

Universal quantum melting of quasiperiodic attractors in driven-dissipative cavities

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Nonlinear classical mechanics has established rich phenomena. These include limit tori defined by toroidal attractors supporting quasiperiodic motion with incommensurate frequencies. We study the fate of such structures in open quantum systems using two coupled driven-dissipative Kerr cavities modeled via the Lindblad master equation. Combining Liouvillian spectral theory with the truncated Wigner approximation, we characterize the quantum-to-classical crossover. In the classical limit, two pairs of purely imaginary Liouvillian eigenvalues signal persistent quasiperiodic modes. Quantum fluctuations induce small negative real parts to these eigenvalues, giving rise to finite lifetimes and leading to the *quantum melting* of the torus. The associated Liouvillian gaps vanish algebraically in the classical limit, indicating a dynamical critical crossover with spontaneous breaking of time-translational symmetry. Quantum trajectory analysis reveals that this melting is driven by fluctuation-induced dephasing. Using a circular-variance-based order parameter, we uncover universal scaling in system size and time. These results establish quantum melting of limit tori as a distinct and robust nonequilibrium critical phenomenon, with clear experimental signatures in trapped ions and superconducting circuits.

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I. INTRODUCTION

Nonlinear classical mechanics has provided a framework for understanding phenomena such as bistability, chaos, synchronization, and robust periodic motion [1–20]. Such dynamics are analyzed using tools like bifurcation theory [21,22], Floquet analysis [23], and Lyapunov spectra [24], which reveal the stability and structure of underlying attractors. Among these attractors are limit tori, which emerge through Neimark-Sacker bifurcations [25–30]. They exhibit quasiperiodic motion: Trajectories densely fill a toroidal manifold due to two or more incommensurate frequencies. The quasiperiodicity and the associated geometric structure make limit tori intrinsically more complex than limit cycles, where a single fundamental frequency yields purely periodic dynamics and a closed orbit in phase space. Quasiperiodic attractors appear across disciplines, including fluid dynamics, nonlinear optics, chemistry, and biology [31–36]. Although limit tori are well understood in classical systems, their quantum analogs have received little theoretical study, with only a few recent works [29].

Recent advances in driven-dissipative quantum platforms, such as superconducting circuits [37], ultracold atomic platforms [30,38], and trapped-ion platforms, have opened new possibilities for investigating nonlinear dissipative quantum dynamics. Examples including quantum synchronization

[10–12,15,16,18–20] and time-crystalline behavior [39–43] show that many-body quantum systems can maintain long-lived coherence despite strong dissipation.

In this work, we address the question of how phenomena established in classical nonlinear mechanics translate into quantum dissipative dynamics. As an intriguing representative example we focus on limit tori. We note that while other phenomena such as limit cycles [16,17,42,44–46] and emergent signatures of quantum chaos [37,47–50] have been explored in the quantum domain, the quantum description, and generally the fate of limit tori and complex attractors, are essentially unexplored. Quantum coherence, fluctuations, and dissipation can drastically reshape or destabilize such structures, raising fundamental questions about their persistence in open quantum systems. We consider the following questions:

(1) Can quasiperiodic attractors survive quantum noise and dissipation?

(2) What are the spectral signatures of quasiperiodic attractors in the Liouvillian?

(3) Do universal laws govern the dephasing and disappearance of quasiperiodic attractors in the quantum regime?

Answering these questions is crucial for bridging classical nonlinear dynamics with the physics of open quantum systems. To address these questions on a concrete example, we develop a quantum description of limit tori and their associated Liouvillian spectrum in a minimal model of two coupled driven-dissipative Kerr cavities. By combining mean-field analysis, Liouvillian spectral theory [51–53], and quantum trajectory methods within the truncated Wigner approximation [54–58], we investigate the emergence, stability, and quantum melting, that is, the quantum-fluctuation-induced

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degradation, of limit tori across the quantum-to-classical crossover. We identify quasiperiodic motion via pairs of complex-conjugate Liouvillian eigenvalues whose real parts vanish in the classical limit, with Liouvillian gaps that close algebraically. This spectral signature reveals universal scaling laws governing the quantum melting and supports its interpretation as a dynamical critical crossover characterized by algebraic softening of Liouvillian phase modes.

The paper is organized as follows. In Sec. II, we introduce the model and theoretical framework. In Sec. III, we present our results on the emergence and melting of quantum limit tori and identify associated universal scaling laws. In Sec. IV, we discuss prospects for experimental realization, outlining a concrete implementation strategy in a trapped-ion platform. Finally, Sec. V summarizes our findings and discusses future directions.

II. THEORETICAL FRAMEWORK

A. Coupled Kerr cavities

We consider a minimal theoretical model of two coupled Kerr cavities, subject to incoherent driving and two-photon loss processes, as illustrated in Fig. 1(a). The Hamiltonian of the system reads

$$\frac{\hat{\mathcal{H}}}{\hbar} = \sum_{k=1,2} \omega_k \hat{a}_k^\dagger \hat{a}_k + \frac{U_k}{2} \hat{a}_k^{\dagger 2} \hat{a}_k^2 - J(\hat{a}_1^{\dagger 2} \hat{a}_2 + \hat{a}_2^{\dagger 2} \hat{a}_1), \quad (1)$$

where $\hat{a}_k$ ($\hat{a}_k^\dagger$) are the bosonic annihilation (creation) operators for cavity k , satisfying the commutation relation $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$, ω_k denotes the bare cavity frequencies, U_k the strength of the on-site Kerr nonlinearities, and J the amplitude of the nonlinear tunneling term. Throughout this work, we refer to the excitations as “photons” for simplicity, though in specific implementations, such as the trapped-ion realization, discussed in Sec. IV, they may represent other bosonic modes, such as phonons. The incoherent pump provides energy to the system, while the two-photon loss is a nonlinear process that damps oscillations at high amplitudes. This combination of dissipative processes is known to stabilize self-sustained limit cycles (LCs) [10].

The dissipative dynamics of the system is given by the Lindblad master equation for the density matrix $\hat{\rho}$ [59,60]:

$$\frac{d\hat{\rho}}{dt} = \mathcal{L}\hat{\rho} = \frac{1}{i\hbar}[\hat{\mathcal{H}}, \hat{\rho}] + \sum_{k=1,2} \gamma_k \mathcal{D}[\hat{a}_k^\dagger] \hat{\rho} + \eta_k \mathcal{D}[\hat{a}_k^2] \hat{\rho}, \quad (2)$$

where $\mathcal{D}[\hat{L}]\hat{\rho} = \hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}(\hat{L}^\dagger\hat{L}\hat{\rho} + \hat{\rho}\hat{L}^\dagger\hat{L})$ denotes the dissipator in Lindblad form associated with jump operator $\hat{L}$. Here γ_k and η_k represent the incoherent pumping and two-photon loss rates, respectively.

B. Spectral analysis of the Liouvillian

The dynamics of an open quantum system is governed by the Liouvillian superoperator $\mathcal{L}$, defined in Eq. (2), which acts on the system’s density matrix. Its spectrum is defined by the eigenvalue equation $\mathcal{L}\hat{\rho}_j = \lambda_j \hat{\rho}_j$, where λ_j are generally complex-valued. According to Spohn’s theorem

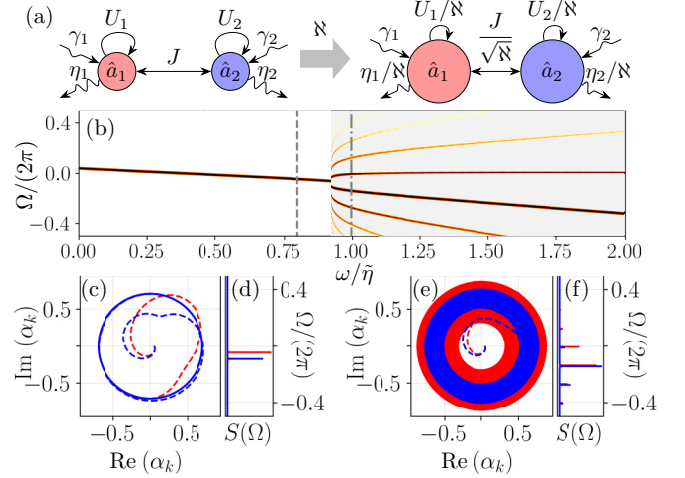


FIG. 1. Coupled driven-dissipative Kerr cavities and emergence of limit tori. (a) Schematic representation of the coupled Kerr cavities, described by Eqs. (1) and (2), and the quantum-to-classical rescaling of the system. The parameter $\aleph$ is a dimensionless scaling parameter that controls the quantum-to-classical crossover; see Eq. (4). (b) Heatmap of the Fourier spectrum $S_k(\Omega)$ of mode $k = 1$, as defined in Eq. (9), obtained from the Gross-Pitaevskii (GP) dynamics, Eq. (3), as a function of ω . The region with a white background corresponds to limit-cycle dynamics, dominated by a single frequency, while the gray background indicates the limit-torus regime, dominated by two fundamental frequencies (and higher harmonics). The gray dashed line marks $\omega = 0.8$ for the limit cycle, shown in (c) and (d), and the dashed-dotted line marks $\omega = 1.0$ for the limit torus, shown in (e) and (f). [(c) and (e)] Phase-space trajectories of the mean fields α_1 (red) and α_2 (blue), obtained from GP equations, Eq. (3). [(d) and (f)] Corresponding Fourier spectra $S_k(\Omega)$ of modes $k = 1$ and $k = 2$, respectively. The trajectories are obtained by numerically integrating the GP equations up to $\eta t = 10^4$, with initial conditions $\alpha_1(0) = 0.05$ and $\alpha_2(0) = 0.05$. Unless stated otherwise, the remaining parameters are $U_1 = U_2 = 0.1$, $J = 0.4$, $\gamma_1 = \gamma_2 = \gamma = 1$, and $\eta_1 = \eta_2 = \eta = 1$.

[61–63], any finite-dimensional open quantum system admits a unique nonequilibrium steady state (NESS) associated with the eigenvalue $\lambda_0 = 0$, while all other eigenvalues have strictly negative real parts. Consequently, any initial state asymptotically relaxes to the NESS. The timescale of this relaxation is set by the Liouvillian gap, defined as the nonzero eigenvalue with the smallest magnitude of the real part [52]. The closing of the Liouvillian gap signals a qualitative change in the long-time dynamics, marking the onset of critical phenomena such as dissipative phase transitions or the emergence of persistent oscillations in the stationary state [42].

C. Quantum-to-classical rescaling

In the limit of large photon occupation, the dynamics of a driven-dissipative system can be described by the Gross-Pitaevskii equation (GPE), a mean-field approach that neglects quantum fluctuations. The GPE is obtained from the adjoint master equation of Eq. (2) by applying the mean-field approximation $\langle \hat{A}\hat{B} \rangle = \langle \hat{A} \rangle \langle \hat{B} \rangle$ and taking the classical limit

$\hbar \rightarrow 0$. It reads

$$\begin{aligned} i\frac{d\alpha_1}{dt} &= \left(\omega_1 + i\frac{\gamma_1}{2} + (U_1 - i\eta_1)|\alpha_1|^2\right)\alpha_1 - 2J\alpha_1^*\alpha_2 \\ i\frac{d\alpha_2}{dt} &= \left(\omega_2 + i\frac{\gamma_2}{2} + (U_2 - i\eta_2)|\alpha_2|^2\right)\alpha_2 - J\alpha_1^2, \end{aligned} \quad (3)$$

where $\alpha_k = \text{Tr}(\hat{a}_k \hat{\rho})$ denotes the expectation value of the annihilation operator. To systematically control the crossover between quantum and classical behavior, we introduce a scaling parameter $\aleph$, which rescales the system parameters according to

$$U_k = \frac{\tilde{U}_k}{\aleph}, \quad J = \frac{\tilde{J}}{\sqrt{\aleph}}, \quad \eta_k = \frac{\tilde{\eta}_k}{\aleph}, \quad (4)$$

as illustrated in Fig. 1(a). Under this transformation, the complex fields scale as $\alpha_k = \tilde{\alpha}_k \sqrt{\aleph}$, leading to mean-field populations $n_k = |\alpha_k|^2 \propto \aleph$. Notably, Eq. (3) remains invariant under these scaling relations, demonstrating that the limit $\aleph \rightarrow \infty$ defines a well-controlled classical limit characterized by an infinite number of photons. This scaling method is commonly used in the study of driven-dissipative quantum systems to systematically explore quantum-to-classical transitions [64–69]. Unless otherwise specified, we consider the symmetric parameter choice $\omega_1 = \omega_2 = \omega = 1$, $\tilde{U}_1 = \tilde{U}_2 = 0.1$, $\tilde{J} = 0.4$, $\gamma_1 = \gamma_2 = \gamma = 1$, and $\tilde{\eta}_1 = \tilde{\eta}_2 = \tilde{\eta} = 1$. These values enable the emergence of complex dynamics, including LCs and limit tori (LT).

D. Truncated Wigner approximation

To capture quantum fluctuations and the mixed character of the density matrix beyond mean-field theory, we employ the truncated Wigner approximation (TWA) [54–58,70–75]. In the TWA, the density operator $\hat{\rho}$ is mapped onto the Wigner quasiprobability distribution $W(\{\alpha_k, \alpha_k^*\})$ over the real and imaginary parts of the complex mode amplitudes α_k . Truncating the dynamical equations for the Wigner distribution at order $\mathcal{O}(\hbar^2)$ yields a Fokker-Planck equation; see Supplemental Material [76], which is equivalent to the ensemble dynamics of stochastic Langevin trajectories.

For the two-cavity model, we initialize the Wigner function with the distribution corresponding to a two-mode coherent state $|\alpha_1(0), \alpha_2(0)\rangle$. The Langevin equations for the rescaled fields $\tilde{\alpha}_k$ take the form:

$$\begin{aligned} i\frac{d\tilde{\alpha}_1}{dt} &= \left[\omega_1 + i\frac{\gamma_1}{2} + (\tilde{U}_1 - i\tilde{\eta}_1)\left(|\tilde{\alpha}_1|^2 - \frac{1}{\aleph}\right)\right]\tilde{\alpha}_1 \\ &\quad - 2\tilde{J}\tilde{\alpha}_1^*\tilde{\alpha}_2 + \sqrt{\frac{\gamma_1}{2\aleph}}\chi_1(t) + \sqrt{\frac{2\tilde{\eta}_1|\tilde{\alpha}_1|^2}{\aleph}}\xi_1(t) \\ i\frac{d\tilde{\alpha}_2}{dt} &= \left[\omega_2 + i\frac{\gamma_2}{2} + (\tilde{U}_2 - i\tilde{\eta}_2)\left(|\tilde{\alpha}_2|^2 - \frac{1}{\aleph}\right)\right]\tilde{\alpha}_2 \\ &\quad - \tilde{J}\tilde{\alpha}_1^2 + \sqrt{\frac{\gamma_2}{2\aleph}}\chi_2(t) + \sqrt{\frac{2\tilde{\eta}_2|\tilde{\alpha}_2|^2}{\aleph}}\xi_2(t), \end{aligned} \quad (5)$$

where χ_k and ξ_k are independent, zero-mean and unit-variance complex Gaussian stochastic variables, characterized by correlation functions $\langle \chi_k(t)\chi_{k'}(t') \rangle = \langle \xi_k(t)\xi_{k'}(t') \rangle = 0$, $\langle \chi_{k'}^*(t)\chi_k(t') \rangle = \delta_{k,k'}\delta(t-t')$, and $\langle \xi_k^*(t)\xi_{k'}(t') \rangle = \delta_{k,k'}\delta(t-t')$.

The noise terms $\chi_k(t)$ and $\xi_k(t)$ account for quantum fluctuations. The expectation value of an observable $\hat{O}$ is approximated by averaging its Weyl symbols $O_W(\{\alpha_k, \alpha_k^*\})$ over N_{traj} stochastic trajectories:

$$\langle \hat{O} \rangle(t) \approx \frac{1}{N_{\text{traj}}} \sum_{\mu=1}^{N_{\text{traj}}} O_W(\{\alpha_{k,\mu}(t), \alpha_{k,\mu}^*(t)\}), \quad (6)$$

where the individual stochastic realizations are labeled by μ . In particular, for symmetrically, i.e., Weyl-ordered products of operators, such as $\mathcal{S}[(\hat{a}_i^\dagger)^n \hat{a}_j^m]$, the expectation value reads

$$\langle \mathcal{S}[(\hat{a}_i^\dagger)^n \hat{a}_j^m] \rangle(t) \approx \frac{1}{N_{\text{traj}}} \sum_{\mu=1}^{N_{\text{traj}}} (\alpha_{i,\mu}^*(t))^n (\alpha_{j,\mu}(t))^m. \quad (7)$$

The system is initialized with each mode prepared in a coherent state of amplitude $\tilde{\alpha}_0 = 0.05$. The initial conditions for the stochastic trajectories are sampled from the corresponding Gaussian Wigner distribution,

$$\tilde{\alpha}_k(t=0) = \tilde{\alpha}_0 + \frac{1}{\sqrt{2\aleph}}\zeta_k, \quad (8)$$

where ζ_k is a unit-variance complex random variable with zero mean, satisfying $\langle \zeta_k \zeta_{k'} \rangle = 0$ and $\langle \zeta_k^* \zeta_{k'} \rangle = \delta_{k,k'}$. Each realization corresponds to a possible measurement outcome, providing a semiclassical picture of the system's quantum dynamics.

III. QUANTUM MELTING OF LIMIT TORI AND ITS UNIVERSALITY

A. Quantum melting of LT dynamics

1. Semiclassical analysis

We begin by analyzing the mean-field dynamics to identify parameter regimes where LC and LT attractors emerge. Figure 1(b) shows the Fourier spectrum of mode $k=1$,

$$S_k(\Omega) = \left| \int_0^\infty d\tau \alpha_k(\tau) e^{i\Omega\tau} \right|, \quad (9)$$

computed from the GPE, as a function of ω . On the left side of the plot, with white background, a single dominant frequency is visible, signaling LC behavior, while on the right, with gray background, the presence of two fundamental incommensurate frequencies signals the LT phase, characterized by quasiperiodic dynamics. A Lyapunov exponent analysis confirms this distinction. The LC phase shows a single zero Lyapunov exponent, while the LT phase is marked by two zero exponents, see Supplemental Material [76]. The phase-space trajectories further illustrate the nature of the attractors. For $\omega = 0.8$, the system exhibits periodic oscillations with well-defined amplitude and frequency; see Fig. 1(c). For $\omega = 1$, the amplitude modulation over multiple periods reveals a toroidal structure; see Fig. 1(e). The ratio of the dominant frequencies is analyzed in the Supplemental Material [76].

2. Quantum transient

Individual stochastic TWA trajectories show that the LT dynamics observed in the mean-field regime persist in the

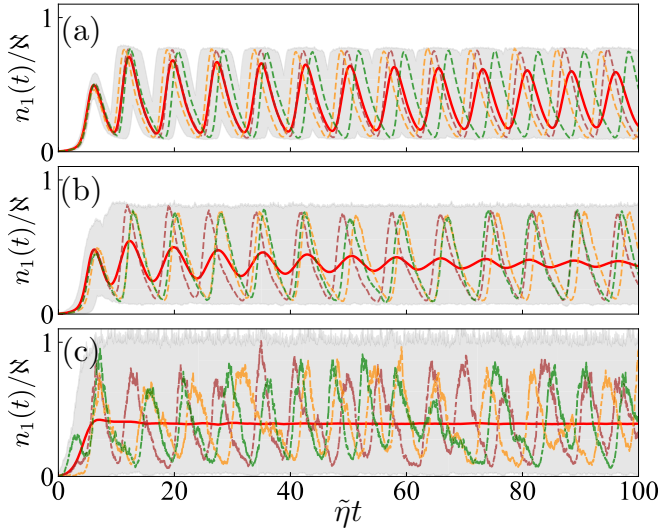


FIG. 2. Dynamics of the rescaled occupation $n_1(t)/N$ of mode 1, in the TWA. Panels (a)–(c) correspond to decreasing values of the scaling parameter: (a) $N = 30\,000$, (b) $N = 4\,000$, and (c) $N = 200$. Dashed lines show the evolution of individual TWA trajectories, while the solid red lines indicate the expectation value of $n_1(t)/N$. The gray area represents the range between the minimal and maximal occupation values across all trajectories. Both modes are initialized in a coherent state centered at $\tilde{\alpha}_0 = 0.05$. The expectation values are averaged over 3×10^4 TWA trajectories. Mode 2 shows qualitatively similar behavior.

quantum regime. Figure 2 shows the time evolution of single-trajectory dynamics, represented by dashed lines. After a rapid transient regime ($\tilde{\eta}t \lesssim 10$), each trajectory settles into the LT attractor and explores its full amplitude range without damping.

The gray-shaded area in Fig. 2 delineates the envelope of minimal and maximal occupations across all trajectories. Although quantum fluctuations are present, each individual trajectory stays within the envelope, maintaining dynamics that is confined to the LT attractor. They remain quasiperiodic, with persistent undamped oscillations. Over time, stochastic noise induces gradual dephasing between trajectories, leading to a decay of phase coherence across the ensemble of single trajectories. This results in an effective damping of ensemble-averaged TWA expectation values, as shown in Fig. 2, where the solid red lines depict the rescaled occupation $n_1(t)/N$ approaching a nonoscillatory steady state. The progressive temporal broadening of the envelope (gray-shaded area) in Fig. 2(a) further reflects the dephasing dynamics. This dephasing-driven melting differs fundamentally from bistable Kerr dynamics, where stochastic quantum trajectories exhibit telegraphic switching between metastable attractors [77,78]; here trajectories remain confined to the LT manifold and relaxation occurs via diffusive phase decoherence. This process constitutes the dominant relaxation mechanism and is governed by the Liouvillian gap: Quantum noise induces random phase drift, effectively melting the LT.

3. Liouvillian spectral signature of LT

As discussed above, at the mean-field level, the two coupled driven-dissipative Kerr cavities in our model undergo a

Neimark-Sacker bifurcation, giving rise to a two-dimensional LT attractor embedded in four-dimensional phase space. This attractor is characterized by exactly two vanishing Lyapunov exponents, one for each neutrally stable direction of the torus, while all other exponents are strictly negative; see Supplemental Material [76]. The resulting quasiperiodic dynamics are governed by two incommensurate frequencies, a hallmark of toroidal motion. These classical features leave a clear imprint on the quantum dynamics. Specifically, the emergence of LTs is encoded in the spectral structure of the Liouvillian superoperator $\mathcal{L}$. The low-lying spectrum encodes two key ingredients: the quasiperiodic motion inherited from mean-field dynamics and the slow dephasing induced by quantum fluctuations.

Figure 3(a) schematically illustrates this structure of the low-lying Liouvillian spectrum. The steady state corresponds to a unique zero eigenvalue $\lambda_0 = 0$, denoted as a red cross, as required by Lindblad formalism. In a LC regime, the Liouvillian spectrum is dominated by a single pair of complex conjugate eigenvalues. The imaginary parts of these eigenvalues determine the oscillation frequency. The small negative real parts set the decay rate, which vanishes in the classical limit where quantum fluctuations are negligible [42,45]. By contrast, the low-lying Liouvillian spectrum of a LT comprises two sets of complex conjugate eigenvalues

$$\lambda_j^\pm = -\Lambda_j \pm i\nu_j \quad \text{for } j = 1, 2, \quad (10)$$

with incommensurate frequencies $\nu_{1,2}$ associated with motion along the two angular directions of the torus. Each direction is subject to quantum-fluctuation-induced dephasing, quantified by a distinct Liouvillian gap Λ_j . In the classical limit, where $\Lambda_j \rightarrow 0$, this leads to two sets of purely imaginary eigenvalues, recovering the undamped quasiperiodic motion.

To probe the resulting quasiperiodicity in cavity 1, we compute its time-resolved emission spectrum via a windowed Fourier transform of the first-order correlation function [79],

$$S_1(t, \Omega) = 2\text{Re} \int_0^\infty d\tau w(t - \tau) \langle \hat{a}_1^\dagger(t_0 + \tau) \hat{a}_1(t_0) \rangle e^{i\Omega\tau}, \quad (11)$$

with window function $w(s)$. This quantity is numerically evaluated using a discrete short-time Fourier transform of the correlation function sampled from $N_{\text{traj}} = 10^5$ TWA trajectories; see Supplemental Material [76]. The resulting spectrogram displays two dominant peaks at constant incommensurate frequencies $\nu_{1,2}$, whose magnitudes decay over time. We numerically extract the corresponding lifetimes; see Supplemental Material [76]. The frequencies $\nu_{1,2}$ coincide with the imaginary parts of the low-lying Liouvillian eigenvalues, while the decay rates are set by the corresponding Liouvillian gaps $\Lambda_{1,2}$. The spectrogram Eq. (11) therefore provides a direct probe of the Liouvillian spectrum. Since the observable considered here is sensitive only to cavity 1, spectral features associated with cavity 2 are not directly resolved. Nevertheless, cavity 2 contributes an additional pair of complex-conjugate eigenvalues to the Liouvillian spectrum, corresponding to its own quasiperiodic motion. Figure 3(b) shows the scaling of Λ_j with scaling parameter N . We find a

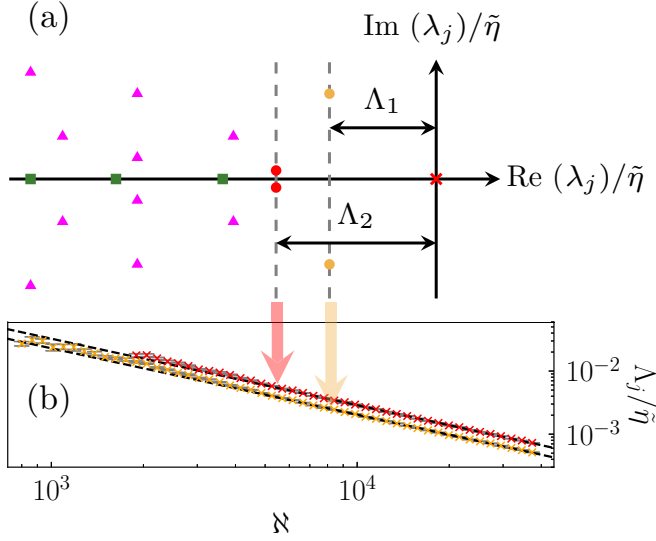


FIG. 3. Liouvillian spectrum analysis. (a) Schematic representation of the spectrum of the Liouvillian. A zero eigenvalue, indicated with a red cross, corresponds to the unique steady state. In the LT regime, two pairs of complex conjugate eigenvalues (red and gold) emerge, each with small real parts that vanish in the classical limit and incommensurate imaginary parts. (b) Scaling of the Liouvillian gaps with $\aleph$. The power-law fits of the two Liouvillian gaps for $5 \times 10^3 \leq \aleph \leq 4 \times 10^4$ is shown by the dashed and dash-dotted lines. The coefficients of the fit are $\Lambda_j/\tilde{\eta} = \aleph^{-a_j} \times b_j$, with $a_1 = 1.054 \pm 0.01$ and $b_1 = 47.144 \pm 5.724$ and $a_2 = 1.052 \pm 0.01$ and $b_2 = 32.922 \pm 3.452$.

power-law behavior,

$$\Lambda_j \propto \aleph^{-a_j}, \quad a_j \approx 1. \quad (12)$$

The exponent a_j therefore characterizes the algebraic softening of Liouvillian phase modes and can be interpreted as a dynamical critical exponent governed by diffusive phase decoherence along the two neutrally stable directions of the torus, rather than an equilibrium critical exponent associated with spatial correlations. In particular, the inverse gaps $1/\Lambda_j$ set the slowest timescales over which quasiperiodic oscillations in single-trajectory observables lose phase coherence, and are directly linked to the relaxation of population oscillations in Fig. 2. The algebraic gap closing reflects diffusive phase decoherence along each angular coordinate, with a diffusion constant scaling as $D \propto \aleph^{-1}$. Accordingly, each Liouvillian gap behaves as $\Lambda_j \approx D \propto \aleph^{-1}$. When probing an observable sensitive to *both* angular directions [e.g., $S_1(\omega)$], the total dephasing rate becomes $\Lambda_1 + \Lambda_2 \approx 2D$, preserving the $\aleph^{-1}$ scaling. This power-law scaling can also be understood from signal-to-noise ratio (SNR) considerations: The coherent amplitude scales as $|\alpha| \propto \sqrt{\aleph}$, while quantum fluctuations remain $\aleph$ independent, leading to $\text{SNR} \sim \aleph$. Since phase diffusion is driven by quantum noise, dephasing rates scale inversely with SNR, yielding $\Lambda_j \propto \aleph^{-1}$. This inverse scaling of the dephasing rate with the SNR can be seen in analogy to the scaling of the Schawlow-Townes laser linewidth in laser theory, where spontaneous emission drives dephasing of the coherent field, yielding a Lorentzian spectrum with width proportional to the inverse intracavity SNR [80,81].

Crucially, the frequencies ν_j remain invariant with increasing $\aleph$, reflecting their mean-field origin. Indeed, they match the frequencies obtained from the spectrum of the GPE, confirming that the Liouvillian spectrum inherits the angular frequencies of the underlying quasiperiodic mean-field dynamics.

Importantly, the extracted scaling exponents $a_j \approx 1$ for the Liouvillian gaps Λ_j are found to be robust across different parameter regimes that support limit tori. This indicates that the exponent $a_j \approx 1$ is a robust and universal feature of the quantum-to-classical crossover governed by diffusive phase decoherence in quasiperiodic open systems.

4. Quantum melting

The quantum melting of the LT structure is further illustrated through Wigner function reconstructions presented in Fig. 4. The reconstructions are obtained from TWA fields α_1 (red) and α_2 (blue) for different times (columns) and scaling parameters $\aleph$ (rows). For reference, the phase-space area of the classical LT obtained from the GPE, and also shown in Fig. 1(e), is overlaid as light red and light blue regions. Initially, the system is prepared according to Eq. (8) for both modes shown in the first column of Fig. 4. During time evolution, the Wigner distribution migrates towards the phase-space region of the LT, see second and further columns, circulating while remaining confined within the toroidal attractor. Remarkably, the distribution is well confined to the phase-space region of the classical LT. Single trajectories remain confined in this structure, consistent with Fig. 2. The noise manifests itself along both temporal and scaling dimensions, depicted horizontally and vertically, respectively. At later times, dephasing at the level of single trajectories induces broadening of the Wigner function over the entire toroidal structure in both modes. Similarly, decreasing $\aleph$ results in a broader distribution, indicating that quantum fluctuations speed up the diffusion process. A visualization of the limit torus dynamics in the full four-dimensional phase space of the system is presented in the Supplemental Material [76]. We also provide an alternative representation of the Wigner function reconstructions shown in Fig. 4 based on two-dimensional Gaussian kernel density estimates; see Supplemental Material [76].

B. Universal scaling

Having established the emergence of LT and their melting via both trajectory dynamics and spectral signatures, we now investigate how coherence degrades as a function of time t and scaling parameter $\aleph$. In the following, we demonstrate that this melting follows universal scaling laws. To this end, we quantify phase diffusion in phase space via the circular variance,

$$R(t) = 1 - \left| \frac{1}{N_{\text{traj}}} \sum_{n=1}^{N_{\text{traj}}} e^{i\theta_n(t)} \right|, \quad (13)$$

where θ_n denotes the phase-space angle of the n th trajectory, and N_{traj} is the total number of trajectories. We note that the scaling parameter $\aleph$ is closely related to the number of photons in the cavities. The metric in Eq. (13) exploits the intrinsic circular symmetry of the system dynamics in phase

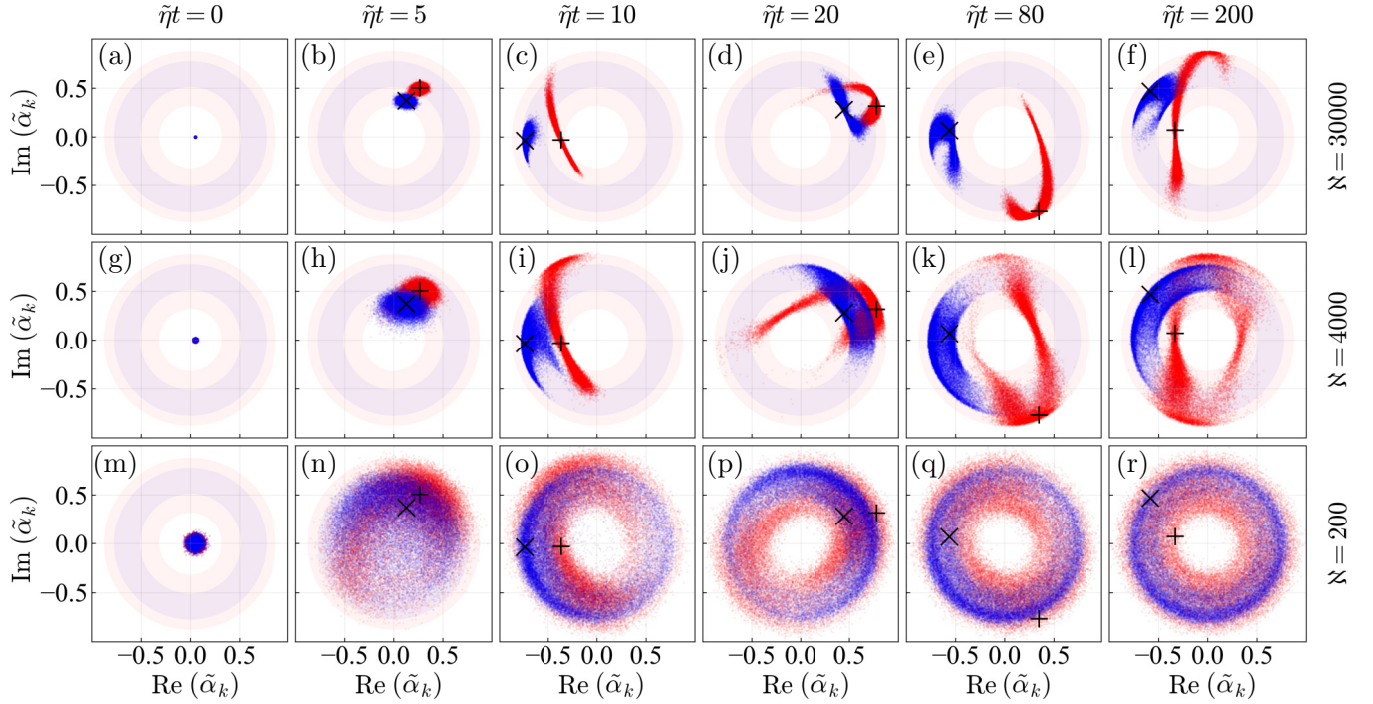


FIG. 4. Wigner function reconstructions based on the TWA fields α_1 (red) and α_2 (blue), shown for different times t (columns) and scaling parameter $\aleph$ (rows). Classical GPE solutions for mode 1 marked with +, and mode 2 marked with $\times$ are shown at corresponding times. The background light red and light blue regions indicate the phase-space area traced by the classical LT attractor, as obtained from GPE simulations. The three values of $\aleph$ match those in Fig. 2, decreasing from the top to the bottom row. Each distribution is obtained from 3×10^4 TWA trajectory realizations.

space and serves as a sensitive measure of phase diffusion [82,83]. For an ideally circular-invariant state, R solely and faithfully captures the dephasing dynamics. However, when the Wigner distribution deviates from perfect circular symmetry, see Fig. 4, oscillations in the mean phase-space angle emerge, thereby introducing an additional time-dependent modulation in R . As the deviation from perfect circular symmetry, and thus the oscillation in R , stem solely from the initialization, and are not associated with the melting of the LT structure, we mitigate their influence by introducing the period-averaged circular variance,

$$\bar{R}(t) = \frac{1}{T} \int_{t-T/2}^{t+T/2} d\tau R(\tau), \quad (14)$$

where T is the oscillation period of the mean phase-space angle. We compute $\bar{R}(\aleph, t)$ in cavity 1 over increasing values of the scaling parameter $\aleph$ and time t , fitting the resulting curves to an exponential relaxation form, $\bar{R}(x) = 1 - \exp(-\delta_{\bar{R}}x)$, with x representing either time or the inverse scaling parameter $1/\aleph$. The relaxation rate $\delta_{\bar{R}}(x)$ thus plays an analogous role to the Liouvillian gap extracted earlier from the emission spectrum: Both quantify the characteristic decay rate associated with the dephasing of coherent dynamics due to quantum noise [84]. The relaxation rate $\delta_{\bar{R}}(x)$ is found to exhibit a power-law dependence, $\delta_{\bar{R}}(x) = cx^d$, with c and d positive constants. Figure 5 illustrates the scaling property of $\delta_{\bar{R}}(x)$. The time dependence of $\bar{R}(\aleph, t)$ is displayed for five fixed values $\aleph_j$, while the dependence on the inverse scaling parameter $1/\aleph$ is shown for five fixed time points t_j .

Rescaling the time and the $1/\aleph$ axis using the coefficients that characterize the power-law dependencies of $\delta_{\bar{R}}(t)$ and $\delta_{\bar{R}}(1/\aleph)$,

$$t \rightarrow t' = (\aleph_j/\aleph_0)^{d_1} t, \quad 1/\aleph \rightarrow 1/\aleph' = (t_j/t_0)^{d_2} 1/\aleph, \quad (15)$$

leads to a scaling collapse of the curves, with $d_1 \approx 1.08$ and $d_2 \approx 0.90$. We find that the two scaling dimensions, time and $\aleph$, are intrinsically linked by the relation

$$t = \alpha \aleph^\beta, \quad (16)$$

with $\alpha = (c_1/c_2)^{1/d_2} = 1.04 \pm 0.01$ and $\beta = d_1/d_2 = -1.196 \pm 0.001$. In Fig. 5, this universality is further demonstrated by a collapse of the data when Eq. (16) is used to rescale the time axis as

$$t' \rightarrow k_1(t')^{k_2}, \quad (17)$$

with $k_1 = (c_1/c_2)^{d_2/d_1-1} t_0^{d_2}/\aleph_0^{d_1}$, $k_2 = -d_2/d_1$. This clearly shows that the rate of phase diffusion is inherently tied to the scaling parameter and quantum fluctuations. The scaling relation demonstrates a universal mechanism for the emergence of classical behavior from quantum dynamics, highlighting the universal nature of dephasing in quantum systems exhibiting LT dynamics.

C. Consistent indicators of quantum melting

We now demonstrate that the observables analyzed in Secs. III A and III B reflect the same underlying mechanism:

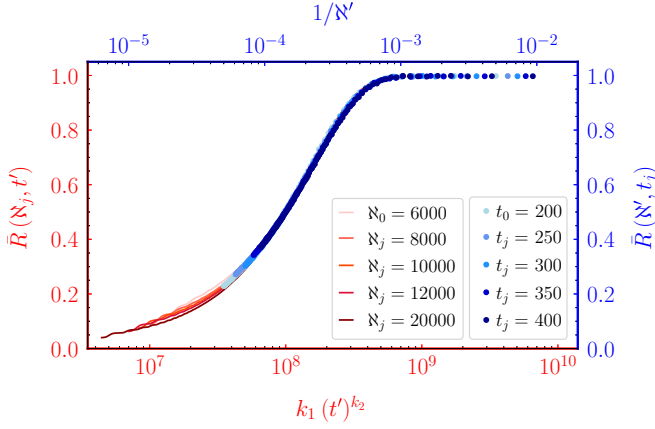


FIG. 5. Universal scaling behavior of the quantum-to-classical crossover. Data collapse of the period-averaged circular variance is achieved by plotting $\bar{R}$ against (i) the rescaled time t' (red curves and axes), $\bar{R}(N_j, t')$, and (ii) the rescaled scaling parameter N' (blue curves and axes), $\bar{R}(N', t_j)$. The time (red) and inverse scaling parameter (blue) axes are rescaled according to Eq. (15), revealing a power-law dependence of the relaxation dynamic along both dimensions. Further rescaling the time axis according to Eq. (17) achieves an overall data collapse, highlighting the universality of the scaling behavior.

quantum-fluctuation-induced dephasing of coherent torus dynamics. While conceptually distinct, the Liouvillian spectral gap and the phase diffusion measured by circular variance yield consistent scaling behavior. Specifically, the Liouvillian gap Λ_j extracted from the emission spectrum defined in Eq. (11) scales as N^{-a_j} , while the relaxation rate $\delta_{\bar{R}}$ of the period-averaged circular variance scales as N^{-d_1} . Both exponents are approximately unity, with $a_j \approx 1.05$ and $d_1 \approx 1.08$, as seen in Eq. (12). This agreement confirms that both observables capture the same fluctuation-driven relaxation process. Importantly, the spectral approach probes dissipation along each angular direction of the torus individually, while the circular variance captures global phase decoherence. The agreement between their scaling exponents therefore implies that quantum dephasing is isotropic across the torus manifold and that the underlying mechanism is universal.

This dual characterization not only strengthens the interpretation of LT melting as a universal, fluctuation-induced crossover but also provides two experimentally accessible probes for identifying this regime. Furthermore, it demonstrates that the universal scaling law, revealed along one angular dimension of the LT via circular variance, applies uniformly across all toroidal directions. This reveals a deep structural invariance of the scaling, highlighting a new universality class for topologically nontrivial attractors in open quantum systems.

D. Quantum melting of LT as an emergent dynamical critical crossover

The above analysis of the Liouvillian spectrum shows that the gradual melting of LT dynamics with increasing quantum fluctuations is governed by the softening of toroidal phase modes. We observe an algebraic Liouvillian gap closing

$\Lambda_j \propto N^{-1}$, while the steady state is unique and remains qualitatively unchanged for finite N . The algebraic gap closing thus does not indicate a dissipative phase transition, but a universal critical-like crossover governed by symmetry breaking in the classical limit and the algebraic restoration of these symmetries at finite N . The algebraic scaling is also distinct from exponential gap closing in metastable systems, as for example observed in dissipative bistable Kerr resonators, where slow dynamics originate from noise-activated switching between metastable attractors. In our case, the slow dynamics stems from diffusive phase motion along the torus manifold rather than activation-driven switching. Below we clarify the microscopic mechanism underlying the universal critical-like crossover.

1. Symmetry structure and soft modes

The Hamiltonian Eq. (1) and the Liouvillian Eq. (2) are both invariant under a global continuous U(1) symmetry,

$$\hat{a}_1 \rightarrow e^{i\phi} \hat{a}_1, \quad \hat{a}_2 \rightarrow e^{2i\phi} \hat{a}_2. \quad (18)$$

Additionally, $\mathcal{L}$ is invariant under continuous time translations, as it is explicitly time independent. In the classical limit, the LT breaks the global U(1) symmetry and the continuous time-translation symmetry, giving rise to two independent phase directions associated with the incommensurate frequencies $\nu_{1,2}$. Applying the open-system Nambu-Goldstone theorem [85] yields two diffusive Goldstone modes. One of these modes is a “mixed” mode, resulting from a nonzero commutator between the U(1) generator and the time-translation generator. The second mode corresponds to phase diffusion along the remaining torus angle. For any finite system parameters and N , the Liouvillian generically possesses a unique, time-translationally invariant steady state [51,52,63], indicating that all spontaneously broken symmetries are eventually restored at long times. The Goldstone modes therefore acquire finite damping rates.

2. Softening of toroidal Goldstone modes

In the classical limit $N \rightarrow \infty$, each incommensurate frequency parametrizes an independent phase direction on the torus arising from the breaking of time-translation symmetry [86,87]. A two-frequency torus therefore supports two independent Goldstone modes, each associated with phase motion along one of the torus angles. Quantum fluctuations restore these symmetries, softening the Goldstone modes into damped modes with finite coherence times. This softening manifests in the Liouvillian spectrum: In the classical limit, each gapless Goldstone mode appears as a purely imaginary pair of eigenvalues,

$$\lambda_j^{\pm} = \pm i\nu_j, \quad (19)$$

characterizing undamped phase motion. At finite N , quantum corrections introduce small negative real parts,

$$\lambda_j^{\pm} = -\Lambda_j \pm i\nu_j, \quad (20)$$

where the Liouvillian gaps Λ_j quantify phase diffusion along the corresponding angular directions and define the finite lifetimes of the soft modes. This generalizes previous results for

limit cycles [45,88], where the breaking of a single continuous time-translation symmetry yields one soft mode. For LTs, the presence of two incommensurate frequencies, and thus two independent phase directions generated by the single broken time-translation symmetry on the LT, gives rise to two distinct soft modes, reflecting the richer quasiperiodic symmetry-breaking structure of the torus.

3. Critical-like behavior and algebraic softening

Dissipative critical phenomena in open quantum systems are commonly classified according to the closing of the Liouvillian gap: (i) exponential closing, typically associated with bistability, metastability, and first-order-like transitions and (ii) algebraic closing, which is linked to diffusive soft modes and continuous (second-order) dynamical criticality [52,66,89]. In our system, $\Lambda_j \propto \aleph^{-1}$, implying diverging coherence times $\tau_j \sim \Lambda_j^{-1}$ and dynamical critical slowing down of quasiperiodic phase modes as the semiclassical limit $\aleph \rightarrow \infty$ is approached [90]. Importantly, this algebraic gap softening does not signal a dissipative phase transition at finite parameters: The Liouvillian retains a unique steady state for all finite $\aleph$ [52,53]. Moreover, the low-lying spectrum does not exhibit quasidegenerate real eigenvalues indicative of metastable sectors or noise-activated switching between competing attractors [77,78] as, for example, observed in bistable Kerr resonators. Instead, the slow dynamics originates from diffusive phase motion along the torus manifold. More precisely, the algebraic closing reflects the gradual softening of diffusive phase (Goldstone-like) modes associated with motion along the angular directions of the torus, which acquire finite damping at finite $\aleph$ due to quantum fluctuations. The system therefore undergoes a continuous, fluctuation-induced deformation from coherent quasiperiodic attractor dynamics to an incoherent, noise-dominated steady state. We thus interpret the observed algebraic scaling as a universal quantum-to-classical critical crossover governed by diffusive phase decoherence rather than a genuine dissipative phase transition.

4. Two-dimensional scaling structure

Despite the system being $0 + 1$ -dimensional, i.e., zero spatial dimensions plus time, we uncover a two-dimensional scaling structure in $(\aleph, t)$. In the absence of spatial degrees of freedom, conventional finite-size scaling does not apply. Instead, the scaling parameter $\aleph$, which controls photon occupation and noise strength, acts as an effective system size governing the dynamical scaling of the Liouvillian soft modes. A genuine dynamical critical point emerges in the limit $\aleph \rightarrow \infty$, even in the absence of spatial structure [90]. Simultaneously, the nontrivial scaling relation $t \sim \aleph^\beta$ shows that time itself becomes a critical scaling dimension. The resulting dynamical scaling collapse signals the emergence of a universal behavior, echoing space-time scaling phenomena seen in Berezinskii-Kosterlitz-Thouless transitions and quasi-long-range order in 2D condensates such as superfluids, where a genuine order arises only asymptotically in a singular limit (here, $\aleph \rightarrow \infty$).

Taken together, these results show that the “melting” of LT dynamics is not a dissipative phase transition in the

conventional sense but a universal quantum-to-classical crossover governed by the algebraic restoration of broken symmetries. The softening of toroidal Goldstone modes provides a unified framework connecting the spectral signatures, trajectory-level phase diffusion, and the emergent scaling structure across $(\aleph, t)$. This identifies a novel class of dynamical universality in open quantum systems, combining topological attractors, symmetry-protected soft modes, and fluctuation-driven decoherence in low-dimensional systems.

E. Robustness against single-photon loss and thermal noise

To ensure experimental relevance, we evaluated the robustness of our results against unavoidable single-photon loss and thermal fluctuations. These effects are incorporated into the master equation and TWA formalism, as detailed in the Supplemental Material [76]. We systematically analyzed how key quantities, such as the Liouvillian gap and the circular variance, are affected by these perturbations. These findings demonstrate that our conclusions remain valid under experimentally realistic levels of dissipation and thermal noise.

IV. EXPERIMENTAL REALIZATION

A. Trapped ions

The driven-dissipative coupled Kerr oscillator system described by Eqs. (1) and (2) can be experimentally realized with trapped ions [10,17]. In this realization, two motional modes of a single ion serve as the bosonic modes [91], offering precise control over system parameters and facilitating the study of quantum-to-classical crossovers.

1. Motional modes and trapping potential

In a linear Paul trap, an ion experiences harmonic confinement along the orthogonal directions. By adjusting the trap potentials, one can achieve nearly degenerate motional frequencies, $\omega_1 \approx \omega_2$, as required for the model. The noninteracting Hamiltonian, including internal electronic states, is given by

$$\frac{\hat{H}_0}{\hbar} = \omega_1 \hat{a}_1^\dagger \hat{a}_1 + \omega_2 \hat{a}_2^\dagger \hat{a}_2 + \sum_j \omega_{eg}^{(j)} |e_j\rangle \langle e_j|, \quad (21)$$

where $\omega_{eg}^{(j)}$ denotes the energy difference between the j th electronic excited state $|e_j\rangle$ and the electronic ground state $|g\rangle$.

2. Engineering Kerr nonlinearity

Anharmonicities in the trapping potential can be introduced with multipole trap designs or by applying tailored static electric fields [92–94], leading to a Kerr-type nonlinearity in the motional modes. In the interaction picture and under the rotating wave approximation (RWA), the quartic potential terms $\propto (\hat{a}_k + \hat{a}_k^\dagger)^4$ yield an effective Kerr interaction:

$$\frac{\hat{V}_I^{\text{RWA}}}{\hbar} \approx \frac{U_k}{2} \hat{a}_k^\dagger \hat{a}_k^\dagger \hat{a}_k \hat{a}_k, \quad (22)$$

where U_k is the Kerr coefficient, tunable via the trap’s electric field configuration.

3. Nonlinear mode coupling

Nonlinear coupling between the two motional modes can be achieved through laser driving that induces two-phonon and single-phonon transitions via an intermediate electronic state $|e_j\rangle$ [17]. This is achieved by applying two laser fields that off-resonantly drive the second red sideband of mode 1 and the first red sideband of mode 2. The laser addressing mode 1 has frequency $\omega_{d,1}$ with detuning from the electronic transition defined as $\delta_1^{(j)} \equiv \omega_{d,1} - \omega_{eg}^{(j)} = -2\omega_1 + \Delta$. The laser addressing mode 2 has frequency $\omega_{d,2}$, with detuning $\delta_2^{(j)} \equiv \omega_{d,2} - \omega_{eg}^{(j)} = -\omega_2 + \Delta$. Here Δ is the common detuning of the drives from the respective sidebands. Under these conditions and within the RWA, the resulting interaction Hamiltonian takes the form:

$$\frac{\hat{V}_I^{\text{RWA}}}{\hbar} \approx -\frac{\eta_{\text{LD},1}^2 \Omega_1}{4} \hat{\sigma}_+^{(j)} \hat{a}_1 \hat{a}_1 e^{-i\Delta t + i\phi_1} + \text{H.c.} \\ + i \frac{\eta_{\text{LD},2} \Omega_2}{2} \hat{\sigma}_+^{(j)} \hat{a}_2 e^{-i\Delta t + i\phi_2} + \text{H.c.}, \quad (23)$$

where $\hat{\sigma}_+^{(j)}$ and $\hat{\sigma}_-^{(j)}$ are the raising and lowering operators between $|g\rangle$ and $|e_j\rangle$, $\eta_{\text{LD},k}$ are the Lamb-Dicke parameters, Ω_k the Rabi frequencies, and ϕ_k the phases of the driving fields.

Under the condition $|\Delta| \gg \Omega_k$, the excited state can be adiabatically eliminated [95,96] leading to an effective nonlinear coupling term:

$$\frac{\hat{H}_{\text{int}}}{\hbar} = J(\hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1 \hat{a}_1), \quad (24)$$

with coupling strength $J = \frac{\eta_{\text{LD},1}^2 \eta_{\text{LD},2} \Omega_1 \Omega_2}{8|\Delta|}$, adjustable via laser parameters. This effective coupling arises at second order through virtual excitation of the electronic state; see Supplemental Material [76].

4. Dissipative processes

Dissipation, as described in Eq. (2), can be engineered using sideband cooling and heating techniques [10]. For each mode individually, incoherent pumping is implemented by driving the first blue sideband, i.e., $+\omega_k$ detuning, resulting in the following:

$$\frac{\hat{V}_I^{\text{RWA}}}{\hbar} \approx \sum_{k=1,2} \frac{\eta_{\text{LD},k} \Omega_k}{2} (i \hat{\sigma}_+^{(j)} \hat{a}_k^\dagger e^{i\phi_k} + \text{H.c.}), \quad (25)$$

while two-phonon loss is achieved by driving the second red sideband, i.e., $-2\omega_k$ detuning:

$$\frac{\hat{V}_I^{\text{RWA}}}{\hbar} \approx -\sum_{k=1,2} \frac{\eta_{\text{LD},k}^2 \Omega_k}{4} (\hat{\sigma}_+^{(j)} \hat{a}_k \hat{a}_k e^{i\phi_k} + \text{H.c.}). \quad (26)$$

Note that the two processes described in Eqs. (25) and (26) can be independently implemented using different internal states $|e_j\rangle$. Under the assumption of rapid decay of the excited state, i.e., decay rate $\Gamma_j \gg \eta_{\text{LD},k} \Omega_k$, adiabatic elimination leads to effective dissipators in the master equation:

$$\gamma_{k,\text{eff}} \mathcal{D}[\hat{a}_k^\dagger] \hat{\rho} \quad \text{and} \quad \eta_{k,\text{eff}} \mathcal{D}[\hat{a}_k^2] \hat{\rho}, \quad (27)$$

with rates

$$\gamma_{k,\text{eff}} = \eta_{\text{LD},k}^2 \Omega_k^2 / \Gamma_j \quad \text{and} \quad \eta_{k,\text{eff}} = \eta_{\text{LD},k}^4 \Omega_k^2 / (4\Gamma_j), \quad (28)$$

controllable via laser intensities; see Supplemental Material [76].

B. Alternative platforms

Alternatively, the model can be realized in circuit QED using two microwave cavities, each coupled to Josephson junction elements to produce on-site Kerr nonlinearities U_k [97]. The nonlinear tunneling J is induced by a parametric pump at frequency $\omega_p = 2\omega_1 - \omega_2$, enabling three-wave mixing between the cavities [98]. Two-photon loss η_k is engineered by coupling each cavity to a lossy auxiliary mode via similar parametric drives, while incoherent pumping γ_k can be emulated either by coupling each cavity to a driven, lossy transmon qubit or by injecting broadband microwave noise into the input line [99].

Cavity optomechanical systems offer another platform, where the two dissipative modes are realized as mechanical oscillators in a “membrane-in-the-middle” setup. Here nonlinear coupling is achieved by coupling both mechanical modes to an additional cavity mode [12,17].

C. Quantum-to-classical scaling

The crossover from quantum to classical behavior, characterized by the scaling parameter $\aleph$, can be probed across a variety of experimental platforms by tuning U_k , J , and the effective dissipation rates according to Eq. (4). As apparent from the rescaling in Eq. (4), the classical regime corresponds to large occupations of the oscillator modes, whereas the quantum regime corresponds to low occupations.

An alternative approach to accessing the quantum-to-classical scaling involves a dual-scaling procedure of system parameters, similarly to the procedure proposed in Ref. [90]. In this case, the scaling relation in Eq. (4) is replaced by the modified relation

$$\omega_k = \tilde{\omega}_k \aleph, \quad J = \tilde{J} \sqrt{\aleph}, \quad \text{and} \quad \gamma_k = \tilde{\gamma}_k \aleph. \quad (29)$$

D. State characterization

Both, trapped ion and circuit QED systems permit direct measurement of the bosonic states through techniques such as Wigner function tomography [100–103], enabling reconstruction of phase-space distributions described in Sec. III B. While single quantum trajectories can be challenging, ensemble measurements provide access to averaged quantities, enabling comparison with the predictions discussed above.

V. CONCLUSIONS

In this work, we have developed a quantum theory of limit tori (LTs), establishing a dynamical crossover that bridges open quantum dynamics and nonlinear classical dynamics. By focusing on a minimal model of two coupled driven-dissipative Kerr cavities, we systematically explored the quantum-to-classical crossover using Liouvillian spectral theory and stochastic unraveling of the density matrix. Our results reveal a universal structure underlying this crossover,

suggesting a broader connection between classical attractor dynamics and dissipative quantum systems.

In the classical limit, we identified robust LTs characterized by quasiperiodic motion on a toroidal phase-space manifold. Each incommensurate frequency of the torus is encoded in the Liouvillian spectrum as a pair of purely imaginary eigenvalues, corresponding to Goldstone-like modes arising from the spontaneous breaking of continuous time-translation symmetry. As quantum fluctuations increase, these eigenvalues acquire finite negative real parts, indicating the onset of quantum phase diffusion and the restoration of time-translation symmetry, a process we term quantum melting.

This quantum melting manifests as a breakdown of persistent quasiperiodicity. Through analysis of individual quantum trajectories, we identified quantum-fluctuation-induced dephasing as the microscopic mechanism driving this transition. Remarkably, even in the presence of strong noise, these trajectories remain confined to a topologically nontrivial toroidal manifold, demonstrating the robustness of the underlying structure.

The crossover is characterized by matching power-law scaling in both the Liouvillian spectral gap and a circular-variance-based order parameter, revealing a two-dimensional dynamical scaling structure in system size and time. Time itself emerges as an effective critical dimension, a hallmark of dynamical criticality reminiscent of Berezinskii-Kosterlitz-Thouless transitions, superfluid crossovers, and scaling phenomena in low-dimensional dissipative systems. This establishes quantum melting as an emergent dynamical critical crossover governed by universal scaling laws.

We further propose concrete implementations of LTs and their quantum melting in trapped ion systems and superconducting circuits, offering clear dynamical signatures and observables for detection.

Our findings contribute to a broader theoretical framework that connects topologically nontrivial attractors in classical nonlinear systems with their quantum counterparts. They also enrich current efforts to understand quantum chaos in open quantum systems [37,47–49], where the complex structure of the Liouvillian spectrum plays a central role in signaling chaotic dynamics and their relation to classical chaos.

Together, these advances offer a path toward classifying, stabilizing, and realizing coherent quasiperiodic behavior in engineered platforms operating at the frontier between classical and quantum nonlinear dynamics. Understanding how quantum fluctuations modify classical attractors will not only deepen our theoretical understanding but also provide insights into the design of robust quantum states for sensing, simulation, and information processing.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

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