

Assessing the dynamical assumptions in Tsirelson inequality tests of nonclassicality in harmonic oscillators

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Macrorealism posits that a system possesses definite properties at all times and that we can discover these properties, in principle, without disturbing the system's subsequent behavior. The Leggett-Garg inequalities are derived under these assumptions and are readily violated by standard quantum mechanics, thereby providing a scheme to test whether demonstrably macroscopic systems can exhibit quantum coherence. Unfortunately, Leggett-Garg tests suffer from the difficult to avoid clumsiness loophole: the difficulty of proving that sequential measurements have not inadvertently disturbed the system. The recently uncovered Tsirelson inequality is derived from the simple dynamical assumption of uniform precession, obeyed by many classical systems, and requires only single-time measurements. However, Tsirelson inequality violations could be explained by a macrorealistic system that merely breaks the dynamical assumption, rather than genuine quantum behavior. By carrying out a quantum-mechanical analysis of the Tsirelson inequality in the harmonic oscillator, we develop a protocol to rule out this possibility by assessing generalized conditions of uniform precession. We show that various measures of uniform precession, some of which are related to Leggett-Garg quantities, are satisfied well enough that the presence of quantum-mechanical interference terms must be implied. We derive several incidental mathematical results relating to violating states of Tsirelson's inequality, concerning dwell time, crossing number, and probability currents, and also consider a group-theoretic analysis of the Tsirelson operator.

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I. INTRODUCTION

A. Tsirelson inequality

Tsirelson wrote a paper with a title that asks an endearingly straightforward question: “How often is the coordinate of a harmonic oscillator positive?” [1]. Tsirelson demonstrated that a simple dynamical assumption about classical oscillators can give rise to a surprising disparity between classical and quantum expectations. Violations of the Tsirelson inequality [2] are associated inherently with the negativity of the Wigner function [3] and present an alluring scheme to assess the quantumness of a system using only single-time measurements on an ensemble of systems.

This work went virtually unnoticed for more than a decade and a half until it was rediscovered by Zaw *et al.*, who extended Tsirelson's work to a broader family of systems undergoing uniform precession, such as finite-dimensional spins in real space [4]. Tsirelson's protocol has been generalized as a viable entanglement witness for coupled harmonic oscillator systems [5,6] and entangled spin ensembles [7]. The most recent work in the field has focused primarily upon further

generalizations of the protocol [8] and computing stricter bounds on maximal quantum violations [9,10]. A recent triumph has been the first experimental realization (a so-called quantumness certification) of Tsirelson violations in precessing nuclear spins [11].

To derive Tsirelson's inequality, we consider a classical harmonic oscillator with time period T . Define the dichotomic variable $Q(t)$ to be the sign function $\text{sgn}[x(t)]$, which is ± 1 depending on the sign of x , and consider its value at three equally spaced times, every one-third of a period ($t_1 = 0$, $t_2 = \frac{1}{3}T$, and $t_3 = \frac{2}{3}T$). A simple dynamical assumption about oscillators is this: The particle will never be found on the same side of the origin three times in a row. Therefore, the sum $Q(0) + Q(\frac{1}{3}T) + Q(\frac{2}{3}T)$ can only assume the value $+1$ or -1 for any given classical trajectory. This is illustrated in Fig. 1.

Defining $A = Q(0) + Q(\frac{1}{3}T) + Q(\frac{2}{3}T)$, it follows that the ensemble average $\langle A \rangle$ over several experimental runs must satisfy the Tsirelson inequality

$$-1 \leq \langle A \rangle \leq 1. \quad (1.1)$$

(Tsirelson originally considered the harmonic oscillator's classical phase-space dynamics, illustrated in Fig. 2, and the probability that a particle is found on one side of the origin.) As the expectation value of a sum equals the sum of expectation values, $\langle A \rangle$ can instead be written in terms of single-time averages

$$\langle A \rangle = \langle Q(0) \rangle + \langle Q(\frac{1}{3}T) \rangle + \langle Q(\frac{2}{3}T) \rangle. \quad (1.2)$$

Thus, the Tsirelson quantity $\langle A \rangle$ can be measured using only single-time measurements on an ensemble of identical systems. Note that, in practice, measuring the dichotomic

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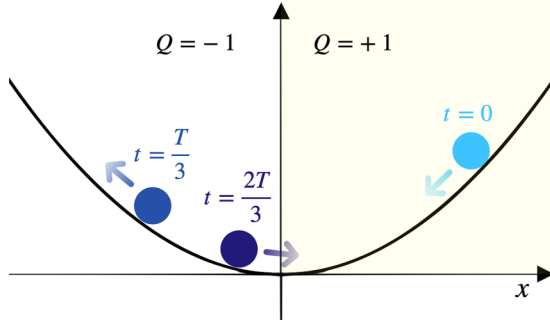


FIG. 1. A classical particle oscillating in a quadratic potential is one example of a system undergoing uniform precession. The dynamical assumption is that Q cannot have the same sign at all three times, irrespective of where in its trajectory the particle starts at time $t = 0$.

variable Q may be done using a light beam that illuminates one half of the harmonic well [12].

The Tsirelson quantity $\langle A \rangle$ may be conveniently written in the Wigner representation [3,9,13–16] in terms of the function $W(p, x, t)$ (and see also Appendix A). It has the form

$$\langle A \rangle = \iint \text{sgn}(x) \sum_i W(p, x, t_i) dx dp. \quad (1.3)$$

In the Heisenberg picture, with a time-independent Wigner function, the Tsirelson quantity is

$$\langle A \rangle = \iint \left[Q(0) + Q\left(\frac{T}{3}\right) + Q\left(\frac{2T}{3}\right) \right] W(p, x) dx dp, \quad (1.4)$$

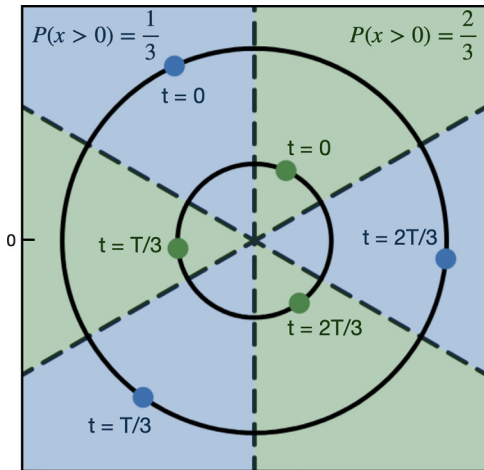


FIG. 2. Classical phase-space depiction of the system's dynamics for three possible measurement times. If the system is initialized in the green region for one of the experimental runs, then it will have a probability of $\frac{2}{3}$ of being found on the right side of the origin at a randomly chosen measurement time $t \in \{0, T/3, 2T/3\}$. Conversely, the probability will be $\frac{1}{3}$ if the particle begins in a blue region. As such, the ensemble average, a convex sum of these probabilities, is bounded between $\frac{1}{3}$ and $\frac{2}{3}$. This is equivalent to Eq. (1.1) up to a scale factor. (Figure has been adapted from Zaw *et al.* [4].)

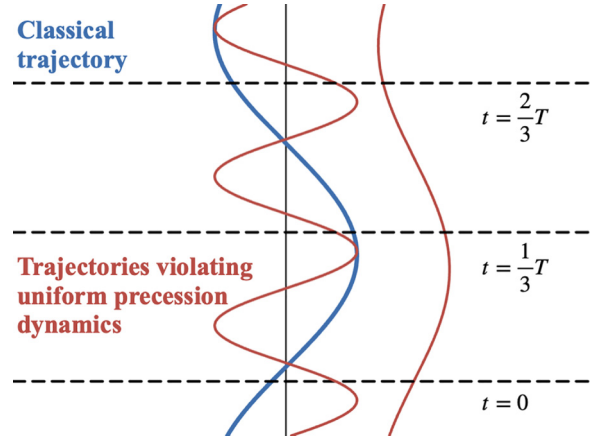


FIG. 3. Unlike the blue classical trajectory, which follows the expected sinusoidal dynamics, the erratic red trajectories, still macro-realistic, either dwell for too long on one side of the origin or oscillate too rapidly, and will violate the Tsirelson inequality since they are measured on the same side three times in a row.

which makes it clear that Tsirelson inequality violations are indeed effected by negativity in the Wigner function.

However, a crucial point which has so far not been addressed in the literature is the validity of the dynamical assumption of uniform precession in a quantum-mechanical context. If truly the Tsirelson test is to be interpreted as a credible certification of quantum behavior, it is imperative to demonstrate that violations are indicative of genuine quantum phenomena and not simply of the failure of its founding dynamical assumption.

A classical system that breaks the dynamical assumption, that is, one which is not precessing uniformly (or not oscillating at all, even), will easily violate the Tsirelson inequality. Figure 3 depicts qualitatively that a Tsirelson violation could be caused by a classical particle that does not obey the expected Newtonian dynamics of oscillation induced by a harmonic potential, either lingering too long on one side of the origin or oscillating faster than the classical frequency.

Uniform precession can be defined perfectly well in the case of a classical oscillator by considering, for instance, the phase-space representation of its trajectory. The condition for uniform precession is then simply that the angular variable $\varphi = \tan^{-1}(x/p)$ evolves linearly in time ($\dot{\varphi} = \text{const}$). However, quantum mechanically, the corresponding operator, related to the arrival time operator, is conjugate to the Hamiltonian and is not self-adjoint [17–20]. As such, the eigenstates of the angular operator cannot be orthogonal, and there is some intrinsic indefiniteness in the angular behavior and hence in the definition of uniform precession; this property is also related to the difficulties of defining a phase operator for the harmonic oscillator [21]. In the Wigner picture, although the Wigner function evolves along the classical path, it is not a probability distribution, since it can be negative, and thus we cannot immediately assert that the evolution in the quantum case respects uniform precession. Also, from an operator perspective, the position and momentum operators of course evolve according to classical equations of motion, so in this sense uniform precession is respected in the dynamics.

However, uniform precession involves confirming that the particle is not on the same side of the origin at three different times, which means specifying three noncommuting positions, so we would not expect to see exact classical behavior. In practice, these features mean that uniform precession can be defined in multiple inequivalent ways in quantum mechanics, as we will see.

B. This paper

The aim of this paper is to find precise sets of conditions characterizing uniform precession in terms of measurable three-time probabilities and then to link them to the Tsirelson inequality. This provides a way to make a clear assessment of the degree to which a Tsirelson inequality violation is caused by the presence of genuine quantum effects or merely explained by a violation of uniform precession.

In order to carry out this aim, we embed the Tsirelson scheme within the more comprehensive Leggett-Garg framework [22], the canonical test of nonclassicality in temporal correlations; this is done in Sec. II. In Sec. III we establish the tools required for quantum-mechanical calculations and discuss quantum violations of the Tsirelson inequality. We present our most important results in Sec. IV on the main protocol to assess uniform precession, broken down into two methods; both of these utilize the Leggett-Garg formalism. In Sec. V we investigate two other quantities, the dwell time and crossing number, that are less strictly related to the Tsirelson inequality but provide further heuristic evidence of the extent to which uniform precession is satisfied. We will show that the former has quantum properties identical to the classical case and the latter approximately so. Moreover, we will argue that these results can be used to rule out alternative classical explanations. We conclude in Sec. VI with a critical assessment of whether the protocol introduced in Sec. IV ultimately sidesteps the problem of noninvasive measurement. We report lengthy technical results and important mathematical proofs in the Appendixes.

II. LEGGETT-GARG FRAMEWORK

The Leggett-Garg inequalities (LGIs) [22,23], a temporal analog to Bell-type inequalities [24,25], must be satisfied by any theory obeying macrorealism, a quintessentially classical worldview that is encapsulated by the following three postulates: macroscopic realism per se, i.e., a macroscopic system exists in a definite state at all times; noninvasive measurability, i.e., the state of the system can be determined, in principle, without affecting its subsequent dynamics; and induction, i.e., the current state does not depend upon future measurements.

Macrorealism pertains to measurements made on a specific data set [26], which we take here to be the set of three single-time averages $\langle Q_i \rangle$, where $Q_i \equiv Q(t_i)$, and the three correlators C_{ij} , the latter measured in a noninvasive way. With possible measurement outcomes $s_i = \pm 1$, where $i = 1, 2, 3$ for measurements taken at times t_1, t_2, t_3 , respectively, the three two-time correlators are defined as

$$C_{ij} \equiv \langle Q_i Q_j \rangle = \sum_{s_i, s_j} s_i s_j p_{ij}(s_i, s_j), \quad (2.1)$$

with $-1 \leq C_{ij} \leq 1$ and $\{i, j\} = \{1, 2\}, \{2, 3\}, \{1, 3\}$ and where $p_{ij}(s_i, s_j)$ is the proportion of outcomes in the data set where a measurement of Q at time t_i yields s_i and at (a later time) t_j yields s_j .

It will be useful to introduce here some mathematical expressions we will require for later analysis. The measured data set fully determines the two-time quasiprobability distribution [27,28]

$$q(s_i, s_j) = \frac{1}{4}(1 + s_i \langle Q_i \rangle + s_j \langle Q_j \rangle + s_i s_j C_{ij}) \quad (2.2)$$

for $s_i, s_j = \pm 1$ and $j > i \in \{1, 2, 3\}$ and partially determines the three-time quasiprobability

$$q(s_1, s_2, s_3) = \frac{1}{8}(1 + s_1 \langle Q_1 \rangle + s_2 \langle Q_2 \rangle + s_3 \langle Q_3 \rangle + s_1 s_2 C_{12} + s_2 s_3 C_{23} + s_1 s_3 C_{13} + s_1 s_2 s_3 D), \quad (2.3)$$

where $D = \langle Q_1 Q_2 Q_3 \rangle$ is the triple correlator but is not measured in the simplest protocols. These two quasiprobabilities are classically positive. Quantum mechanically, they are equivalent to the expressions

$$q(s_1, s_2) = \text{Re}\{\text{Tr}[P_{s_2}(t_2)P_{s_1}(t_1)\rho]\}, \quad (2.4)$$

$$q(s_1, s_2, s_3) = \text{Re}\{\text{Tr}[P_{s_3}(t_3)P_{s_2}(t_2)P_{s_1}(t_1)\rho]\}, \quad (2.5)$$

with projection operators $P_{s_i}(t_i) = \frac{1}{2}(1 + s_i \hat{Q}_i)$ (we avoid using hats for quantum operators except when they are needed to avoid ambiguity between classical and quantum variables). We also give here the analogous sequential measurement probabilities and their moment expansions:

$$p_{12}(s_1, s_2) = \text{Tr}[P_{s_2}(t_2)P_{s_1}(t_1)\rho P_{s_1}(t_1)] \\ = \frac{1}{4}(1 + s_1 \langle Q_1 \rangle + s_2 \langle Q_2^{(1)} \rangle + s_1 s_2 C_{12}), \quad (2.6)$$

$$p_{123}(s_1, s_2, s_3) = \text{Tr}[P_{s_3}(t_3)P_{s_2}(t_2)P_{s_1}(t_1)\rho P_{s_1}(t_1)P_{s_2}(t_2)] \\ = \frac{1}{8}(1 + s_1 \langle Q_1 \rangle + s_2 \langle Q_2^{(1)} \rangle + s_3 \langle Q_3^{(12)} \rangle + s_1 s_2 C_{12} + s_2 s_3 C_{23}^{(1)} + s_1 s_3 C_{13}^{(2)} + s_1 s_2 s_3 D), \quad (2.7)$$

where the superscripts denote the presence of an earlier or intermediate measurement that has been summed out. More specifically, the quantity $\langle Q_2^{(1)} \rangle$ is defined as [29]

$$\langle Q_2^{(1)} \rangle = \sum_{s_1, s_2} s_2 p_{12}(s_1, s_2), \quad (2.8)$$

and similarly for $Q_3^{(12)}$, $C_{13}^{(2)}$, and $C_{23}^{(1)}$. This is therefore the average of the operator $\sum_{s_1} P_{s_1}(t_1) \hat{Q}_2 P_{s_1}(t_1)$. Note that there are two more candidate three-time quasiprobabilities that we will explore, obtained from Eq. (2.3) by replacing C_{23} with $C_{23}^{(1)}$ or replacing C_{13} with $C_{23}^{(2)}$ (see Appendix H).

Turning now to the conditions for macrorealism, the three postulates of macrorealism imply that there exists an underlying joint probability for the given data set, which implies that a set of 12 two-time and four three-time Leggett-Garg (LG) inequalities hold. Specifically, the 12 quasiprobabilities (2.2), for time pairs (t_1, t_2) , (t_1, t_3) , and (t_2, t_3) , are non-negative (LG2s) and the following four LG3 inequalities are

satisfied:

$$L_1 = \frac{1}{4}(1 + C_{12} + C_{23} + C_{13}) \geq 0, \quad (2.9)$$

$$L_2 = \frac{1}{4}(1 - C_{12} - C_{23} + C_{13}) \geq 0, \quad (2.10)$$

$$L_3 = \frac{1}{4}(1 - C_{12} + C_{23} - C_{13}) \geq 0, \quad (2.11)$$

$$L_4 = \frac{1}{4}(1 + C_{12} - C_{23} - C_{13}) \geq 0. \quad (2.12)$$

These 16 conditions are not only necessary for macrorealism but are also sufficient, as is shown by the LG version of Fine's theorem [30–32]. In quantum mechanics, the LG2 and LG3 inequalities can be violated up to the value $-\frac{1}{8}$, the Lüders bound.¹

We also note that the four LG3 quantities may be written as

$$L_1 = q(+, +, +) + q(-, -, -), \quad (2.13)$$

$$L_2 = q(+, -, +) + q(-, +, -), \quad (2.14)$$

$$L_3 = q(+, -, -) + q(-, +, +), \quad (2.15)$$

$$L_4 = q(+, +, -) + q(-, -, +), \quad (2.16)$$

from which we see that, when non-negative, the LG3s describe a probability distribution for changes of sign. Equations (2.13)–(2.16) are the probabilities for zero or two sign changes and the two different possibilities for a single sign change, respectively.

This means that if the LG3s are satisfied but the LG2s are not, then even though macrorealism is violated for the full data set, it can still be satisfied for the data subset of the sign-change variables $\chi_1 = s_1 s_2$ and $\chi_2 = s_2 s_3$, which are ± 1 depending on whether or not the sign of Q has flipped between measurement times. We will make extensive use of this in Sec. IV, where we are only interested in the sign-change probabilities as a quantification of uniform precession.

Violations of the LGIs have been observed experimentally in a myriad of physical systems, such as superconducting qubits [35,36], nuclear spins [37–39], millimeter-scale crystals [40], photons [41], and (recently) neutron interferometry [42,43]. The Leggett-Garg inequalities also have far-reaching applications to other fields [44–48]. A more exhaustive account of the experimental tests of macrorealism to date is provided in the extensive review articles in Refs. [33,49].

However, a significant issue for the Leggett-Garg scheme is that macrorealism per se is conjoined inextricably with noninvasive measurability. The so-called clumsiness loophole [50] is that any recorded violations could always be ascribed instead to the unintended invasiveness of the measurement process [51]. The standard way to render the measurements noninvasive is to use ideal negative result measurements [52,53]. A convenient refinement of this protocol is described in Ref. [54], where it is shown that the correlators such as C_{12} can always be measured in a context where the no-signaling-in-time condition is satisfied. This is the condition that $\langle Q_2^{(1)} \rangle$, defined in Eq.(2.8), is the same as $\langle Q_2 \rangle$, i.e., measurements

at t_1 have no disturbing effect on the average of Q at time t_2 . Alternative noninvasive measurement methods exist, such as the waiting detector protocol [55–57]. If the noninvasive measurement condition is implemented convincingly, then violations of the LGIs imply the failure of macrorealism per se.

The derivation of the Leggett-Garg inequalities solely from the postulates of macrorealism is completely independent of the system's equations of motion. It is this indifference to the underlying dynamics that necessitates sequential measurements in order to obtain a quantum-classical disparity. Alternatively, the introduction of a well-chosen and experimentally testable dynamical assumption can supplant the requirement for noninvasive measurement. A few examples of such assumptions have appeared in the LG literature and typically consist of assuming that certain transition probabilities are zero [55,58,59] or can be simplified in some way [33,60,61]. The key point is that the assumptions are chosen so that the correlators simplify to single-time measurement averages. Furthermore, the dynamical assumption is either implemented through the design of the experiment (e.g., the choice of Hamiltonian, where possible) or checked experimentally through separate control experiments. Dynamical assumptions are often classically motivated, and the Tsirelson scheme is an example par excellence, with its simplifying dynamical assumption of uniform precession.

The related single-crossing dynamical assumption, studied in the context of a waiting detector model in Ref. [55] and Chap. 6 of Ref. [57], assumes one crossing of the origin over a chosen time interval and would be valid for a harmonic oscillator over half a period. In fact, this is then mathematically equivalent to the assumption of uniform precession over a full period, a proof of which we provide in Appendix B.

However, does a Tsirelson inequality violation truly imply genuine quantum behavior and the failure of macrorealism, or could a stubborn macrorealist now argue that it stems purely from faulty dynamics? We saw that even trajectories with a defined position at all times can produce spurious Tsirelson inequality violations (Fig. 3). So, just as separating realism from noninvasive measurability has proved to be the greatest hassle in LG tests, do we now run into a similar problem in Tsirelson tests of separating the failure of realism from this newfound dynamical assumption?

Our strategy in Sec. IV to assess the dynamical assumption will be to use the Leggett-Garg framework to rewrite the Tsirelson quantity in terms of various probabilities that can be interpreted as measures of uniform precession.

III. TSIRELSON VIOLATIONS IN THE QUANTUM HARMONIC OSCILLATOR

A. Expansion in energy eigenbasis

In order to discuss the Tsirelson quantity quantum mechanically, we promote the variable Q to be an operator in the Heisenberg picture, undergoing unitary time evolution as $\hat{Q}(t) = U^\dagger \hat{Q} U$, where $U = \exp(-itH)$ with a standard quantum harmonic oscillator (QHO) Hamiltonian H . We can write $\hat{Q} = 2\theta(\hat{x}) - 1$, where $\theta(\hat{x}) = \int_0^\infty dx |x\rangle \langle x|$ is the Heaviside operator (and $|x\rangle$ are position eigenstates). For convenience, we will work exclusively with units chosen such that

¹This is for so-called Lüders measurements, which simply measure the value of Q . More negative values can be obtained by measuring finer-grained quantities (von Neumann measurements) and then coarse graining their probabilities to find probabilities of Q [33,34].

$\hbar = 1$; further, it will be convenient to normalize the angular frequency of oscillations to unity, $\omega = 1$. The Tsirelson inequality is then $-1 \leq \langle A \rangle \leq 1$, where now

$$\hat{A} \equiv \hat{Q} + U^\dagger \hat{Q} U + (U^\dagger)^2 \hat{Q} U^2, \quad (3.1)$$

with $U = \exp(-\frac{2\pi}{3}iH)$, recalling that, by choosing units of $\omega = 1$, one full period corresponds to $T = 2\pi$. We will refer to $\hat{A}$ hereafter as the Tsirelson operator. Quantum systems can violate this inequality, up to a maximal value of $\langle A \rangle \approx 1.26$, so Tsirelson found [1].

We can compute the Tsirelson quantity for an arbitrary state of the QHO Hilbert space formed as a superposition of energy eigenstates $|n\rangle$ from the ground state upward up to a finite cutoff:

$$|\Psi\rangle = \sum_{n=0}^N c_n |n\rangle, \quad c_n \in \mathbb{C}. \quad (3.2)$$

The single-time average $\langle Q \rangle = \langle \Psi | [2\theta(\hat{x}) - 1] | \Psi \rangle$ can be written as

$$\langle \Psi | [2\theta(\hat{x}) - 1] | \Psi \rangle = \sum_{n,k=0}^N c_n^* Q_{nk} c_k, \quad (3.3)$$

where we have defined the matrix elements $Q_{nk} = \langle n | [2\theta(\hat{x}) - 1] | k \rangle$. A tractable expression for Q_{nk} is derived in Appendix C and is given by

$$\frac{2\pi^{-1/2}}{(k-n)\sqrt{2^{n+k}n!k!}} [kH_{k-1}(0)H_n(0) - nH_{n-1}(0)H_k(0)], \quad (3.4)$$

where $H_n(x)$ are the standard Hermite polynomials.

So, for a state in the form of Eq. (3.2),

$$\langle Q(t) \rangle = \langle \Psi | \hat{Q} | \Psi \rangle = \sum_{n,k=0}^N c_k^* c_n e^{-i(n-k)t} Q_{nk}. \quad (3.5)$$

Therefore, for an arbitrary state up to an energy truncation of N , the Tsirelson quantity is given by

$$\langle A \rangle = \sum_{n,k=0}^N c_k^* c_n (1 + e^{-i(2\pi/3)(n-k)} + e^{-i(4\pi/3)(n-k)}) Q_{nk}. \quad (3.6)$$

Violations of the Tsirelson inequality are manifestly associated with negativity in the Wigner function. A corollary is that a single coherent state, which has a completely positive Wigner function, will not violate the Tsirelson inequality; neither will a single energy eigenstate, for it is trivially of strictly odd or even parity. A proof, in terms of probability currents, that definite-parity states do not violate the Tsirelson inequality is provided in Sec. V A. Perhaps more surprising is that a superposition of at least three energy eigenstates is required to produce a violation. We provide a full proof of this fact in Appendix E.

B. The simplest violating state

The Tsirelson operator is Hermitian ($\hat{A} \equiv \hat{Q}_1 + \hat{Q}_2 + \hat{Q}_3 = \hat{A}^\dagger$), and so its eigenvalues for a truncated $(N+1)$ -dimensional subspace of the Hilbert space are real. The

greatest eigenvalue corresponds to the largest possible Tsirelson violation, $\langle A \rangle > 1$, for a superposition $|\psi\rangle = \sum_{n=0}^N c_n |n\rangle$ of QHO energy eigenstates up to $|N\rangle$. Likewise, the lowest eigenvalue corresponds to the most severe violation of the classical lower bound, $\langle A \rangle = -1$, in the subspace. The matrix elements of $\hat{A}$ in the energy eigenbasis can be written in terms of the familiar Q_{nk} ,

$$A_{nk} \equiv \langle n | A | k \rangle = Q_{nk} (1 + e^{i(2\pi/3)(n-k)} + e^{i(4\pi/3)(n-k)}), \quad (3.7)$$

for $n, k = 0, 1, \dots, N$.

By explicitly diagonalizing the Tsirelson operator $\hat{A}$ in an increasingly large subspace, which is detailed in Appendix D, we find that the greatest Tsirelson violation in the $N = 6$ subspace is given by

$$|\Psi_Z\rangle = \frac{4}{\sqrt{42}}|0\rangle - \frac{1}{\sqrt{2}}|3\rangle + \sqrt{\frac{5}{42}}|6\rangle, \quad (3.8)$$

with an upper-bound Tsirelson violation $\langle A \rangle = 1.1195$, or equivalently by swapping the sign of the odd-parity eigenstate,

$$|\Psi'_Z\rangle = \frac{4}{\sqrt{42}}|0\rangle + \frac{1}{\sqrt{2}}|3\rangle + \sqrt{\frac{5}{42}}|6\rangle, \quad (3.9)$$

with a lower-bound Tsirelson violation $\langle A \rangle = -1.1195$. This agrees exactly with the work of Ref. [6], for which we label Eq. (3.8) the Zaw-Scarani state. We will use this as a simple candidate state for later analysis because the $N = 6$ cutoff is the first point at which violations to Tsirelson's inequality become possible (a direct consequence of Appendix E and Sec. III C).

This is not an isolated violating state. Figure 4 depicts a modified Bloch sphere representation of the set of all possible states constructed in the subspace spanned by the $|0\rangle$, $|3\rangle$, and $|6\rangle$ energy eigenstates with real coefficients, that is,

$$|\Psi\rangle = a_0|0\rangle + a_3|3\rangle + a_6|6\rangle, \quad \text{where } a_0, a_3, a_6 \in \mathbb{R}. \quad (3.10)$$

Clearly, there exists a sizable region of states over which a Tsirelson violation can be achieved to varying extent. This is good news for experimental realizations of the protocol, because it means even quantum states that are prepared imperfectly can in principle produce demonstrable Tsirelson violation. Another important observation is that every upper-bound-violating state is paired with a commensurate lower-bound-violating state ($\langle A \rangle \rightarrow -\langle A \rangle$), which is obtained by swapping the sign of the odd-parity coefficients (e.g., $c_3 \rightarrow -c_3$).

C. Structure of the Tsirelson operator

This section establishes a link between the Tsirelson operator and the group C_3 . We then apply this connection to find an alternative explanation for a result previously noted in Ref. [4].

The group C_3 has three elements which we denote $\{e, r, r^2\}$. It is an Abelian group whose generator is r [62]. From the Cayley table of the group, outlined in Table I, it is straightforward to determine that $\{\mathbb{1}, U, U^2\}$ forms a faithful representation of C_3 under multiplication with U acting as the generator.

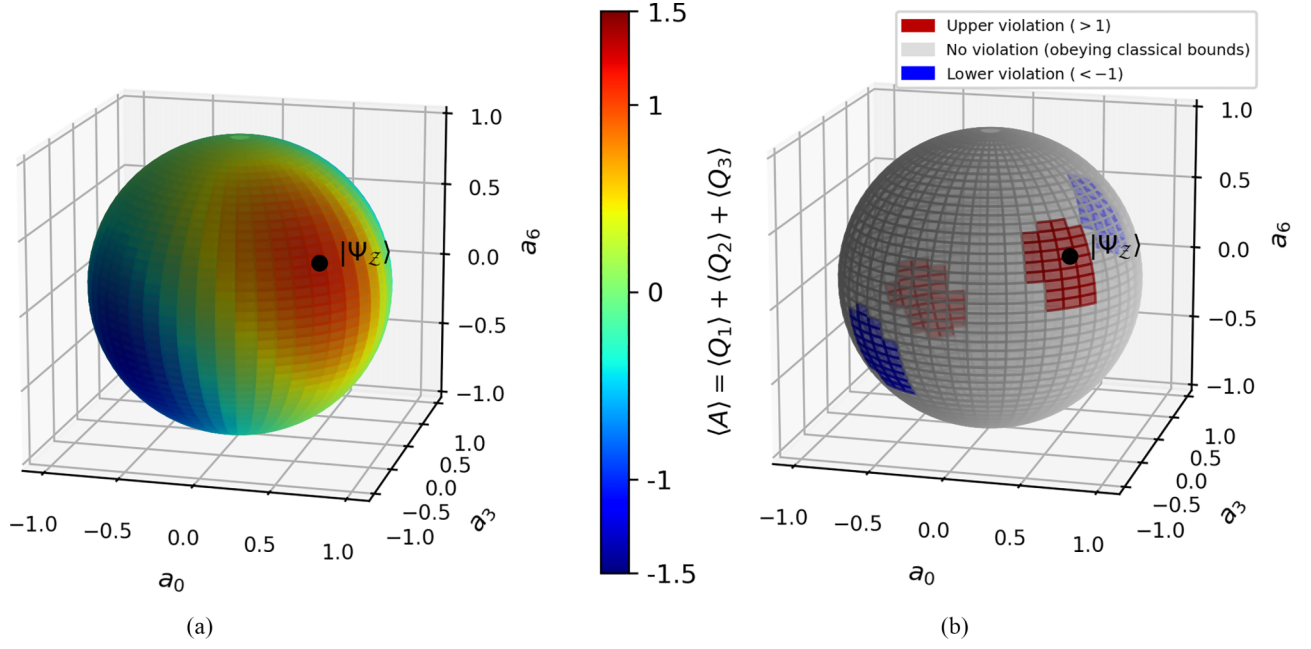


FIG. 4. Representation of all states in the subspace described by Eq. (3.10), where the axes represent the (real) coefficients. (a) A colormap is used to represent the value of the Tsirelson quantity when each of these states is taken to be the initial state. The Zaw-Scarani state, labeled $|\Psi_Z\rangle$, has the greatest violation in this subspace and lies in the center of the upper-bound-violating region. (b) Only the violating (nonclassical) regions are highlighted, showing that each upper violation is paired with a lower violation.

From this observation, we can relate the Tsirelson operator to the twirl of the sign operator,

$$\text{Twirl}(\hat{Q}) = \frac{1}{3}\hat{A} = \frac{1}{3}[\mathbb{I}\hat{Q}\mathbb{I} + U^\dagger\hat{Q}U + (U^2)^\dagger\hat{Q}(U^2)]. \quad (3.11)$$

The $\text{Twirl}(\cdot)$ operation is the averaging of a matrix over the actions of a group, in this case C_3 [63]. It is a useful operation because it extracts the component of the matrix that transforms under the identity representation of the group. Note that $U^\dagger \text{Twirl}(\hat{Q})U = (1) \text{Twirl}(\hat{Q}) \Rightarrow [\text{Twirl}(\hat{Q}), U] = 0$.

Because the two normal operators commute, they cannot mix each other's eigenspaces. The action of U on the basis states $|n\rangle$ clearly partitions the Hilbert space $\mathcal{H}$ into three degenerate eigenspaces corresponding to its three eigenvalues λ_j : eigenspace for $\lambda_1 = 1$,

$$\mathcal{H}_0 = \text{span}\{|n\rangle : n \equiv 0 \pmod{3}\}; \quad (3.12)$$

eigenspace for $\lambda_2 = e^{-i(2\pi/3)}$,

$$\mathcal{H}_1 = \text{span}\{|n\rangle : n \equiv 1 \pmod{3}\}; \quad (3.13)$$

and eigenspace for $\lambda_3 = e^{-i(4\pi/3)}$,

$$\mathcal{H}_2 = \text{span}\{|n\rangle : n \equiv 2 \pmod{3}\}. \quad (3.14)$$

TABLE I. Cayley table of C_3 . The three elements of the group are $\{e, r, r^2\}$, where e is the identity.

	e	r	r^2
e	e	r	r^2
r	r	r^2	e
r^2	r^2	e	r

Therefore, the eigenvectors of $\text{Twirl}(Q)$, and hence A , must be made up exclusively of basis vectors in a single degenerate subspace. This property is illustrated in Fig. 5 by representing the Tsirelson operator as a graph, where each subgraph corresponds to one of the subspaces. The bipartite nature of the subgraphs visualizes the underlying symmetric spectrum of the operator.

IV. ASSESSING THE DYNAMICAL ASSUMPTION

A. General framework

We now arrive at the main issue, which is assessing the degree to which the dynamical assumptions behind the Tsirelson inequality hold. As discussed, like many other clas-

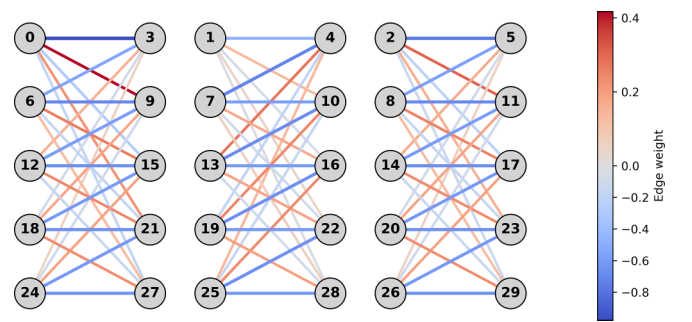


FIG. 5. The Tsirelson operator in the energy eigenbasis with any finite energy cutoff can be interpreted as the adjacency matrix of a simple weighted graph. This graph is illustrated with a cutoff at $n = 29$, where the nodes are energy eigenstates and the edges represent nonzero matrix elements of the Tsirelson operator. The color represents the value of the matrix element.

sical phenomena, uniform precession and its closely related assumptions have no unique definition in quantum theory. We will therefore explore a number of inequivalent implementations which entail different measurement techniques to check it. Our first method involves demonstrating an explicit connection between $\langle A \rangle$ and the probability of zero sign changes, L_1 [Eqs. (2.9) and (2.13)], which should be zero classically. Our second method stems from the interesting observation that a weaker condition of setting merely the difference $q(+, +, +) - q(-, -, -)$ to be zero also leads to the Tsirelson inequality, as we will show, and thus this may also be considered a measure of uniform precession.

Specifically, we will implement the following procedure in a number of different ways.

- (1) Declare a data set, comprising measurements of $\langle A \rangle$ (the Tsirelson test itself) as well as subsidiary quantities to gauge the dynamical assumption of uniform precession (UP).
- (2) Write the Tsirelson quantity in the form

$$\frac{1}{2}(1 + \langle A \rangle) = (\text{positive term}) + (\text{UP-violating term}) + (\text{quantum interference term}). \quad (4.1)$$

The specific form on the left-hand side is used because $\frac{1}{2}(1 \pm \langle A \rangle)$ are the probabilities for $A = \pm 1$ when it can take only values ± 1 , as is the case classically (since uniform precession holds). This gives a convenient scale against which to compare violations of the Tsirelson inequality and of uniform precession.

- (3) Establish whether or not the UP-violating term is capable of dominating the quantum interference terms in terms of explaining the Tsirelson inequality violation, in other words, whether or not a classical model with no quantum-mechanical interferences could account for the violation. If not, the failure of macrorealism per se must also be implied.

B. Method 1: Macrorealistic description of sign changes

We will quantify the breaking of the dynamical assumption of uniform precession through the LG3 quantity L_1 [Eq. (2.9)], corresponding to the quasiprobability sum $q(+, +, +) + q(-, -, -)$, which should be zero classically.

Step 1. The full data set comprises all three correlators C_{12} , C_{23} , and C_{13} measured noninvasively and $\langle A \rangle$ via a Tsirelson test.

Step 2. We find, using the probability moment expansions in Eqs. (2.3) and (2.6), that the Tsirelson quantity can be written as

$$\frac{1}{2}(1 + \langle A \rangle) = \sum_{j>i} p_{ij}(+, +) - L_1 + I, \quad (4.2)$$

where the sum of two-time measurement probabilities

$$\sum_{j>i} p_{ij} \equiv p_{12}(+, +) + p_{23}(+, +) + p_{13}(+, +) \quad (4.3)$$

is manifestly positive, and the interference term is given by

$$I = \frac{1}{4}(\langle Q_2 \rangle - \langle Q_2^{(1)} \rangle) + \frac{1}{4}(\langle Q_3 \rangle - \langle Q_3^{(2)} \rangle) + \frac{1}{4}(\langle Q_3 \rangle - \langle Q_3^{(1)} \rangle). \quad (4.4)$$

This constitutes the explicit form of Eq. (4.1).

The candidate state we consider is the parity-flipped Zaw-Scarani state (3.9) with a (rescaled) lower-bound Tsirelson violation

$$\frac{1}{2}(1 + \langle A \rangle) = -0.0598 < 0, \quad (4.5)$$

which forms the left-hand side of the key formula [Eq. (4.2)].

Step 3. Finally, we ask the following question: Is it possible that a nonzero no-crossing probability (that breaks uniform precession) explains the full Tsirelson violation? Referring to the crucial Eq. (4.2), we have calculated the left-hand side term $\frac{1}{2}(1 + \langle A \rangle) = -0.0598$ and the second term on the right-hand side, $L_1 = 0.0421$ (from Appendix F). Importantly, because the first term on the right-hand side, $\sum_{j>i} p_{ij}$, is positive, then no matter its exact value, it is mathematically impossible for $\sum_{j>i} p_{ij} - 0.0421$ to reach the value -0.0598 . Thus, it is immediately clear that $I \neq 0$ must be implied (specifically, $I < 0$), that is, nonzero quantum interferences are present. We conclude that the violation of uniform precession is *not* sufficient to explain the Tsirelson inequality violation. This is a crucial result, showing that this method has proven successful in identifying that uniform precession is satisfied well enough that quantum behavior must be implied.

We also note that Eq. (4.2) can instead be written in terms of the two-time quasiprobabilities [Eq. (2.2)] with no interferences:

$$\frac{1}{2}(1 + \langle A \rangle) = q_{12}(+, +) + q_{23}(+, +) + q_{13}(+, +) - L_1. \quad (4.6)$$

Each of the two-time quasiprobabilities satisfies the the Lüders bound; but, in fact, the sum $q_{12}(+, +) + q_{23}(+, +) + q_{13}(+, +)$ is itself bounded below by $-\frac{1}{8}$, as is readily shown by writing the sum purely in terms of $\langle A \rangle$ and $\langle A^2 \rangle$ using the moment expansions [Eq. (2.2)]:

$$\sum_{j>i} q_{ij}(+, +) = \frac{1}{8}(\langle (A + 2)^2 \rangle - 1). \quad (4.7)$$

From Eqs. (4.6) and (4.7) we see that the Tsirelson violation $\frac{1}{2}(1 + \langle A \rangle)$ will itself respect the Lüders bound if uniform precession is satisfied exactly ($L_1 = 0$), but in general can exceed this bound, as is the case with the maximally violating state in the full infinite-dimensional QHO Hilbert space, which Tsirelson found to be $\frac{1}{2}(1 + \langle A \rangle) = -0.13 < -0.125$ [1].

We now proceed with some further analysis on the breaking of uniform precession in this state. We already noted above that $L_1 > 0$, and we also found that $L_2 = L_3 = L_4 = 0.3193 > 0$. This complete satisfaction of the LG3s is not some esoteric feature unique to this simple candidate state we have chosen; in fact, we find in Appendix I that *any* Tsirelson-violating state constructed in one of the three subspaces given by Eqs. (3.12)–(3.14) will satisfy all four LG3 inequalities.

Recall from Eqs. (2.13)–(2.16) that the LG3 quantities describe a probability distribution of sign changes. Therefore, there exists a valid classical (macrorealistic) description of crossing probabilities for the state. That there exists a macrorealistic description of sign-change probabilities is a crucial point: We now have the basis to claim that L_1 represents a valid probabilistic description of the breaking of uniform precession.

TABLE II. Various means by which the dynamical assumption of uniform precession can be assessed through three-time probability distributions in Method 2 of Sec. IV. A tick indicates that the UP violation is too small to produce the overall Tsirelson violation, which must therefore be produced, in part, by the quantum-mechanical interference terms.

Measure of UP	Interference term I	$2\Delta p$	UP satisfied sufficiently? ($2 \Delta p < \frac{1}{2} 1 + \langle A \rangle = 0.0598$?)
Δq	0	-0.0598	✗
Δp_{123}	$\frac{1}{2}(\langle Q_2 \rangle - \langle Q_2^{(1)} \rangle) + \frac{1}{2}(\langle Q_3 \rangle - \langle Q_3^{(12)} \rangle)$	-0.0191	✓
Δq_A	$\frac{1}{2}(\langle Q_3 \rangle - \langle Q_3^{(2)} \rangle)$	-0.0785	✗
Δq_B	$\frac{1}{2}(\langle Q_2 \rangle - \langle Q_2^{(1)} \rangle) + \frac{1}{2}(\langle Q_3 \rangle - \langle Q_3^{(1)} \rangle)$	-0.0573	✓

We also note, at this point, that simple superpositions of coherent states, such as the three-headed cat state, will violate the Tsirelson quantity [5]. For example,

$$|\Psi\rangle = N(|\sqrt{2}\rangle + |e^{2\pi i/3}\sqrt{2}\rangle + |e^{4\pi i/3}\sqrt{2}\rangle), \quad (4.8)$$

with the appropriate normalization factor N , yields a Tsirelson violation of $\langle A \rangle = -1.0795 < -1$. Thus, the method of assessing UP with an LG quantity can also be used upon such states, which may be of greater interest experimentally, for Gaussian state superpositions can be easier to create in a laboratory setting [64].

C. Method 2: Three-time sequential measurements

Our second method uses a dynamical assumption involving $\Delta p = p(+, +, +) - p(-, -, -)$, where p is a probability to be specified, which weakens the original assumption for deriving Tsirelson's inequality [1,2,4]. This is of interest because, classically, the Tsirelson quantity A can take values ± 3 and ± 1 , but setting $\Delta p = 0$ prevents the values ± 3 from contributing to the computation of the average $\langle A \rangle$, and hence trivially $-1 \leq \langle A \rangle \leq 1$.

The probabilities could be specified to involve three-time projective measurements Δp_{123} , corresponding to Eq. (2.7), or we could consider the quasiprobability Δq , corresponding to Eq. (2.3). We could also use the two quasiprobability distributions mentioned in the discussion surrounding Eq. (2.8), which lie between Δp_{123} and Δq . We denote these distributions Δq_A and Δq_B , which we define fully in Appendix H. Appendix H also details the computational approach to calculating all such measurement probabilities. An important point is that all of these four distributions are identical to a macrorealist but differ quantum mechanically by various interference terms.

Step 1. We take the data set to be Δp measured via various three-time measurements and the Tsirelson quantity $\langle A \rangle$ measured with single-time measurements.

Step 2. We find an alternative way in which the Tsirelson quantity can be expressed, using this time the moment expansion (2.7):

$$\frac{1}{2}(1 + \langle A \rangle) = \frac{1}{2}(1 - D) + 2\Delta p + I. \quad (4.9)$$

Here I is the interference term, and its values for each of the four cases are given in Table II. Both classically and quantum mechanically, the triple correlator satisfies $|D| \leq 1$, which renders the first term $\frac{1}{2}(1 - D)$ manifestly non-negative, so that Eq. (4.9) is the explicit form of Eq. (4.1) in this method.

Step 3. We compare the size of the UP violation, $2|\Delta p|$, with the size of the Tsirelson violation, $\frac{1}{2}|1 + \langle A \rangle|$, noting

once again that the interference terms would vanish in a purely classical model of the violation. Specifically, $2|\Delta p| = \frac{1}{2}|1 + \langle A \rangle|$ marks the threshold that determines whether or not there is some plausible value of D that can satisfy Eq. (4.9) with $I = 0$.

Table II catalogs our numerical results with this approach, considering each of the four aforementioned distributions. We compare the extent of a lower-bound violation that would be possible from only the breaking of the dynamical assumption in a classical model with the true value of the Tsirelson violation, using the same state as before [Eq. (3.9)]. With the three-time sequential measurement probability Δp_{123} as the chosen measure of UP, we calculate that $2|\Delta p_{123}| = 0.0191 < 0.0598$, showing once again that the breaking of the dynamical assumption falls significantly short of accounting for the full extent of the Tsirelson violation, thereby implying the presence of quantum-mechanical interference terms.

In the case of the distribution Δq_B , UP is satisfied only marginally; when the quasiprobability distribution Δq is instead used, which produces no interference terms, the UP violation is exactly commensurate with the Tsirelson violation. This is to be expected, since $\Delta q = 0$ is a sufficient condition to derive the Tsirelson inequality, which property follows directly from the moment expansion form of Eq. (2.3).

An alluring question is whether it is possible, by some variation of the parameters for the quantum state, to render $\Delta p_{123} = 0$ exactly, while keeping within a Tsirelson-violating region (such as that depicted in Fig. 4), which may serve the poetic justice of “total satisfaction” of the dynamical assumption, whereby the Tsirelson violation is solely due to effects quantum mechanical in origin. In fact, unlike in Method 1, wherein setting $L_1 = 0$ leads trivially to the satisfaction of the Tsirelson inequality, it is possible in this case to set $\Delta p_{123} = 0$. We find by using a constrained optimization procedure via Lagrange multipliers that a simple state, still within the $N = 6$ subspace, albeit now with complex coefficients, can be found to produce $\langle A \rangle = 1.0756$ with $\Delta p_{123} = 0$. The computational procedure is documented in Appendix J.

We also note that perfect sign-operator measurements are an idealization for any real experimental procedure and comment on the effect of smoothed projectors in Appendix K.

V. HEURISTIC NOTIONS OF UNIFORM PRECESSION

In this section we describe an alternative method of assessing uniform precession which is somewhat heuristic and indirect but wherein the measurements involved are simpler and in some cases are, appealingly, single-time measurements. We consider measurements of two simple quantities during a

given time interval: the dwell time in $x > 0$ and the number of crossings of the origin. Physical measurements will reveal the quantum-mechanical values of these quantities and, we will show, if sufficiently close to the corresponding classical values will significantly constrain possible violations of uniform precession in a trajectory model. This in turn gives heuristic indications of the smallness of the various forms of the probabilities $p(+, +, +)$ and $p(-, -, -)$ used in Methods 1 and 2 above. We also relate the Tsirelson inequality to the probability current.

A. Dwell time

We mentioned previously that dynamical assumptions are often classically motivated. One potential notion of how well the dynamical assumption of uniform precession is satisfied is the dwell time, which is classically given by the integral

$$T_D(\tau) = \frac{1}{2\pi} \int_0^\tau \theta(x(t)) dt, \quad (5.1)$$

where we highlight the explicit dependence of θ upon the time-dependent position $x(t)$. The dwell time is a measure of the fraction of time the particle spends in $x > 0$. Classically, this value should be $\frac{1}{2}$ over a full period ($\tau = 2\pi$). This can readily be turned into a self-adjoint operator $\hat{T}_D(\tau)$ [65]. The average quantum-mechanical dwell time $\langle T_D(\tau) \rangle$ can be ascertained experimentally by measuring $\langle \theta(\hat{x}(t)) \rangle$ at multiple times.

For an arbitrary state $|\psi\rangle = \sum_n c_n |n\rangle$, in terms of energy eigenstates, we can write

$$\langle T_D(2\pi) \rangle = \sum_n |c_n|^2 \langle n | \theta(\hat{x}(0)) | n \rangle, \quad (5.2)$$

because $|n\rangle$ are stationary states, meaning $\langle n | \theta(\hat{x}(t)) | n \rangle = \langle n | \theta(\hat{x}(0)) | n \rangle$. Also, as energy eigenstates have definite parity, $\langle n | \theta(\hat{x}(0)) | n \rangle = \frac{1}{2}$. Therefore,

$$\langle T_D(2\pi) \rangle = \frac{1}{2}. \quad (5.3)$$

We see that the dwell time, integrated over a full period, is exactly the classical value. By this measure, classical and quantum oscillators are indistinguishable.

The dwell time operator is diagonal over a full period (and in fact is proportional to the identity), which can be understood as due to the fact that $\hat{T}_D(2\pi)$ is $\text{Twirl}[\theta(\hat{x})]$ over a representation of the group $U(1)$, and thus there is no spread in its distribution. Within the subspace $\mathcal{H}_0$, the dwell time operator is also diagonal when integrated to one-third or two-thirds of a period. The latter case is particularly informative, since this represents integrating between the first and final measurement times. We have found that $\langle \hat{T}_D(4\pi/3) \rangle = \frac{1}{3}$ exactly, which, remarkably, also falls within the range of classically allowed values. Another observation is that looking at the dwell time over any cycle, starting at any time, yields the same results.

It is quite striking that the quantum dwell time agrees with the classical result and yet Tsirelson violations still occur. In fact, Ref. [4] considered alternatives to the Tsirelson scheme with a greater number of measurement times during a period and found that quantum violations faded away as the number of measurement times was increased (limiting to the dwell

time result which has no quantum violations). This perspective motivates viewing the Tsirelson quantity as a Riemann sum approximation to the dwell time integral. Upon performing such an analysis (see Appendix L), we have found that the Tsirelson quantity is related to the probability density current $J(x, t) = \text{Im}(\psi^* \partial_x \psi)$ at the origin,

$$\langle A \rangle = [6\langle T_D(2\pi) \rangle - 3] + 2\pi J(0, \tau) = 2\pi J(0, \tau), \quad (5.4)$$

where τ is some time on the interval $[0, 2\pi]$, as the result was obtained using the mean value theorem. Thus, we see that the dependence of $\langle A \rangle$ upon the average dwell time drops out, since $\langle T_D(2\pi) \rangle$ assumes the classical value, and we see that a Tsirelson violation necessitates a large (i.e., greater than $1/2\pi$) probability current at the origin. Consequently, this relation restricts the choice of violating states. Conversely, Eq. (5.4) provides a condition guaranteeing Tsirelson satisfaction. Specifically, if $|J(0, \tau)| \leq 1/2\pi$ for all $\tau \in [0, 2\pi]$, then $|\langle A \rangle| \leq 1$. In Appendix L we also consider classicalizations of the quantum phase-space distribution and the resulting classical probability current. These classicalized currents were consistently lower than the corresponding quantum currents, indicating a connection to the above condition on Tsirelson satisfaction.

B. Crossing number

Another simple property of classical uniform precession, which is closely linked to Tsirelson's dynamical assumption, is that trajectories cross the origin exactly twice over a period (barring the singular case of a particle sitting at rest at the phase-space origin). Classically, we can compute the number of crossings over a time interval τ using

$$N_c(\tau) = \int_0^\tau dt |p(t)| \delta(x(t)), \quad (5.5)$$

where $N_c(2\pi) = 2$.

Likewise, we can define the Hermitian quantum operator

$$\hat{N}_c(\tau) = \frac{1}{4} \int_0^\tau dt [\{|\hat{p}|, \delta(\hat{x})\} + \text{sgn}(\hat{p})\{|\hat{p}|, \delta(\hat{x})\} \text{sgn}(\hat{p})], \quad (5.6)$$

suppressing the time dependence of $\hat{x}$ and $\hat{p}$. We note that, written in this form, the first term picks up only symmetric states and the second term only antisymmetric states when $\tau = 2\pi$.

Explicit calculations indicate an asymptotic tendency towards the classical value of two crossings in the limit of high-energy eigenstates, which is depicted in Fig. 6. Moreover, we calculate for the Zaw-Scarani state an average crossing number of $\langle N_c(2\pi) \rangle_Z \approx 1.49$, with the possible measurement outcomes $\langle 0 | \hat{N}_c(2\pi) | 0 \rangle = \langle 3 | \hat{N}_c(2\pi) | 3 \rangle = \sqrt{2}$ and $\langle 6 | \hat{N}_c(2\pi) | 6 \rangle \simeq 2.033$ [as the energy eigenstates are also eigenstates of $\hat{N}_c(2\pi)$]. So the spread of possible measurement outcomes is around 0.6. Over a third of a period, between measurement times, $\langle N_c(2\pi/3) \rangle_Z \approx 0.50$. These results indicate that the number of crossings in the quantum case for the Zaw-Scarani state is significantly smaller than the classical case.

A potential issue is that measuring the crossing number requires a two-time measurement, which could raise concerns

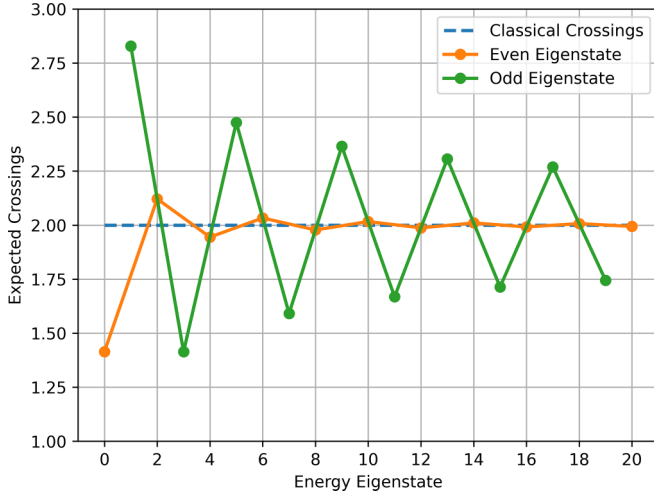


FIG. 6. Plot of the average crossing number $\langle n | N_c(2\pi) | n \rangle$ over a full period for the first 21 energy eigenstates. We see that the even eigenstates converge much more rapidly to the classical value of 2.

of invasiveness once again. However, we note that, classically,

$$|p|\delta(x) = \sqrt{p^2 + x^2}\delta(x) = \sqrt{2E}\delta(x) \quad (5.7)$$

in a fixed energy state. In turn, the first term of Eq. (5.6) can be taken to be proportional to $\sqrt{2E}|\psi(0)|^2$, where we include the nonzero ground-state energy contribution in E . This provides an alternative expression for the crossing number for fixed energy states which only involves a single-time measurement. Using this form, we find that $\langle 0 | \hat{N}_c | 0 \rangle \approx 1.7725$ and $\langle 6 | \hat{N}_c | 6 \rangle \approx 1.997$. These results are reasonably close to those obtained above and exhibit the same qualitative features. However, this method only works for the symmetric states, and we note that states like the Zaw-Scarani state contain a large antisymmetric part.

C. Implications for trajectory models

We have seen that the quantum-mechanical properties of the dwell time are identical to its classical properties and that, as for the crossing number, its quantum-mechanical properties are qualitatively the same as classical and quantitatively similar. These features suggest that measurements of the dwell time and crossing number may be used to give indirect but useful bounds on the size of uniform precession violations. These quantities are in principle related to the three-time probabilities used in Sec. IV, but any such relation will be nonunique, and we have not been able to find a particularly useful one. We proceed using a more heuristic analysis.

We will suppose that there exists an underlying probability distribution on a set of configuration-space trajectories $x(t)$ which are close to classical trajectories but could potentially produce a Tsirelson inequality violation by violating uniform precession. A trajectory model could produce nonzero values of $p(+, +, +)$ and $p(-, -, -)$ if some fraction of the trajectories are on the same side of the x axis at all three measurement times. We argue that the dwell time and crossing number results limit such alternative explanations.

The properties of the dwell time, which hold for any time interval covering a single period and for any initial state,

indicate that the underlying trajectories must spend equal times in $x > 0$ and $x < 0$. Hence, over two-thirds of a period, each trajectory cannot spend more than half a period on one side and so must cross at least once between the first and third measurement times. This means that there can be no trajectories remaining on one side for two-thirds of a period. So the only trajectories contributing to $p(+, +, +)$ and $p(-, -, -)$ must, in order to be on the same side at all three measurement times, cross at least twice during the first third of a period or during the second third, or both. To rule out such wiggly trajectories, we can appeal to the crossing operator results which indicate that highly erratic trajectories have very low probability.

This argument gives an independent understanding as to why uniform precession should hold reasonably well. We have not found a more precise quantitative account to compare with the results of Table II.

D. Classical versus quantum dynamics

Finally, we make some general remarks about dynamics in classical versus quantum theories and why we might expect, on general grounds, that the violation of uniform precession is small.

The predictions of quantum theory are very different from the predictions of classical physics and there are a number of reasons for this. Quantum theory allows a much richer variety of initial states than does classical physics, for example, through Wigner functions that can be negative versus classical phase-space probabilities. In general, there is also a wide variety of possible single-time measurements that can be made that contribute to quantum behavior. For two-time sequential measurements, the noncommutativity of measurements also contributes.

However, when assessing the degree to which the dynamics of a quantum-mechanical point-particle system is behaving classically, we are only interested in a very limited class of measurements, namely, coarse-grained position or phase-space measurements at different times. For such measurements, the dynamics of classical and quantum systems actually look much more similar. There are standard results showing that wave packets and more generally phase-space localized quasiprojectors evolve to a good approximation along classical paths (see, for example, Ref. [66] and references therein). These results can be used to show that the probabilities for histories of coarse-grained position or phase-space measurements are strongly peaked around classical paths.

This near-classical behavior is modified if the measurements significantly chop the evolving wave function, as is the case in this paper (and, for multiple measurements, if they are applied at time intervals shorter than the energy time $\hbar/\langle E \rangle$ [66], but this is not significant here). The magnitude of the nonclassical effects producing the chopping can be seen in the specific model of Refs. [67,68], which exhibited LG2 violations for measurements of $\text{sgn}(x)$ for a harmonic oscillator with a Gaussian initial state. This was produced by the LG2 corresponding to the quasiprobability $q(-, +)$, which we write explicitly in the Wigner representation

$$q(-, +) = \int dp dq W_{-+}(p, q) W_{\rho}(p, q), \quad (5.8)$$

where W_ρ is the Wigner function of the Gaussian initial state, which is clearly non-negative, and W_{-+} is the Wigner-Weyl transform of the operator $\frac{1}{2}\{\theta(\hat{x}(t_1)), \theta(\hat{x}(t_2))\}$, given in Refs. [1,68]. The classical analog of $W_{-+}(p, q)$ is clearly $\theta(x(t_1))\theta(x(t_2))$, which is trivially non-negative; but $W_{-+}(p, q)$ has regions of negativity which arise from the noncommutativity of the two operators, and this is the only source of nonclassicality. (Physically, this is related to the diffraction-in-time effect [68–70].) Most significantly, the violation produced is only 22% of the maximum possible violation (the Lüders bound). By contrast, near-maximal LG violations are possible in this system with, for example, an initial state consisting of the first excited state of the harmonic oscillator, which has a Wigner function with substantial regions of negativity. (More speculatively, we also note that the size of the uniform precession violation here is roughly 4%, which is peculiarly similar to the estimated bound on the so-called quantum backflow effect [20,71–73].)

This example indicates why we might expect that, for the very limited set of measurements used to check proximity to classical behavior, the nonclassical effects arising from those measurements will generally not match the level of nonclassicality arising from the choice of initial state. These assertions could be checked experimentally using control experiments in which the dynamics of a system prepared in a set of classical-like states are measured.

So, in summary, there are general physical and experimentally testable reasons why we might expect the violation of a dynamical assumption like uniform precession to be small compared to the more significant nonclassical effects that can arise from the initial state.

VI. SUMMARY AND DISCUSSION

The Tsirelson inequality makes clear that, by introducing a classically motivated dynamical assumption, the prospect of detecting quantum behavior in temporal correlations need not be plagued by potentially invasive sequential measurements. In this paper we have placed the Tsirelson inequality within the wider framework of Leggett-Garg tests for macrorealism, thereby yielding a method for an assessment of the dynamical assumption.

In Sec. III we analyzed violating states in an energy-truncated subspace and formalized the Tsirelson operator as the twirl of the sign operator over the group elements of the three-element cyclic group. This perspective directly explained its eigenspace structure and opens up the possibility for more tools from quantum information to provide further insight into the Tsirelson operator. We provided proofs that a superposition of at least three energy eigenstates of the QHO is required to produce a Tsirelson violation and that, in relevant subspaces, all three correlators are equal for the specific measurement time interval of two-thirds of a period. The violations possible with simple states such as the Zaw-Scarani state—and, moreover, superpositions of coherent states—are certainly promising and within the reach of experimentalists in the coming years.

However, as we noted in Sec. I, there is a crucial loophole in the Tsirelson inequality: particularly, that violations could ensue from the failure of the dynamical assumption

of uniform precession in a classical system, meaning that a Tsirelson violation is not a sufficient condition for genuine quantum behavior. Therefore, in this paper we have developed a subsidiary protocol to disentangle violations of macrorealism from the breaking of the dynamical assumption, to which end we have embedded the Tsirelson inequality within the wider Leggett-Garg formalism.

In Sec. IV we broke apart the Tsirelson quantity into separate terms that quantified the breaking of uniform precession and quantum interferences such that their sizes may be compared. Recognizing that uniform precession can be defined quantum mechanically in several inequivalent ways in terms of measurable probabilities, we investigated two methods. In the first method, we proved the surprising fact that violating eigenstates of the Tsirelson operator satisfy all four LG3 inequalities, meaning that there invariably exists a macrorealistic description of sign changes, which can then be used to assess uniform precession through an LG3 test. In the second method, we considered a generalized dynamical assumption with the difference of no-sign-change probabilities (perhaps less desirable than the first method, for it involves three sequential measurements). In all cases, we found success with the protocol, demonstrating that quantum interference terms must be implied, elevating the Tsirelson test from a simple test of nonclassical behavior to a more robust demonstration of quantumness.

In Sec. V, by considering the heuristic notions of dwell times and the crossing number operator, we presented further qualitative evidence that uniform precession is satisfied well in states that violate the Tsirelson inequality. We found that the quantum-mechanical dwell time obeys the classical bound exactly and that the total number of origin crossings is approximately the classical value. On their own, these stand as interesting mathematical results, further elucidating the various ways in which uniform precession is and is not equivalent in the classical and quantum cases. However, we also further interpreted these quantities in the context of underlying trajectory models, noting that they greatly restrict the extent to which underlying trajectories can be erratic. We also came upon the global property that Tsirelson violations are linked to the probability currents through the origin, whereby weakly flowing states, or the classicalized currents of quantum states, never produce a violation.

We conclude with a discussion on the key issue of whether or not the need for sequential measurements has really been eliminated—a virtue of the Tsirelson inequality we so admired in the first place—if, at the very least, a Leggett-Garg test is still required to assess uniform precession. Is noninvasive measurement then reinvoked as an untoward assumption alongside macrorealism per se and uniform precession? At one level, this paper can be regarded as a purely quantum-mechanical analysis of Tsirelson violations, and we have found that they arise largely from the presence of interferences and a genuine failure of macrorealism, but with a small contribution stemming from UP violations.

However, an ultimate goal for Tsirelson tests will be to probe systems not knowing beforehand whether they are quantum or not. Then, to an experimentalist, who should wish to implement this protocol by collecting a data set of the Tsirelson quantity supplemented with a Leggett-Garg test

for the relevant probabilities, the problem of reinvoking a Leggett-Garg test, in order that we should further conclude that macrorealism per se has been violated, is possibly harder to justify. However, we may appeal here to the heuristic results of Sec. V, which provide evidence that the LG3 quantity L_1 should be small and which support the idea that violations of uniform precession are not in general sufficiently large to be the primary cause of Tsirelson inequality violations. In some cases, we were partially successful in showing how these quantities may be measured through only single-time measurements. In the original Leggett-Garg scheme, there is no reason to expect *a priori* that L_1 should be small; here, however, we have gathered evidence that it should indeed be small (and positive), and so if an experiment finds otherwise, there is reason to suspect that measurement invasiveness may be largely to blame.

The results of this paper open up several new research directions. Future work could consider an anharmonic oscillator, multiple coupled oscillators and links to field theories, entanglement, alternative measures of crossings, and stronger quantum-mechanical definitions of uniform precession, among other things. We have concentrated in this work upon the case of the harmonic oscillator, but it would be interesting to conduct this same analysis and assessment of uniform precession for spin systems; subsequent comparison with the results herein obtained could be insightful. We have considered the use smoothed projectors, but these order of magnitude arguments need to be bolstered by a more detailed calculation, similar to that of Ref. [67]. The analysis of Tsirelson's inequality and the breaking of uniform precession could be extended to incorporate suspected connections with the quantum backflow limit. This could all provide further illumination on the true nature of violations of Tsirelson's intriguing inequality.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: TSIRELSON QUANTITY FROM THE WIGNER FUNCTION

The Tsirelson quantity can also be written in terms of the Wigner quasiprobability distribution $W_\rho(x, p)$ as

$$\langle A \rangle = \text{Tr}(\rho \hat{A}) = \iint dx dp W_A(x, p) W_\rho(x, p), \quad (\text{A1})$$

where $W_A(x, p)$ [and likewise $W_\rho(x, p)$] is the phase-space representation of the operator $\hat{A}$, obtained from its Weyl

transform:

$$W_A(x, p) = \int_{-\infty}^{\infty} dy e^{-ipy/\hbar} \left\langle x + \frac{y}{2} \left| \hat{A} \right| x - \frac{y}{2} \right\rangle. \quad (\text{A2})$$

This formula is valid for any Hermitian operator.

APPENDIX B: SINGLE-CROSSING DYNAMICAL ASSUMPTION

An alternative set of inequalities can be formulated under the single-crossing assumption (SCA), which is an alternative dynamical assumption that instead asserts that $p(+, -, +) = p(-, +, -) = 0$, forbidding that the particle cross the origin more than once. A very rapidly oscillating system could cross many times in between measurements while maintaining $p(+, -, +) = p(-, +, -) = 0$, but we assume that the relative amplitudes of these erratic trajectories are small. In fact, similar inequalities can be obtained using the polytope method along with dynamical constraints over a full period as shown in Ref. [2].

The SCA can be imposed in the same way as was uniform precession in the original Tsirelson derivation, by setting $p(+, -, +) = p(-, +, -) = 0$. Using the moment expansions (2.3), it is straightforward to obtain

$$-1 \leq \langle Q_1 \rangle - \langle Q_2 \rangle + \langle Q_3 \rangle \equiv \langle A_{\text{SCA}} \rangle \leq 1. \quad (\text{B1})$$

One of the most interesting features of the original Tsirelson inequality, we found, was that it could be derived from the weaker condition that $p(+, +, +) = p(-, -, -)$. In fact, there is an exact analog of this for the SCA, whereby Eq. (B1) follows directly from the weaker condition $p(+, -, +) = p(-, +, -)$.

Classically, one of the simplest examples of a system for which the SCA is valid would be a free particle. Acted upon by no external forces, a free particle will only ever cross a predefined origin at most once. In the case of a uniformly precessing system, such as the harmonic oscillator, this assumption holds true over half a period. We choose measurement times $t_j = \frac{jT}{6}$, $j = 0, 1, 2$, where T is the time period of the oscillator.

We repeat the same analysis as before, defining an operator

$$\hat{A}_{\text{SCA}} = \hat{Q}_{t=0} - \hat{Q}_{t=1/6} + \hat{Q}_{t=1/3}, \quad (\text{B2})$$

where we have set the time period to unity for convenience. What we will now proceed to show is that imposing the SCA over half a period is mathematically identical to imposing uniform precession over a full period, in that

$$\langle A_{\text{SCA}} \rangle = \langle A \rangle. \quad (\text{B3})$$

Since $\langle A \rangle = \langle Q_{t=0} \rangle + \langle Q_{t=1/3} \rangle + \langle Q_{t=2/3} \rangle$ and $\langle A_{\text{SCA}} \rangle = \langle Q_{t=0} \rangle - \langle Q_{t=1/6} \rangle + \langle Q_{t=1/3} \rangle$, we see that these two quantities only differ by one term. If indeed it is true that $\langle A \rangle = \langle A_{\text{SCA}} \rangle$, then it is implied that $\langle Q_{t=2/3} \rangle = -\langle Q_{t=1/6} \rangle$. In fact, the more general result holds true that $\langle Q_t \rangle = -\langle Q_{t+1/2} \rangle \forall t$, that is to say, the single-time average of the dichotomic variable Q merely flips sign after half a period.

We can understand why this is the case by considering what happens to the energy eigenfunctions after exactly half a period. Undergoing unitary time evolution in accordance with the QHO Hamiltonian, the eigenstate $|n\rangle$ accrues a relative

phase of $e^{-i\pi n}$ after half a period. As such, the odd- n eigenstates will pick up a minus sign, while the even- n states are unaffected. In the language of wave functions, it is simple to see that this means a reflection across the y axis after exactly half a period. Therefore, quantum mechanically, the quantities $\langle A \rangle$ and $\langle A_{\text{SCA}} \rangle$ are identical.

APPENDIX C: MATRIX ELEMENTS IN AN ENERGY EIGENBASIS

Let

$$|\Psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \quad \hat{Q} = 2\theta(\hat{x}) - \mathbb{1}, \quad (\text{C1})$$

where $\theta(\hat{x})$ is the Heaviside projector on $x > 0$. To obtain explicit matrices we truncate the Hilbert space at a finite excitation N :

$$|\Psi\rangle = \sum_{n=0}^N c_n |n\rangle, \quad \mathbf{c} = (c_0, \dots, c_N)^T. \quad (\text{C2})$$

For any operator $\hat{O}$ its expectation value reduces to a quadratic form,

$$\langle \Psi | \hat{O} | \Psi \rangle = \sum_{n,k=0}^N c_n^* O_{nk} c_k = \mathbf{c}^\dagger \mathbf{O} \mathbf{c}, \quad O_{nk} := \langle n | \hat{O} | k \rangle. \quad (\text{C3})$$

Hence the task is to evaluate the matrix $Q_{nk} = \langle n | \hat{Q} | k \rangle$. Using the position representation and energy eigenstate wave functions $\Psi_n(x)$,

$$Q_{nk} = \int_{-\infty}^{\infty} \Psi_n(x) \Psi_k(x) [2\theta(\hat{x}) - 1] dx, \quad (\text{C4})$$

using the property that the energy eigenstates for the one-dimensional harmonic oscillator can always be taken to be real to drop the complex conjugate on $\Psi_n(x)$. To make the integral tractable, we split the integral at the origin and use the overlap formula in the Appendix of Ref. [67],

$$F_{nk} \equiv \int_0^{\infty} \Psi_n(x) \Psi_k(x) dx = \frac{\Psi'_k(0) \Psi_n(0) - \Psi'_n(0) \Psi_k(0)}{2(k-n)} \quad (n \neq k), \quad (\text{C5})$$

where $F_{nk} = \langle n | \theta(\hat{x}) | k \rangle$ and $F_{nn} = \frac{1}{2}$. Therefore, we find

$$Q_{nk} = 2F_{nk}, \quad Q_{nn} = 0. \quad (\text{C6})$$

To evaluate Eq. (C5), we use the closed-form expressions for the eigenstates of the harmonic oscillator in terms of the Hermite polynomials $H_n(x)$,

$$\Psi_n(x) = \frac{\pi^{-1/4}}{\sqrt{2^n n!}} \exp\left(-\frac{1}{2}x^2\right) H_n(x). \quad (\text{C7})$$

Consequently, using the identities for the derivatives of Hermite polynomials,

$$H'_0(0) = 0, \quad H'_n(0) = 2nH_{n-1}(0) \quad (n \geq 1), \quad (\text{C8})$$

we find

$$Q_{nk} = \frac{2\pi^{-1/2}}{(k-n)\sqrt{2^{n+k}n!k!}} [kH_{k-1}(0)H_n(0) - nH_{n-1}(0)H_k(0)] \quad (n \neq k). \quad (\text{C9})$$

Finally, since $U|n\rangle = e^{-2\pi i n/3}|n\rangle$,

$$A_{nk} = Q_{nk}(1 + e^{(2\pi i/3)(n-k)} + e^{(4\pi i/3)(n-k)}). \quad (\text{C10})$$

Our numerical implementation of the sign operator and Tsirelson operator on sympy, the symbolic python MATH package, also employed the identities

$$H_{2m}(0) = (-2)^m (2m-1)!!, \quad H_{2m+1}(0) = 0. \quad (\text{C11})$$

We verified this implementation by checking that matrix elements of the F_{nk} agreed with those cited in Ref. [67] and our implementation of the Tsirelson matrix was verified by checking some known violating states.

APPENDIX D: DIAGONALIZATION OF THE TSIRELSON OPERATOR IN AN ENERGY EIGENBASIS

Note that when $\tau = \frac{2\pi}{3}$, the term

$$1 + e^{i\tau(n-k)} + e^{2i\tau(n-k)} = 1 + \cos\left(\frac{2\pi}{3}(n-k)\right) + \cos\left(\frac{4\pi}{3}(n-k)\right) + i \sin\left(\frac{2\pi}{3}(n-k)\right) + i \sin\left(\frac{4\pi}{3}(n-k)\right) \quad (\text{D1})$$

in Eq. (3.7) vanishes unless the difference $(n-k)$ is a multiple of 3. Moreover, the Zaw-Scarani state produces the largest

violating state not only in the $N=6$ subspace, but also in the $N=7$ and $N=8$ subspaces. For $N=9$, on the other hand,

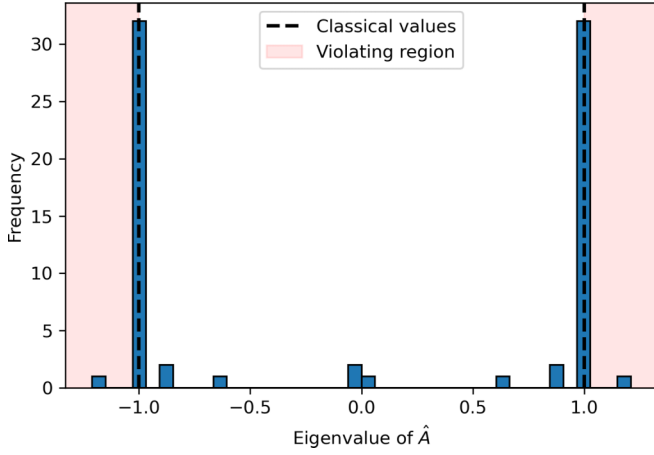


FIG. 7. Histogram of the spectrum of eigenvalues of $\hat{A}$ plotted by using an energy truncation of $E = 70\hbar\omega$. The y axis refers to the number of eigenvalues found within a given range.

we can achieve a superior violation of $\langle A \rangle \approx 1.12$ with the state

$$|\Psi_9\rangle \approx 0.6277|0\rangle - 0.7064|3\rangle + 0.3255|6\rangle + 0.0304|9\rangle. \quad (\text{D2})$$

We resort to an approximate numerical specification of the coefficients, because neat analytic forms are intractable. Importantly, any numerical errors are encapsulated in the floating point specification of these coefficients, and the remaining coefficients (c_1 , c_2 , c_4 , etc.) really are exactly and identically zero.

Figure 7 shows the results of a spectral decomposition of the Tsirelson operator in a high-energy truncated subspace. This serves as a good illustration that the classical values $A = \pm 1$ are accumulation points [1], but violations past this value are indeed possible. None of our computed eigenvalues exceed approximately 1.26, which agrees with the value of the maximum possible violation quoted in Tsirelson's original paper [1]. A more accurate estimate of the true upper and lower bounds on the spectral norm of $\hat{A}$ has been obtained in the recent work of Ref. [9]. Figure 8 illustrates that considering a progressively larger subspace from the low-lying energy states upward does indeed lead to an asymptotic convergence towards Tsirelson's quoted value (up to an arbitrary scaling

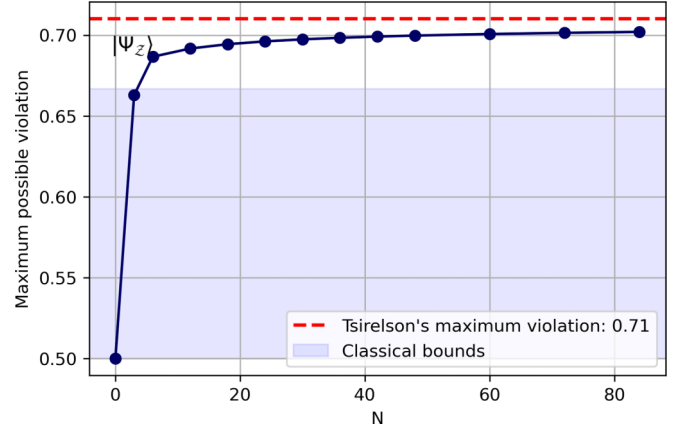


FIG. 8. Computed maximal eigenvalue of $\hat{A}$ in an increasingly large subspace, tending towards a finite maximal violation. Note, importantly, that we rescale the eigenvalues λ as $\frac{1}{6}(\lambda + 3)$. This is only an arbitrary scaling choice to see agreement with the definition of the operator considered by Tsirelson in Ref. [2]. The Zaw-Scarani state $|\Psi_Z\rangle$ is pointed out at $N = 6$.

factor) for the maximum violation in the full QHO Hilbert space.

We have also computed the eigenvectors of $\hat{A}$ in increasingly larger subspaces, cataloged in Table III, and have found that they all obey the same pattern of being composed solely of energy eigenstates that are multiples of 3.

APPENDIX E: SUPERPOSITIONS OF TWO ENERGY EIGENSTATES

As established in Ref. [1], single energy eigenstates of the harmonic oscillator cannot violate Tsirelson's inequality due to their definite parity. The next logical step is to find violations produced by a superposition of two eigenstates. However, as we show now, any superposition of two energy eigenstates is insufficient to produce a violation.

1. Proof strategy

We aim to show that the magnitude of $\langle A \rangle$ is less than 1 for the state

$$|\psi_2\rangle = \alpha|n\rangle + \beta|k\rangle, \quad (\text{E1})$$

TABLE III. Explicit computation of maximally violating states in a truncated energy subspace ($|\Psi_N\rangle = \sum_{n=0}^N c_n|n\rangle$). The only nonzero coefficients are c_{3j} , where $j \in \mathbb{N}$.

N	c_0	c_3	c_6	c_9	c_{12}	c_{15}	c_{18}	$\langle A \rangle$
3	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	0	0	0	0	0.9772
6	$\frac{4}{\sqrt{42}}$	$-\frac{1}{\sqrt{2}}$	$\sqrt{\frac{5}{42}}$	0	0	0	0	1.1195
9	0.6277	-0.7064	0.3255	0.0304	0	0	0	1.1200
12	-0.6529	0.6600	-0.1591	-0.2538	0.2200	0	0	1.1500
15	-0.6509	0.6591	-0.1592	-0.2561	0.2259	-0.0083	0	1.1500
18	-0.6655	0.6354	-0.0999	-0.2771	0.1451	0.1396	-0.1615	1.1661
$\vdots$				$\vdots$				$\vdots$
90								1.2131

$|\Psi\rangle \rightarrow \text{true eigenstate of } A$

where $|\alpha|^2 + |\beta|^2 = 1$. The expectation value of $\hat{A}$ for $|\psi_2\rangle$ is

$$\langle A \rangle = |\alpha|^2 \underbrace{\langle n|A|n \rangle}_0 + |\beta|^2 \underbrace{\langle k|A|k \rangle}_0 + 2 \operatorname{Re}(\alpha^* \beta) \langle n|A|k \rangle. \quad (\text{E2})$$

Expanding $\langle A \rangle$ in terms of Q_{nk} yields

$$\langle A \rangle = 2 \operatorname{Re}(\alpha^* \beta) Q_{nk} (1 + e^{(2\pi i/3)(n-k)} + e^{(4\pi i/3)(n-k)}). \quad (\text{E3})$$

The sum of exponentials is 3 when $n - k = 0 \bmod 3$ and 0 otherwise. So $\langle A \rangle$ is clearly maximized when $(n - k)$ is a multiple of 3. Moreover, $|\alpha|^2 + |\beta|^2 = 1$ puts a constraint on the maximum value of $\operatorname{Re}(\alpha^* \beta)$. Intuitively, the interference term will be maximized if the state is an equal superposition of $|n\rangle$ and $|k\rangle$. Therefore, the maximum value of $\operatorname{Re}(\alpha^* \beta)$ is $\frac{1}{2}$. To be precise, consider the following short proof. Let $\alpha = ce^{i\theta}$ and $\beta = se^{i\phi}$. Then $\operatorname{Re}(\alpha^* \beta) = cs \cos(\phi - \theta)$. The $\cos(\phi - \theta)$ is maximized if $\phi = \theta$. Thus we can choose $\alpha = c$ and $\beta = s$, up to a global phase. Then the constraint $c^2 + s^2 = 1$ means we can parametrize c and s as $c = \cos(\xi)$ and $s = \sin(\xi)$. Therefore, $cs = \frac{1}{2} \sin(2\xi)$, which is maximized if $\sin(2\xi) = 1$. Hence, $\max(cs) = \max[\operatorname{Re}(\alpha^* \beta)] = \frac{1}{2}$. Thus, $\langle A \rangle$ for an equal superposition of $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|n\rangle + |k\rangle)$ with $n - k = 0 \bmod 3$ is

$$\langle A \rangle = 2\left(\frac{1}{2}\right) Q_{nk}(3) = 3Q_{nk}. \quad (\text{E4})$$

We see that the problem of showing that $|\langle A \rangle| < 1$ reduces to showing that

$$|Q_{nk}| < \frac{1}{3} \forall n, k \text{ s.t. } k - n = 3m \text{ for } m \in \mathbb{N}, \quad (\text{E5})$$

where we only need to consider $m \geq 0$ because Q_{nk} is symmetric. One might conjecture that the Q_{03} matrix element will be the largest in magnitude that meets the $k - n = 3m$ condition. If that holds then our proof would be complete, as $|Q_{03}| \approx 0.32 < \frac{1}{3}$ [computed using Eq. (C9)]. Therefore, the strategy we choose to take is to show two things about Q_{nk} : (1) The largest matrix element, in magnitude, on a diagonal, i.e., for fixed m , $|Q_{n,n+2m+1}|$ is $|Q_{0,2m+1}|$,

$$Q_{nk} = \begin{pmatrix} 0 & Q_{01} & 0 & Q_{03} & \cdots \\ Q_{10} & 0 & Q_{12} & 0 & \cdots \\ 0 & Q_{21} & 0 & Q_{23} & \cdots \\ Q_{30} & 0 & Q_{32} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (\text{E6})$$

and (2) the nonzero matrix elements decrease in magnitude across the 0th row, i.e., $|Q_{0,2m+1}|$ is a decreasing function

in m ,

$$Q_{nk} = \begin{pmatrix} 0 & Q_{01} & 0 & Q_{03} & \cdots \\ Q_{10} & 0 & Q_{12} & 0 & \cdots \\ 0 & Q_{21} & 0 & Q_{23} & \cdots \\ Q_{30} & 0 & Q_{32} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (\text{E7})$$

We already have established that the only nonzero matrix elements of the Q_{nk} are those with opposite parity, and we ignore elements of the same parity when claiming these sequences are decreasing. Once those two statements are proven, we will have shown that Q_{01} is the largest matrix element. This does not satisfy the condition $k - n = 3m$ though, and so we take the next largest matrix element Q_{03} , thereby completing the proof.

2. Diagonals of the Q_{nk} matrix

Showing the first statement is slightly complicated by the fact that the even n and odd n matrix elements have different formulas. When n is even (and hence k is odd) the relevant formula [obtained using Eqs. (C9) and (C11)] is

$$|Q_{nk}^{\text{even}}| = \left(\sqrt{\frac{2}{\pi}} \frac{(n-1)!!}{\sqrt{n!}} \right) \frac{(k+1)!}{(k-n)\sqrt{k!} 2^{(k+1)/2} \left(\frac{k+1}{2}\right)!}. \quad (\text{E8})$$

When n is odd (and hence k is even) the relevant formula is

$$|Q_{nk}^{\text{odd}}| = \left(\sqrt{\frac{2}{\pi}} \frac{n!!}{\sqrt{n!}} \right) \frac{k!}{(k-n)\sqrt{k!} 2^{k/2} \left(\frac{k}{2}\right)!}. \quad (\text{E9})$$

To resolve this complication we consider the odd and even indexed subsequences separately. We define the sequence $\{a_n\}_{n=0}^\infty$, where $a_n = |Q_{n,n+2m+1}|$ for fixed m . Then we just need to show the even indexed subsequence is decreasing ($a_{2j+2} < a_{2j}$) and that when the next element of the sequence is odd indexed it is less than the previous even-indexed element ($a_{2j+1} < a_{2j}$).

a. Even subsequence is decreasing

Consider the fraction a_{2j+2}/a_{2j} and show it is less than 1:

$$\frac{a_{2j+2}}{a_{2j}} = \left| \frac{Q_{2j+2,2j+2m+3}}{Q_{2j,2j+2m+1}} \right|, \quad (\text{E10})$$

$$|Q_{2j+2,2j+2m+3}| = \frac{(2j+1)!!}{\sqrt{(2j+2)!}} \frac{(2j+2m+4)!}{(2m+1)\sqrt{(2j+2m+3)!} 2^{j+m+2} (j+m+2)!}, \quad (\text{E11})$$

$$|Q_{2j,2j+2m+1}| = \frac{(2j-1)!!}{\sqrt{(2j)!}} \frac{(2j+2m+2)!}{(2m+1)\sqrt{(2j+2m+1)!} 2^{j+m+1} (j+m+1)!}. \quad (\text{E12})$$

Therefore,

$$\frac{a_{2j+2}}{a_{2j}} = \frac{(2j+1)}{\sqrt{(2j+2)(2j+1)}} \frac{(2j+2m+4)(2j+2m+3)}{\sqrt{(2j+2m+3)(2j+2m+2)2(j+m+2)}} \quad (\text{E13})$$

$$= \sqrt{\frac{(4j^2 + 4jm + 8j) + 2m + 3}{(4j^2 + 4jm + 8j) + 4m + 4}}, \quad (\text{E14})$$

$$\frac{a_{2j+2}}{a_{2j}} < 1. \quad (\text{E15})$$

■

b. The next odd element is less than the preceding even element

Using a very similar argument to above, consider the fraction a_{2j+1}/a_{2j} and show it is less than 1:

$$\frac{a_{2j+1}}{a_{2j}} = \left| \frac{Q_{2j+1,2j+2m+2}}{Q_{2j,2j+2m+1}} \right|, \quad (\text{E16})$$

$$|Q_{2j+1,2j+2m+2}| = \frac{(2j+1)!!}{\sqrt{(2j+1)!}} \frac{(2j+2m+2)!}{(2m+1)\sqrt{(2j+2m+2)!}2^{j+m+1}(j+m+1)!}, \quad (\text{E17})$$

$$|Q_{2j,2j+2m+1}| = \frac{(2j-1)!!}{\sqrt{(2j)!}} \frac{(2j+2m+2)!}{(2m+1)\sqrt{(2j+2m+1)!}2^{j+m+1}(j+m+1)!}. \quad (\text{E18})$$

Therefore,

$$\frac{a_{2j+1}}{a_{2j}} = \left(\frac{(2j+1)!!}{(2j-1)!!} \right) \sqrt{\frac{(2j)!}{(2j+1)!}} \sqrt{\frac{(2j+2m+1)!}{(2j+2m+2)!}} \quad (\text{E19})$$

$$= \sqrt{\frac{(2j+1)}{(2j+1)+2m+1}}, \quad (\text{E20})$$

$$\frac{a_{2j+1}}{a_{2j}} < 1. \quad (\text{E21})$$

■

3. The 0th row of the Q_{nk} matrix is decreasing

The second thing to show is that $Q_{0,2m+1}$ is a decreasing function in m . To accomplish this, consider the fraction $|Q_{0,2m+3}/Q_{0,2m+1}|$ and show it is less than 1:

$$Q_{0,2m+3} = \left(\sqrt{\frac{2}{\pi}} \frac{(-1)!!}{\sqrt{0!}} \right) \times \frac{(2m+4)!}{(2m+3)\sqrt{(2m+3)!}2^{m+2}(m+2)!}, \quad (\text{E22})$$

$$Q_{0,2m+1} = \left(\sqrt{\frac{2}{\pi}} \frac{(-1)!!}{\sqrt{0!}} \right) \times \frac{(2m+2)!}{(2m+1)\sqrt{(2m+1)!}2^{m+1}(m+1)!}, \quad (\text{E23})$$

where $(-1)!! = 1$. Therefore,

$$\left| \frac{Q_{0,2m+3}}{Q_{0,2m+1}} \right| = \frac{2m+1}{\sqrt{(2m+3)(2m+2)}} < \frac{2m+1}{\sqrt{(2m+1)(2m+1)}}, \quad (\text{E24})$$

$$\left| \frac{Q_{0,2m+3}}{Q_{0,2m+1}} \right| < 1. \quad (\text{E25})$$

■

APPENDIX F: CALCULATION OF CORRELATORS IN THE QUANTUM HARMONIC OSCILLATOR

Consider a general state

$$|\psi\rangle = \sum_{n=0}^N c_n |n\rangle, \quad c_n \in \mathbb{C}. \quad (\text{F1})$$

The measurement operator in the Heisenberg picture is given by $\hat{Q}(t) = e^{iHt}[2\theta(\hat{x}) - 1]e^{-iHt}$, where we choose $t_2 - t_1 = \tau$ and $t_1 = 0$, such that

$$\hat{Q}_j = e^{iH(j-1)\tau}[2\theta(\hat{x}) - 1]e^{-iH(j-1)\tau}. \quad (\text{F2})$$

Then

$$C_{12} = \frac{1}{2} \langle \psi | \{\hat{Q}_1, \hat{Q}_2\} | \psi \rangle. \quad (\text{F3})$$

Therefore,

$$C_{12} = \frac{1}{2} \sum_{n,k=0}^N c_n^* c_k \langle n | \hat{Q}_1 e^{iH\tau} \hat{Q}_1 e^{-iH\tau} | k \rangle + \frac{1}{2} \sum_{n,k=0}^N c_n^* c_k \langle n | e^{iH\tau} \hat{Q}_1 e^{-iH\tau} \hat{Q}_1 | k \rangle. \quad (\text{F4})$$

We can write

$$e^{iH\tau} = \sum_{m=0}^{\infty} e^{im\tau} |m\rangle \langle m|, \quad (\text{F5})$$

so that

$$C_{12} = \frac{1}{2} \sum_{n,k=0}^N c_n^* c_k \langle n | \hat{Q}_1 \sum_{m=0}^{\infty} e^{im\tau} |m\rangle \langle m| \hat{Q}_1 e^{-ik\tau} |k\rangle + \frac{1}{2} \sum_{n,k=0}^N c_n^* c_k \langle n | e^{in\tau} \hat{Q}_1 \sum_{m=0}^{\infty} e^{-im\tau} |m\rangle \langle m| \hat{Q}_1 |k\rangle, \quad (\text{F6})$$

which can be reduced to a form containing only the $Q_{nk} \equiv \langle n | \hat{Q}_1 | k \rangle$ matrix elements:

$$\begin{aligned} & \frac{1}{2} \sum_{n,k=0}^N \sum_{m=0}^{\infty} e^{i(m-k)\tau} c_n^* c_k Q_{nm} Q_{mk} \\ & + \frac{1}{2} \sum_{n,k=0}^N \sum_{m=0}^{\infty} e^{i(n-m)\tau} c_n^* c_k Q_{nm} Q_{mk}. \end{aligned} \quad (\text{F7})$$

The matrix $\mathbf{J}$ is real and symmetric, and so we can combine terms as

$$\begin{aligned} & \frac{1}{2} \sum_{n,k=0}^N \sum_{m=0}^{\infty} e^{i(m-k)\tau} c_n^* c_k Q_{nm} Q_{mk} \\ & + \frac{1}{2} \left(\sum_{n,k=0}^N \sum_{m=0}^{\infty} e^{i(m-k)\tau} c_n^* c_k Q_{nm} Q_{mk} \right)^*, \end{aligned} \quad (\text{F8})$$

that is,

$$C_{12} = \text{Re} \left(\sum_{n,k=0}^N \sum_{m=0}^{\infty} e^{i(m-k)\omega\tau} c_n^* c_k Q_{nm} Q_{mk} \right). \quad (\text{F9})$$

In fact, for states like the Zaw-Scarani state with support entirely on a given $\mathcal{H}_k$, it is sufficient to just compute C_{12} as, with a measurement time spacing of a third of a period, $C_{12} = C_{13} = C_{23} = C$. We prove this statement in Appendix G. For more general states, the calculation procedure for C_{23} and C_{13} is very similar to Eq. (F9).

APPENDIX G: ALL THREE CORRELATORS ARE EQUAL ON $\mathcal{H}_k$

For some state $|\psi_k\rangle \in \mathcal{H}_k$, the correlators C_{ij} can be written as

$$C_{ij} = \frac{1}{2} \langle \psi_k | \{\hat{Q}_i, \hat{Q}_j\} | \psi_k \rangle. \quad (\text{G1})$$

Now the expectation value $F_{ijk} = \frac{1}{2} \langle \psi_k | \{\hat{Q}_i, \hat{Q}_j\} | \psi_k \rangle$, $i < j$, can be expanded as

$$\begin{aligned} F_{ijk} &= \text{Re} \langle \psi_k | (U^{i-1})^\dagger \hat{Q}_1 U^{i-1} (U^{j-1})^\dagger \hat{Q}_1 U^{j-1} | \psi_k \rangle \\ &= \text{Re} \langle \psi_k | \omega^{k(i-1)} \hat{Q}_1 U^{i-j} \hat{Q}_1 \omega^{k(j-1)} | \psi_k \rangle, \end{aligned} \quad (\text{G2})$$

by leveraging the unitarity of U and its action on $|\psi_k\rangle$. We then group some terms with the aim of showing $F_{12k} = F_{23k} = F_{13k}$,

$$F_{ijk} = \text{Re} \omega^{k(j-i)} \langle \psi_k | \hat{Q}_1 U^{i-j} \hat{Q}_1 | \psi_k \rangle. \quad (\text{G3})$$

Clearly, $F_{12k} = \text{Re} \omega^k \langle \psi_k | \hat{Q}_1 U^{-1} \hat{Q}_1 | \psi_k \rangle = F_{23k}$, but it also quickly follows that $F_{12k} = F_{13k}$. Note that

$$F_{13k} = \text{Re} \omega^{2k} \langle \psi_k | \hat{Q}_1 U^{-2} \hat{Q}_1 | \psi_k \rangle, \quad (\text{G4})$$

where $\omega^{2k} = (\omega^k)^*$, $U^{-2} = U$, and $\langle \psi_k | \hat{Q}_1 U^{-2} \hat{Q}_1 | \psi_k \rangle = \langle \psi_k | \hat{Q}_1 U^{-1} \hat{Q}_1 | \psi_k \rangle^*$. So

$$\begin{aligned} F_{13k} &= \text{Re} (\omega^k \langle \psi_k | \hat{Q}_1 U^{-1} \hat{Q}_1 | \psi_k \rangle)^* \\ &= \text{Re} \omega^k \langle \psi_k | \hat{Q}_1 U^{-1} \hat{Q}_1 | \psi_k \rangle = F_{12k} = F_{23k}. \end{aligned} \quad (\text{G5})$$

Hence, by pairing Eqs. (G1) and (G5), we have shown that for a given quantum state, with support entirely on $\mathcal{H}_k$, all

the correlators are equal when the measurement spacing is $\Delta t = 2\pi/3$.

APPENDIX H: CALCULATION OF SEQUENTIAL MEASUREMENT PROBABILITIES

We consider the initial density operator

$$\rho = |\psi\rangle\langle\psi| = \sum_{n,k=0}^N c_n^* c_k |k\rangle\langle n| \quad (\text{H1})$$

and define the Heisenberg-picture projection operators $P_+(t) = \theta(\hat{x}(t))$ and $P_-(t) = 1 - \theta(\hat{x}(t))$, in order to make more concise the notation for the time-evolved Heaviside projector onto the $+x$ axis:

$$P_+(t) \equiv \theta(\hat{x}(t)) = \int_0^\infty e^{-iHt} |x\rangle\langle x| e^{iHt} dx. \quad (\text{H2})$$

Then we have the simple expression

$$p_{123}(s_1, s_2, s_3) = \text{Tr}[P_{s_3}(t_3) P_{s_2}(t_2) P_{s_1}(t_1) \rho P_{s_1}(t_1) P_{s_2}(t_2)]. \quad (\text{H3})$$

In order to compute these probabilities numerically, however, it is easiest to dissociate the time dependence of the operators such that we recover expressions in terms of the familiar Q_{nk} matrix elements, that is,

$$\begin{aligned} p_{123}(+, +, +) &= \text{Tr}[e^{2iH\tau} P_+(t_1) e^{-iH\tau} P_+(t_1) e^{-iH\tau} \\ &\quad \times P_+(t_1) \rho P_+(t_1) e^{iH\tau} P_+(t_1) e^{-iH\tau}], \end{aligned} \quad (\text{H4})$$

recognizing that $P_+(t_1)$ is simply the Heaviside operator without any time evolution. Then writing

$$e^{iH\tau} = \sum_{m=0}^{\infty} e^{im\tau} |m\rangle\langle m| \quad (\text{H5})$$

and setting $\omega = \hbar = 1$ as always, we define the matrix elements

$$L_{nk} \equiv \langle n | (\hat{Q}_1 + 1) | k \rangle = Q_{nk} + \delta_{nk}, \quad (\text{H6})$$

where δ_{nk} is the Kronecker delta symbol, such that

$$\begin{aligned} p_{123}(+, +, +) &= \frac{1}{32} \sum_{m,p,q,r=0}^{\infty} \sum_{n,k=0}^N c_n^* c_k e^{i\tau(q+r-m-p)} \\ &\quad \times L_{rm} L_{mp} L_{pk} L_{nq} L_{qr}, \end{aligned} \quad (\text{H7})$$

recalling that $P_+(t_1) = \frac{\hat{Q}_1 + 1}{2}$. Similarly,

$$\begin{aligned} p_{123}(-, -, -) &= \frac{1}{32} \sum_{m,p,q,r=0}^{\infty} \sum_{n,k=0}^N c_n^* c_k e^{i\tau(q+r-m-p)} \\ &\quad \times L'_{rm} L'_{mp} L'_{pk} L'_{nq} L'_{qr}, \end{aligned} \quad (\text{H8})$$

where $\mathbf{L}' \equiv \mathbf{1} - \mathbf{Q}$. Computationally, we have chosen a finite cutoff value N to truncate the infinite sum over m, p, q , and r .

There are also two other quasiprobability distributions, as elucidated in the discussion surrounding Eq. (2.8). We construct these by beginning from Eq. (2.5) and either replacing C_{13} by $C_{13}^{(2)}$ (the first possibility) or replacing C_{23} by $C_{23}^{(1)}$ (the

second). [Doing both would effectively yield Eq. (2.7).] Thus, we define

$$q_A(+, +, +) = \frac{1}{2} \text{Tr}[P_+(t_3)P_+(t_2)\{P_+(t_1), \rho\}P_+(t_2)]. \quad (\text{H9})$$

Classically, this distribution is identical to $p_{123}(+, +, +)$. In this case,

$$q_A(+, +, +) - q_A(-, -, -) \equiv \Delta q_A = \frac{1}{4}(\langle A \rangle + D + I), \quad (\text{H10})$$

where the interference terms would only stem from the second measurement: $I = \langle Q_3^{(2)} \rangle - \langle Q_3 \rangle$. Further,

$$q_B(+, +, +) = \frac{1}{2} \text{Tr}[P_+(t_3)\{P_+(t_2), P_+(t_1)\rho P_+(t_1)\}]. \quad (\text{H11})$$

For reference, this is readily computed in terms of unevaluated Heaviside operators

$$\begin{aligned} q_B(+, +, +) &= \frac{1}{2} \text{Tr}[e^{2iH\tau} \theta(\hat{x}) e^{-2iH\tau} \{e^{iH\tau} \theta(\hat{x}) e^{-iH\tau}, \theta(\hat{x}) \rho \theta(\hat{x})\}], \\ &= \frac{1}{2} \text{Tr}[e^{2iH\tau} \theta(\hat{x}) e^{-iH\tau} \theta(\hat{x}) e^{iH\tau} \theta(\hat{x}) \rho \theta(\hat{x})], \end{aligned} \quad (\text{H12})$$

which can be computed straightforwardly in terms of the previously defined L_{nk} matrix elements, for example,

$$\begin{aligned} &\frac{1}{2} \text{Tr}[e^{2iH\tau} \theta(\hat{x}) e^{-iH\tau} \theta(\hat{x}) e^{iH\tau} \theta(\hat{x}) \rho \theta(\hat{x})] \\ &= \frac{1}{32} \sum_{n,k,p,q,r} c_n^* c_k e^{i\tau(2p+r-q)} L_{pq} L_{qm} L_{mk} L_{np}. \end{aligned} \quad (\text{H13})$$

APPENDIX I: TSIRELSON VIOLATION IMPLIES LG3 SATISFACTION ON $\mathcal{H}_k$

1. Outline and preliminaries

This appendix is devoted to proving that if Tsirelson's inequality is violated by a given quantum state, with support entirely in $\mathcal{H}_k$, then that state will satisfy all four LG3s [Eqs. (2.9)–(2.12)] if the measurement times t_1 , t_2 , and t_3 are the same as in the Tsirelson scheme.

To show this claim, one simply needs to show two statements hold true. The conjunction of these two statements would imply that all four LG3s are greater than zero.

(1) A Tsirelson violation implies satisfaction of the first LG3, i.e., $L_1 > 0$. We straightforwardly show this in Appendix 12. In fact, this result holds for arbitrary quantum states.

(2) For quantum states in $\mathcal{H}_0$, $\mathcal{H}_1$, and $\mathcal{H}_2$, the first Leggett-Garg quantity is always the smallest of the four. Consequently, all four LG3s are positive.

2. Tsirelson violation implies $L_1 > 0$

Consider a state that violates Tsirelson's inequality such that

$$|\langle Q_1 + Q_2 + Q_3 \rangle| > 1. \quad (\text{I1})$$

Now consider the Cauchy-Schwarz inequality applied to the left-hand side of Eq. (I1),

$$\langle (Q_1 + Q_2 + Q_3)^2 \rangle \geq (\langle Q_1 + Q_2 + Q_3 \rangle)^2. \quad (\text{I2})$$

Using Eqs. (I1) and (I2), we can write

$$\langle (Q_1 + Q_2 + Q_3)^2 \rangle > 1. \quad (\text{I3})$$

Finally, simply expanding out the expectation value nets

$$1 + C_{12} + C_{13} + C_{23} > 0. \quad (\text{I4})$$

3. Inequality L_1 is the smallest LG3 on $\mathcal{H}_0$, $\mathcal{H}_1$, and $\mathcal{H}_2$

Recall from Appendix G that for $|\psi_k\rangle \in \mathcal{H}_k$, $C_{12} = C_{13} = C_{23} = C$. Therefore, on these subspaces, $L_1 = 1 + 3C$ and the rest are $1 - C$. So, by showing that C_{12} is negative we will have shown that L_1 is the smallest LG3. Note that

$$C_{12} = \frac{1}{2} \langle \psi_k | \hat{Q}(\omega^{-k}U + \omega^kU^\dagger) \hat{Q} | \psi_k \rangle, \quad (\text{I5})$$

where $\hat{Q}|\psi_k\rangle$ can be decomposed into $\hat{Q}|\psi_k\rangle = |\phi_k\rangle + |\phi_\perp\rangle$, $|\phi_k\rangle \in \mathcal{H}_k$, and $\langle \phi_k | \phi_\perp \rangle = 0$. Hence, as $|\phi_\perp\rangle$ is constructed out of orthogonal energy eigenstates to $|\phi_k\rangle$, we can write

$$\begin{aligned} C_{12} &= \frac{1}{2} \langle \phi_k | (\omega^{-k}U + \omega^kU^\dagger) | \phi_k \rangle \\ &\quad + \frac{1}{2} \langle \phi_\perp | (\omega^{-k}U + \omega^kU^\dagger) | \phi_\perp \rangle. \end{aligned} \quad (\text{I6})$$

Motivated by the roots of unity representation of the cyclic group, note that

$$(\mathbb{I} + \omega^{-k}U + \omega^kU^\dagger) |\phi_k\rangle = 3 |\phi_k\rangle \Rightarrow (\omega^{-k}U + \omega^kU^\dagger) |\phi_k\rangle = 2 |\phi_k\rangle \quad (\text{I7})$$

and

$$(\mathbb{I} + \omega^{-k}U + \omega^kU^\dagger) |\phi_\perp\rangle = 0 \Rightarrow (\omega^{-k}U + \omega^kU^\dagger) |\phi_\perp\rangle = -|\phi_\perp\rangle. \quad (\text{I8})$$

Thus, we can write Eq. (I6) as

$$C_{12} = \langle \phi_k | \phi_k \rangle - \frac{1}{2} \langle \phi_\perp | \phi_\perp \rangle. \quad (\text{I9})$$

By substituting in the condition $\langle \phi_\perp | \phi_\perp \rangle + \langle \phi_k | \phi_k \rangle = 1$ we find

$$C_{12} = \frac{3}{2} \langle \phi_k | \phi_k \rangle - \frac{1}{2}. \quad (\text{I10})$$

Reintroducing $|\psi_k\rangle$ nets an inequality including the spectral norm of an operator

$$C_{12} \leq \frac{3}{2} \|P_k \hat{Q} P_k\|^2 - \frac{1}{2}, \quad (\text{I11})$$

where P_k is the projection operator onto $\mathcal{H}_k$. We can trivially extend this upper bound by writing

$$C_{12} \leq \frac{3}{2} \max_{k \in \{0,1,2\}} \|P_k \hat{Q} P_k\|^2 - \frac{1}{2}. \quad (\text{I12})$$

However,

$$\max_{k \in \{0,1,2\}} \|P_k \hat{Q} P_k\| = \|\text{Twirl}(\hat{Q})\| = \frac{1}{3} \|\hat{A}\|. \quad (\text{I13})$$

Therefore,

$$C_{12} \leq \frac{1}{6} \|\hat{A}\|^2 - \frac{1}{2}. \quad (\text{I14})$$

Finally, using the upper bound on $\|\hat{A}\|$ from Ref. [9],

$$C_{12} \leq \frac{1}{6}(1.39)^2 - \frac{1}{2} \leq 0. \quad (\text{I15})$$

APPENDIX J: CONSTRAINED OPTIMIZATION PROCEDURE FOR METHOD 2

We seek a state that violates the Tsirelson inequality but satisfies the condition $p_{123}(+, +, +) = p_{123}(-, -, -)$, or $\Delta p_{123} = 0$, exactly. We can reframe this question as a constrained optimization problem where the function we wish to maximize is

$$\langle A \rangle \equiv \langle Q_1 + Q_2 + Q_3 \rangle, \quad (\text{J1})$$

which can be written as a function of the coefficients in the energy basis expansion of the initial state:

$$\langle A \rangle = f(c_n). \quad (\text{J2})$$

Recall that

$$p_{123}(+, +, +) = \frac{1}{32} \sum_{m,p,q,r=0}^{\infty} \sum_{n,k=0}^N c_n^* c_k e^{i\tau(q+r-m-p)} \times L_{rm} L_{mp} L_{pk} L_{nq} L_{qr}, \quad (\text{J3})$$

where the L_{nk} matrix elements are defined as in Eq. (H6), with a similar expression for $p_{123}(-, -, -)$. Defining

$$\Gamma_{nk} = L_{rm} L_{mp} L_{pk} L_{nq} L_{qr}, \quad (\text{J4})$$

the constraint can be rewritten in the simple form

$$\Delta p_{123} \equiv 2 \sum_{n,k \text{ with opposite parity}}^N c_n^* c_k \Gamma_{nk} = 0 \quad (\text{J5})$$

or, more algorithmically,

$$\sum_{j=1}^{\lfloor (N+1)/2 \rfloor} \sum_{i=0}^{N-(2j-1)} \text{Re}(c_i^* c_{i+2j-1} \Gamma_{i,i+2j-1}) = 0. \quad (\text{J6})$$

Thus, the objective function that we will optimize is

$$L(c_n, \lambda) = f(c_n) - \lambda \left(\sum_{j=1}^{\lfloor (N+1)/2 \rfloor} \sum_{i=0}^{N-(2j-1)} \text{Re}(c_i^* c_{i+2j-1} \Gamma_{i,i+2j-1}) \right), \quad (\text{J7})$$

where the constraint is introduced as a Lagrange multiplier. We also introduce a constraint on the norm of the vector of being 1.

We find that the state

$$\begin{aligned} & (-0.281 + 0.52i)|0\rangle + (0.096 - 0.127i)|1\rangle \\ & + (-0.024 + 0.102i)|2\rangle + (0.423 - 0.539i)|3\rangle \\ & + (-0.066 + 0.068i)|4\rangle + (0.012 - 0.067i)|5\rangle \\ & + (-0.25 + 0.262i)|6\rangle \end{aligned} \quad (\text{J8})$$

readily produces a Tsirelson violation of $\langle A \rangle = 1.0756$ with $\Delta p_{123} = 0$. More accurately, Δp_{123} is not precisely and

analytically zero for the quoted state, since the numerical computation was done using a truncated energy expansion. However, we find that slightly tweaking the state parameters can lead to states with Δp_{123} slightly above or below zero, and so it is almost certain that, in principle, there exists a state with $\Delta p_{123} = 0$ exactly.

APPENDIX K: SMOOTHED PROJECTORS

We have modeled the measurements of position using the step function $\theta(\hat{x})$, but of course realistic measurements and experimental procedures, e.g., involving a light source localized in $x > 0$ (as alluded to in Sec. I), will not be an exact step function but rather will interpolate smoothly between 0 and 1 in a region of size ϵ around $x = 0$. The effect of such smoothing was calculated explicitly in Ref. [67], and it was shown that LG violations in the harmonic oscillator for measurements of $\text{sgn}(x)$ are significantly reduced when the smoothing scale ϵ is of order 1 (in the dimensionless units we are using here). This effect is related to the well-known fact that the Wigner function is rendered positive when smoothed over phase-space regions of size of order $\hbar$ (noting that when calculating probabilities in the Wigner picture, the smoothing on step functions can be switched onto the Wigner function of the initial state). We therefore also expect a reduction in the violation of the Tsirelson inequality due to smoothing if the smoothing scale ϵ approaches 1, so we need $\epsilon \ll 1$ to maintain a violation.

The smoothing will also affect the probabilities $p(+, +, +)$ and $p(-, -, -)$ characterizing uniform precession and we therefore need to revisit what uniform precession means in this case. To assess this we use a simplified but instructive smoothing function $\theta_\epsilon(x)$, which agrees with the usual $\theta(x)$ function except that it takes the value $\frac{1}{2}$ for $-\epsilon/2 < x < \epsilon/2$. Also, for simplicity, we use a classical analysis with a state concentrated around $x = 0$ and $p = 0$.

Then there are two cases. If the width of the state is less than ϵ , then most of the state is concentrated in the smoothing region and we have effectively $\theta_\epsilon(x) = \frac{1}{2}$. We would then have $p(+, +, +)$ be of order $\frac{1}{8}$ and likewise $p(-, -, -)$, and hence L_1 is of order $\frac{1}{4}$. This means that any values of L_1 less than $\frac{1}{4}$ may be taken to mean that uniform precession is satisfied.

However, if the width of the state is much greater than ϵ , then only a small fraction of the state will be in the smoothing region and the above effect will be proportionately reduced. For the Zaw-Scarani state, plotted in Fig. 9(a), the probability $p(\delta)$ of finding the particle within any interval $[-\delta, \delta]$ around the origin is plotted in Fig. 9(b). The condition for uniform precession is then that L_1 is less than $\frac{1}{4}p(\delta)$ where $\delta = \epsilon/2$ (so that the interval coincides with the smoothing region). As noted above, the interesting regime for a Tsirelson violation is $\epsilon \ll 1$, and from the plot Fig. 9(b) one can see that $p(\delta)$, and hence L_1 , will also be $\ll 1$ in this case. Hence the regime most relevant to us is actually very close to the case of exact projective measurements considered in this paper. More precise estimates are not possible without knowledge of the experimental parameter ϵ and would require a more detailed calculations similar to those of Ref. [67], which would also need to include an assessment of the possible reduction of the

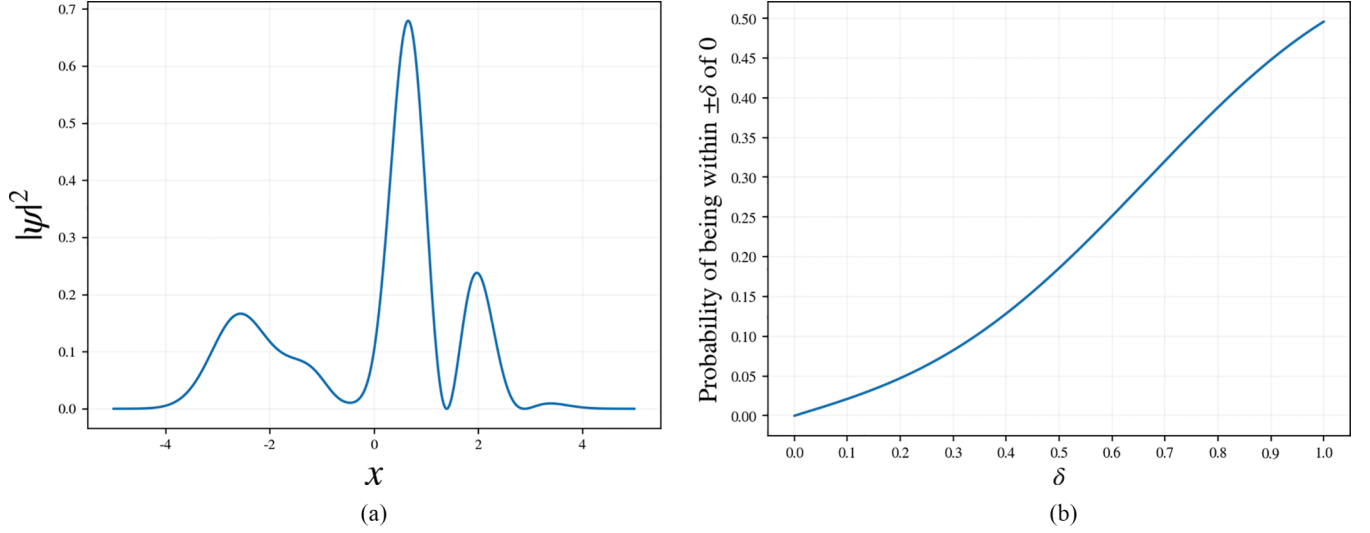


FIG. 9. (a) Plot of $|\psi(x)|^2$ against the dimensionless position for the Zaw-Scarani state $|\Psi_Z\rangle = \frac{4}{\sqrt{42}}|0\rangle - \frac{1}{\sqrt{2}}|3\rangle + \sqrt{\frac{5}{42}}|6\rangle$. (b) For the Zaw-Scarani state, this plot shows the probability $p(\delta)$ of measuring the particle to be in the interval $(-\delta, \delta)$ against δ , where δ is the dimensionless position.

Tsirelson violation produced by the smoothing; but we will not do this here.

APPENDIX L: PROBABILITY CURRENTS

1. Relating the Tsirelson quantity to the quantum current

Consider the limit definition of the integral to write $\langle T_D(2\pi) \rangle$ as a Riemann sum

$$\langle T_D(2\pi) \rangle = \frac{1}{2\pi} \left[\lim_{N \rightarrow \infty} \frac{2\pi}{N+1} \sum_{n=0}^N \left\langle \theta \left(x \left(\frac{2\pi n}{N+1} \right) \right) \right\rangle \right] \quad (\text{L1})$$

$$= \frac{1}{3} \sum_{n=0}^2 \left\langle \theta \left(x \left(\frac{2\pi n}{3} \right) \right) \right\rangle + \frac{\varepsilon}{2\pi}, \quad (\text{L2})$$

where ε refers to an error term. The equality certainly holds for some $\varepsilon \in \mathbb{R}$.

Assuming $\langle \theta(x(t)) \rangle$ is continuously differentiable in t , then Taylor's theorem applies to $\langle \theta(x(t)) \rangle$ and we can borrow some well-known results from numerical analysis to quantify our error term. Specifically, we notice that this amounts to a left-end-point approximation to the integral for the dwell time. Therefore, we can use the formula for the error term [74],

$$E_L = \frac{(2\pi)^2}{6} \frac{d\langle \theta(x) \rangle}{dt} \Big|_{t=\tau}, \quad \tau \in [0, 2\pi], \quad (\text{L3})$$

to state that

$$\langle T_D(2\pi) \rangle = \frac{1}{3} \sum_{n=0}^2 \left\langle \theta \left(x \left(\frac{2\pi n}{3} \right) \right) \right\rangle + \frac{E_L}{2\pi} \quad (\text{L4})$$

$$= \frac{1}{3} \sum_{n=0}^2 \left\langle \theta \left(x \left(\frac{2\pi n}{3} \right) \right) \right\rangle + \frac{2\pi}{6} \frac{d\langle \theta(x) \rangle}{dt} \Big|_{t=\tau}, \quad (\text{L5})$$

where $\tau \in [0, 2\pi]$. The sum can be related to the Tsirelson quantity because $\langle \theta(x(t)) \rangle = \frac{\langle Q(t) \rangle + 1}{2}$ and $\langle A \rangle = \sum_{n=0}^2 \langle Q \rangle$. So

$$\langle T_D(2\pi) \rangle = \frac{1}{6} \left[\sum_{n=0}^2 \left\langle Q \left(\frac{2\pi n}{3} \right) \right\rangle + 3 \right] + \frac{2\pi}{6} \frac{d\langle \theta(x) \rangle}{dt} \Big|_{t=\tau} \quad (\text{L6})$$

$$= \frac{1}{6} (3 + \langle A \rangle) + \frac{2\pi}{6} \frac{d\langle \theta(x) \rangle}{dt} \Big|_{t=\tau}. \quad (\text{L7})$$

Hence,

$$\begin{aligned} \langle A \rangle &= [6\langle T_D(2\pi) \rangle - 3] - 2\pi \frac{d\langle \theta(x) \rangle}{dt} \Big|_{t=\tau} \\ &= -2\pi \frac{d\langle \theta(x) \rangle}{dt} \Big|_{t=\tau}. \end{aligned} \quad (\text{L8})$$

This expression thereby provides an alternative explanation for why $\langle A \rangle = 0$ for energy eigenstates (stationary states). The expectation value of the Heaviside operator has no time dependence for energy eigenstates, so those derivatives are 0 for any value of τ .

Note that because $\langle \theta(x(t + \pi)) \rangle = 1 - \langle \theta(x(t)) \rangle$, the derivatives of $\langle \theta(x(t)) \rangle$ simply flip sign after half a period. Hence, without loss of generality, we may drop the negative signs on our expressions, since τ is some unknown time on the interval $[0, 2\pi]$:

$$\langle A \rangle = 2\pi \frac{d\langle \theta(x) \rangle}{dt} \Big|_{t=\tau}. \quad (\text{L9})$$

We can rewrite Eq. (L9) as

$$\langle A \rangle = 2\pi \frac{d}{dt} \int_0^\infty |\psi(x, \tau)|^2 dx. \quad (\text{L10})$$

Then, using the one-dimensional continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0, \quad (\text{L11})$$

where $\rho = |\psi|^2$ is the probability density and $J = \text{Im}(\psi^* \partial_x \psi)$ is the probability density current [75], we can substitute the current into the Tsirelson quantity:

$$\langle A \rangle = -2\pi \int_0^\infty \frac{\partial J(x, \tau)}{\partial x} dx \quad (\text{L12})$$

$$= 2\pi J(0, \tau). \quad (\text{L13})$$

Therefore, assuming $J(\infty, \tau) \rightarrow 0$, we have directly related the Tsirelson quantity to the probability current at the origin. Specifically,

$$\exists \tau \in [0, 2\pi] \text{ s.t. } \langle A \rangle = 2\pi J(0, \tau). \quad (\text{L14})$$

From Eq. (L14) we can make the following claim: If $|J(0, \tau)| \leq \frac{1}{2\pi} \forall \tau \in [0, 2\pi]$, then $|\langle A \rangle| \leq 1$, that is to say, if the current at the origin stays sufficiently low, then we can guarantee the state will not violate the Tsirelson inequality. Notably, $J(0, t)$ could be measured by taking the time derivative of measured values of $\langle \theta(x(t)) \rangle$, which only requires single-time measurements.

This result also provides some further insight into why completely even or odd wave functions cannot violate the Tsirelson inequality. Expanding a general state as a superposition of energy eigenstate wave functions $\psi_n(x)$, the current at the origin can be written as

$$J(0, t) = \text{Im} \sum_{n,m} c_n^* c_m \psi_n(0) \psi_m'(0) e^{i(n-m)t}. \quad (\text{L15})$$

Even functions have 0 slope at $x = 0$ and odd functions are 0 at $x = 0$. As such, if the wave function in question can be decomposed into exclusively even energy eigenfunctions or exclusively odd energy eigenfunctions, then $J(0, t) = 0 \forall t \in [0, 2\pi]$, so $\langle A \rangle = 0$.

2. Tsirelson satisfaction condition on $\mathcal{H}_0$

We will refer to the dwell time over $\frac{2}{3}$ of a period as $T_{2/3}$. Consider the integral approximation

$$T_{2/3} = \frac{1}{6} \left(\sum_{j=1}^3 \langle \theta(x(t_j)) \rangle + \langle \theta(x(t_1)) \rangle \right) + \frac{1}{2\pi} \sum_{j=1}^3 E_j \quad (\text{L16})$$

$$= \frac{1}{12} \langle A \rangle + \frac{1}{12} \langle Q_1 \rangle + \frac{1}{3} + \frac{1}{2\pi} \sum_{j=1}^3 E_j. \quad (\text{L17})$$

Therefore,

$$\langle A \rangle = 12T_{2/3} - 4 - \langle Q_1 \rangle - \frac{6}{\pi} \sum_{j=1}^3 E_j, \quad (\text{L18})$$

where both E_1 and E_2 are left-end-point errors with $E_1 = \frac{(2\pi/3)^2}{2} \frac{d\langle \theta(x) \rangle}{dt} \Big|_{\tau_1}$ and $\tau_1 \in [0, 2\pi/3]$ and with $E_2 = \frac{(\pi/3)^2}{2} \frac{d\langle \theta(x) \rangle}{dt} \Big|_{\tau_2}$ and $\tau_2 \in [2\pi/3, \pi]$. However, E_3 is a right-end-point error with $E_3 = -\frac{(\pi/3)^2}{2} \frac{d\langle \theta(x) \rangle}{dt} \Big|_{\tau_3}$ and $\tau_3 \in$

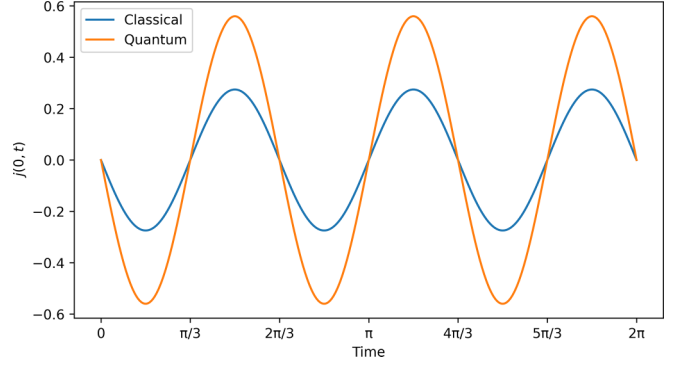


FIG. 10. Plot of the probability density current for the Zaw-Scarani state at the origin over a period compared to the classical probability density current from the classical time evolution of the Husimi- Q representation of the Zaw-Scarani state. Notably, the classicalized current is consistently lower than the quantum current.

$[\pi, 4\pi/3]$. So, expanding and simplifying the error term contribution,

$$\sum_{j=1}^3 E_j = \frac{\pi^2}{18} [4J(\tau_1) + J(\tau_2) - J(\tau_3)]. \quad (\text{L19})$$

Therefore,

$$\langle A \rangle = 12T_{2/3} - 4 - \langle Q_1 \rangle - \frac{\pi}{3} [4J(\tau_1) + J(\tau_2) - J(\tau_3)]. \quad (\text{L20})$$

However, by noting that $-J(\tau_3) = +J(\tilde{\tau}_3)$ and $\tilde{\tau}_3 \in [0, \pi/3]$, the sum of the three currents can be reformulated as a weighted average over the domain $[0, \pi]$. Pairing that with an application of the intermediate value theorem allows us to write

$$\langle A \rangle = 12T_{2/3} - 4 - \langle Q_1 \rangle - 2\pi J(\tau), \quad \tau \in [0, \pi]. \quad (\text{L21})$$

For $|\psi\rangle \in \mathcal{H}_0$, we have that $T_{2/3} = \frac{1}{3}$ and $\langle Q_1 \rangle = \frac{1}{3} \langle A \rangle$. Therefore, the expression for $\langle A \rangle$ simplifies to

$$\langle A \rangle = -\frac{3\pi}{2} J(\tau), \quad \tau \in [0, 2\pi/3], \quad (\text{L22})$$

where the domain of τ was decreased as the current is periodic over $2\pi/3$. Once again, an application of the sign flipping property of the current allows us to say that

$$\text{for } |\psi\rangle \in \mathcal{H}_0, \exists \tau \in [0, 2\pi/3] \text{ s.t. } \langle A \rangle = \frac{3\pi}{2} J(\tau). \quad (\text{L23})$$

An immediate corollary of this is that

$$\text{for } |\psi\rangle \in \mathcal{H}_0, \text{ if } |J(\tau)| \leq \frac{2}{3\pi} \forall \tau \in [0, 2\pi/3] \text{ then } |\langle A \rangle| \leq 1. \quad (\text{L24})$$

3. Classical and quantum probability current

Peculiarly, when we computed the classical probability current, using

$$j_c(0, t) = \iint dx dp p(t) \delta(x(t)) W_c(x, p), \quad (\text{L25})$$

for any natural classicalization of the phase-space distribution, $W_c(x, p)$, associated with $|\psi_Z\rangle \in \mathcal{H}_0$, we found the classical current was lower in magnitude than the quantum current across the entire period. Specifically, from the Wigner function of the Zaw-Scarani state, $W_Z(x, p)$, we constructed the following classical distributions: (1) $W_c(x, p) \propto |\langle \alpha | \psi_Z \rangle|^2$ (the Husimi- Q distribution), (2) $W_c(x, p) \propto |W_Z(x, p)|$, (3) $W_c(x, p) \propto W_Z(x, p) + |W_Z(x, p)|$,

and (4) $W_c(x, p) = |\psi_Z(x)|^2 |\phi_Z(p)|^2$, where $\phi_Z(p)$ is the momentum-space wave function of the Zaw-Scarani state. In Fig. 10 we illustrate the resulting classical current, where the classicalization strategy chosen is using the Husimi- Q distribution, and compare it to the quantum current. This result perhaps suggests a connection between the current-based bound on the Tsirelson quantity in Eq. (L24) and classicalized currents of violating quantum states.

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