\lambda_{\text{rot}}, t_2)$ do not amplify. Weak macroscopic realism posits that after sufficient amplification, at time t_m , the outcomes of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ are determined. We symbolize these values by $\tilde{\lambda}_{\theta}^A$ and $\tilde{\lambda}_{\phi}^B$. According to wLR, the values exist probabilistically as in the hidden-variable model $\mathcal{P}_B$ of Eq. (7.38) at the earlier time t_2 , after the settings are fixed.

We have demonstrated in the previous section how the projected state is deduced by retrodiction based on deterministic relations, and knowledge of the initial state $Q(\lambda, t_0)$, only (Results VIII.1a and VIII.e). Yet, as we show in this section, the projected state is sufficient to explain Bell violations in the Q -based model, where the measurement is carried out by amplification H_{amp} .

The test of a Bell inequality involves the preparation of the Bell state. The fields are then spatially separated and the settings locally adjusted as in Fig. 22, in four sets of runs, to measure $E(\theta, \phi)$, $E(\theta', \phi)$, $E(\theta, \phi')$, and $E(\theta', \phi')$ as given in (7.5). To gain insight into the dynamics that leads to the Bell nonlocality, we first analyze the simulation where one adjusts the settings at the two locations, A and B, *sequentially*. This corresponds to an alternative Scenario II for a Bell test, where the system is prepared with respect to the measurement basis θ and ϕ at a time t_0 . Then there are three types of runs: (1) where settings remain fixed at θ and ϕ ; (2) two sets of runs with a change of setting at one site only, to measure $E(\theta, \phi')$ and $E(\theta', \phi)$; (3) a set of runs with a change of setting at both sites, to measure $E(\theta', \phi')$.

A. Preparation: Measurement basis of $\hat{x}_A$ and $\hat{x}_B$

In the Bell test, we consider first that the system is prepared at time t_0 , with respect to the measurement basis $\hat{x}_A$ and $\hat{x}_B$. We ask what the Q model indicates if there is measurement of $\hat{x}_A$ and $\hat{x}_B$, without a change of settings. This corresponds to Fig. 28, with $\theta = \phi = 0$ (and $t_0 = t_2$).

In Sec. VII E, Results VII.10 and VII.11, we have extended the premises wMR(1) and wLR(1) of weak macroscopic realism (wMR) and weak local realism (wLR), given by Definitions (6) and (10) in Sec. I B, and by Results IV.6 and IV.14,

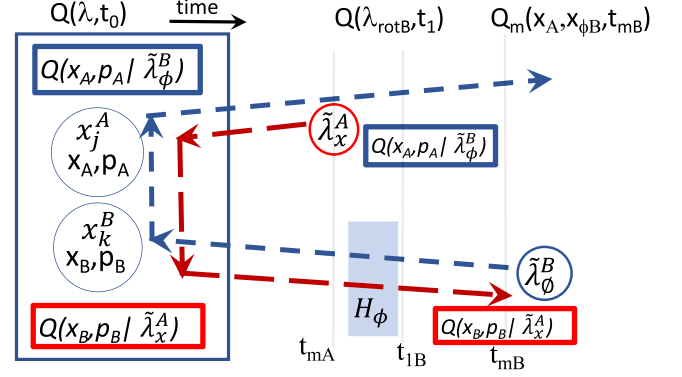


FIG. 29. Mutual projection and no-signaling. Depiction of the constraints on the system A of Fig. 24 due to the change of setting at B to ϕ at time t_{1B} , while the setting at A is fixed at x . At the time t_{mB} , after amplification at B, the value $\tilde{\lambda}_{\phi}^B$ determines the outcome for $\hat{x}_{\phi B}$. The amplitudes at B conditioned on branch $\tilde{\lambda}_{\phi}^B$ correlate with certain amplitudes for A at t_0 , as determined by $Q(\lambda, t_0)$. This ensures (blue dashed arrows) the projected state at A is given as $Q(x_A, p_A / \tilde{\lambda}_{\phi}^B)$. There is also a projection from A to B (red dashed arrows) as in Fig. 24. The mutual projection is consistent with the density of amplitudes at t_{mB} , as given by the marginal $Q_m(x_A, x_{\phi B}, t_{mB})$ of the Q function of the amplified state. The change of setting at B induces a change (blue dashed arrows) to the hidden interference terms $\mathcal{I}nt_{AB}$ of the Q distribution $Q(\lambda_{\text{rotB}}, t_{1B})$, but does not change the observed probabilities [given by $Q(x_A, p_A / \tilde{\lambda}_{\phi}^B)$] over what can be explained by a local transformation at B. Hence, in the Q model, the value $\tilde{\lambda}_x^A$ giving the outcome of $\hat{x}_A$ is unchanged, which implies *no-signaling*.

so that they apply to the bipartite system. This implies that once the measurement settings are fixed, as in Fig. 28 at time t_2 , and an amplification H_{amp} has taken place at each site, then the values for the outcomes of $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ (given by $x_{\theta j}^A$ and $x_{\phi k}^B$) are determined, as in the branches of Fig. 20. Assuming wLR, the system is viewed as being in an HV state where there is a certain probability for outcomes, at the time t_2 when settings are fixed.

B. Changing the setting at one location: Projection and no-signaling

Figure 24 shows the system prepared at time t_2 for measurements $\hat{x}_A$ and $\hat{x}_B$, where $\theta = \phi = 0$. Suppose the measurement setting is then changed at one location, B, to $\phi \neq 0$, with the setting at A fixed. We ask: does the change of setting at B influence the system, or outcome, at A i.e., is there a nonlocal effect at A? We show that the simulations modeling the Bell violations are consistent with no-signaling.

1. Mutual consistency of projection

First, we note that there are mutual constraints on the two systems, due to projection, depicted by the red and blue dashed lines in Fig. 29. If there is a change of setting to ϕ at B, then there are constraints imposed on system B by the fixed setting at A (Result VIII.1, Sec. VIII A) as depicted in Fig. 24, and by the red dashed line in Fig. 29. The state for system B conditioned on the branch $\tilde{\lambda}_x^A$ of A is the projected state, distribution $Q(x_B, p_B / \tilde{\lambda}_x^A)$, given by Eq. (4.12).

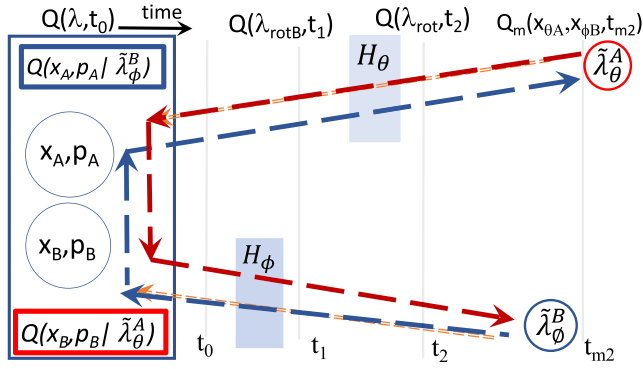


FIG. 30. Depiction of the mutual consistency of projection. There is a change of setting ϕ at B and a change of setting θ at A . The Q function changes after each change of setting, as described in Eqs. (7.32) and (7.35). The projected states $Q(x_A, p_A | \tilde{\lambda}_\theta^A)$ and $Q(x_B, p_B | \tilde{\lambda}_\phi^B)$ are mutually consistent with the Q function $Q(\lambda, t_0)$. Each is derived based only on the knowledge of the local setting (and outcome) and the initial state $Q(\lambda, t_0)$. The distributions $Q(x_A, p_A | \tilde{\lambda}_\theta^A)$ and $Q(x_B, p_B | \tilde{\lambda}_\phi^B)$ are hence independent of the time order of the setting-changes, H_ϕ and H_θ . Any Bell-nonlocal effect is demonstrated after changes of setting occur at both locations. This avoids time-order paradoxes, which arise when in one frame of reference the setting is changed at B first, and in another frame of reference the setting is changed at A first (refer to Sec. IX.C).

However, there is also a constraint to A from B (blue dashed line in Fig. 29). After amplification at B , at time t_{mB} , there is a value, symbolized by $\tilde{\lambda}_\phi^B$, for the outcome of measurement $\hat{x}_{\phi B}$. The state at A conditioned on the outcome associated with $\tilde{\lambda}_\phi^B$ at B is determined by the initial Q function and any relevant deterministic relations, as explained in Sec. VIII.C. This defines the projected state

$$Q(x_A, p_A | \tilde{\lambda}_\phi^B) \quad (9.1)$$

at A , which is the Q function for the reduced state of A given the outcome $\tilde{\lambda}_\phi^B$ at B (Result VIII.1c).

$Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ and $Q(x_B, p_B | \tilde{\lambda}_\theta^A)$ are mutually consistent with $Q(\lambda, t_0)$ and the Q function at time t_{mB} . Figure 29 depicts the mutual consistency.

Definition: Mutual consistency of projection. The mutual consistency of the projected probability distributions $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ and $Q(x_B, p_B | \tilde{\lambda}_\theta^A)$, as depicted in Fig. 29, means that they coexist for the same system, independently of the time-order of the setting changes θ and ϕ at A and B . The distributions are defined consistently with the Q function describing the bipartite system at all times of the dynamics, including at time t_0 .

Result IX.1a: Mutual Consistency of projection. The projected probability distributions $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ and $Q(x_B, p_B | \tilde{\lambda}_\theta^A)$ depicted in Fig. 29 are mutually consistent. We can generalize the consistency to the case depicted in Figs. 5 and 30, where a setting change θ has occurred at A after time t_0 and prior to time t_{mA} depicted in Fig. 29, so that the projected distribution $Q(x_B, p_B | \tilde{\lambda}_\theta^A)$ is defined (Result VIII.1e).

Proof. From Results VIII.1a and VIII.1e given Sec. VIII, we see that the evaluation of $Q(x_B, p_B | \tilde{\lambda}_\theta^A) \equiv Q(x_B, p_B | x_{\theta j}^A)$

requires the knowledge of the setting θ at A and of the initial Q function $Q(\lambda, t_0)$. The evaluation does not require knowledge of ϕ , the setting at B . It is hence established locally at A , independent of whether a setting change ϕ has already occurred at B . The distribution $Q(x_B, p_B | \tilde{\lambda}_\theta^A)$ gives predictions of probabilities at B conditioned on the outcome $x_{\theta j}^A$, for all choices of setting ϕ at B that may occur after time t_{mA} , or that have occurred after time t_0 . This is because the probability distribution for the measurement $\hat{x}_{\phi B}$ at B is given by transforming from coordinates x_B, p_B to $x_{\phi B}$ and $p_{\phi B}$, as explained in Results VIII.1b and AH.2. A similar result applies to $Q(x_A, p_A | \tilde{\lambda}_\phi^B) \equiv Q(x_A, p_A | x_{\phi k}^B)$ where $x_{\phi k}^B$ is the outcome at B .

The proof of causal consistency (Results IX.9 and AH.1) demonstrates how the projected distributions are consistent with $Q(\lambda, t_0)$. Details are given by Results AH in Appendix H. ■

Comment. From the above, we see how the Q -based model allows a mutual consistency of projected states, where the initial state at time t_0 is given as a probabilistic distribution over variables $\lambda = (x_A, p_A, x_B, p_B)$.

Result IX.1b: Paradox of the mutual consistency of projection. A paradox seemingly arises. Consider Fig. 30, where there are two changes of settings, θ at A and ϕ at B . We see from Result VIII.1e and the related Comments that the projected distribution $Q(x_B, p_B | \tilde{\lambda}_\theta^A) \equiv Q(x_B, p_B | x_{\theta j}^A)$ conditioned on an outcome $x_{\theta j}^A$ at A is based on a set $\mathcal{D}_1$ of *deterministic relations* implied by the asymmetric local hidden variable (LHV) model $\mathcal{P}_{\text{asym}, B}$ given by $\rho_{\text{mix}, B}$ as in Result VII.3, which correctly describes the outcomes of $\hat{x}_{\theta A}$ and $\hat{x}_B$ (refer Result VII.13). These deterministic relations determine the correlations between outcomes of $\hat{x}_{\theta A}$ and $\hat{x}_B$, but assume one setting change only, in this case at A . The deterministic relations $\mathcal{D}_1$ are reversible, allowing inference of the $Q(x_B, p_B | x_{\theta j}^A)$ at the initial time.

The paradox is the following: how can the second projected distribution $Q(x_A, p_A | x_{\phi k}^B)$ be consistent with $Q(x_B, p_B | x_{\theta j}^A)$, since the derivation of $Q(x_A, p_A | x_{\phi k}^B)$ assumes an LHV model $\mathcal{P}_{\text{asym}, A}$ based on $\rho_{\text{mix}, A}$ defined by Result VII.3, which also assumes a change of setting at *one* location, in this case B , only? A second set $\mathcal{D}_2$ of deterministic relations is used for deriving $Q(x_A, p_A | x_{\phi k}^B)$, but both sets of deterministic relations are consistent with LHV models. How is the existence of the mutually compatible projected states consistent with the violation of LHV models, as implied by Bell's theorem?

Resolution of the paradox. The resolution is that the projected state for B given the outcome at A is inferred for the system B as it is *defined at the initial time t_0* , prior to any change of setting at B (Fig. 31). Similarly, the projected state of A given an outcome at B is inferred to describe the system A at the time t_0 . The deterministic relations $\mathcal{D}_1$ and $\mathcal{D}_2$ (Results VII.13) provide a set of relations for outcomes in the Q model, but the relations *connect only the initial outcomes* (x_j^A or x_k^B) *of one system to the final outcomes* (x_ϕ^B or x_θ^A) *of the other*, and are hence defined consistently over both setting-changes. They provide relations that are weaker than what could be deduced from the full LHV models $\mathcal{P}_{\text{asym}, A}$ and $\mathcal{P}_{\text{asym}, B}$, because they do not distinguish between mixtures and superposition states.

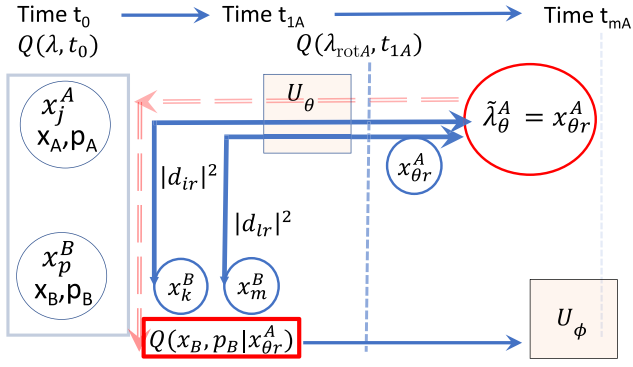


FIG. 31. Deterministic relations and the mutual consistency of projection. The diagram depicts the evaluation of the projected distribution $Q(x_B, p_B | x_{\theta r}^A)$ for the system B conditioned on an outcome $x_{\theta r}^A$, as in Fig. 27, and how this is consistent with evaluation of $Q(x_A, p_A | x_{\phi s}^B)$ at B . We consider that $Q(\lambda, t_0)$ represents the entangled state $|x_i^A\rangle|x_k^B\rangle + |x_l^A\rangle|x_m^B\rangle$. For certain θ , outcomes x_k^B and x_m^B can both be consistent with a specific outcome $x_{\theta r}^A$ at A , according to deterministic relations $\mathcal{D}_1$ depicted by the solid thick blue arrows (Result IX.1). The relations $\mathcal{D}_1$ are $x_k^B \leftrightarrow x_{\theta r}^A$ with probability $|d_{ir}|^2$; $x_m^B \leftrightarrow x_{\theta r}^A$ with probability $|d_{lr}|^2$. Hence, retrodiction implies the outcome at B given outcome $x_{\theta r}^A$ at A would be either x_k^B or x_m^B . Knowledge of $Q(\lambda, t_0)$ and $Q(\lambda_{\text{rotA}}, t_{1A})$ allows inference that the projected state $Q(x_B, p_B | x_{\theta r}^A)$ is a superposition $|\psi_B\rangle = d_{ij}|x_k^B\rangle + d_{lj}|x_m^B\rangle$. Similarly, deterministic relations $\mathcal{D}_2$ imply that for certain choices of ϕ , the projected state $Q(x_A, p_A | x_{\phi s}^B)$ for system A given an outcome $x_{\phi s}^B$ at B is a superposition $|\psi_A\rangle = d_{ks}|x_i^A\rangle + d_{ms}|x_l^A\rangle$. Each projected state is deduced from deterministic relations, and knowledge of the initial state $Q(\lambda, t_0)$ as well as the setting at the other site, and is sufficient to explain Bell violations.

We illustrate by considering that $Q(\lambda, t_0)$ represents the entangled state $|\psi(t_0)\rangle = c_1|x_i^A\rangle|x_k^B\rangle + c_2|x_l^A\rangle|x_m^B\rangle$ ($c_1, c_2 \neq 0$) defined in Comment (1) of Result VIII.1e. Changing the basis at A gives

$$|\psi\rangle \rightarrow \sum_r |x_{\theta r}^A\rangle (c_1 d_{ir} |x_k^B\rangle + c_2 d_{lr} |x_m^B\rangle),$$

where the d_{ir} are defined in Appendix E, implying a projected state at B of $(c_1 d_{ir} |x_k^B\rangle + c_2 d_{lr} |x_m^B\rangle)$. Taking $c_1 = c_2$, the relations $\mathcal{D}_1$ are $x_k^B \leftrightarrow x_{\theta r}^A$ with probability $|d_{ir}|^2$; $x_m^B \leftrightarrow x_{\theta r}^A$ with probability $|d_{lr}|^2$. The relations are based on the model $\mathcal{P}_{\text{asym},B}$ (Result VII.3) where system B is in a mixed state and cannot account for superpositions states at B . For certain choices of θ , the outcomes x_k^B and x_m^B can both be consistent with a specific outcome $x_{\theta r}^A$ at A . Hence, simple retrodiction implies the outcome at B could be either x_k^B or x_m^B .

The relations $\mathcal{D}_1$ can also appear for a state $\rho_{\text{mix},AB}$ which is a mixture of $|x_i^A\rangle|x_k^B\rangle$ and $|x_l^A\rangle|x_m^B\rangle$. Knowledge of $Q(\lambda, t_0)$ and $Q(\lambda_{\text{rotA}}, t_{1A})$ allows inference that the projected state $Q(x_B, p_B | x_{\theta r}^A)$ is a superposition $|\psi_B\rangle = c_1 d_{ir} |x_k^B\rangle + c_2 d_{lr} |x_m^B\rangle$, rather than a mixture of $|x_k^B\rangle$ and $|x_m^B\rangle$, where here we assume d_{ir}, d_{lr} are nonzero for the choice of θ . The deterministic relations $\mathcal{D}_1$ themselves provide a weak framework only, and do not provide the distinction between the superposition and mixture.

If we consider a change of setting to ϕ at B , then we would write

$$|\psi\rangle \rightarrow \sum_s (c_1 d_{ks} |x_i^A\rangle + c_2 d_{ms} |x_l^A\rangle) |x_{\phi s}^B\rangle.$$

Given an outcome $x_{\phi s}^B$ at B , the possible correlated values x_i^A and x_l^A at A , at the initial time t_0 . The deterministic relations $\mathcal{D}_2$ are $x_i^A \leftrightarrow x_{\phi s}^B$ with probability $|d_{ks}|^2$; $x_l^A \leftrightarrow x_{\phi s}^B$ with probability $|d_{ms}|^2$. For certain choices of ϕ , such that $d_{ks}, d_{ms} \neq 0$, the projected state $Q(x_A, p_A | x_{\phi s}^B)$ for system A given an outcome $x_{\phi s}^B$ at B is a superposition $|\psi_A\rangle = d_{ks}|x_i^A\rangle + d_{ms}|x_l^A\rangle$. The projected states are mutually consistent, and consistent with $Q(\lambda, t_0)$: This is evident since the $|\psi\rangle$ can be written simultaneously with respect to both bases [refer to Comments (1) and (3) of Result VIII.1e], and the deterministic relations do not specify the full nature of the projected state. The superposition nature of the projected states that arise for an entangled state $Q(\lambda, t_0)$ ensures the correlations for future choices of setting can differ from those predicted, for example, by a nonentangled mixed state $\rho_{\text{mix},AB}$ or ρ_{mix}^{AB} defined in Comment (3), allowing a violation of Bell inequalities.

Comment. The choice of setting-change that enables the distinction between the entangled and nonentangled state requires $d_{ij}, d_{lj} \neq 0$, $d_{kq}, d_{mq} \neq 0$. This is to create a superposition of the eigenstates in the original measurement basis, consistent with Results VII.5, VII.7, and VII.8.

2. No-signaling

The following Result complements the earlier Result V.2a (“no macroscopic retrocausality”) of Sec. V. It shows consistency with the second premise of wMR [Definition (6b) in Sec. IB], completing the earlier Results IV.6 and VIII.2 for wMR. We assume spacelike-separated systems A and B .

Result IX.2: Weak macroscopic realism: Premise wMR(2): No-signaling. Without loss of generality, we assume that the system is prepared for measurements $\hat{x}_A$ and $\hat{x}_B$ at time t_0 , as in Fig. 29. The system A is amplified, so that at time t_{mA} the value λ_x^A gives the outcome for A if $\hat{x}_A$ is to be measured. The second premise of wMR posits that the value λ_x^A for the outcome at A is not changed by interactions or dynamics that may then occur (after t_{mA}) at system B . Hence, a change of setting ϕ at B does not change the measurable outcome λ_x^A for $\hat{x}_{\theta A}$. (NB: We can generalize to consider the system prepared for measurements $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ at time t_0 .)

Proof. We assume the frame F in which both systems are stationary. The setting at A is fixed, so the simulation is consistent with the outcome λ_x^A of A being unchanged if we apply the assumption of Result IV.8. It has been shown in the literature (i.e., refer to Ref. [108]) that this assumption, which applies after measurement settings are fixed, is consistent with Bell nonlocality. The Result can also be proved directly, as in the Corollary of Result IX.3 below. ■

Comment. The Result IX.2 explains no-signaling: *No-signaling* is defined (Sec. IB) as the conditional independence of the outcome at A and the setting at B , given the setting at A [53]. We consider the setup in Fig. 29. In the simulation and Q model, weak macroscopic realism holds, and the value of λ_x^A gives the outcome for measurement $\hat{x}_A$. This value is

not changed by the change of setting at B . Hence, there is no signaling.

Comment. While the branch value $\tilde{\lambda}_x^A$ does not change, there can be changes in hidden interference terms. The detailed mechanism for nonlocality in the Q model is explained in Result IX.4 below.

Result IX.2 can be generalized to hold at the earlier time t_{1A} (which corresponds to time t_0 in Fig. 29), just after the unitary operation determining the setting at A [108,120]. We hence complete Results IV.14 and VIII.3 for weak local realism, by showing consistency with the second premise of wLR [Definition (10b) of Sec. IB].

Result IX.3: Weak local realism: Premise wLR(2). We consider the system at the time t_{1A} , when the unitary operation U_θ^A that determines the setting to be θ at A has been carried out. In Fig. 29, this corresponds to the time $t_{1A} \equiv t_0$, and it is taken that $\theta = 0$, so the system A is prepared for the measurement $\hat{x}_{\theta A} \equiv \hat{x}_A$. We assume there is subsequently a change of setting ϕ at B . There are two versions of weak local realism (wLR): deterministic and probabilistic. The Q -model simulation of the Bell nonlocality is consistent with both versions of the second premise wLR(2) as given by Definition (10b) in Sec. IB.

(i) For the first version, Premise wLR(2) posits that at time t_{1A} the value $x_{\theta j}^A$ for the outcome of the measurement $\hat{x}_{\theta A}$ is determined, and this value is unchanged by any subsequent interactions such as H_ϕ occurring at B .

(ii) The second version of Premise wLR(2) posits that the system A is described by hidden variables $\{\lambda_\theta^A\}$ at the time t_{1A} . There is a probability $P(x_{\theta j}^A | \{\lambda_\theta^A\})$ for an outcome $x_{\theta j}^A$ of the measurement at A , given the hidden-variable state $\{\lambda_\theta^A\}$. For the second version, Premise wLR(2) posits that these probabilities are unchanged by any subsequent interactions at B occurring after time t_{1A} .

Proof. Consider the first version of wLR: Result VII.3 (Sec. VII) shows that the probabilities for the joint outcomes of $\hat{x}_A$ and $\hat{x}_{\phi B}$ are indistinguishable from those of a nonentangled state $\rho_{\text{mix},A}$ [Eq. (7.26)], for which there is a fixed predetermined value λ_A for the outcome of system A which is unchanged by a local change of setting ϕ occurring at B . The quantum predictions (and those of the simulation) arising from a change of basis at the single location B only can hence be modeled by an LHV theory, denoted model $\mathcal{P}_{\text{asym},A}$, the details of which are given in Appendix F. Hence, there is no inconsistency with the deterministic version of Premise wLR(2), as given by the prediction for the joint probabilities.

Consider the second version of wLR: The consistency with the probabilistic version of Premise wLR(2) is illustrated for an entangled state in Result VII.5. Without loss of generality, we consider $\theta = \phi = 0$. The hidden variables $\{\lambda\}$ for both systems are the x_A, p_A, x_B, p_B of $Q(\lambda, t_0)$. We define $\{\lambda_\theta^A\}$ as x_A, p_A , since here $\theta = 0$ (refer to Results VII.11 and VIII.3). For the given $\{\lambda_\theta^A\}$ at time t_0 , the probabilities of outcomes at A are defined by the Q function for system A , which is the marginal $Q_A(x_A, p_A, t_0) = \int Q(\lambda, t_0) dx_B dp_B$ (refer to Appendix H). The HV model $\mathcal{P}_{\text{sup}}$ can be developed as in Sec. IVF, to define a probability $P(x_j^A | \{\lambda_\theta^A\})$ of an outcome x_j^A given $\{\lambda_\theta^A\}$. The change in setting to ϕ at B induces a new $Q(\lambda_{\text{rot}B}, t_1)$ [i.e., as in Eq. (7.32)] with a change in $\mathcal{I}nt_{AB}$ [Eq. (7.33)] but which does not transform the variables x_A ,

p_A . The change of setting at B is equivalent to a change of coordinates from x_B, p_B to $x_{\phi B}, p_{\phi B}$, but this does not change $Q(x_A, p_A)$, and does not change the probabilities for A , if the setting at A is fixed. Hence, there is consistency with Premise wLR(2). ■

Corollary. Proof of consistency with Premise wMR(2). We prove consistency of the Q model with Premise wMR(2). The above proof for probabilistic wLR is general and applies to any system, including where Premise wMR(1) can be applied. We can hence consider the time t_m at which the system A has been amplified, so that the branches corresponding to the amplified eigenvalues Gx_j^A are distinct, as in Figs. 2 and 9, so that any given amplitude $x_A(t_m)$ is associated with just one of the branches (note Results IV.6 and 7). In the proof above, we define the variables $\{\lambda_\theta^A\} \equiv \{x_A(t_m), p_A(t_m)\}$, where we assume $\theta = 0$ without loss of generality, and note that for each pair, the value of $x_A(t_m)$ defines a particular branch $\mathcal{B}_{x_A(t_m)}$, and hence a particular outcome, which we denote by $x_j^A(x_A(t_m))$. Hence, the $P(x_j^A | \{\lambda_\theta^A\})$ is either 1 or 0, depending on whether $x_j^A = x_j^A(x_A(t_m))$. This is a special case of the above proof, which implies the probability $P(x_j^A | \{\lambda_\theta^A\})$ does not change when a change of setting ϕ is implemented. Hence, in the Q model, the branch and outcome does not change at A . ■

3. Projection as a mechanism for Bell nonlocality

Result IX.4: Microscopic mechanism for Bell nonlocality, in the presence of no-signaling. In the Q model, the change of setting to ϕ at B (while keeping the setting at A fixed) leads to changes in the unobservable hidden interference terms of the reduced Q function for A . There is no immediate observable nonlocal effect. The simulation, which accounts for the forward-backward trajectories, is consistent with the above results and those in Sec. VII B. The hidden interference terms for the state at A can lead to an observable Bell nonlocality, if there is a further change of setting at A .

Proof. The mechanism is depicted in Figs. 32 and 30. The state at time t_0 is given by the Q function $Q(\lambda, t_0)$ written with respect to the measurement basis, for $\hat{x}_A$ and $\hat{x}_B$. There is then a change of setting at B to ϕ . Assuming the system B is then amplified, since the $\hat{x}_{\phi B}$ is to be measured, the trajectories $x_{\phi B}(t_{m1})$ conditioned on a branch symbolized by $\tilde{\lambda}_\phi^B$ can be traced to the time t_0 (blue dashed arrows in Fig. 32). This requires first tracing to time t_1 , to infer distribution $Q_{\text{loop}}(\lambda_{\text{rot}B}, t_1 | \tilde{\lambda}_\phi^B)$ (refer to Result VIII.1e and Fig. 27). A deterministic transformation then combines the $x_{\phi B}(t_1)$ with the $p_{\phi B}(t_1)$ according to the inverse of (7.14) to infer the postselected distribution $Q_{\text{loop}}(\lambda, t_0 | \tilde{\lambda}_\phi^B)$, where $\lambda = (x_A, p_A, x_B, p_B)$ at the initial time t_0 . The projected distribution $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ for system A can be evaluated by integrating over x_A and p_B . This corresponds to the projected state for A given the outcome $\tilde{\lambda}_\phi^B$ and gives the predictions for future measurements at A , with the setting fixed at ϕ , at B (Results VIII.1b and VIII.1c). Where there is entanglement, we find that this distribution $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ for the state A can contain hidden interference terms that can lead to Bell nonlocality.

First, to explain the no-signaling, we know from Sec. VII (Results VII.3 and VII.5) that the change of setting at B

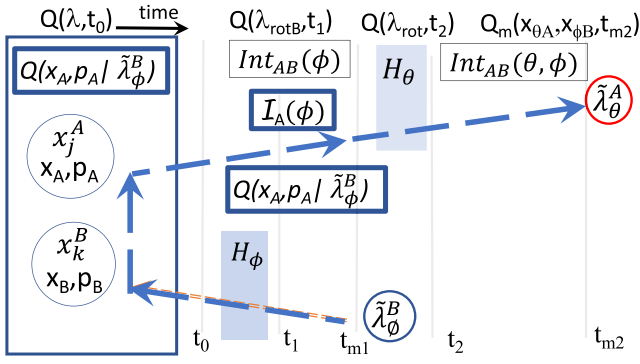


FIG. 32. Mechanism for Bell nonlocality. Depiction of Bell nonlocality, when there is a change of setting at B to ϕ , and then a change of setting θ at A . The change of setting ϕ at B induces a change to the state, given by $Q(\lambda_{\text{rotB}}, t_1)$. However, the observable correlations if made at the time t_1 can be explained as a local transformation at B [as induced for the nonentangled state $\rho_{\text{mix},A}$ of (7.26)]. However, the Q function $Q(\lambda, t_0)$ for the entangled state is different to that of $\rho_{\text{mix},A}$. This difference can appear as unobservable interference terms $\text{Int}_{AB}(\phi)$ in $Q(\lambda_{\text{rotB}}, t_1)$ [refer to Eq. (7.33)], and also in the projected distribution $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ which describes the state for A conditioned on the outcome $\tilde{\lambda}_\phi^B$ after the setting change at B (blue reversed dashed lines): For certain ϕ , the $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ is a superposition state, with hidden interference terms $I_A(\phi)$ not present for the system prepared in $\rho_{\text{mix},A}$. The difference can manifest with a further change of setting θ at A , for certain choices of θ , when Bell nonlocality can arise, detectable in the joint correlations $\langle \tilde{\lambda}_\theta^A \tilde{\lambda}_\phi^B \rangle$ at time t_{m2} .

changes the hidden interference term Int_{AB} [Eq. (7.33)]. This change, however, does not manifest in the *measured* probabilities, if the setting at A is fixed. The new state $Q(\lambda_{\text{rotB}}, t_1)$ has predictions for joint probabilities $P(x_A, x_{\phi B})$ identical to those of a nonentangled state $\rho_{\text{mix},A}$ [Eq. (7.26)] for which an LHV model $\mathcal{P}_{\text{asym},A}$ (Appendix F) is possible. Consistent with this, measurements of $\hat{x}_A$ on the system A given by the reduced state $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ will give results indistinguishable from the state $\rho_{\text{mix},A}$. This explains consistency with no-signaling: There is no observable Bell nonlocality at this stage.

However, different to $\rho_{\text{mix},A}$, the projected distribution $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ for the Bell state has extra interference terms that reflect the system A is in a superposition, rather than a mixture, of eigenstates $|x_j\rangle$ of $\hat{x}_A$. We see from the analysis of Eq. (7.33) that the integration over p_B can lead to nonzero integrals if the rotation of coordinates due to the change of setting ϕ combines both x_B and p_B . Hence, different to $\rho_{\text{mix},A}$, interference terms arising from Int_{AB} can be present in $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$. This occurs for certain choices of ϕ only.

In short, the system A as given by the projected distribution $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ is a superposition rather than a mixture. However, in order to distinguish the superposition from the mixture, a further change of setting at A is required. This is because the Q function (which is a probability distribution for the hidden amplitudes λ) for the superposition differs from that of the mixture only by the presence of the hidden interference terms (refer to Result III.1). Hence, the superposition effects ultimately give different predictions to those of the LHV model $\mathcal{P}_{\text{asym},A}$, leading to Bell nonlocality, but only if a further change of setting at A occurs. ■

Comment. The projected distribution $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ is sufficient to determine the Bell nonlocality, for future measurements at A . From Result VIII.1a, we see that the projected distribution $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ is evaluated based on knowledge of the initial Q function $Q(\lambda, t_0)$, using *continuous individual backward* trajectories, applying first the deterministic relation $x_{\phi k}^B \leftrightarrow Gx_{\phi k}^B$ due to H_{amp}^B (refer to Appendix H), ensuring consistency with deterministic relations $\mathcal{D}$ due to correlations (Result VII.13), and then the deterministic transformation (7.14) due to the setting change at B . This continuity cannot apply in the forward direction, however, where there is a discontinuity at the time t_1 .

Comment. As depicted in Fig. 29, the trajectories and projected distribution $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ are consistent with the Q function $Q(\lambda_{\text{rotB}}, t_1)$ for the joint system at time t_1 , after the change of setting at B . This leads to the following results:

Result IX.5a: Forward-backward stochastic simulation of Bell nonlocality. The Bell nonlocality can be described by the forward-backward stochastic trajectories consistently with the dynamics of the Q function summarized by the results in Sec. VII. The proof of causal consistency for the bipartite system is given in Result IX.7 below.

Result IX.5b: Consistent with Result VII.4, based on the dynamics of the Q function, the stochastic solutions show that the Bell nonlocality emerges when there is a change of measurement setting ϕ and θ , at both locations A and B , respectively.

C. Bell nonlocality, causal consistency, and resolution of paradoxes

In summary, the Bell nonlocality arises because of changes to the interference terms Int_{AB} when there is a change of setting at both locations. A careful study of the final joint probabilities as in Sec. VII B after changes of settings at A and B shows there is a conversion of hidden interference terms Int_{AB} into terms $\text{Int}_{AB}(\theta, \phi)$ which contribute to the *observable* probabilities on measurement [i.e., Eqs. (7.35) and (7.36)]. We summarize the following results about nonlocality.

Weak local reality. The Bell simulation is consistent with the assumptions of weak local realism and weak macroscopic realism, presented as the three premises in Definitions (6) and (10) in Sec. IB. The consistency has been demonstrated in the earlier Results (IV.6, IV.14, VII.10, VII.11, VIII.2, VIII.3, IX.2, IX.3) and is summarized in Figs. 33 and 34.

Result IX.6: Causal order in the Bell experiment: *There is no retrocausality, where the future affects the past.* This is an extension of Results V.2a and V.2b. Suppose the change of setting ϕ at B occurs at time t_1 , illustrated in Figs. 33 and 34. The change of setting at B does not change the value of the macroscopic variables $\tilde{\lambda}_x^A$ or $\tilde{\lambda}_x^B$ that are defined *prior to* or *at time* t_1 . There is hence no genuine retrocausality, where a future change of setting affects past macroscopic properties. We have seen from the dynamics outlined in Result IX.4 that the *impact on A of the change of setting at B occurs only for future events, after further setting changes at A*. A change of setting at A is required for any change to macroscopic values (i.e., $\tilde{\lambda}_x^A$) at A . This is illustrated in Figs. 33 and 35, where we see that the macroscopic values $\tilde{\lambda}_x^A$, $\tilde{\lambda}_x^B$, $\tilde{\lambda}_\theta^A$, and $\tilde{\lambda}_\phi^B$ only change when there is a local interaction.

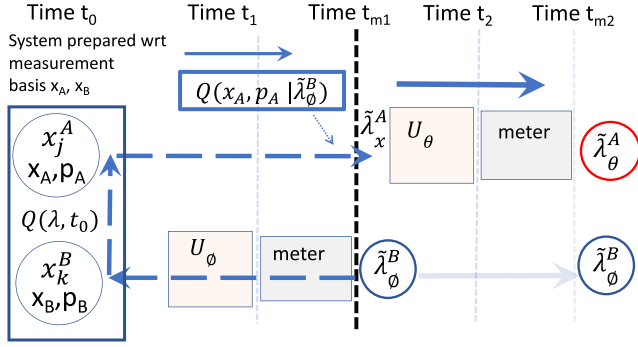


FIG. 33. Bell nonlocality is consistent with the three premises of weak macroscopic realism. The diagram sketches the relations based on the trajectories of the simulation of measurements $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ on spacelike-separated systems A and B, with all systems stationary in a frame F . Here, U_θ and U_ϕ represent the unitary operations that determine the settings θ and ϕ . The meter corresponds to the amplification H_{amp} . The first premise [premise wMR(1)] asserts that the outcome of the measurement $\hat{x}_{\phi B}$ is determined at time t_{m1} (vertical black dashed line), after the amplification due to the meter at B. The value for the outcome is denoted $\tilde{\lambda}_\phi^B$. The second premise [premise wMR(2)] asserts that this value is not changed by any later interaction at A (i.e., due to the meter at A) at a time $t > t_{m1}$ (indicated by a faint blue horizontal arrow). The third premise [premise wMR(3)] asserts that, at time t_{m1} , a projected distribution $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ is defined for system A, which determines the outcomes of $\hat{x}_{\theta A}$ of system A (for any future change of setting θ at A) conditioned on the outcome $\tilde{\lambda}_\phi^B$ at B (indicated by dashed blue arrows).

Result IX.7 :Causal consistency. This is the bipartite extension of Result V.1 (Sec. VB). At any given time t , the Q function $Q(\lambda, t)$ is defined as a probability distribution, which uniquely represents the quantum state and which hence evolves causally in the forward-time direction. The distribution $Q(\lambda, t)$ is consistent with the probability densities of the forward-backward trajectories. The causal consistency is depicted in Fig. 3. We extend the proof presented for Result V.1 to bipartite systems in Appendix H.

Result IX.8: Resolving the paradox of the future boundary condition: Timing of emergence of Bell nonlocality. The Bell violations are not put in by hand, as part of the future boundary condition: In the causal model (proposed in Sec. V), the correlations that lead to detection of Bell nonlocality do not arise from the future boundary condition at the time t_f , as indicated in Fig. 35. They are determined at a time t_{Bell} prior to t_f .

So, when do the Bell correlations emerge? We see from Result VII.12 that the correlation between the outcomes $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ is determined at the time $t_{\text{Bell}} = t_2$, after the interactions H_θ^A and H_ϕ^B (refer to Fig. 22). This follows from Results III.1 and VII.2, and the proposed causal and HV model, $\mathcal{P}_B$ (Sec. VIID), which apply after both settings θ and ϕ are fixed. The correlation between the outcomes $\hat{x}_{\theta A}$ and $\hat{x}_{\phi B}$ is directly determined by the correlation between the means, $x_{\theta j}^A$ and $x_{\phi k}^B$, of the Gaussians in the Q function $Q(\lambda_{\text{rot}}, t_2)$, defined at time t_2 . We refer to this time as the “time of emergence of Bell nonlocality.”

However, we also see that, if the setting changes are sequential, a hidden Bell nonlocality emerges at an earlier time,

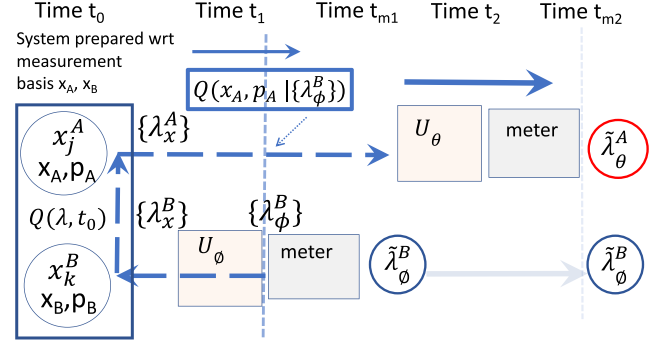


FIG. 34. Bell nonlocality is consistent with the three premises of weak local realism. The first premise [premise wLR(1)] asserts that the outcome (or, at least, the probabilities for outcomes) of the measurement $\hat{x}_\phi^B$ at B is determined at time t_1 , after the operation U_ϕ (vertical light blue dashed line at time t_1). In the simulation, based on the hidden variable model $\mathcal{P}_B$ [Eq. (7.38)], the probabilities of outcomes for $\hat{x}_\phi^B$ are determined by variables $\{x_{\phi B}, p_{\phi B}\}$ which we symbolize by $\{\lambda_\phi^B\}$. The second premise [premise wLR(2)] asserts that the outcome (or else, the probabilities of outcomes) are not changed by any later interaction (i.e., U_θ) at A. The third premise [premise wLR(3)] posits that at time t_1 , a projected Q function $Q(x_A, p_A | \{\lambda_\phi^B\})$ is defined, which determines the state of system A (for any future change of setting θ at A) conditioned on the variables $\{\lambda_\phi^B\}$ at B (blue dashed arrows).

after the change of just one setting (Result IX.4). This is evident by Eqs. (7.24)–(7.35) (Sec. VII B) and is depicted in Fig. 35. First, we assume wMR. According to premise wMR(3), the distribution $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ for outcomes at A given an outcome $\tilde{\lambda}_\phi^B$ at B is determined at (or by) the time t_{m1} , when system B has been amplified (and hence a branch $\tilde{\lambda}_\phi^B$ and

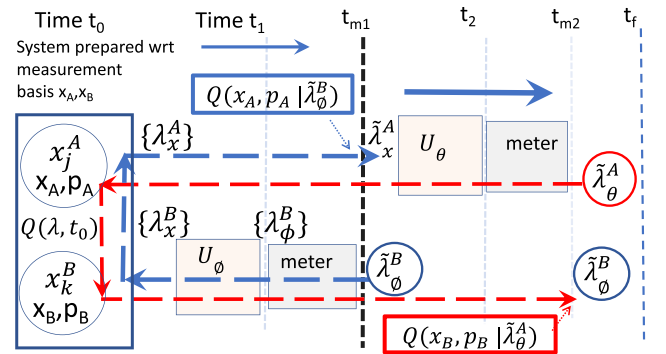


FIG. 35. Resolving the paradox of the future boundary condition: The timing of the emergence of Bell nonlocality. The system is as described for Fig. 33. A hidden nonlocality emerges at (or by) the time t_{m1} (vertical black dashed line), prior to the specification of the future boundary at time $t_f > t_{m2}$. According to premise wMR(3), the Q function $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$, which determines the state of system A (for any future change of setting θ induced by U_θ at A) conditioned on the outcome $\tilde{\lambda}_\phi^B$, is determined at (or by) time t_{m1} (blue dashed lines). Hence, the nonlocality is determined at time t_{m1} , when the conditional state $Q(x_A, p_A | \tilde{\lambda}_\phi^B)$ is specified, but *only over both operations* U_ϕ and U_θ at time t_{m2} does an observable Bell nonlocality emerge. The *values* for the outcomes that demonstrate the nonlocality are hence determined at (or by) the time $t_{m2} < t_f$.

outcome specified). Since $Q(x_A, p_A|\tilde{\lambda}_\phi^B)$ fully determines the outcomes for A , which will exhibit the Bell nonlocality after a further setting change θ is made, it can also be argued that the nonlocality has emerged at (or by) this time $t_{\text{Bell}} = t_{m1}$. On the other hand, if we assume wLR, the distribution for outcomes at A can be defined earlier, at the time t_1 when the setting at B is fixed, so that $t_{\text{Bell}} = t_1$.

This leaves us with two final results.

Result IX.9: Paradox of time-order: Lorentz invariance. At first glance, the concepts of weak macroscopic realism and weak local realism appear paradoxical. In Figs. 33 and 34, we have considered that the setting of A is changed to θ , after that of B , but for spacelike-separated systems such as A and B , there is no specified time order of events, which can change in different frames. According to wMR, the value of outcome at B is fixed and determined (as $\tilde{\lambda}_\phi^B$) at time t_{m1} , and this conditions the outcome at A at the later time t_{m2} after the setting change θ at A . But in the different frame, the outcome at A is fixed and determined first, prior to that of B . Hence, there is a paradox.

The paradox may be explained by the symmetry and mutual consistency allowed by the projected states of the Q function (Fig. 30). The outcome at A (say) is determined not only by the nonlocal setting change at B , but by the local setting change at A (and vice versa). The projected states given by $Q(x_A, p_A|\tilde{\lambda}_\phi^B)$ and $Q(x_A, p_A|\tilde{\lambda}_\theta^B)$ (blue and red dashed lines) in Fig. 30 are mutually consistent, being independent of the time order of the setting interactions H_θ and H_ϕ . They are consistent with each other and the initial state $Q(\lambda, t_0)$, and with the final joint probabilities determined by the marginal $Q(x_{\theta A}, x_{\phi B}, t_{mA})$ of the final Q function of the amplified state, which is also independent of the time order of the settings.

Result IX.10: Breakdown of Bell's assumptions: Failure of statistical independence or local causality? In the Introduction, we ask what assumptions of Bell's local hidden variable (LHV) model break down, according to the Q -based model? Bell's local hidden variable condition is

$$P_{++}^{AB}(\theta, \phi) = \int d\lambda \rho(\lambda) P_A(+|\lambda, \theta) P_B(+|\lambda, \phi), \quad (9.2)$$

as defined by Eq. (7.3), where $P_{++}^{AB}(\theta, \phi)$ is the probability of obtaining +1 at both sites with settings θ and ϕ ; $\rho(\lambda)$ is the distribution for hidden variables λ , and $P_A(+|\lambda, \theta)$ [$P_B(+|\lambda, \phi)$] is the probability for +1 at A [B], given λ and θ [ϕ]. Bell nonlocality is the breakdown of the condition (9.2).

Bell's assumption may break down due to a failure of the existence of a distribution $\rho(\lambda)$ that defines hidden variables λ for the system, as it exists at a time t_0 prior to the implementation of the setting transformations. Such a distribution is *statistically independent* of any future choice of measurement settings. Another possibility is a breakdown of the assumption of the factorization in the integrand (*local causality*).

In the simulations, the hidden variables λ are the coordinates x_A , p_A , x_B , and p_B of the Q function. The interactions H_θ and H_ϕ that determine the measurement settings at A and B , respectively, are local and deterministic. The amplitudes x_θ^A , x_ϕ^B of the simulation, if directly measurable, would lead to moments satisfying Bell's condition, with $\rho(\lambda) \equiv Q(\lambda, t_0)$ where $\lambda \equiv (x_A, p_A, x_B, p_B)$. However, this is not the case, because of the hidden noise, which is not measured, associated

with noise inputs at the future boundary, and also because of the hidden interference (Sec. III B).

The Bell violations require changes of settings θ and ϕ at both sites, so that the Bell HV state $\{\lambda\}$ is considered with respect to rotations to two different bases, one for each system A and B . After both unitary rotations, we have seen from Results VII.5 and IX.4 that interference terms that are hidden at the original time t_0 can contribute to probabilities for observable outcomes. For certain conditions, the contribution can be nonlocal (Result VII.6), implying

$$P_{++}^{AB}(\theta, \phi) \neq \int d\lambda Q(\lambda, t_0) P_A(+|\lambda, \theta) P_B(+|\lambda, \phi). \quad (9.3)$$

It is possible to deduce the reason for the breakdown of the condition (9.3) in the Q model, by evaluating from the simulation a probability $P_{++}^{AB}(\lambda, \theta, \phi)$ for outcomes given the settings θ and ϕ , and given a set of values of the hidden variables λ , where $\lambda \equiv (x_A, p_A, x_B, p_B)$ are defined by the Q function. We can write

$$P_{++}^{AB}(\theta, \phi) = \int d\lambda Q(\lambda, t_0) P_{++}^{AB}(\lambda, \theta, \phi). \quad (9.4)$$

The evaluation is explained in Sec. VII D (Result VII.9b), where a probability $P_{\text{Bell}}(\mathcal{S}_A, \mathcal{S}_B|\lambda, \theta, \phi)$ is deduced for spin outcomes $\mathcal{S}_A$ and $\mathcal{S}_B$ at A and B , given the hidden variables $\lambda \equiv (x_A, p_A, x_B, p_B)$ and settings θ and ϕ . We obtain $P_{++}^{AB}(\lambda, \theta, \phi) = P_{\text{Bell}}(1, 1|\lambda, \theta, \phi)$. This provides a nonlocal model $\mathcal{P}_{\text{Bell}}$ to explain the violations, with a statistically independent distribution $\rho(\lambda) = Q(\lambda, t_0)$ that describes the system at time t_0 , prior to the dynamics associated with the choice of settings. The causal consistency of the model is depicted in Fig. 3 and proved in Appendix H. At any time t , the Q function is defined as a fixed distribution, consistent with the forward-backward trajectories. The locality assumption implies that it is possible to write $P_{++}^{AB}(\lambda, \theta, \phi) = P_A(+|\lambda, \theta) P_B(+|\lambda, \phi)$. Hence, the Q -based model implies a breakdown of the locality assumption.

X. CONCLUSIONS

A summary of the main results of this paper has been given in the Introduction. The simulations we present are motivated by the Q function in quantum optics and contribute toward an explanation of how a hidden variable theory may complete quantum mechanics, while remaining consistent with the predictions of quantum mechanics, in particular, allowing violation of Bell inequalities. In the model used in this paper, the measurement proceeds as amplification, which we show plays a role similar to decoherence: The interference terms that distinguish a superposition from a mixed state decay with amplification, and hence do not contribute to directly measured probabilities (Results III.1 and VII.2).

In Secs. II–IV, we prove and demonstrate an equivalence of the evolution of the Q function $Q(x, p, t)$ (which represents the quantum state) to a forward-backward stochastic dynamics of amplitudes, x and p . The Q -based model is presented in Sec. IV, with a derivation of Born's rule and the role of the Wigner function. We treat entangled systems, modeling EPR and Bell correlations, in Secs. VI–IX. A causal (“cause-and-effect”) model for the dynamics is given in Secs. V and

IX, along with a proof of causal consistency and explanation of why there is no retrocausality at a macroscopic level (Results V.1, V.2, IX.6–IX.9, and Appendix H).

The collapse of the wave function is analyzed by examining a postselected state conditioned on an outcome x_j (Results IV.9 and IV.10). We include in Sec. VIII a treatment of projection, analyzing a Schrödinger-cat state where a system is entangled with a meter. We conclude that the system is inferred to be in an eigenstate, because the observer records the value of an amplified variable x , losing information about the complementary variable p (Result VIII.1). We examine the forward-backward dynamics of the variables associated with a given branch and outcome x_j , thus deducing “hidden loops” (Result IV.12).

The simulations are consistent with a Q -based model of reality in which three weak local realistic premises (*weak local realism*, wLR) hold (Results IV.14, VII.11, VIII.3, IX.3). There is a partial relaxation of EPR’s and Bell’s local realistic premises so that elements of reality exist but are defined for the system as it exists *after* interaction with devices (i.e., a polarizer or phase-shifter) that adjust measurement settings. Based on the observation of branches in the simulation, there is also consistency with *weak macroscopic realism* (wMR): predetermined values for outcomes emerge at a time t_m , after sufficient amplification (Results IV.6–IV.8, V.2, VII.10, VIII.2, IX.2, and Results VI.1–VI.2). The simulation demonstrates how the weak local realistic premises are consistent with Bell nonlocality (Figs. 33 and 34). By analyzing the dynamics of the Q function and of the trajectories (Secs. VII B and IX), a mechanism for Bell nonlocality is given in terms of hidden interference terms that contribute to measurable probabilities only after a change of measurement setting θ and ϕ at both sites (Results VII.5, IX.4). For certain conditions on θ and ϕ , the contribution can lead to a failure of Bell’s local-hidden-variables assumptions (Results VII.6–VII.8). The mechanism explains how Bell nonlocality can be consistent with macroscopic realism and no-signaling (Results IV.6–IV.8, V.2, IX.2–IX.4). The solutions of the forward-backward equations for entangled systems show consistency with the mechanism for Bell nonlocality, based on a model for projection (Sec. IX).

The theory presented in this paper explains why there is no genuine retrocausality, which implies that a system is changed by a future event. The realist variables λ defined for a macroscopic system are (according to the wLR and wMR premises) unchanged by a future event, and unchanged by a spacelike-separated event (Results IV.6–IV.8, V.2, IX.2, and IX.3). The causal consistency as depicted in Figs. 3 and 5 explains the mechanism by which this is possible. The Q function $Q(\lambda, t)$ evolves causally in the forward-time direction, and yet is consistent with the future boundary condition and counterpropagating trajectories (Figs. 33 and 34). The paradox of the future boundary condition (Fig. 35) is resolved by the nature of the causal (cause-and-effect) model based on the simulation, which involves causal deterministic relations for amplified variables (once settings are fixed) and a type of retrocausality (meaning “backward-in-time” propagation) for random noise inputs at the vacuum level (Secs. V, VII, and IX).

Interpretations. An interesting question is whether the Q -based model of reality favors any particular interpretation of Bell nonlocality. There are real values $\hat{\lambda}$ posited for the outcome of the measurement that are defined for the system as it exists at the time *after* the measurement settings have been adjusted and after amplification, which supports a macroscopic-realist viewpoint. There is no contradiction with the known violation of Leggett-Garg inequalities because the derivation of those inequalities makes the additional assumption of noninvasive measurability [101]. There is also no contradiction with the existence of macroscopic superposition states, because the Q -based reality model defines distributions of amplitudes (“states”) for the systems that are not defined as states for the system in standard quantum mechanics. Moreover, we have constructed (nonlocal) hidden-variable (HV) models based on the simulation (Results IV.13, VII.9). The hidden variables λ in these models are defined for the system at a time t_0 after preparation of the state of the system but before the choice of settings which are considered independent of λ , similar to Bell’s model (but apparently different to explanations of nonlocality based on superdeterminism).

A possible explanation of Bell nonlocality is that the hidden-variable distribution $\rho(\lambda)$ in the Bell model (9.2) is not statistically independent of future settings. The causal consistency provided by the Q model (refer to Results V.1, IX.7, and Appendix H) shows that the distribution $\rho(\lambda) \equiv Q(\lambda, t_0)$, where $\lambda \equiv (x_A, p_A, x_B, p_B)$ is fixed at time t_0 (Figs. 5 and 35), contrary to this explanation. The hidden-variable model $\mathcal{P}_{\text{Bell}}$ (Result VII.9, Sec. VII D) gives predictions for the outcomes, conditioned on a set of hidden variables $\lambda = (x_A, p_A, x_B, p_B)$ that describe the system at the time t_0 (prior to selection of the measurement settings). We conclude that in the Q model, the violation arises due to a failure of the Bell-locality assumption.

Experiments. Finally, we consider experimental tests that might be performed. The interaction H_{amp} of Eq. (2.2) is that of parametric amplification [128], which has been experimentally realized [129,130]. The states studied in this paper are superpositions of squeezed or coherent states which have also been realized experimentally [150–158]. Hence, the measurement model can be verified experimentally. A feature would be to demonstrate the vanishing of the interference terms in the Q function, as the system is amplified.

The prediction of causal consistency can be tested, as depicted in Figs. 3 and 5. It is feasible to perform a simulation that matches boundary conditions given by the experimental Q functions, which are measurable by tomography [159]. The density of amplitudes $x(t)$ and $p(t)$ in the simulation must match the distribution given by an experimentally measured Q function $Q(x, p, t)$, regardless of the choice of time t_f of the future boundary condition.

A second experiment might illustrate a hidden loop and the cat paradox (Fig. 4). A hidden loop is defined so as to involve amplitudes that are not amplified (and hence not observed) in the measurement, and a distribution Q_{loop} that cannot be described by a quantum wave function (Sec. IV E). The Q_{loop} for an individual branch of the superposition can be constructed from the simulation that is based on experimental Q functions. If Q_{loop} shows a reduction of variances below the Heisenberg

uncertainty bound, then, within the constraints of the model, this signifies a hidden loop.

The simulations give an explanation of the EPR paradox, in terms of the “weak elements of reality” that exist after sufficient amplification, as calculated in Fig. 20 [109,142]. The simulations might be useful to analyze questions raised by Schrödinger about the EPR argument, as in the recent experiment of Colciaghi *et al.*, which reported EPR correlations for Bose-Einstein condensates [123].

ACKNOWLEDGMENTS

This publication was made possible through the support of Grant No. 62843 from the John Templeton Foundation. The opinions expressed in this publication are those of the author(s) and do not necessarily reflect the views of the John Templeton Foundation. This research has also been supported by the Australian Research Council Grants schemes under Grants No. DP180102470 and No. DP190101480, and the authors thank NTT Research for their financial and technical support.

APPENDIX A: STOCHASTIC PATH INTEGRAL THEOREMS

1. Definitions and notation

We make use of methods that were first used for diffusive path integrals [160,161], here generalized to forward-backward trajectories. For M bosonic modes, the phase-space vector λ is a $2M$ -dimensional real vector, and $\mathbf{p}, \mathbf{x}$ are M -dimensional real vectors with $\lambda = (\mathbf{x}, \mathbf{p})$. For the measurement interactions treated in this paper, the generalized Fokker-Planck equation (GFPE) satisfied by a Q function is

$$\dot{Q}(\lambda, t) = \mathcal{L}(\lambda)Q(\lambda, t). \quad (\text{A1})$$

The differential operator $\mathcal{L}(\lambda)$ has first- and second-order differential terms $\mathcal{L}_1(\lambda)$ and $\mathcal{L}_2(\lambda)$, each with forward and backward components $\mathcal{L}_{np}(\mathbf{p})$, $\mathcal{L}_{jx}(\mathbf{x})$, so that

$$\mathcal{L}(\lambda) = \mathcal{L}_1(\lambda) + \mathcal{L}_2(\lambda), \quad (\text{A2})$$

$$= \sum_n [\mathcal{L}_{np}(\mathbf{p}) - \mathcal{L}_{nx}(\mathbf{x})], \quad (\text{A3})$$

where $\mathcal{L}_{jp,x}$ are defined as

$$\mathcal{L}_{1p} = - \sum_j \partial_j^p a_j^i(\mathbf{p}), \quad (\text{A4})$$

$$\mathcal{L}_{2p} = \frac{1}{2} \sum_j \partial_j^p \partial_j^p d^j. \quad (\text{A5})$$

Here $\mathcal{L}_x$ is identical to $\mathcal{L}_p$ except for the substitution of x for p , and we define $\partial_j^p \equiv \partial/\partial p^j$, $\partial_j^x \equiv \partial/\partial x^j$. For the Hamiltonians in this paper, we have $\mathbf{a}_p = \mathbf{a}_p(\mathbf{p})$ and $\mathbf{a}_x = \mathbf{a}_x(\mathbf{x})$, with diagonal diffusion $\mathbf{d}$.

For use in the theorems proved below, we define a phase-space trajectory and stochastic path integral as follows: First, the trajectory is a sequence $[\lambda(t_0), \dots, \lambda(t_N)] \equiv [\lambda_0, \dots, \lambda_N]$, with a total action of

$$S_{kn} = S_{kn}^p + S_{kn}^x = \sum_{m=k+1}^n S_m. \quad (\text{A6})$$

The “one-step” action for the n th step is given by $S_n = S_n^p + S_n^x$, with S_n^p having the same form as S_n^x except for the changed superscripts, where:

$$S_n^p = S^p(\lambda_{n-1}, \lambda_n) = \sum_j \left[\frac{\epsilon}{2d^j} |v_n^{j,p}|^2 + \ln(\sqrt{\mathcal{N}^j}) \right]. \quad (\text{A7})$$

This includes a path-integral normalization factor, $\mathcal{N}^j = (2\pi\epsilon d^j)$, and relative velocity fields $\mathbf{v} \equiv (\mathbf{v}^p, \mathbf{v}^x)$ which are functions of neighboring coordinates:

$$\mathbf{v}_n^p \equiv \frac{1}{\epsilon} (\mathbf{p}_n - \mathbf{p}_{n-1}) - \mathbf{a}^p(\mathbf{p}_{n-1}),$$

$$\mathbf{v}_n^x \equiv \frac{1}{\epsilon} (\mathbf{x}_{n-1} - \mathbf{x}_n) - \mathbf{a}^x(\mathbf{x}_n). \quad (\text{A8})$$

The N -step total path probability $\mathcal{P}$ for a path $\underline{\lambda} = (\lambda_0, \dots, \lambda_N)$ at times $t_k = t_0 + k\epsilon$, with $k = 0, \dots, N$, is

$$\mathcal{P}[\underline{\lambda}] = e^{-S_{0N}}. \quad (\text{A9})$$

The two-time propagator G is then defined as

$$G(\lambda, t_k | \tilde{\lambda}, \tilde{t}) = \lim_{\epsilon \rightarrow 0} \int \delta(\lambda - \lambda_k) \delta(\tilde{\lambda} - \lambda_{0N}) \mathcal{P}[\underline{\lambda}] d\underline{\lambda}. \quad (\text{A10})$$

Here, $d\underline{\lambda} = \prod_{n=0}^N d\lambda_n = \prod_{n=0}^N \prod_{k=1}^M dx_n^k dp_n^k$, $\lambda_{0N} = (\mathbf{x}_N, \mathbf{p}_0)$, while $\tilde{\lambda} = (\mathbf{x}_f, \mathbf{p}_i)$ gives the initial momentum $\mathbf{p}_i$ and final position $\mathbf{x}_f$ at times $\tilde{t} = (t_i, t_f) = (t_0, t_N)$.

Marginals have the notation $P(\mathbf{x}, t) = \int Q(\lambda, t) d\mathbf{p}$, with the variables that are integrated removed from the arguments. Boundary conditions are imposed on the final marginal $P(\mathbf{x}, t_f)$ and initial conditional, $P(\mathbf{p}|\mathbf{x}, t_i) = Q(\lambda, t_i) P^{-1}(\mathbf{x}, t_i)$. The remainder of this section shows the equivalence of the differential and path-integral approaches, both analytically and numerically.

2. Path-integral equivalence theorems

Theorem A1. Any path-integral Q function Q_{pi} obtained from multiplication by the joint probabilities at the boundaries satisfies the Q function dynamical equations, where

$$Q_{pi}(\lambda, t) = \int G(\lambda, t | \tilde{\lambda}, \tilde{t}) P(\tilde{\lambda}, \tilde{t}) d\tilde{\lambda}. \quad (\text{A11})$$

The quantum Q function solution $Q = Q_{pi}$ is found upon solving simultaneous equations for the joint distribution $P(\tilde{\lambda}, \tilde{t})$, and hence for Q_{pi} , where

$$\int Q_{pi}(\lambda, t_f) d\mathbf{p} = P(\mathbf{x}, t_f), \quad (\text{A12})$$

and:

$$\frac{Q_{pi}(\lambda, t_i)}{\int Q_{pi}(\lambda, t_i) d\mathbf{p}} = P(\mathbf{p}|\mathbf{x}, t_i). \quad (\text{A13})$$

Proof. We wish to show that $Q_{pi} = Q$, by proving that Q_{pi} satisfies the GFPE (A1), and has an initial condition that corresponds to the Q function for the initial quantum state. From linearity, proving that $Q_{pi}(\lambda)$ satisfies the GFPE (A1) is achieved by verifying this for the propagator, $G(\lambda, t | \tilde{\lambda}, \tilde{t})$. To show this, define advanced and retarded propagators for λ at

$t = t_j$ with $\tilde{\mathbf{p}} = \mathbf{p}_0$ at $t_i = t_0$ and $\tilde{\mathbf{x}} = \mathbf{x}_N$ at $t_f = t_N$, as

$$G^r(\mathbf{p}_j, t | \tilde{\mathbf{p}}, t_i) = \lim_{\epsilon \rightarrow 0} \int e^{-S_{0j}^p} \prod_{n=1}^{j-1} d\mathbf{p}_n, \\ G^a(\mathbf{x}_j, t | \tilde{\mathbf{x}}, t_f) = \lim_{\epsilon \rightarrow 0} \int e^{-S_{jN}^x} \prod_{n=j+1}^{N-1} d\mathbf{x}_n. \quad (\text{A14})$$

Due to its path-integral construction, G^r satisfies a forward Kolmogorov equation in $\mathbf{p}$ [162,163]. By reversing the sign of t , we find that G^a also satisfies a forward Kolmogorov equation in $\mathbf{x}$, but in the negative time direction:

$$\dot{G}^r(\mathbf{p}, t | \tilde{\mathbf{p}}, t_i) = \mathcal{L}^p G^r(\mathbf{p}, t | \tilde{\mathbf{p}}, t_i), \\ \dot{G}^a(\mathbf{x}, t | \tilde{\mathbf{x}}, t_f) = -\mathcal{L}^x G^a(\mathbf{x}, t | \tilde{\mathbf{x}}, t_f). \quad (\text{A15})$$

The normalization of the advanced Gaussian propagator terms in the path integral for $\mathbf{x}_n$ with $n < j$ means that all these past time factors integrate to unity. Similarly, the retarded propagator is independent of $\mathbf{p}_n$ for future time points $n > j$, as these also integrate to give unity. As a result, we can write that

$$G^r(\mathbf{p}, t_k | \tilde{\mathbf{p}}, t_0) = \lim_{\epsilon \rightarrow 0} \int e^{-S_{0N}^p} \delta(\mathbf{p} - \mathbf{p}_k) \delta(\tilde{\mathbf{p}} - \mathbf{p}_0) d\mathbf{p}, \\ G^a(\mathbf{x}, t_k | \tilde{\mathbf{x}}, t_f) = \lim_{\epsilon \rightarrow 0} \int e^{-S_{0N}^x} \delta(\mathbf{x} - \mathbf{x}_k) \delta(\tilde{\mathbf{x}} - \mathbf{x}_N) d\mathbf{x}. \quad (\text{A16})$$

From these results and the definitions above, it follows that the total propagator factorizes as $G(\lambda, t | \tilde{\lambda}, \tilde{t}) = G^r(\mathbf{p}, t | \tilde{\mathbf{p}}, t_0) G^a(\mathbf{x}, t | \tilde{\mathbf{x}}, t_f)$. Hence, using the chain rule for differentiation and Eq (A15), the required time-evolution can be written as

$$\dot{G}(\lambda, t | \tilde{\lambda}) = \dot{G}^r(\mathbf{p}, t | \tilde{\mathbf{p}}, t_0) G^a(\mathbf{x}, t | \tilde{\mathbf{x}}, t_f) \\ + G^r(\mathbf{p}, t | \tilde{\mathbf{p}}, t_0) \dot{G}^a(\mathbf{x}, t | \tilde{\mathbf{x}}, t_f) \\ = (\mathcal{L}^p - \mathcal{L}^x) G(\lambda, t | \tilde{\lambda}, \tilde{t}). \quad (\text{A17})$$

Due to linearity, any integral of G over its boundary values also satisfies the GFPE. Therefore, the path integral construction must obey the required GFPE,

$$\dot{Q}_{pi}(\lambda, t) = [\mathcal{L}_p(\mathbf{p}) - \mathcal{L}_x(\mathbf{x})] Q_{pi}(\lambda, t). \quad (\text{A18})$$

Provided the joint probability $P(\tilde{\lambda}, \tilde{t})$ satisfies the boundary equations, one can verify that at the initial time, $Q_{pi}(\lambda, t_0) = Q(\lambda, t_0)$. Due to the uniqueness of the solutions to a first-order differential equation, Q_{pi} is equal to the quantum-mechanical Q function solution for all times.

As a further check, in the short-time limit of $t_0 = t_f$ we find that

$$\lim_{t_0, t_f \rightarrow t} G(\lambda, t | \tilde{\lambda}) = \delta(\lambda - \tilde{\lambda}), \quad (\text{A19})$$

hence, as $t_0, t_f \rightarrow t$, one has that

$$\lim_{t_0, t_f \rightarrow t} Q_{pi}(\lambda, t) = \int \delta(\lambda - \tilde{\lambda}) P(\tilde{\lambda}, t') d\tilde{\lambda} \\ = P(\mathbf{p} | \mathbf{x}, t') P(\mathbf{x}, t'). \quad (\text{A20})$$

From the definition of the conditional probability, this is the initial Q function.

Theorem A2. The path-integral solution corresponds to a time-symmetric stochastic differential equation (TSSDE) or forward-backward SDE, with a conditional initial distribution in $\mathbf{p}$ and final marginal distribution in $\mathbf{x}$:

$$p^j(t) = p^j(t_0) + \int_0^t a_p^j(t') dt' + \int_0^t dw_p^j, \\ x^j(t) = x^j(t_f) + \int_t^{t_f} a_x^j(t') dt' + \int_t^{t_f} dw_x^j. \quad (\text{A21})$$

Here $\mathbf{x}(t_f)$ is distributed as $P(\mathbf{x}, t_f)$, while $\mathbf{p}(t_0)$ is distributed conditionally as $C(\mathbf{p} | \mathbf{x}(t_0), t_0)$, and $[dw^\mu] = (d\mathbf{w}_x, d\mathbf{w}_p)$ are independent real Gaussian noises correlated as

$$\langle dw^\mu dw^\nu \rangle = \delta^{\mu\nu} d\epsilon, \quad (\text{A22})$$

where $\mu = 1, \dots, 2M$ and $d^{j+M} = d^j$.

Proof. The TSSDE is obtained by discretizing the equation for times $t_k = t_0 + k\epsilon$, with $k = 0, \dots, N$, and then taking the limit of $\epsilon \rightarrow 0$. We define $[\lambda] = [\lambda_0, \lambda_1, \dots, \lambda_N]$ as the stochastic path. The discretized solutions are then given as the simultaneous solutions of the equations, for $k = 1, \dots, N$,

$$\mathbf{p}_k = \mathbf{p}_{k-1} + \mathbf{a}^p(\mathbf{p}_{k-1})\epsilon + \Delta_k^p, \\ \mathbf{x}_{k-1} = \mathbf{x}_k + \mathbf{a}^x(\mathbf{x}_k)\epsilon + \Delta_k^x. \quad (\text{A23})$$

To obtain an equivalent path integral to the TSSDE, we first obtain the N -step trajectory probability density, conditioned on random noises $[\Delta]$. This is a product of Dirac delta functions,

$$\mathcal{G}([\lambda] | [\Delta]) = \prod_{j=1}^N \delta^{2M}(\epsilon \mathbf{v}_j - \Delta_j), \quad (\text{A24})$$

which gives a normalized probability conditioned on a specific noise vector $[\Delta]$. Solving for the resulting set of trajectory values $[\lambda]$ that satisfy the delta-function constraints is straightforward in the parametric amplifier case, due to the decoupling of forward and backward equations.

In a Fourier transform representation with $\mathbf{k}_j \equiv (k_j^1, k_j^2, \dots, k_j^{2M})$, one can expand the delta functions as

$$\mathcal{G}_n([\lambda] | [\Delta]) = \prod_{j=1}^N \int \frac{d\mathbf{k}_j}{(2\pi)^{2M}} e^{-i\mathbf{k}_j(\mathbf{v}_j \epsilon - \Delta_j)}. \quad (\text{A25})$$

The $2M$ real Gaussian noises Δ_k at each step in time, for $k > 0$, are distributed as

$$P(\Delta_k) = \frac{1}{(2\pi\epsilon d)^M} e^{-|\Delta_k|^2/(2\epsilon d)}. \quad (\text{A26})$$

On integration over $\mathbf{k}_j$ one obtains the path probability result as defined previously:

$$\mathcal{G}_n([\lambda]) = \int \mathcal{G}_n([\lambda] | [\Delta]) P([\Delta]) d[\Delta]. \quad (\text{A27})$$

By construction, the initial values of $\mathbf{x}_N$ and $\mathbf{p}_0$ are sampled according to the required joint and conditional probabilities. Hence, the probability of a TSSDE solution is Q_{pi} , which is equal to the quantum average Q from Theorem I. Our definitions are similar to those in the mathematics literature [137], except for the use of a conditional probabilistic boundary.

APPENDIX B: NUMERICAL χ^2 TESTS

The analytic theorems obtained in Appendix A were verified numerically using χ^2 tests in several cases described here, with details that are given below. The forward-backward stochastic equations were integrated by first propagating x backward in time, generating a conditional sample, then propagating p forward in time.

All trajectories plotted use 40 sample trajectories to provide an intuitive demonstration of how they behave. To verify the quantitative accuracy of the trajectory probability distributions, large numbers of samples were generated and plotted using binning methods.

These samples were statistically tested with χ^2 methods [164,165] to compare them with analytic solutions. The test cases used $N_s = 2 \times 10^6$ sample trajectories to obtain good statistics for the numerically sampled distributions,

$$p_{ijk}^{\text{samp}} = \frac{N_{ijk}}{N_s}. \quad (\text{B1})$$

Here N_{ijk} is the number of trajectories in the bin at x_i , p_j , and sampled time t_k . Such verification requires binning on a three-dimensional (x, p, t) grid to obtain numerical estimates of the integrated analytic probabilities p_{ijk} , where

$$p_{ijk} = \int_A d\delta x d\delta p Q(x_i + \delta x, p_j + \delta p, t_k). \quad (\text{B2})$$

This was evaluated by numerical integration of the analytic Q distribution, using a two-dimensional Simpson's rule integration in each bin.

To treat the dynamics of the Q function, time averages were evaluated to give a definitive overall result. Due to the correlations inside each trajectory, the *range* of fluctuations of time-averaged statistics is reduced compared with one-time tests, which exactly follow the χ^2 distribution. Individual tests at each time-point are in agreement with χ^2 statistics, and will be reported elsewhere.

We therefore define

$$\bar{\chi}^2 = \frac{1}{N_t} \sum_{i,j,k} \frac{\langle [p_{ijk} - p_{ijk}^{\text{samp}}]^2 \rangle}{\langle \sigma_{ijk}^2 \rangle}, \quad (\text{B3})$$

where

$$\sigma_{ijk}^2 = p_{ijk}/N_s \quad (\text{B4})$$

is the expected probability variance for N_s total samples and N_t time points, with a phase-space bin area of A .

As an example, the simulations described in Fig. 1 used 30 time-steps of $gdt = 0.1$, combined with a midpoint stochastic integration method for improved accuracy [166]. No significant discretization error improvements were found with smaller time-steps. Comparisons were made between the analytic and numerically sampled $Q(x, p, t)$ distributions with $dx = 0.02$ and $dp = 0.05$. This gave an average of ≈ 55 100 comparison grid-points at each time step, after discarding bins with nonsignificant sample populations of $N < 10$ [167].

In a typical test with 2×10^6 sample trajectories and $A = 10^{-3}$, the time-averaged statistical error was $\bar{\chi}^2 = 55.2 \times 10^3$, with 1.7×10^6 valid comparisons. There were an average

of $k = 55.1 \times 10^3$ significant points per time step. This shows that $\bar{\chi}^2$ is within the expected range of $\langle \chi^2 \rangle = k \pm \sqrt{2k}$.

Hence, as expected from the path-integral and stochastic theorems, there is excellent agreement between the analytic Q -function probability from quantum theory and the ensemble-averaged stochastic trajectories.

APPENDIX C: CALCULATION OF PROPERTIES AND Q FUNCTIONS

Q function of a squeezed state. The calculation of the Q function [Eq. (3.3)] of a squeezed state uses

$$S(r)|0\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} (-\tanh r)^n \frac{\sqrt{(2n)!}}{2^n n!} |2n\rangle$$

and $\langle 0|D(\beta - \alpha) = \langle \gamma|$ where $\gamma = \alpha - \beta$ to deduce that

$$\begin{aligned} \langle \alpha|D(\beta)S(r)|0\rangle &= \langle 0|D(-\alpha)D(\beta)S(r)|0\rangle \\ &= e^{(-\alpha\beta^* + \alpha^*\beta)/2} \langle 0|D(\beta - \alpha)S(r)|0\rangle \\ &= \frac{e^{(-\alpha\beta^* + \alpha^*\beta)/2}}{\sqrt{\cosh r}} e^{-|\gamma|^2/2} e^{[-(\tanh r)\gamma^*2/2]}. \end{aligned}$$

Q function of the amplified state. For the calculation of Eq. (3.13), we use $e^{-iH_{\text{amp}}t/\hbar} = S(-gt)$, implying squeezing in $\hat{p}$. Also $S(r)D(\beta) = D(\alpha)S(r)$ where $\alpha = \beta \cosh r - \beta^* \sinh r$. For β real, $\alpha = e^{-r}\beta$. Hence $S(-gt)D(\frac{x_1}{2}) = D(\frac{Gx_1}{2})S(-gt)$ and $S(-gt)S(r) = S(-gt + r)$ where $G = e^{gt}$ is the amplification factor.

Properties of the eigenstates of $\hat{x}$. It is easy to prove from the identities above that the eigenstates $|x_j\rangle$ defined by (3.2) as highly squeezed states in $\hat{x}$ satisfy $|\langle x_j|x_l\rangle|^2 = e^{-|x_j - x_l|^2} e^{2r}$, becoming orthogonal (since $r \rightarrow \infty$) for $j \neq l$. Completeness is assumed in the limit where the eigenvalues x_j form a continuous spectrum in analogy with the position representation and noting that the overlap functions (i.e., $\langle x_j^A|\alpha\rangle$) approach those (i.e., $\langle x|\alpha\rangle$) of the position representation as $r \rightarrow \infty$.

The Q function and quantum coefficients. We note that the quantum state $|\psi(t_0)\rangle$ can be expanded in terms of the eigenstates $|x_j\rangle$, as $|\psi(t_0)\rangle = \sum_j c_j |x_j\rangle$, where c_j are probability amplitudes, and the Q function is derived accordingly. The values x_j are an important part of the description of the model, since they are amplified by H_{amp} . The $|\psi(t_0)\rangle$ and hence the c_j are defined uniquely by $Q(\lambda, t_0)$ and can in principle be determined by $Q(\lambda, t_0)$. For example, for a given $Q(\lambda, t_0)$, the $|c_j|$ and x_j can be determined by applying H_{amp} in the limit of t . From Sec. III, we see that the Q function is of form $Q = \sum_i |c_i|^2 \mathcal{G}_i + \text{Int}$, where $\mathcal{G}_i$ are the known Gaussian functions involving x_j , and Int is an interference term, which can hence be deduced from knowledge of $|c_j|$ and x_j . For a simple two-state superposition as in Eq. (3.7), this implies the phase for c_j .

Eigenstates of rotated quadratures. The Q function of a rotated squeezed state can also be calculated. We consider

$$|\beta_j, z\rangle_{sq} = D(\beta_j)S(z)|0\rangle, \quad (\text{C1})$$

where $D(\beta_j) = e^{\beta_j \hat{a}^\dagger - \beta_j^* \hat{a}}$ and $S(z) = e^{\frac{1}{2}(z^* \hat{a}^2 - z \hat{a}^{\dagger 2})}$ are the displacement and squeezing operators, where z and β_j are complex numbers. We define the rotated quadrature phase

amplitudes as $\hat{x}_\theta = \hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}$ and $\hat{p}_\theta = (\hat{a}e^{i\theta} - \hat{a}^\dagger e^{-i\theta})/i$ i.e., $\hat{x}_\theta = \hat{x} \cos \theta + \hat{p} \sin \theta$ and $\hat{p}_\theta = -\hat{x} \sin \theta + \hat{p} \cos \theta$. For the state with squeezed fluctuations in $\hat{x}_\theta = \hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}$, we note that $z = re^{2i\theta}$ where r is a real, positive number. Hence $(\Delta \hat{x}_\theta)^2 = e^{-2r}$ and $(\Delta \hat{p}_\theta)^2 = e^{2r}$. We choose the direction of β_j to be along the direction of squeezing, so that $\beta_j = x_{\theta j} e^{i\theta}/2$. The eigenstate of $\hat{x}_\theta$ is denoted $|x_{\theta j}\rangle_\theta$ and is thus defined as the limiting squeezed state

$$|x_{\theta j}\rangle_\theta \equiv \lim_{r \rightarrow \infty} \left| \frac{x_{\theta j}}{2} e^{i\theta}, z \right\rangle_{sq}. \quad (\text{C2})$$

Where the meaning is clear, we abbreviate the notation to $|x_{\theta j}\rangle$. The Q function of $|x_{\theta j}\rangle$ is $Q(\alpha) = \frac{1}{\pi} |\langle \alpha | x_{\theta j} \rangle|^2$ where $\alpha = (x + ip)/2$ but we can calculate in the rotated axis, with amplitude $\alpha_{\text{new}} = (x_\theta + ip_\theta)/2$ and $x_\theta = x \cos \theta + p \sin \theta$, $p_\theta = -x \sin \theta + p \cos \theta$. Hence, the Q function $Q_{x_{\theta j}}(x_\theta, p_\theta)$ of the eigenstate $|x_{\theta j}\rangle$ is

$$Q_{x_{\theta j}}(x_\theta, p_\theta) = \frac{e^{-p_\theta^2/2\sigma_{p_\theta}^2} e^{-(x_\theta - x_{\theta j})^2/2\sigma_{x_\theta}^2}}{2\pi\sigma_{x_\theta}\sigma_{p_\theta}},$$

where $\sigma_{p_\theta}^2 = 1 + e^{2r}$ and $\sigma_{x_\theta}^2 = 1 + e^{-2r}$.

Extensions to general superpositions. The results of Eqs. (3.7)–(3.21) can be extended to the superposition $\sum_j c_j |x_j\rangle$ where $|x_j\rangle$ are modeled as highly squeezed states. The Q function is

$$Q(x, p, t) = N \frac{e^{-p^2/2\sigma_p^2(t)}}{2\pi\sigma_x(t)\sigma_p(t)} \left(\sum_j |c_j|^2 e^{-(x-G(t)x_j)^2/2} + \sum_{P \in \{j,k\}} 2|c_j c_k| \mathcal{I}_{jk}(t) \right), \quad (\text{C3})$$

where we sum over the pairs P given by all sets of the nonordered pair j and k . Here $\mathcal{I}_{jk}(t) = e^{-[(x-x_k)^2 + (x-x_j)^2]/4} \mathcal{F}_{jk}(t)$ and $(c_j = |c_j| e^{i\theta_j})$

$$\mathcal{F}_{jk}(t) = \cos(\theta_j - \theta_k) \cos[pG(t)(x_j - x_k)/2] + \sin(\theta_j - \theta_k) \sin[pG(t)(x_j - x_k)/2]. \quad (\text{C4})$$

We see that the calculation of the marginal $Q(x, t)$ will involve the integrals over p as given by Eq. (3.18), so that the interference terms will not contribute to $Q(x, 0)$. Result III.1 follows. The extension to mixed states is straightforward.

Similarly, the proof of Result VII.5 in Sec. VII can be generalized to hold for any arbitrary state $|\psi\rangle$ at time t_0 , by expanding in terms of the measurement basis, as $|\psi\rangle = \sum_{jk} c_{jk} |x_j^A\rangle |x_k^B\rangle$. Here, we assume the $\{|x_j^K\rangle\}$ form a complete orthogonal set for the states of system K (proved in Appendix C). The Q function of $|\psi\rangle$ is a sum of Gaussians and an interference term $\mathcal{I}_{AB}$. Similar to Eq. (C3), the $\mathcal{I}_{AB}$ is a sum of terms $\mathcal{I}_{AB,P}$ each involving a pair P of product eigenstates, labeled by $P \in \{\{j, k\}, \{j', k'\}\}$, where either $k \neq k'$, $j \neq j'$, or both. The calculation of the marginal $Q_m(x_A, x_B, t)$ involves integrals over p_A and p_B as in Eq. (7.31), for each pair. The integral is zero, so that the interference terms do not contribute to $Q_m(x_A, x_B, t)$. The same result applies when there is a rotation of setting at B , as in Eq. (7.32). With a rotation of setting at both A and B , each of the interference terms $\mathcal{I}_{AB,P}$ can contribute a nonzero value to the marginal

$Q(x_{\theta A}, x_{\phi B}, t)$ because the integration over $p_{\theta A}$ and $p_{\phi B}$ is not necessarily zero. This can only occur for the specified rotations, as evident by Eq. (7.36). The extension to mixed states is straightforward.

APPENDIX D: TRANSFORMATION IN Q FUNCTION DUE TO MEASUREMENT SETTINGS

The Q function $Q(\lambda_{\text{rot}}, t_1)$ of the new state at time t_2 after the interactions due to the H_θ and H_ϕ is that obtained by rotating the coordinates x_A, p_A, x_B, p_B in the original Q function, to new coordinates $x_{\theta A}, p_{\theta A}, x_{\phi B}, p_{\phi B}$. We see that

$$\alpha e^{i\theta} = (x_A + ip_A) e^{i\theta} = x'_A + ip'_A, \quad (\text{D1})$$

where $x'_A = x_A \cos \theta - p_A \sin \theta$ and $p'_A = x_A \sin \theta + p_A \cos \theta$. To obtain the Q function at time t_1 , we hence transform an initial Q function $Q(x_A, p_A)$ by substituting α for $\alpha e^{i\theta}$, which means substituting x_A and p_A for x'_A and p'_A , i.e., $x_A \rightarrow x_A \cos \theta - p_A \sin \theta$ and $p_A \rightarrow p'_A = x_A \sin \theta + p_A \cos \theta$ i.e., $x_A = x_{\theta A} \cos \theta - p_{\theta A} \sin \theta$ and $p_A = x_{\theta A} \sin \theta + p_{\theta A} \cos \theta$. The x_A and p_A in the rotated Q function are identified as $x_{\theta A}$ and $p_{\theta A}$, so that in terms of the original coordinates,

$$\begin{aligned} x_{\theta A} &= x_A \cos \theta + p_A \sin \theta, \\ p_{\theta A} &= -x_A \sin \theta + p_A \cos \theta, \end{aligned} \quad (\text{D2})$$

consistent with Eq. (7.1). Similarly, the change of setting at B induced by the interaction H_ϕ amounts to a rotation of coordinates

$$\begin{aligned} x_{\phi B} &= x_B \cos \phi + p_B \sin \phi, \\ p_{\phi B} &= -x_B \sin \phi + p_B \cos \phi, \end{aligned} \quad (\text{D3})$$

consistent with (7.2). Hence, the adjustment of measurement settings is carried out in the simulation by changing the Q function $Q(\lambda, t_0)$ to obtain a new Q function $Q(\lambda_{\text{rot}}, t_2)$ where $\lambda_{\text{rot}} = (x_{\theta A}, p_{\theta A}, x_{\phi B}, p_{\phi B})$.

APPENDIX E: TRANSFORMATION OF COORDINATES IN Q

The system interacts with the device so that the amplitudes are transformed as above in Appendix D. The transformation proceeds by substituting $x_B = x_{\phi B} \cos \phi - p_{\phi B} \sin \phi$ and $p_B = x_{\phi B} \sin \phi + p_{\phi B} \cos \phi$. (Note the Jacobian of the transformation is unity.) To transform the Gaussian $Q_G(x_B, p_B) = \frac{e^{-p_B^2/2\sigma_p^2}}{2\pi\sigma_x\sigma_p} e^{-(x_B - x_k^B)^2/2\sigma_x^2}$ with variances $\sigma_x^2 = 1 + e^{-2r}$ and $\sigma_p = 1 + e^{2r}$ where $r \rightarrow \infty$, we note that the term $\frac{1}{\sigma_x\sqrt{2\pi}} e^{-(x_B - x_k^B)^2/2}$ dominates. Since $x_k^B = x_k^B (\cos^2 \phi + \sin^2 \phi)$, this term becomes, after some manipulation,

$$\begin{aligned} &e^{-(x_B - x_k^B)^2/2} \\ &\rightarrow e^{-\frac{(x_{\phi B} - x_k^B \cos \phi)^2}{2\sigma_{x_{\phi B}}^2}} e^{-\frac{(p_{\phi B} + x_k^B \sin \phi)^2}{2\sigma_{p_{\phi B}}^2}} e^{-\frac{(x_{\phi B} - x_k^B \cos \phi)(p_{\phi B} + x_k^B \sin \phi)}{\sigma_{x_{\phi B}}\sigma_{p_{\phi B}}}}, \end{aligned}$$

where $\sigma_{x_{\phi B}}^2 = 1/\cos^2 \phi$ and $\sigma_{p_{\phi B}}^2 = 1/\sin^2 \phi$. The variances $\sigma_{x_{\phi B}}^2$ and $\sigma_{p_{\phi B}}^2$ of the transformed bivariate Gaussian are such

that neither is now necessarily infinite. Hence, provided $\phi \neq n\pi/2$, where n is an integer, it is possible that neither the variance of $\sigma_{p_{\phi B}}$ and $\sigma_{x_{\phi B}}$ will approach ∞ . We note that we have written the expressions (7.35) as above, in the limit of $r \rightarrow \infty$, which leads to functions that are not normalizable. The full correct solutions take into account the term $\frac{1}{\sigma_p \sqrt{2\pi}} \exp(-p_B^2/2\sigma_p^2)$, allowing for the finite variance of p . To illustrate, we consider the transformation $f_{x_k^B}(x_{\phi B}, p_{\phi B})$ of the full Gaussian $Q_G(x_B, p_B) = \frac{e^{-p_B^2/2\sigma_p^2}}{2\pi\sigma_x\sigma_p} e^{-(x_B-x_k^B)^2/2\sigma_x^2}$ distribution, as it is passed through the phase-shifter H_ϕ^B , modeling the rotations from x_B, p_B to $x_{\phi B}, p_{\phi B}$. The evolution of the $x_B(t)$ and $p_B(t)$ is plotted in Fig. 21, where the equalization of the variances in the $x_B(t)$ and $p_B(t)$ is evident for certain times t . The joint probability density of the $x_B(t)$ and $p_B(t)$ in Fig. 21 is given by the transformed function $f_{x_k^B}(x_{\phi B}, p_{\phi B})$.

The coefficients of transformation can be calculated. We rewrite the eigenstate $|x_k^A\rangle$ is the new basis $\{|x_{\theta l}^A\rangle\}$ as $|x_k^A\rangle = \sum_l d_{kl} |x_{\theta l}^A\rangle$ where $d_{kl} = \langle x_{\theta l}^A | x_k^A \rangle$ since the eigenstates are orthogonal and complete. Then for any single-mode state $|\psi\rangle = \sum_k c_k |x_k^A\rangle$, we find $|\psi\rangle = \sum_l f_l |x_l^A\rangle$ in the new basis, where $f_l = \sum_k c_k d_{kl}$. The $\langle x_{\theta l}^A | x_k^A \rangle$ are readily calculated.

APPENDIX F: RESULT VII.3 AND ASYMMETRIC LHV MODEL $\mathcal{P}_{\text{asym},A}$

Following the text, we consider the system prepared with respect to basis $\theta = \phi = 0$, and consider that there is then a change of measurement setting at one site, B , only. We hence rotate the basis at B , so that the system is prepared for the final coupling to the meter (along with amplification) to give the outcomes for a new measurement $\hat{B}_\phi$ ($\phi \neq 0$). Here, the predictions show no inconsistency with Bell's local hidden variable condition. The simulation is consistent with the premise that there is no impact on the observed value of $\hat{\lambda}_x^A$ at A by a change of setting at B .

We prove as follows: The initial system is expressed in terms of the prepared basis:

$$\begin{aligned} |\psi_{\text{Bell}}\rangle &= \sum_{ij} c_{ij} |i\rangle_A |j\rangle_B \\ &\rightarrow c_{11} |+\rangle |+\rangle + c_{12} |+\rangle |-\rangle \\ &\quad + c_{21} |-\rangle |+\rangle + c_{22} |-\rangle |-\rangle. \end{aligned} \quad (\text{F1})$$

Here $|i\rangle$ and $|j\rangle$ are eigenstates of measurements $\hat{A}$ and $\hat{B}$, respectively, with eigenvalues λ_i^A and λ_j^B , and the c_{ij} are probability amplitudes. We have generalized our notation to allow that the measurements may be (for example) of spin, as in the standard Bell formulation. The last line gives the example of the expansions for a spin Bell state $= \frac{1}{\sqrt{2}}(|+\rangle_A |+\rangle_B + |-\rangle_A |-\rangle_B)$ written in the basis of Pauli spins σ_z^A and $\hat{\sigma}_z^B$, the $|\pm\rangle_K$ being the Pauli spin- z eigenstates with outcome ± 1 for system $K \in \{A, B\}$. For convenience, we denote $|s_A\rangle_A |s_B\rangle_B$ where $s_K = \pm 1$ by $|s_A\rangle |s_B\rangle$, dropping the subscripts where the meaning is clear. In the Result VII.3, the eigenstates of $\hat{A}$ are $|i\rangle \equiv |x_i^A\rangle$. We denote generally the eigenstates of the rotated measurement $\hat{B}_\phi$ to be $|k\rangle_B$ with eigenvalues λ_k^B .

With the rotation at B , we expand in the new basis: $|j\rangle = \sum_k d_{jk} |k\rangle_B$. Thus, the state becomes

$$|\psi_{\text{Bell}}\rangle_\phi = \sum_{ij} c_{ij} |i\rangle_A \left[\sum_k d_{jk} |k\rangle_B \right],$$

which simplifies to

$$\begin{aligned} |\psi_{\text{Bell}}\rangle_\phi &= \sum_i \sum_k \sum_j c_{ij} d_{jk} |i\rangle_A |k\rangle_B \\ &= \sum_i \sum_k e_{ik} |i\rangle_A |k\rangle_B, \end{aligned} \quad (\text{F2})$$

where $e_{ik} = \sum_j c_{ij} d_{jk}$. Looking at the example of the Bell state, the Q function for this rotated state can be written as a sum of the four new terms plus new interference terms $\mathcal{I}_{ntAB}(\phi)$. The joint probability for outcomes λ_i and λ_k for the system in state $|\psi_{\text{Bell}}\rangle$ is

$$\begin{aligned} |e_{ik}|^2 &= \left| \sum_j c_{ij} d_{jk} \right|^2 = |c_{11}d_{11} + c_{12}d_{21}|^2 + \dots \\ &= |c_{11}d_{11} + c_{12}d_{21}|^2 + \dots \end{aligned} \quad (\text{F3})$$

We compare this result for $|\psi\rangle$ with the system prepared in a certain *mixed* state $\rho_{\text{mix},A}$, in which A can be viewed as being in one of the eigenstates $|i\rangle_A$ with a definite probability $|f_i|^2 = \sum_j |c_{ij}|^2$. This means that A can be viewed as having a definite outcome $\tilde{\lambda}^A$ for measurement $\hat{A}$. We define the joint probability for outcomes of λ_i and λ_k to be P_{ik} . We show that after the change of setting at B , there is no difference between the probabilities P_{ik} as predicted by $|\psi_{\text{Bell}}\rangle$ and $\rho_{\text{mix},A}$. The system is consistent with the interpretation of having a definite value $\tilde{\lambda}^A$ for the outcome of $\hat{A}$ throughout an interaction H_ϕ giving a change of setting at B .

To show this, we rewrite (F1) as

$$|\psi_{\text{Bell}}\rangle = \sum_i f_i |i\rangle_A |\psi_i^B\rangle, \quad (\text{F4})$$

where $|\psi_i^B\rangle = \sum_j [c_{ij}/f_i] |j\rangle_B$. We note that $\sum_j |c_{ij}|^2 / |f_i|^2 = 1$. We consider the mixed (nonentangled) state

$$\rho_{\text{mix},A} = \sum_i |f_i|^2 |i\rangle \langle i| \psi_i^B \langle \psi_i^B|. \quad (\text{F5})$$

The rotation ϕ at B for the state $|\psi_i^B\rangle$ gives

$$\begin{aligned} |\psi_i^B\rangle_\phi &= \sum_j \frac{c_{ij}}{f_i} \sum_k d_{jk} |k\rangle_B = \sum_k \sum_j \frac{c_{ij}}{f_i} d_{jk} |k\rangle_B \\ &\equiv \sum_k g_k |k\rangle_B. \end{aligned} \quad (\text{F6})$$

Hence

$$\rho_{\text{mix},A,\phi} = \sum_i |f_i|^2 \sum_{k,k'} g_k g_{k'}^* |i\rangle \langle i| |k\rangle \langle k'|. \quad (\text{F7})$$

The joint probability of outcomes λ_i^A and λ_k^B for the mixed state $\rho_{\text{mix},A}$ is

$$|f_i|^2 |g_k|^2 = \left| \sum_j c_{ij} d_{jk} \right|^2,$$

in agreement with (F3), those of $|\psi_{\text{Bell}}\rangle$.

Now we ask what happens to the Q function, as the change of setting occurs at B ? The Q function for $\rho_{\text{mix},A,\phi}$ includes interference terms $\mathcal{I}_{\text{mix}}(\phi)$ (arising from off-diagonal elements in the new basis) but the $|\psi_{\text{Bell}}\rangle$ has extra interference terms, since its density matrix has extra terms arising from the fact that the system A is in a superposition (i.e., originally entangled with B). Hence (even though the probabilities for outcomes λ_i^A and λ_k^B on rotation of basis at B are the same for $|\psi_{\text{Bell}}\rangle$ and $\rho_{\text{mix},A,\phi}$), there is a difference between the hidden interference terms $\mathcal{I}_{\text{ntAB}}$ of $|\psi_{\text{Bell}}\rangle$ and $\rho_{\text{mix},A,\phi}$. However, because the change of setting at B induces only changes to the hidden terms $\mathcal{I}_{\text{ntAB}}$ of Q (which are not amplified by H_{amp}), we see that the simulation is consistent with the premise that there is no impact on the value of $\tilde{\lambda}_x^A$ at A by a change of setting at B (no-signaling).

LHV model. The result leads to an LHV model, which we denote by $\mathcal{P}_{\text{asym},A}$. We assume amplification H_{amp}^A at A , so that there is the branch given by $\tilde{\lambda}_x^A$. Here, we designate the branch value as $\lambda_x^A \equiv \tilde{\lambda}_x^A$. We can track the amplitudes back to the origin, as in Sec. IV D. This models projection as explained in Sec. VIII. The state for B can be inferred, as $Q(x_B, p_B | \tilde{\lambda}_i^A)$ (Fig. 24). The HV model is that of the mixed state $\rho_{\text{mix},A,\phi}$, in which the system is in a state with a definite outcome $\lambda_i^A \equiv x_i^A$ with a fixed probability $|f_i|^2$ (in the above $|i\rangle_A \equiv |x_i^A\rangle$ and $\hat{A} \equiv \hat{x}_A$). The state at A with outcome $\tilde{\lambda}_i^A \equiv x_j^A$ is coupled to the state $|\psi_i^B\rangle$ at B , which can be a superposition state for B . In the Q -based model, this state $|\psi_i^B\rangle$ is given by the variables x_B, p_B and the Q distribution $Q(x_B, p_B | \tilde{\lambda}_i^A)$.

APPENDIX G: STANDARD BELL EXAMPLE

Bell violations require adjustments of measurement settings θ and ϕ . It is worth reviewing the role of the measurement settings in standard Bell experiments. The choice of settings θ at A and ϕ at B involves interacting the systems locally with a physical device, i.e., a Stern-Gerlach analyzer is orientated so that the spin component $\hat{\sigma}_\theta$ is measured at A and $\hat{\sigma}_\phi$ is measured at B . After passing through the analyzers, the systems A and B are prepared with respect to the measurement basis, for measurements $\hat{\sigma}_\theta$ and $\hat{\sigma}_\phi$, respectively. The Bell state is

$$|\psi_{\text{Bell}}\rangle = \frac{1}{\sqrt{2}} \{ |+\rangle_A |-\rangle_B - |-\rangle_A |+\rangle_B \}, \quad (\text{G1})$$

where $|\pm\rangle_K$, $K=A,B$, are the spin eigenstates of $\hat{\sigma}_z$ for the system denoted K . The probabilities for outcomes are calculated from (G1), by rewriting $|\psi_{\text{Bell}}\rangle$ in terms of the measurement basis, corresponding to the eigenstates of $\hat{\sigma}_{\theta,A}$ and $\hat{\sigma}_{\phi,B}$ (we take $\hat{\sigma}_\theta = \hat{\sigma}_z \cos \theta + \hat{\sigma}_x \sin \theta$). For example, the

state after a single rotation ϕ at B is

$$|\psi_{\text{Bell}}\rangle_\phi = \frac{1}{\sqrt{2}} |+\rangle_A \left\{ \cos \frac{\phi}{2} |-\rangle_{\phi B} + \sin \frac{\phi}{2} |+\rangle_{\phi B} \right\} - \frac{1}{\sqrt{2}} |-\rangle_A \left\{ -\sin \frac{\phi}{2} |-\rangle_{\phi B} + \cos \frac{\phi}{2} |+\rangle_{\phi B} \right\}, \quad (\text{G2})$$

where $|+\rangle_{\phi B} = \cos \frac{\phi}{2} |+\rangle + \sin \frac{\phi}{2} |-\rangle$ and $|-\rangle_{\phi B} = -\sin \frac{\phi}{2} |+\rangle + \cos \frac{\phi}{2} |-\rangle$. The probabilities for outcomes $+-$, $++$, $--$, $-+$ are $\frac{1}{2} \cos^2 \frac{\phi}{2}$, $\frac{1}{2} \sin^2 \frac{\phi}{2}$, $\frac{1}{2} \sin^2 \frac{\phi}{2}$, and $\frac{1}{2} \cos^2 \frac{\phi}{2}$, respectively. We may compare with predictions for the nonentangled mixed states

$$\rho_{\text{mix}}^{(AB)} = \frac{1}{2} \{ |+\rangle_A |-\rangle_B \langle -|_B \langle +|_A + |-\rangle_A |+\rangle_B \langle +|_B \langle -|_A \}. \quad (\text{G3})$$

The state after the rotation ϕ at B is

$$\rho_{\text{mix},\phi} = \frac{1}{2} \{ \rho_+^{(A)} \rho_-^{(B)} + \rho_-^{(A)} \rho_+^{(B)} \}, \quad (\text{G4})$$

where $\rho_\pm^{(A)} = |\pm\rangle_A \langle \pm|_A$ and $\rho_\pm^{(B)} = |\pm\rangle_{\phi B} \langle \pm|_{\phi B}$, which gives the same predictions for the probabilities as $|\psi_{\text{Bell}}\rangle_\phi$.

However, after a *further* change of setting (basis) to θ at A , the predictions between the entangled and nonentangled states diverge. For the Bell state, the state in the measurement basis becomes

$$|\psi_{\text{Bell}}\rangle_{\phi,\theta} = \cos \left(\frac{\theta - \phi}{2} \right) \frac{1}{\sqrt{2}} \{ |+\rangle_{\theta A} |-\rangle_{\phi B} - |-\rangle_{\theta A} |+\rangle_{\phi B} \} - \sin \left(\frac{\theta - \phi}{2} \right) \frac{1}{\sqrt{2}} \{ |+\rangle_{\theta A} |+\rangle_{\phi B} + |-\rangle_{\theta A} |-\rangle_{\phi B} \}, \quad (\text{G5})$$

which gives probabilities $\cos^2[(\theta - \phi)/2]$ (and $\sin^2[(\theta - \phi)/2]$) for outcomes $++$ and $--$ (and $+-$ and $-+$), respectively, which are *different* from those of the mixed state $\rho_{\text{mix},AB}$. This is seen by writing both $|\psi_{\text{Bell}}\rangle_\phi$ and $\rho_{\text{mix},A,\phi}$ in the new basis for A , and is evident by the fact that $|\psi_{\text{Bell}}\rangle$ violates a Bell inequality, whereas $\rho_{\text{mix},AB}$ does not. The interference terms that are present in the density operator (when written in the original basis) for the Bell state $|\psi_{\text{Bell}}\rangle$, but not for the mixed state $\rho_{\text{mix}}^{(AB)}$, do not manifest as a difference in predictions between $|\psi_{\text{Bell}}\rangle$ and $\rho_{\text{mix}}^{(AB)}$ when there are no changes of settings, or only one change of setting. The violation of the Bell inequality requires a change of settings at both sites A and B .

APPENDIX H: CAUSAL CONSISTENCY (BIPARTITE CASE) AND PROJECTION

1. Causal consistency

Result AH:1. We give the proof of causal consistency as stated in Result IX.9 and depicted in Fig. 5 (Secs. I and IX).

The simulation for the general superposition state $|\psi\rangle = \sum_{j,k} c_{jk} |x_j^A\rangle |x_k^B\rangle$ begins at the future boundary condition, with values $x_A(t)$ and $x_B(t)$ emanating from particular branches x_j^A , x_k^B with probability $|c_{jk}|^2$. For each branch-pair x_j^A , x_k^B , the

distribution of $x(t_f)$ is given by the Gaussian

$$\frac{e^{-(x_A - Gx_j^A)^2/2}}{\sqrt{2\pi}} \frac{e^{-(x_B - Gx_j^B)^2/2}}{\sqrt{2\pi}}$$

where $G = e^{gt_f}$. We first consider just one branch-pair, x_j^A and x_k^B . At time t , the distribution is

$$\frac{e^{-(x_A - e^{gt} x_j^A)^2/2}}{\sqrt{2\pi}} \frac{e^{-(x_B - e^{gt} x_j^B)^2/2}}{\sqrt{2\pi}}$$

(where $t_0 < t < t_f$). The distribution at time t_0 is

$$Q_{jk}(x_A, x_B) = \frac{e^{-(x_A - x_j^A)^2/2}}{\sqrt{2\pi}} \frac{e^{-(x_B - x_j^B)^2/2}}{\sqrt{2\pi}},$$

a set of values for x_A and x_B that we denote by $\mathcal{B}_{jk}$. At time t_0 , there is a connection with a set of values for p_A, p_B determined by $Q_0(p_A, p_B | x_A, x_B)$ [Eq. (4.12)]. The joint distribution associated with the branch-pair $\tilde{\lambda}_x^A \equiv x_j^A, \tilde{\lambda}_x^B \equiv x_k^B$ is (refer to Sec. IV D)

$$Q_{\text{loop}}(\mathbf{x}, \mathbf{p}, t_0 | x \in \mathcal{B}_{jk}) \equiv Q_0(\mathbf{x}, \mathbf{p} | x \in \mathcal{B}_j), \quad (\text{H1})$$

where $\mathbf{x} = (x_A, x_B)$ and $\mathbf{p} = (p_A, p_B)$.

We take the simple case of the superposition of $|x_j^A\rangle|x_k^B\rangle$ and $|x_l^A\rangle|x_m^B\rangle$ [Eq. (7.27)] where there are just two branch pairs. We take $|c_1| = |c_2|$ and $x_j^A = x_1^A, x_k^B = x_1^B, x_l^A = -x_j^A, x_m^B = -x_k^B$. Hence, it is convenient to denote $\mathcal{B}_{jk}$ and $\mathcal{B}_{lm}$ by $\mathcal{B}_+$ and $\mathcal{B}_-$, respectively. Denoting the distribution of the branch $\mathcal{B}_\pm$ at t_0 by

$$Q_\pm(\mathbf{x}) = \frac{e^{-(x_A \mp x_1^A)^2/2}}{\sqrt{2\pi}} \frac{e^{-(x_B \mp x_1^B)^2/2}}{\sqrt{2\pi}},$$

we consider $\mathcal{B}_+$ and find

$$Q_0(\mathbf{x}, \mathbf{p} | x \in \mathcal{B}_+) = Q_+(\mathbf{x})Q_0(\mathbf{p} | \mathbf{x}). \quad (\text{H2})$$

We use $Q_0(\mathbf{p} | \mathbf{x}) = Q(\mathbf{x}, \mathbf{p}, t_0)/Q_0(\mathbf{x})$ where $Q_0(\mathbf{x}) = \sum_\pm Q_\pm(\mathbf{x})$. Here $Q(\mathbf{x}, \mathbf{p}, t_0)$ is given by (7.28) which here simplifies to

$$Q(\mathbf{x}, \mathbf{p}, t_0) = N \frac{e^{-p_A^2/2\sigma_p^2}}{\sqrt{2\pi}\sigma_p} \frac{e^{-p_B^2/2\sigma_p^2}}{\sqrt{2\pi}\sigma_p} \{Q_+(\mathbf{x}) + Q_-(\mathbf{x}) + 2g(\mathbf{x})\mathcal{F}_{AB}\}, \quad (\text{H3})$$

where $g(\mathbf{x}) = e^{-(x_1^A)^2/2} e^{-(x_1^B)^2/2} e^{-x_A^2/2} e^{-x_B^2/2}$ and $\mathcal{F}_{AB} = \cos[p_A \Delta_A / \sigma_{x_A} + p_B \Delta_B / \sigma_{x_B}]$, $\Delta_A = x_1^A / \sigma_{x_A}$ and $\Delta_B = x_1^B / \sigma_{x_B}$. A similar expression can be obtained for $\mathcal{B}_-$. We note that (on extending the calculation of Eq. (3.14) to two modes, refer to Appendix I below), the Q function $Q(\mathbf{x}, \mathbf{p}, t)$ of the amplified system is given by $Q(\mathbf{x}, \mathbf{p}, t_0)$ on substituting $\sigma_{p_A} \rightarrow \sigma_{p_A}(t)$, $\sigma_{p_B} \rightarrow \sigma_{p_B}(t)$, $x_1^A \rightarrow G(t)x_1^A$, $x_1^B \rightarrow G(t)x_1^B$.

The densities for $x_A(t)$, $x_B(t)$ and $p_A(t)$, $p_B(t)$ at a given time $t_0 \leq t \leq t_f$ can be evaluated, for each branch $\mathcal{B}_+$ and $\mathcal{B}_-$, by examining $Q_{\text{loop}}(\mathbf{x}, \mathbf{p}, t | x \in \mathcal{B}_\pm)$. The value of x_j deterministically amplified to $G(t)x_j$, so that

$$Q_\pm(\mathbf{x}) \rightarrow Q_{G\pm}(\mathbf{x}) = \frac{e^{-(x_A \mp G(t)x_1^A)^2/2}}{\sqrt{2\pi}} \frac{e^{-(x_B \mp G(t)x_1^B)^2/2}}{\sqrt{2\pi}}, \quad (\text{H4})$$

and, similarly in $\mathcal{I}nt_{AB}$, we replace $x_1 \rightarrow Gx_1$, and $\sigma_p \rightarrow \sigma_p(t)$. Hence, denoting $Q_{\text{loop}}(\mathbf{x}, \mathbf{p}, t | x \in \mathcal{B}_\pm)$ by

$Q_{\text{loop}\pm}(\mathbf{x}, \mathbf{p}, t)$ for simplicity, we find

$$Q_{\text{loop}\pm}(\mathbf{x}, \mathbf{p}, t) = N_t Q_{G\pm}(\mathbf{x}) \frac{e^{-p_A^2/2\sigma_p^2(t)}}{\sqrt{2\pi}\sigma_p(t)} \frac{e^{-p_B^2/2\sigma_p^2(t)}}{\sqrt{2\pi}\sigma_p(t)} \times \left(1 + 2g_G(\mathbf{x}) \frac{\mathcal{F}_{AB}(t)}{\sum_\pm Q_{G\pm}(\mathbf{x})}\right), \quad (\text{H5})$$

where $g_G(\mathbf{x}) = e^{-[G(t)x_1^A]^2/2} e^{-[G(t)x_1^B]^2/2} e^{-x_A^2/2} e^{-x_B^2/2}$, $G(t) = e^{gt}$, $\mathcal{F}_{AB}(t) = \cos[p_A \Delta_A / \sigma_{x_A} + p_B \Delta_B / \sigma_{x_B}]$ where $\Delta_A = G(t)x_1^A / \sigma_{x_A}$ and $\Delta_B = G(t)x_1^B / \sigma_{x_B}$ and N_t is a normalization factor. The Q function $Q(\mathbf{x}, \mathbf{p}, t)$ for the state $|\psi(t)\rangle = e^{-iH_{\text{amp}}^A t/\hbar} e^{-iH_{\text{amp}}^B t/\hbar} |\psi\rangle$ at time t corresponds to the joint density given as the average of that for the branches. Hence,

$$Q(\mathbf{x}, \mathbf{p}, t) = \frac{1}{2} \sum_\pm Q_{\text{loop}\pm}(\mathbf{x}, \mathbf{p}, t). \quad (\text{H6})$$

Evaluating,

$$Q(\mathbf{x}, \mathbf{p}, t) = N_t \frac{e^{-p_A^2/2\sigma_p^2(t)}}{2\sqrt{2\pi}\sigma_p(t)} \frac{e^{-p_B^2/2\sigma_p^2(t)}}{\sqrt{2\pi}\sigma_p(t)} \times [Q_{G+}(\mathbf{x}) + Q_{G-}(\mathbf{x}) + 2g_G(\mathbf{x})\mathcal{F}_{AB}(t)] \quad (\text{H7})$$

reduces to the Q function $Q(\mathbf{x}, \mathbf{p}, t)$ defined above (refer to Appendix I) as required. The extension to the more general superposition $|\psi\rangle = \sum_{j,k} c_{jk} |x_j^A\rangle|x_k^B\rangle$ follows along the same lines. Assuming the eigenstates $\{|x_j^K\rangle\}$ form a complete set for a single-mode system $K \in \{A, B\}$, this completes the first part of the proof.

In the second part of the proof, we consider the dynamics of the measurement settings. Figure 5 illustrates the operations that adjust the measurement settings θ and ϕ , prior to amplification. Here, causal consistency follows because the associated dynamics is deterministic and reversible, given by equations (D2) and (D3). To give details, we consider two scenarios (refer to Fig. 5).

Case I. First, we consider that the changes of settings occur at each site between the times t_0 and t_2 , as in Fig. 22, where amplification takes place after time t_2 . We wish to prove causal consistency between these times. The dynamics in terms of the amplitudes is given by the transformations (7.15) and (7.16) (Fig. 21) as explained in Appendix E. Different sequences for the setting changes are possible, but the amplitudes in each case are defined by the deterministic transformations, or as a sequence of them. For successive deterministic transformations, it is assumed that the trajectories flow continuously, so that the output of one transformation is the input of the other. This defines amplitudes $x_A(t)$, $p_A(t)$, $x_B(t)$, $p_B(t)$ at a time $t_0 \leq t \leq t_2$. For each transformation, the amplitudes are the variables of a probability density function. The probability density function for such transformed variables is known (from ordinary calculus) to be found by expressing the x_A , p_A , x_B , p_B in terms of the $x_A(t)$, $p_A(t)$, $x_B(t)$, $p_B(t)$, and substituting in the initial density function, which is given by $Q(\mathbf{x}, \mathbf{p}, t_0)$ (the Jacobian is unity, refer to Sec. VII and Appendix E). Since this corresponds to the procedure used to evaluate the evolved Q function (Sec. VII), we confirm causal consistency.

This concludes the proof for the system depicted in Fig. 22. We note that we prove the equivalence at any one time t , where we consider two time intervals, from t_0 to t_2 , and then from t_2 to t_f . We place the boundary condition at time t_2 , just prior to amplification, and hence define the hidden loop at the time t_2 [$Q_{\text{loop}}(x, p, t_2)$]. We note that a distribution $Q(x_{\theta A}, p_{\theta A}, x_{\phi B}, p_{\phi B})$ at a time t_2 defines a unique distribution $Q(x_A, p_A, x_B, p_B)$ at time t_0 (and vice versa) in accordance with the transformations (7.15) and (7.16). We could also give a proof where the hidden loop is at the time t_0 , prior to the setting dynamics.

Case II. The second scenario is to consider where a setting change occurs at one site (B say), while amplification occurs simultaneously at A from time t_0 to t_f . The system at time t_0 is prepared in the state $Q(\lambda, t_0)$. The interactions given by Hamiltonians H_{ϕ}^B and H_{amp}^A are then applied, up to a time t_f . We consider a time t , such that $t_0 < t < t_f$. The Q function is given by $Q(\mathbf{x}, \mathbf{p}, t)$ defined above but we put $G = 1$ in the expressions relating to B , and the amplitudes of B are rotated (refer to Sec. VII and Appendix E). We find

$$Q(\mathbf{x}, \mathbf{p}, t) \rightarrow N_t \frac{e^{-p_A^2/2\sigma_{p_A}^2(t)} e^{-p_B(t_0)^2/2\sigma_{p_B}^2(t_0)}}{2\sqrt{2\pi}\sigma_{p_A}(t) \sqrt{2\pi}\sigma_{p_B}(t_0)} \times [Q_{G_A+}(\mathbf{x}) + Q_{G_A-}(\mathbf{x}) + 2g_{G_A}(\mathbf{x})\mathcal{F}_{AB_0}(t)], \quad (\text{H8})$$

where $[G(t) = e^{g^t}]$

$$Q_{G_A\pm}(\mathbf{x}) = \frac{e^{-(x_A \mp G(t)x_1^A)^2/2} e^{-(x_B(t_0) \mp x_1^B)^2/2\sigma_{x_B}^2(t_0)}}{\sqrt{2\pi} \sqrt{2\pi}},$$

$$g_{G_A}(\mathbf{x}) = e^{-(G(t)x_1^A)^2/2} e^{-(x_1^B)^2/2\sigma_{x_B}^2(t_0)} \times e^{-x_A^2/2} e^{-x_B(t_0)^2/2\sigma_{x_B}^2(t_0)},$$

and $\mathcal{F}_{AB_0}(t) = \cos[p_A \Delta_A + p_B(t_0) \Delta_{B_0}/\sigma_{x_B}]$ where $\Delta_A = G(t)x_1^A$ and $\Delta_{B_0} = x_1^B/\sigma_{x_B}(t_0)$. The variances $\sigma_{x_A}^2(t)$, $\sigma_{p_A}^2(t)$, $\sigma_{x_B}^2(t_0)$ and $\sigma_{p_B}^2(t_0)$ are given below in Appendix I [$\sigma_{x_A}^2(t) = 1$]. In the expression above for $Q(\mathbf{x}, \mathbf{p}, t)$, we use

$$x_B(t_0) = x_B(t) \cos \phi - p_B(t) \sin \phi,$$

$$p_B(t_0) = x_B(t) \sin \phi + p_B(t) \cos \phi \quad (\text{H9})$$

[noting that we simplify to write $x_B(t) \equiv x_B$, $p_B(t) \equiv p_B$ in the final Q function $Q(\mathbf{x}, \mathbf{p}, t)$]. We consider the branches emanating from t_f , as in Eqs. (H1)–(H7) above, but here, for a branch pair x_j^A, x_k^B , the distribution of $x(t_f)$ is given by

$$\frac{e^{-[x_A - G(t)x_j^A]^2/2} e^{-[x_B(t_0) - x_j^B]^2/2}}{\sqrt{2\pi} \sqrt{2\pi}}.$$

The solution for $Q_{\text{loop}\pm}(\mathbf{x}, \mathbf{p}, t_0)$ is as above, where we note $\sigma_p \equiv \sigma_{p_A}(t_0) = \sigma_{p_B}(t_0)$. Using the deterministic relations $x_1^A \rightarrow G(t)x_1^A$, $\sigma_{p_A} \rightarrow \sigma_{p_A}(t)$, and (H9) as above, the distribution for the $Q_{\text{loop}\pm}(\mathbf{x}, \mathbf{p}, t)$ becomes

$$Q_{\text{loop}\pm}(\mathbf{x}, \mathbf{p}, t) = N_t Q_{G_A\pm}(\mathbf{x}) \frac{e^{-p_A^2/2\sigma_{p_A}^2(t)} e^{-p_B(t_0)^2/2\sigma_{p_B}^2(t_0)}}{\sqrt{2\pi}\sigma_{p_A}(t) \sqrt{2\pi}\sigma_{p_B}(t_0)} \times \left(1 + 2g_{G_A}(\mathbf{x}) \frac{\mathcal{F}_{AB_0}(t)}{\sum_{\pm} Q_{G_A\pm}(\mathbf{x})}\right). \quad (\text{H10})$$

Following the proof above, averaging over both branches, we deduce equivalence to the probability density $Q(\mathbf{x}, \mathbf{p}, t)$. ■

2. Projection

Figure 5 also depicts certain results about the projected states. We explain and prove these results.

First, note that for the bipartite system with Q function $Q(\lambda, t_0)$, the Q function for the system A is given by the marginal $Q(x_A, p_A, t_0) = \int Q(\lambda, t_0) dx_B dp_B$. To illustrate, the bipartite state can be expanded as $|\psi\rangle = \sum_{ij} c_{ij} |x_i^A\rangle |x_j^B\rangle$, where c_{ij} are probability amplitudes, where the $|x_j^B\rangle$ form an orthogonal set (Appendix C). The reduced state for system A is $|\psi_A\rangle = \sum_i d_i |x_i^A\rangle$ where $d_i = \sum_k c_{ik}$, for which the Q function is

$$Q(x_A, p_A, t) = \frac{1}{\pi} \sum_i \sum_k d_i d_k^* \langle \alpha | x_i^A \rangle \langle x_k^A | \alpha \rangle. \quad (\text{H11})$$

The Q function of the bipartite state is $Q(\lambda, t) = |\langle \alpha | \langle \beta | \psi \rangle|^2 / \pi^2$, which simplifies to

$$Q(\lambda, t) = \frac{1}{\pi^2} \sum_i \sum_k \langle \alpha | x_i^A \rangle \langle x_k^A | \alpha \rangle \times \sum_j c_{ij} \langle \beta | x_j^B \rangle \sum_l c_{kl}^* \langle x_l^B | \beta \rangle. \quad (\text{H12})$$

Integrating over x_B and p_B , we find $\int Q(\lambda, t) dx_B dp_B \rightarrow Q(x_A, p_A, t)$ as required.

Result AH 2. The postselected and projected distributions defined before and after settings are fixed are related by a simple transformation of coordinates.

Proof. We examine Fig. 22, where the system is prepared in state $Q(\lambda, t_0)$ at time t_0 . Let us assume the measurement setting is changed to ϕ at B first, and then at time t_1 , is changed to θ at A . At time t_2 , both settings are fixed at ϕ and θ , respectively. A postselected distribution $Q_{\text{loop}}(\{x_{\phi k}^B\}, \lambda_{\text{rot}2}, t_2 | x_{\theta j}^A)$ can be defined at time t_2 , from a branch B_j^A based on the outcome $x_{\theta j}^A$ for system A . Here, $\lambda_{\text{rot}2} \equiv (x_{\theta A}, p_{\theta A}, x_{\phi B}, p_{\phi B})$ are the variables defined after the transformations that determine the settings. We recall that the postselected distribution must be consistent with the deterministic relations $\mathcal{D}$ that determine the correlation between outcomes $x_{\phi k}^B$ for system B and the outcome $x_{\theta j}^A$ at A , which are defined in the causal model at the time t_2 , given the fixed settings θ and ϕ (refer to Sec. VIII). With this specified, we simplify the notation as in the main text, denoting $Q(\{x_{\phi k}^B\}, \lambda_{\text{rot}2}, t_2 | x_{\theta j}^A) \equiv Q(\lambda_{\text{rot}2}, t_2 | x_{\theta j}^A)$. From $Q_{\text{loop}}(\lambda_{\text{rot}2}, t_2 | x_{\theta j}^A)$, a projected distribution $Q_{\phi}(x_{\phi B}, p_{\phi B} | x_j^A)$ can be determined for system B at time t_2 , where

$$Q_2(x_{\phi B}, p_{\phi B} | x_{\theta j}^A) = \int dx_{\theta A} dp_{\theta A} Q_{\text{loop}}(\lambda_{\text{rot}2}, t_2 | x_{\theta j}^A). \quad (\text{H13})$$

Mathematically, it is always possible to transform to different coordinates x_A, p_A for system A , and to x_B, p_B for system B , as defined by (D2) and (D3) (Jacobian factors are unity). A coordinate transformation for A corresponds to the amplitudes $x_A(t_1)$ being combined with $p_A(t_1)$ according to the inverse deterministic transformation (D2), which physically is due to the setting interaction H_{θ}^A . Hence, the trajectories for the branch B_j^A can be traced back to the time t_1 , prior to

the setting change at A . This defines a probability density function $Q_{\text{loop}}(\lambda_{\text{rot1}}, t_1 | x_{\theta j}^A)$ in terms of amplitudes λ at time t_0 , so that $\lambda_{\text{rot1}} = (x_A, p_A, x_{\phi B}, p_{\phi B})$. The projected distribution for system B is (refer to Result VIII.1e)

$$Q_{t_1}(x_{\phi B}, p_{\phi B} | x_{\theta j}^A) = \int dx_A dp_A Q_{\text{loop}}(\lambda_{\text{rot1}}, t_1 | x_{\theta j}^A). \quad (\text{H14})$$

Yet, we can mathematically transform the coordinates for B to define

$$Q_{t_0}(x_B, p_B | x_{\theta j}^A) = \int dx_A dp_A Q_{\text{loop}}(\lambda_{\text{rot1}}, t_1 | x_{\theta j}^A), \quad (\text{H15})$$

hence making a variable change from $(x_{\phi B}, p_{\phi B})$ to (x_B, p_B) . Hence $Q_{t_1}(x_{\phi B}, p_{\phi B} | x_{\theta j}^A)$ can be transformed to a density function $Q_{t_0}(x_B, p_B | x_{\theta j}^A)$ for coordinates (x_B, p_B) . Since the transformation corresponds to the dynamics of the amplitudes as given by H_{ϕ}^B , given by the inverse of (D3), this corresponds to the projected distribution for amplitudes at time t_0 prior to the change of setting from $\phi = 0$ to ϕ at B .

This result is used in the diagrams of Fig. 5 and in Sec. IX, where the projected distributions $Q(x_A, p_A | \tilde{\lambda}_{\phi}^B)$ and $Q(x_B, p_B | \tilde{\lambda}_{\phi}^A)$ are drawn in at an unspecified time t , but with reference to system A and B , respectively. The time t is unspecified because the distribution, i.e., $Q(x_A, p_A | \tilde{\lambda}_{\phi}^B)$ that applies at time t_0 can be readily transformed to apply to the amplitudes at a later time, i.e., t_2 , by transforming the variables x_A, p_A in the density function. ■

Result AH.3: The projected distribution, for example $Q(x_A, p_A | \tilde{\lambda}_{\phi}^B)$, defines the predictions for outcomes at A for any future setting adjustment θ at A . This follows from the proof above, since one can make a transformation to coordinates $x_{\theta A}, p_{\theta A}$ based on knowledge of $Q(x_A, p_A | \tilde{\lambda}_{\phi}^B)$, and this defines the projected distribution for outcomes of $\hat{x}_{\theta A}$ given the outcome defined by $\tilde{\lambda}_{\phi}^B$.

Mutual consistency. Result IX.3 is stated in Sec. IX.

Proof. Following the proof for Result IX.3, the evaluation of $Q(x_A, p_A | \tilde{\lambda}_{\phi}^B)$ requires the knowledge of the setting ϕ at B and of the initial Q function $Q(\lambda, t_0)$. The evaluation does not require knowledge of θ , the setting at A . The distribution $Q(x_A, p_A | \tilde{\lambda}_{\phi}^B)$ gives predictions of probabilities for all choices of setting θ at A , as explained in Result AH.2 above. The proof of Result AH.2 shows how the projected distributions defined from $Q_{\text{loop}}(\lambda_{\text{rot2}}, t_2 | x_{\theta j}^A)$ and $Q_{\text{loop}}(\lambda, t_0 | x_{\theta j}^A)$ are evaluated consistently with each other. The proof of causal consistency (Result AH.1) shows how $Q_{\text{loop}}(\lambda_{\text{rot2}}, t_2 | x_{\theta j}^A)$ and $Q_{\text{loop}}(\lambda, t_0 | x_{\theta j}^A)$ are consistent with $Q(\lambda, t_0)$. ■

APPENDIX I: CALCULATION OF Q FUNCTION: AMPLIFIED BIPARTITE STATE

Consider the state

$$\begin{aligned} |\psi\rangle &= |x_j^A\rangle |x_k^B\rangle + |x_l^A\rangle |x_m^B\rangle \\ &\rightarrow \left| \frac{x_j^A}{2}, r \right\rangle_{sq} \left| \frac{x_k^B}{2}, r \right\rangle_{sq} + \left| \frac{x_l^A}{2}, r \right\rangle_{sq} \left| \frac{x_m^B}{2}, r \right\rangle_{sq}. \end{aligned} \quad (\text{I1})$$

Here, $|x_j^A\rangle$ and $|x_k^B\rangle$ are eigenstates of $\hat{x}_A$ and $\hat{x}_B$, respectively, which we approximate by highly squeezed states in $\hat{x}_A$ and $\hat{x}_B$ [Eq. (3.2)]. With amplification at each site, the state becomes

$|\psi(t)\rangle = e^{-iH_{\text{amp}}^A t/\hbar} e^{-iH_{\text{amp}}^B t/\hbar} |\psi\rangle$. Applying the techniques of Appendix C, we find

$$\begin{aligned} |\psi(t)\rangle &= N \left(\left| G(t) \frac{x_j^A}{2}, r' \right\rangle_{sq} \left| G(t) \frac{x_k^B}{2}, r' \right\rangle_{sq} \right. \\ &\quad \left. + \left| G(t) \frac{x_l^A}{2}, r' \right\rangle_{sq} \left| G(t) \frac{x_m^B}{2}, r' \right\rangle_{sq} \right) \end{aligned} \quad (\text{I2})$$

(N is a normalization constant). Hence, the calculation of the Q function is as for that of (II), on substituting in the final result $x_j^A \rightarrow G(t)x_j^A, x_l^A \rightarrow G(t)x_l^A, x_k^B \rightarrow G(t)x_k^B, x_m^B \rightarrow G(t)x_m^B$, and $\sigma_x \rightarrow \sigma_x(t), \sigma_p \rightarrow \sigma_p(t)$. The variances are given by (3.6): $\sigma_{x_K}^2(t) = 1 + e^{-2r+2gt} = 1 + G(t)^2[\sigma_{x_K}^2(0) - 1]$ and $\sigma_{p_K}^2(t) = 1 + e^{2r-2gt} = 1 + [\sigma_{p_K}^2(0) - 1]/G(t)^2$ ($K \in \{A, B\}$), from which we see that $\sigma_{x_K}^2(t) = 1$ given the system is initially in the eigenstate of $\hat{x}_K$.

APPENDIX J: FINE-TUNING

The simulations of this paper give a model in which no-signaling is explained consistently with the observation of Bell nonlocality. The result does not, however, conflict with results that exclude classical causal models that are not fine-tuned: In Ref. [53], it is stated that an “observed statistical independence between variables should not be explained by fine-tuning of the causal parameters. In other words, all conditional independences (CIs) should be a consequence of the causal structure alone, not a result of the causal–statistical parameters taking some particular set of values.” The authors of Ref. [53] showed that in order for a causal model to explain Bell violations consistently with no-signaling, it is necessary that there be a fine-tuning of the causal parameters. The causal parameters include the measurement settings θ and ϕ .

We have seen that the simulations of this paper support Results IV.9 and IX.3, which imply no-signaling. An interaction H_{ϕ}^B (as in a change of setting) at B can change unobservable hidden terms in the Q function, without changing the outcome $\hat{\lambda}_x^A$ for the setting $\theta \equiv 0$ (Fig. 29). This implies no-signaling.

However, with an interaction H_{θ}^A at A so that $\theta \neq 0$ (i.e., over two rotations), Bell nonlocality can emerge (Fig. 32). Thus, no-signaling does not rule out Bell nonlocality. We see from the analysis of interference terms for the Bell state that specific values of θ and ϕ contribute to the observable probabilities that violate the Bell inequality. The value of θ needs to be adjusted from zero in order to observe the nonlocality, consistent with the conclusions of Ref. [53]. It is not the case that there can be assumed to be no impact on A due to the change of setting at B , in the sense that there is a change to the interference term in the Q function. We summarize with the following result.

Result A: Fine-tuning. There is a fine-tuning of the causal parameters (the measurement settings) in order to observe the violation of the Bell inequality consistently with the observation of no-signaling.

Proof. No-signaling is defined as the condition that there is no change to the outcome of A due to a change of setting at B , with the setting at A fixed [53]. This corresponds to fixing the

setting of θ at A , which we take without loss of generality to be $\theta = 0$ so that we assume $\hat{x}$ is measured at A (as in Figs. 28 and 29). Result IX.4 states that the outcome given by λ_x^A in the model is unchanged as the setting ϕ at B is changed. To

detect Bell nonlocality, the simulations (Results VII.4–VII.7) show that we require adjusting to a nonzero setting $\theta \neq 0$ at A (Fig. 32). Hence, a fine-tuning of the causal parameters is required. ■

-
- [1] J. S. Bell, *Physics* (Long Island City, N. Y.) **1**, 195 (1964).
 - [2] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, *Rev. Mod. Phys.* **86**, 419 (2014).
 - [3] J. F. Clauser and A. Shimony, *Rep. Prog. Phys.* **41**, 1881 (1978).
 - [4] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics: Collected Papers on Quantum Philosophy* (Cambridge University Press, Cambridge, 2004).
 - [5] D. Pegg, *Phys. Lett. A* **78**, 233 (1980).
 - [6] J. G. Cramer, *Phys. Rev. D* **22**, 362 (1980).
 - [7] D. T. Pegg, *Phys. Scr.* **T12**, 14 (1986).
 - [8] J. G. Cramer, *Rev. Mod. Phys.* **58**, 647 (1986).
 - [9] K. Schwarzschild, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* **1903**, 132 (1903).
 - [10] A. D. Fokker, *Eur. Phys. J. A* **58**, 386 (1929).
 - [11] H. M. Tetrode, *Eur. Phys. J. A* **10**, 317 (1922).
 - [12] P. A. M. Dirac, *Rev. Mod. Phys.* **17**, 195 (1945).
 - [13] J. A. Wheeler and R. P. Feynman, *Rev. Mod. Phys.* **17**, 157 (1945).
 - [14] J. A. Wheeler and R. P. Feynman, *Rev. Mod. Phys.* **21**, 425 (1949).
 - [15] N. Bohr, *Essays 1958-1962 on Atomic Physics and Human Knowledge* (Ox Bow Press, Woodbridge, 1987), Vol. 3.
 - [16] X.-s. Ma, J. Kofler, and A. Zeilinger, *Rev. Mod. Phys.* **88**, 015005 (2016).
 - [17] J. A. Wheeler, in *Mathematical Foundations of Quantum Theory*, edited by A. R. Marlow (Academic Press, New York, 1978).
 - [18] J. A. Wheeler and W. H. Zurek, *Quantum Theory and Measurement* (Princeton University Press, Princeton, 2014), Vol. 40.
 - [19] P. H. Eberhard and R. R. Ross, *Found. Phys. Lett.* **2**, 127 (1989).
 - [20] Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, *Phys. Rev.* **134**, B1410 (1964).
 - [21] M. O. Scully and K. Drühl, *Phys. Rev. A* **25**, 2208 (1982).
 - [22] M. O. Scully, B.-G. Englert, and H. Walther, *Nature (London)* **351**, 111 (1991).
 - [23] U. Mohrhoff, *Am. J. Phys.* **64**, 1468 (1996).
 - [24] B.-G. Englert, M. O. Scully, and H. Walther, *Am. J. Phys.* **67**, 325 (1999).
 - [25] Y.-H. Kim, R. Yu, S. P. Kulik, Y. Shih, and M. O. Scully, *Phys. Rev. Lett.* **84**, 1 (2000).
 - [26] S. P. Walborn, M. O. T. Cunha, S. Pádua, and C. H. Monken, *Phys. Rev. A* **65**, 033818 (2002).
 - [27] V. Jacques, E. Wu, F. Grosshans, F. Treussart, P. Grangier, A. Aspect, and J.-F. Roch, *Science* **315**, 966 (2007).
 - [28] H. Price, *Stud. Hist. Philos. Sci. B* **39**, 752 (2008).
 - [29] N. Argaman, *Am. J. Phys.* **78**, 1007 (2010).
 - [30] Y. Aharonov and L. Vaidman, in *Time in Quantum Mechanics* (Springer-Verlag, Berlin, Heidelberg, 2008), p. 399.
 - [31] R. Ionicioiu and D. R. Terno, *Phys. Rev. Lett.* **107**, 230406 (2011).
 - [32] J.-S. Tang, Y.-L. Li, X.-Y. Xu, G.-Y. Xiang, C.-F. Li, and G.-C. Guo, *Nat. Photonics* **6**, 600 (2012).
 - [33] A. Peruzzo, P. Shadbolt, N. Brunner, S. Popescu, and J. L. O'Brien, *Science* **338**, 634 (2012).
 - [34] F. Kaiser, T. Coudreau, P. Milman, D. B. Ostrowsky, and S. Tanzilli, *Science* **338**, 637 (2012).
 - [35] X.-S. Ma, J. Kofler, A. Qarry, N. Tetik, T. Scheidl, R. Ursin, S. Ramelow, T. Herbst, L. Ratschbacher, A. Fedrizzi, *et al.*, *Proc. Natl. Acad. Sci. USA* **110**, 1221 (2013).
 - [36] R. Ionicioiu, T. Jennewein, R. B. Mann, and D. R. Terno, *Nat. Commun.* **5**, 4997 (2014).
 - [37] A. G. Manning, R. I. Khakimov, R. G. Dall, and A. G. Truscott, *Nat. Phys.* **11**, 539 (2015).
 - [38] S.-B. Zheng, Y.-P. Zhong, K. Xu, Q.-J. Wang, H. Wang, L.-T. Shen, C.-P. Yang, J. M. Martinis, A. N. Cleland, and S.-Y. Han, *Phys. Rev. Lett.* **115**, 260403 (2015).
 - [39] R. Chaves, G. B. Lemos, and J. Pienaar, *Phys. Rev. Lett.* **120**, 190401 (2018).
 - [40] R. Rossi Jr., *Phys. Rev. A* **96**, 012106 (2017).
 - [41] A. S. Rab, E. Polino, Z.-X. Man, N. Ba An, Y.-J. Xia, N. Spagnolo, R. Lo Franco, and F. Sciarrino, *Nat. Commun.* **8**, 915 (2017).
 - [42] W. Qin, A. Miranowicz, G. Long, J. Q. You, and F. Nori, *npj Quantum Inf.* **5**, 58 (2019).
 - [43] E. Polino, I. Agresti, D. Poderini, G. Carvacho, G. Milani, G. B. Lemos, R. Chaves, and F. Sciarrino, *Phys. Rev. A* **100**, 022111 (2019).
 - [44] R. Kastner, *Found. Phys.* **49**, 717 (2019).
 - [45] H.-L. Huang, Y.-H. Luo, B. Bai, Y.-H. Deng, H. Wang, Q. Zhao, H.-S. Zhong, Y.-Q. Nie, W.-H. Jiang, X.-L. Wang, *et al.*, *Phys. Rev. A* **100**, 012114 (2019).
 - [46] S. Yu, Y.-N. Sun, W. Liu, Z.-D. Liu, Z.-J. Ke, Y.-T. Wang, J.-S. Tang, C.-F. Li, and G.-C. Guo, *Phys. Rev. A* **100**, 012115 (2019).
 - [47] K. B. Wharton and N. Argaman, *Rev. Mod. Phys.* **92**, 021002 (2020).
 - [48] B. R. La Cour and T. W. Yudichak, *Phys. Rev. A* **103**, 062213 (2021).
 - [49] M. Thenabadu and M. D. Reid, *Phys. Rev. A* **105**, 062209 (2022).
 - [50] T. J. Herzog, P. G. Kwiat, H. Weinfurter, and A. Zeilinger, *Phys. Rev. Lett.* **75**, 3034 (1995).
 - [51] O. Oreshkov, F. Costa, and Č. Brukner, *Nat. Commun.* **3**, 1092 (2012).
 - [52] R. Chaves and T. Fritz, *Phys. Rev. A* **85**, 032113 (2012).
 - [53] C. J. Wood and R. W. Spekkens, *New J. Phys.* **17**, 033002 (2015).
 - [54] E. G. Cavalcanti, *Phys. Rev. X* **8**, 021018 (2018).

- [55] J. Pearl and E. Cavalcanti, *Quantum* **5**, 518 (2021).
- [56] F. Costa and S. Shrapnel, *New J. Phys.* **18**, 063032 (2016).
- [57] J.-M.A. Allen, J. Barrett, D. C. Horsman, C. M. Lee, and R. W. Spekkens, *Phys. Rev. X* **7**, 031021 (2017).
- [58] S. Shrapnel and F. Costa, *Quantum* **2**, 63 (2018).
- [59] S. Shrapnel, *Brit. J. Phil. Sc.* **70**, 1 (2019).
- [60] J. Barrett, R. Lorenz, and O. Oreshkov, *Nat. Commun.* **12**, 885 (2021).
- [61] J. Pienaar, *Phys. Rev. A* **101**, 012104 (2020).
- [62] R. L. Ingraham, *Phys. Rev. A* **50**, 4502 (1994); **51**, 4295 (1995).
- [63] M. Weilenmann and R. Colbeck, *Quantum* **4**, 236 (2020).
- [64] G. Chiribella, *Phys. Rev. A* **86**, 040301(R) (2012).
- [65] G. Chiribella, G. M. D'Ariano, P. Perinotti, and B. Valiron, *Phys. Rev. A* **88**, 022318 (2013).
- [66] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, *Phys. Rev. Lett.* **120**, 040405 (2018).
- [67] M. Araújo, F. Costa, and J. Brukner, *Phys. Rev. Lett.* **113**, 250402 (2014).
- [68] M. Araújo, C. Branciard, F. Costa, A. Feix, C. Giarmatzi, and J. Brukner, *New J. Phys.* **17**, 102001 (2015).
- [69] O. Oreshkov and C. Giarmatzi, *New J. Phys.* **18**, 093020 (2016).
- [70] C. Giarmatzi, in *Rethinking Causality in Quantum Mechanics* (Springer, Cham, Switzerland, 2019), pp. 125–150.
- [71] M. Gachechiladze, N. Miklin, and R. Chaves, *Phys. Rev. Lett.* **125**, 230401 (2020).
- [72] P. J. Daley, K. J. Resch, and R. W. Spekkens, *Phys. Rev. A* **105**, 042220 (2022).
- [73] M. Araújo, P. A. Guérin, and J. Baumeler, *Phys. Rev. A* **96**, 052315 (2017).
- [74] K. Wharton, *Information* **5**, 190 (2014).
- [75] S. Weinstein, [arXiv:1707.03114](https://arxiv.org/abs/1707.03114).
- [76] J. Barrett, R. Lorenz, and O. Oreshkov, [arXiv:1906.10726](https://arxiv.org/abs/1906.10726).
- [77] Y. Guo, Z. Liu, H. Tang, X.-M. Hu, B.-H. Liu, Y.-F. Huang, C.-F. Li, G.-C. Guo, and G. Chiribella, *Phys. Rev. Lett.* **132**, 160201 (2024).
- [78] G. Chiribella and Z. Liu, *Commun. Phys.* **5**, 190 (2022).
- [79] K. Goswami, C. Giarmatzi, M. Kewming, F. Costa, C. Branciard, J. Romero, and A. G. White, *Phys. Rev. Lett.* **121**, 090503 (2018).
- [80] S. Friederich, *Europhys. Lett.* **150**, 40001 (2025).
- [81] H. Price and K. Wharton, [arXiv:2406.04571](https://arxiv.org/abs/2406.04571).
- [82] W. S. DeWitt and B. H. Feintzeig, [arXiv:2507.13593](https://arxiv.org/abs/2507.13593).
- [83] D. Bohm, *Phys. Rev.* **85**, 166 (1952).
- [84] W. Struyve, *Rep. Prog. Phys.* **73**, 106001 (2010).
- [85] B. F. Toner and D. Bacon, *Phys. Rev. Lett.* **91**, 187904 (2003).
- [86] V. Scarani, J.-D. Bancal, A. Suarez, and N. Gisin, *Found. Phys.* **44**, 523 (2014).
- [87] J.-D. Bancal, S. Pironio, A. Acín, Y.-C. Liang, V. Scarani, and N. Gisin, *Nat. Phys.* **8**, 867 (2012).
- [88] W. Struyve and H. Westman, *Proc. R. Soc. London, Ser. A* **463**, 3115 (2007).
- [89] K. B. Wharton, *Found. Phys.* **40**, 313 (2010).
- [90] K. Wharton, *Entropy* **20**, 410 (2018).
- [91] S. Donadi and S. Hossenfelder, *Phys. Rev. A* **106**, 022212 (2022).
- [92] D. Almada, K. Ch'ng, S. Kintner, B. Morrison, and K. Wharton, *Int. J. Quant. Found* **2**, 1 (2016).
- [93] R. I. Sutherland, *Stud. Hist. Philos. Sci. B* **39**, 782 (2008).
- [94] H. Price, *Time's Arrow and Archimedes' Point: New Directions for the Physics of Time* (Oxford University Press, 1996).
- [95] V. Vilasini and R. Colbeck, *Phys. Rev. A* **106**, 032204 (2022).
- [96] V. Vilasini and R. Colbeck, *Phys. Rev. Lett.* **129**, 110401 (2022).
- [97] G. Castagnoli, *Phys. Rev. A* **104**, 032203 (2021).
- [98] G. Castagnoli, *Found. Phys.* **48**, 333 (2018).
- [99] G. Castagnoli, E. Cohen, A. Ekert, and A. Elitzur, *Found. Phys.* **49**, 1200 (2019).
- [100] S. Hossenfelder and T. Palmer, *Front. Phys.* **8**, 139 (2020).
- [101] A. J. Leggett and A. Garg, *Phys. Rev. Lett.* **54**, 857 (1985).
- [102] E. Schrödinger, *Naturwissenschaften* **23**, 823 (1935).
- [103] K. Husimi, *Proc. Physical Math. Soc. Jpn.* **22**, 264 (1940).
- [104] P. D. Drummond and M. D. Reid, *Phys. Rev. Res.* **2**, 033266 (2020).
- [105] P. D. Drummond and M. D. Reid, *Entropy* **23**, 749 (2021).
- [106] S. Friederich, *Br. J. Philos. Sci.* **75**, 769 (2024).
- [107] P. D. Drummond, *Phys. Rev. Res.* **3**, 013240 (2021).
- [108] J. Fulton, R. Y. Teh, and M. Reid, *Phys. Rev. A* **110**, 022218 (2024).
- [109] M. D. Reid and P. D. Drummond, [arXiv:2205.06070](https://arxiv.org/abs/2205.06070) [quant-ph]; [arXiv:2303.02373](https://arxiv.org/abs/2303.02373).
- [110] E. P. Wigner, *Am. J. Phys.* **31**, 6 (1963).
- [111] J. Bell, *Phys. World* **3**, 33 (1990).
- [112] S. Weinberg, *New York Review Books* **64**, 51 (2017).
- [113] M. Born, *Science* **122**, 675 (1955).
- [114] M. F. Pusey, J. Barrett, and T. Rudolph, *Nat. Phys.* **8**, 475 (2012).
- [115] E. P. Wigner, *Phys. Rev.* **40**, 749 (1932).
- [116] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, *Phys. Rep.* **106**, 121 (1984).
- [117] D. Deutsch, *Phys. Rev. D* **44**, 3197 (1991).
- [118] M. Ringbauer, M. A. Broome, C. R. Myers, A. G. White, and T. C. Ralph, *Nat. Commun.* **5**, 4145 (2014).
- [119] J. Fulton, M. Thenabadu, R. Y. Teh, and M. D. Reid, *Entropy* **26**, 11 (2024).
- [120] R. R. Joseph, M. Thenabadu, C. Hatharasinghe, J. Fulton, R.-Y. Teh, P. D. Drummond, and M. D. Reid, *Phys. Rev. A* **110**, 022219 (2024).
- [121] C. Hatharasinghe, M. Thenabadu, P. D. Drummond, and M. D. Reid, *Entropy* **25**, 1620 (2023).
- [122] M. Thenabadu and M. D. Reid, *Phys. Rev. A* **105**, 052207 (2022).
- [123] P. Colciaghi, Y. Li, P. Treutlein, and T. Zibold, *Phys. Rev. X* **13**, 021031 (2023).
- [124] B. Yurke and D. Stoler, *Phys. Rev. Lett.* **57**, 13 (1986).
- [125] M. D. Reid, *Physics* **16**, 92 (2023).
- [126] M. Thenabadu, G.-L. Cheng, T. L. H. Pham, L. V. Drummond, L. Rosales-Zárate, and M. D. Reid, *Phys. Rev. A* **102**, 022202 (2020).
- [127] H. Jeong, M. Paternostro, and T. C. Ralph, *Phys. Rev. Lett.* **102**, 060403 (2009).
- [128] H. P. Yuen, *Phys. Rev. A* **13**, 2226 (1976).
- [129] L.-A. Wu, H. Kimble, J. Hall, and H. Wu, *Phys. Rev. Lett.* **57**, 2520 (1986).
- [130] B. Yurke, L. R. Corruccini, P. G. Kaminsky, L. W. Rupp, A. D. Smith, A. H. Silver, R. W. Simon, and E. A. Whittaker, *Phys. Rev. A* **39**, 2519 (1989).

- [131] R. J. Glauber, in *Frontiers in Quantum Optics*, Malvern Physics Series, edited by E. Pike and S. Sarkar (Adam Hilger, Bristol and Boston, 1986), Vol. 534.
- [132] M. Raymer and S. Yang, *J. Mod. Opt.* **39**, 1221 (1992).
- [133] P. D. Drummond and M. Hillery, *The Quantum Theory of Nonlinear Optics* (Cambridge University Press, Cambridge, 2014).
- [134] L. Arnold, *Stochastic Differential Equations: Theory and Applications* (John Wiley and Sons, New York, 1974), p. 228.
- [135] C. W. Gardiner, *Handbook of Stochastic Methods*, 2nd ed. (Springer-Verlag, Berlin, 1985), p. 442.
- [136] H. Risken, *The Fokker-Planck Equation*, 2nd ed. (Springer-Verlag, Berlin, 1996).
- [137] J. Ma, P. Protter, and J. Yong, *Probab. Theory Relat. Fields* **98**, 339 (1994).
- [138] G. J. Milburn and C. A. Holmes, *Phys. Rev. Lett.* **56**, 2237 (1986).
- [139] A. Einstein, B. Podolsky, and N. Rosen, *Phys. Rev.* **47**, 777 (1935).
- [140] M. D. Reid, *Phys. Rev. A* **40**, 913 (1989).
- [141] M. D. Reid, P. D. Drummond, W. P. Bowen, E. G. Cavalcanti, P. K. Lam, H. A. Bachor, U. L. Andersen, and G. Leuchs, *Rev. Mod. Phys.* **81**, 1727 (2009).
- [142] C. McGuigan, R. Teh, P. Drummond, and M. Reid, [arXiv:2412.00406](https://arxiv.org/abs/2412.00406).
- [143] U. Leonhardt and J. A. Vaccaro, *J. Mod. Opt.* **42**, 939 (1995).
- [144] A. Gilchrist, P. Deuar, and M. D. Reid, *Phys. Rev. Lett.* **80**, 3169 (1998).
- [145] A. Gilchrist, P. Deuar, and M. D. Reid, *Phys. Rev. A* **60**, 4259 (1999).
- [146] K. Banaszek and K. Wódkiewicz, *Phys. Rev. Lett.* **82**, 2009 (1999).
- [147] J. S. Bell, *Rev. Mod. Phys.* **38**, 447 (1966).
- [148] J. S. Bell, EPR correlations and EPW distributions, *John S. Bell on the Foundations of Quantum Mechanics* (World Scientific, Singapore, 2001), Chap. 18.
- [149] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, *Phys. Rev. Lett.* **84**, 2722 (2000).
- [150] A. Ourjoumtsev, H. Jeong, R. Tualle-Brouiri, and P. Grangier, *Nature (London)* **448**, 784 (2007).
- [151] F. Fröwis, P. Sekatski, W. Dür, N. Gisin, and N. Sangouard, *Rev. Mod. Phys.* **90**, 025004 (2018).
- [152] M. Brune, E. Hagley, J. Dreyer, X. Maitre, A. Maali, C. Wunderlich, J. M. Raimond, and S. Haroche, *Phys. Rev. Lett.* **77**, 4887 (1996).
- [153] C. Monroe, D. M. Meekhof, B. E. King, and D. J. Wineland, *Science* **272**, 1131 (1996).
- [154] Z. Leghtas, G. Kirchmair, B. Vlastakis, M. H. Devoret, R. J. Schoelkopf, and M. Mirrahimi, *Phys. Rev. A* **87**, 042315 (2013).
- [155] T. Palomaki, J. Teufel, R. Simmonds, and K. W. Lehnert, *Science* **342**, 710 (2013).
- [156] A. Omran, H. Levine, A. Keesling, G. Semeghini, T. T. Wang, S. Ebadi, H. Bernien, A. S. Zibrov, H. Pichler, S. Choi, *et al.*, *Science* **365**, 570 (2019).
- [157] H.-Y. Ku, N. Lambert, F.-J. Chan, C. Emary, Y.-N. Chen, and F. Nori, *npj Quantum Inf.* **6**, 98 (2020).
- [158] E. M. Wright, D. F. Walls, and J. C. Garrison, *Phys. Rev. Lett.* **77**, 2158 (1996).
- [159] O. Landon-Cardinal, L.C.G. Govia, and A. A. Clerk, *Phys. Rev. Lett.* **120**, 090501 (2018).
- [160] N. Wiener, *Acta Math.* **55**, 117 (1930).
- [161] R. Graham, *Z. Phys. B: Condens. Matter Quanta* **26**, 397 (1977).
- [162] R. Stratonovich, *SIAM J. Control* **4**, 362 (1966).
- [163] R. Graham, *Z. Phys. B: Condens. Matter Quanta* **26**, 281 (1977).
- [164] K. Pearson, *Philos. Mag.* **50**, 157 (1900).
- [165] P. D. Drummond, B. Opanchuk, A. Dellios, and M. D. Reid, *Phys. Rev. A* **105**, 012427 (2022).
- [166] P. D. Drummond and I. K. Mortimer, *J. Comput. Phys.* **93**, 144 (1991).
- [167] A. L. Rukhin, J. Soto, J. R. Nechvatal, M. E. Smid, E. B. Barker, S. D. Leigh, M. Levenson, M. Vangel, D. L. Banks, *et al.*, A statistical test suite for random and pseudorandom number generators for cryptographic applications, Tech. Rep. (2010).