

Quantum simulation of fermionic non-Abelian lattice gauge theories in $(2 + 1)$ dimensions with built-in gauge protection

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Recent advancements in the field of quantum simulation have significantly expanded the potential for applications, particularly in the context of lattice gauge theories (LGTs). Maintaining gauge invariance throughout a simulation remains a central challenge, especially for large-scale non-Abelian LGTs with dynamical matter, which are particularly complex in terms of engineering for experiments. Gauge-symmetry breaking is inevitable in established rishon-based schemes for alkaline-earth-like atoms (AELAs) and controlling the magnitude of its effect is an open challenge. Here we first construct a minimal model to quantum simulate non-Abelian LGTs ensuring that the gauge constraints are met and explicitly derive their unambiguous non-Abelian nature. Second, we present a proposal for a gauge protection scheme using native interactions in AELAs enabling the simulation of toy models of non-Abelian $U(2)$ LGTs with dynamical fermionic matter in $(2 + 1)$ dimensions on large scales. Due to the simplicity of the gauge protection mechanism, based on a Zeeman shift in combination with superexchange interactions, our scheme can be naturally included in other rishon-based quantum simulation protocols. Third, we extend our approach to a fully scalable, hybrid digital-analog simulator for $U(N)$ LGTs based on Rydberg AELA with variable rishon number. The proposed general mechanism for gauge protection provides a promising path towards the simulation of non-Abelian LGTs relevant to particle physics.

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I. INTRODUCTION

Over the past century, the development of gauge theories has provided a comprehensive framework for the description of strongly interacting systems, including the Standard Model of particle physics as well as emergent effective theories in condensed matter systems [1–4]. Despite the numerous advancements, ranging from quantum spin liquids to high- T_C superconductivity and lattice quantum chromodynamics (QCD) [5–10], gauge theories still pose significant challenges, particularly in addressing nonperturbative phenomena and at high (fermionic) matter density [11].

The structure of LGTs [12] allows us to apply classical numerical simulations, such as Monte Carlo [13–15] or tensor network methods [11,16,17], due to (1) the spatially discretized formulation and (2) the ability to formulate models with finite-dimensional link Hilbert spaces [18]. However, large-scale numerical studies in- and out-of-equilibrium remain challenging at strong coupling and beyond $(1 + 1)$

dimensions. The development of quantum technologies has spurred significant growth in the field of quantum simulation of LGTs, with numerous theoretical proposals and promising experimental demonstrations emerging to validate these approaches [19–29]. Today, a variety of platforms, both analog and digital, are being employed as quantum simulators, including ultracold atoms in optical lattices [30–44], Rydberg atoms [45,46], superconducting qubits [47–57], and trapped ions [58–62], including numerous Abelian and non-Abelian proposals.

The long-standing goal of achieving large-scale quantum simulations of non-Abelian (LGTs) beyond $(1 + 1)$ dimensions remains a formidable challenge, since the required local symmetries do not arise naturally but have to be engineered. One main challenge are gauge-breaking terms that need to be suppressed over experimental timescales, before quantum devices can reliably and quantitatively predict outcomes in complex gauge theories such as QCD. Hence, the development of methods in toy models with non-Abelian gauge structure constitutes a significant pathway towards robust large-scale quantum simulations, allowing one to study qualitative features and suitable probes of non-Abelian LGTs. The construction of such models is generally more difficult than in the Abelian cases since multiple Gauss's laws with noncommuting generators have to be fulfilled. Moreover, large-scale systems in dimensions greater than $(1 + 1)$ dimensions are often susceptible to gauge-breaking errors,

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which can undermine the accuracy of simulations. This challenge has been tackled from various perspectives, leading to a range of proposed solutions, including the attempt to eliminate gauge-noninvariant subspaces [25,32,58,63] or the energetic protection against gauge-breaking errors; the latter involves implementing a gauge protection mechanism, designed to constrain the simulation's dynamics to the gauge-invariant sector of the Hilbert space [41,43,64–67].

In this work we propose a scheme for the quantum simulation of non-Abelian $U(N)$ LGTs in $(2+1)$ dimensions with dynamical, fermionic matter in a platform of ultracold atoms. Previous approaches adopted a *bottom-up* strategy, consisting in engineering microscopic processes that fulfill non-Abelian symmetries ensuring that gauge-violating terms remain sufficiently controlled [43,67,68]. Here we develop a general *top-down* approach to engineer gauge-invariant couplings of matter while maintaining the non-Abelian gauge symmetry of the system, enforcing a gauge-protection scheme. In the limit of strong gauge protection, we show that the dynamics is governed by a tunable effective non-Abelian $U(N)$ LGT with timescales realistic for current experiments.

The aim of our work is to extend previous models, proposed and explored in Refs. [42–44], by considering explicit gauge protection mechanisms that enforce the gauge invariance of the simulation. To this end, we propose two schemes with increasing complexity of the emergent non-Abelian LGT based on a combination of optical lattices with (1) two-body $SU(N)$ -invariant Hubbard interactions in AELAs as well as (2) periodically driven Rydberg atoms. More specifically, in this work we focus on an intrinsic and so far unresolved issue in AELA platforms, namely, spin-exchange processes that arise at the microscopic level and act as a source of gauge-symmetry breaking. We show how their effect can be suppressed through suitable protection mechanisms. These schemes are theoretically scalable and gauge-invariant by construction, and hence pave the way for the large-scale quantum simulation of non-Abelian LGTs with gauge-matter couplings.

The paper is structured as follows. In Sec. II we review the rishon formalism of the quantum link model we consider, specify the Gauss's laws of the theory and describe the general mechanism of our proposed scheme. In Sec. III we describe the Hamiltonian that allows us to protect against gauge-breaking terms and we derive the gauge-invariant effective theory. Then in Sec. IV we study a minimal experimentally relevant setup and demonstrate that our proposed scheme contains dynamics intrinsic to a non-Abelian gauge structure. In Secs. V and VI we propose two experimental protocols to implement our scheme in quantum simulators based on AELAs and on a hybrid Rydberg platform. We conclude with a discussion and summary in Sec. VII.

II. LATTICE FORMALISM

The implementation of a LGT in a quantum simulator requires the truncation of a typically infinite-dimensional local link Hilbert space to a finite-dimensional Hilbert space. The so-called rishon formulation of quantum link models [18,69–71] is an efficient and quantum-simulator-friendly way to represent non-Abelian LGTs with finite-dimensional link

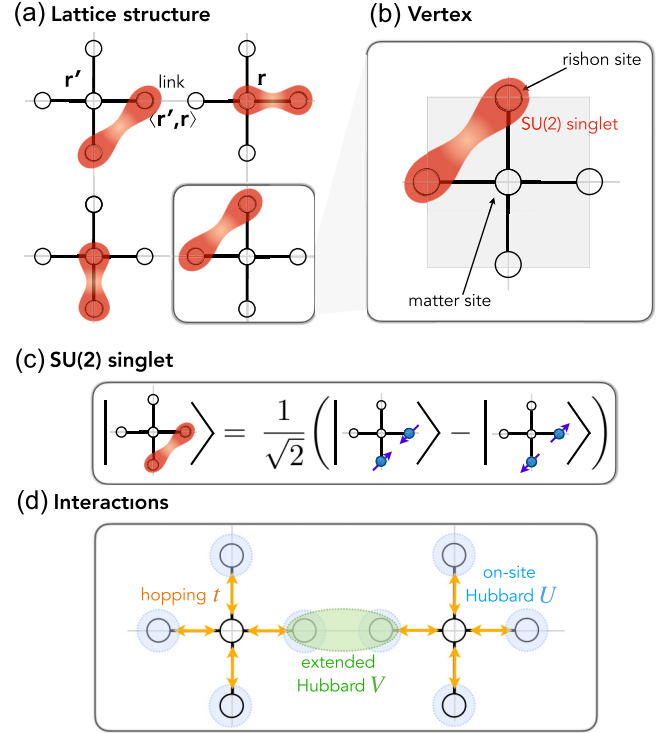


FIG. 1. Rishon-formulation of non-Abelian $U(N)$ quantum-link models. (a) The general lattice structure of the rishon model on a $(2+1)$ -dimensional square geometry showing an allowed singlet covering for the case $N = 2$. (b) The Gauss's law constraints enforce total color singlets on each vertex as well as a conserved number of rishons $\mathcal{N}$ per link; here we consider $\mathcal{N} = 1$ for an $SU(2)$ gauge structure, where the color singlet is given by a spin-1/2 singlet, shown in (c). (d) To energetically stabilize the rishon number constraint, we propose to use $SU(N)$ -invariant on-site Hubbard interactions U on rishon sites (blue) and extended Hubbard interactions V (green) between rishon sites. The weak tunneling t of fermions within a vertex induces the perturbative gauge-invariant dynamics.

Hilbert spaces while maintaining the underlying local symmetry.

Rishon formulation

Rishons are anticommuting fermionic operators used to represent the link degrees of freedom, for instance, the electric and gauge fields in a $U(1)$ LGT. Rather than a realistic physical field, they work as mathematical tools to describe the Hilbert space of a color charged gauge field; hereby configurations in the rishon Hilbert space can be identified uniquely as a configuration in the gauge theory.

For the scope of our analysis, we consider a square lattice with vertices $\mathbf{r}$; see Fig. 1(a). The links are denoted with $\langle \mathbf{r}, \mathbf{r}' \rangle$ and are located between neighboring vertices $\mathbf{r}$ and $\mathbf{r}'$. Each link is associated with two rishon sites, and hence one vertex consists of a central matter site and the adjacent four rishon sites on the connecting links; see Fig. 1(b).

We define the generators of the $SU(N)$ LGT as [69]

$$\hat{G}_{\mathbf{r}}^a = \sum_{\alpha, \beta=1}^N \left[\hat{\psi}_{\mathbf{r}}^{\alpha\dagger} \lambda_{\alpha\beta}^a \hat{\psi}_{\mathbf{r}}^{\beta} + \sum_{\mathbf{k}} \sum_{\mathbf{v}=\pm\mathbf{k}} \hat{c}_{\mathbf{r},\mathbf{v}}^{\alpha\dagger} \lambda_{\alpha\beta}^a \hat{c}_{\mathbf{r},\mathbf{v}}^{\beta} \right] \quad (1)$$

and that of the $U(1)$ LGT as [69]

$$\hat{G}_{\mathbf{r}} = \sum_{\alpha=1}^N \left[\hat{\psi}_{\mathbf{r}}^{\alpha\dagger} \hat{\psi}_{\mathbf{r}}^{\alpha} - \sum_{\mathbf{k}} (\hat{E}_{\mathbf{r},\mathbf{r}+\mathbf{k}}^{\alpha} - \hat{E}_{\mathbf{r}-\mathbf{k},\mathbf{r}}^{\alpha}) - q_{\mathbf{r}}^0 \right], \quad (2)$$

where $\hat{\psi}_{\mathbf{r}}^{\alpha\dagger}$ is the fermionic matter operator with color α , $\beta = 1, \dots, N$ at site $\mathbf{r}$; $\hat{c}_{\mathbf{r}\pm\mathbf{k}}^{\alpha\dagger}$ is the fermionic rishon operator acting on the rishon site adjacent to vertex $\mathbf{r}$ in the direction $\mathbf{k} = (1, 0), (0, 1)$; $\lambda_{\alpha\beta}^a$ are the $N \times N$ matrices that represent the $a = 1, \dots, (N^2 - 1)$ generators of the corresponding $SU(N)$ Lie algebra [69]. The constant $q_{\mathbf{r}}^0$ depends on the choice of the vacuum sector and corresponds to a local background charge. Further, the $U(1)$ electric field associated with color α is defined as

$$\hat{E}_{\mathbf{r},\mathbf{r}+\mathbf{k}}^{\alpha} = \frac{1}{2} (\hat{c}_{\mathbf{r}+\mathbf{k},-\mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r}+\mathbf{k},-\mathbf{k}}^{\alpha} - \hat{c}_{\mathbf{r},+\mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r},+\mathbf{k}}^{\alpha}), \quad (3)$$

which measures the population imbalance of rishons with color α on a link $(\mathbf{r}, \mathbf{r} + \mathbf{k})$.

In the rishon model, two constraints are imposed in the gauge-invariant subspace: (1) the total number $\mathcal{N}$ of rishons per link has to be conserved, where

$$\mathcal{N} = \sum_{\alpha} (\hat{c}_{\mathbf{r}',-\mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r}',-\mathbf{k}}^{\alpha} + \hat{c}_{\mathbf{r},+\mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r},+\mathbf{k}}^{\alpha}) \quad (4)$$

with $\mathbf{r}' = \mathbf{r} + \mathbf{k}$ and (2) the non-Abelian $SU(N)$ Gauss's laws enforce local color singlets on each vertex

$$\hat{G}_{\mathbf{r}}|\varphi\rangle = 0 \quad \forall a, \mathbf{r}. \quad (5)$$

In addition, we can impose a $U(1)$ conservation law

$$\hat{G}_{\mathbf{r}}|\varphi\rangle = 0 \quad \forall \mathbf{r} \quad (6)$$

to obtain the enlarged gauge group $U(N) = SU(N) \times U(1)$. Consequently, any reliable quantum simulation of the non-Abelian LGT we aim to construct has to obey dynamics governed by the above constraints.

III. MODEL AND ENERGETIC GAUGE PROTECTION

In this article, we develop two closely related schemes for simulating $(2+1)$ -dimensional non-Abelian LGTs. Both start by ensuring the gauge constraints are satisfied, by introducing large gauge-protection energy scales adequate for the underlying interactions. Subsequently, we introduce perturbations that explicitly break the above constraints, but—as we derive—these terms induce (1) effective and manifestly gauge-invariant couplings and (2) superexchange terms that protect against subleading gauge-breaking terms. The latter introduce an energy penalty that suppresses the accumulation of lower order gauge-violating processes whose number typically increases with system size, hence providing a scalable scheme for quantum simulators. The robustness of the protected subspace provides a necessary condition for extending the scheme to larger systems, but while it ensures that the relevant constrained sector remains well defined as the system complexity increases, actual scalability also depends on additional experimental details such as coherence times, control precision, and the ability to maintain the required energy hierarchies in extended geometries.

In our *top-down* approach, where gauge-invariant dynamics is generated perturbatively, the minimal coupling term is

the dominant one in the effective gauge theory, and it describes the gauge-invariant tunneling of $U(N)$ gauge charges coupled to the color charged gauge fields, as we show in this section. We note that in this approach it is inherently difficult to engineer four-body plaquette terms [72]. Other potentially complementary schemes for strong, nonperturbative plaquette interactions have been proposed in the context of $U(1)$ LGTs and toric-code simulations; see, e.g., Refs. [39,45,73,74]. We point out that existing methods like the linear gauge protection allow one to stabilize large system simulations of Abelian theories [75], but they are not sufficient for the implementation of non-Abelian symmetries and therefore need to be replaced with different strategies.

The general approach we consider has been introduced and initially studied by Banerjee *et al.* [43] and Surace *et al.* [44]. However, as we explicitly show in Sec. V, any experimentally *realistic* large-scale setup has to be protected against gauge-symmetry breaking errors, which arise from inherent atomic scattering processes in $SU(N)$ Hubbard models. Such gauge-breaking terms were not included in the analysis of Ref. [43] and only partially suppressed in the recent scheme of Ref. [44].

In our proposed scheme, we start with the construction of a Hamiltonian that enforces the correct rishon number by energetically isolating states with the desired number from the unphysical ones: this energy protection term is set to be the most relevant scale of the system Hamiltonian. Additionally, the singlet protection is obtained by suppressing spin-exchange interactions between neighboring vertices using a combination of a strong external (magnetic) field and weaker spin-exchange interactions that appear at lower order. In this way, the resulting Hamiltonian can be made inherently gauge invariant. Subsequently, the dynamics is promoted by additional subdominant terms, such as intravertex tunneling, to achieve second-order dynamical processes within the gauge-invariant sector, through virtual gauge-symmetry breaking. In the analysis below, we consider sectors with one rishon per link, but our scheme generalizes to different rishon numbers as well.

Our scheme is complementary but fundamentally different from the one proposed in Ref. [41]: here the rishon-number constraint is actively, perturbatively broken and the spin-exchange mechanism, which contributes as main dynamical protection strategy in Ref. [41], appears as a lower-order protection term. The conceptual ingredients underlying our construction—namely, the rishon formulation, gauge-invariant superexchange dynamics, and energetic protection of gauge symmetry—build on previous work Refs. [41,43,44]. The central point of the present analysis is that, for the microscopic architecture considered here, intrinsic exchange processes appear that are not part of the ideal gauge-invariant target dynamics. Making these terms explicit is essential in order to assess the stability of gauge symmetry and to formulate the appropriate protection mechanism.

A. Rishon-number protection mechanism

Let us define a microscopic model which energetically separates states with different rishon numbers in order to realize the constraints discussed in Sec. II. In particular, we introduce

the Hamiltonian

$$\begin{aligned} \hat{H}_0 = & V \sum_{\alpha\beta} \sum_{\mathbf{r},\mathbf{k}} \hat{n}_{\mathbf{r},\mathbf{k}}^{\alpha} \hat{n}_{\mathbf{r}+\mathbf{k},-\mathbf{k}}^{\beta} + U \sum_{\alpha<\beta} \sum_{\mathbf{r},\mathbf{k}} \sum_{\mathbf{v}=\pm\mathbf{k}} \hat{n}_{\mathbf{r},\mathbf{v}}^{\alpha} \hat{n}_{\mathbf{r},\mathbf{v}}^{\beta} \\ & + \delta \sum_{\alpha} \sum_{\mathbf{r}} \hat{n}_{\mathbf{r},m}^{\alpha} + U_m \sum_{\alpha<\beta} \sum_{\mathbf{r}} \hat{n}_{\mathbf{r},m}^{\alpha} \hat{n}_{\mathbf{r},m}^{\beta}, \end{aligned} \quad (7)$$

where we used the notation $\hat{n}_{\mathbf{r},m}^{\alpha} = \hat{\psi}_{\mathbf{r}}^{\alpha\dagger} \hat{\psi}_{\mathbf{r}}^{\alpha}$ and $\hat{n}_{\mathbf{r},\mathbf{k}}^{\alpha} = \hat{c}_{\mathbf{r},\mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r},\mathbf{k}}^{\alpha}$. The first term $\propto V$ is an $SU(N)$ -invariant extended Hubbard interaction [Fig. 1(d)], which acts as an interaction between two rishons at opposite sides of a link. The second term $\propto U$ is an on-site $SU(N)$ -invariant Hubbard interaction between rishons. Further, we introduce a chemical potential on the matter site $\propto \delta$ that will allow us later to tune the resonance condition for the effective hopping terms [42]. The Hamiltonian separates states with different rishon numbers through energy terms proportional to V and δ : this allows one to energetically split the Hilbert space into sectors with well-defined rishon numbers per link. The term $\propto U_m$ is an additional on-site Hubbard interaction between matter excitations. The absence of hopping and spin-exchange interactions between adjacent vertices makes this idealized model (7) inherently gauge invariant [43]. When going beyond one-rishon per link simulations, we additionally need to impose the condition $V \approx U$ in order to maintain the energy resonance between states in the gauge-invariant sector; e.g., when considering a model with two rishons per link, a state with two rishon excitations localized on one side (rishon site) of the link must be near resonant with a state with one rishon excitation on each side of the link.

Thus far, the Hamiltonian (7) does not have any dynamics. Hence, a state initialized in the target rishon-number sector and with total spin singlet per vertex remains static and gauge-invariant. As we argue next, an intravertex perturbation t , that does not couple to the spin (or color) on neighboring vertices but that breaks the rishon number constraint, induces gauge-invariant dynamics [42,43]. In the limit of weak perturbation $|t| \ll \delta, V$, the eigenstates are still approximately described by well-defined rishon number sectors $\mathcal{N}$, such that the effective model satisfies the required gauge constraints up to controlled errors [65,66], which we analyze numerically for some realistic cases in Sec. V.

This approach requires the energy of the rishon number sectors to depend nonlinearly on the rishon number $\mathcal{N}$ to avoid a resonance between, e.g., a gauge-breaking state characterized by two rishon excitations on the same link and zero rishon excitations on another link and a gauge-invariant state correctly characterized by two links with one rishon excitation each. We point out that the Hamiltonian we introduce in Eq. (7) allows us to stabilize any rishon number sector with $\mathcal{N} \leq 2N$ and therefore can be considered a general scheme to explore gauge theories with increasing number of rishons per link.

B. Gauge-invariant effective theory

Next, we introduce a perturbative Hamiltonian term

$$\hat{H}_t = -t \sum_{\alpha} \sum_{\mathbf{r},\mathbf{k}} \sum_{\mathbf{v}=\pm\mathbf{k}} (\hat{c}_{\mathbf{r},\mathbf{v}}^{\alpha\dagger} \hat{\psi}_{\mathbf{r}}^{\alpha} + \text{H.c.}), \quad (8)$$

which describes tunneling of amplitude t between nearest-neighbor sites within a vertex; see Fig. 1(d). Hence, it preserves the total color singlet constraint in each vertex [43]. First, we consider the resulting effective model, at second order $O(t^2)$, for a scheme with exactly one singlet per vertex and one rishon per link. In this case, the effective interactions describe correlated *intravertex* processes that conserve the total number of particles on each vertex,

$$\hat{N}_{\mathbf{r}} = \sum_{\alpha} \left(\hat{n}_{\mathbf{r},m}^{\alpha} + \sum_{\mathbf{k}} \sum_{\mathbf{v}=\pm\mathbf{k}} \hat{n}_{\mathbf{r},\mathbf{v}}^{\alpha} \right), \quad (9)$$

imposing the additional $U(1)$ gauge constraint.

In the limit of $|t| \ll V$, we perform a Schrieffer-Wolff transformation and obtain a gauge-invariant hopping given by [42]

$$\hat{H}_{\text{eff}}^t = -t_{\text{eff}} \sum_{\alpha\beta} \sum_{\mathbf{r},\mathbf{k}} (\hat{\psi}_{\mathbf{r}}^{\alpha\dagger} \hat{c}_{\mathbf{r},\mathbf{k}}^{\alpha} \hat{c}_{\mathbf{r}+\mathbf{k},-\mathbf{k}}^{\beta\dagger} \hat{\psi}_{\mathbf{r}+\mathbf{k}}^{\beta} + \text{H.c.}), \quad (10)$$

with effective tunneling amplitude

$$t_{\text{eff}} = t^2 \left(\frac{1}{\delta - V} + \frac{1}{-\delta} \right). \quad (11)$$

For $|t| \ll \delta < V$, the coefficient scales $\propto t^2/\delta$, and we achieve gauge-invariant dynamics in the energy-protected manifold. This approach similarly allows one to derive the manifestly gauge-invariant tunnelings for any filling and rishon-number sector, enabling the implementation of theories beyond the simplest instance of a rishon link model treated above.

C. Gauge-breaking interactions

So far we followed the strategies proposed in earlier works [42,43] working with an idealized unperturbed Hamiltonian $\hat{H}_0$ that does not violate gauge invariance. However, the model in Eq. (7) is fine-tuned to a gauge-invariant point in the following sense: it requires strong intervertex density-density interactions $\propto V$ which commute with the local $SU(N)$ Gauss's laws of both involved vertices. In a realistic experimental setup, using, e.g., AELAs, we may indeed assume that local interactions, such as the V term in Eq. (7) are $SU(N)$ -invariant to a very good approximation, but the interactions commute with the total $SU(N)$ spin formed by the two involved rishon sites together and therefore couple particles of adjacent vertices.

Hence, another term in the microscopic Hamiltonian that we have to consider in any generic, i.e., non-fine-tuned, situation involves spin-exchange processes between neighboring rishon sites¹

$$\hat{H}_{\text{ex}} = V_{\text{ex}} \sum_{\alpha\beta} \sum_{\mathbf{r},\mathbf{k}} \hat{c}_{\mathbf{r},\mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r}+\mathbf{k},-\mathbf{k}}^{\beta\dagger} \hat{c}_{\mathbf{r},\mathbf{k}}^{\beta} \hat{c}_{\mathbf{r}+\mathbf{k},-\mathbf{k}}^{\alpha}. \quad (12)$$

In particular, one should generally expect $|V_{\text{ex}}|$ to be comparable in strength to the stabilizing interaction V in Eq. (7), which

¹In the specific case of AELAs, this term is only nonzero for interactions between two different orbitals, otherwise it vanishes due to the $SU(N)$ symmetric nature of the interactions.

we assume to be a large energy scale in the problem [76,77]. In fact, Eq. (12) does not commute with the $SU(N)$ Gauss's laws of the individual vertices $\mathbf{r}$ and $\mathbf{r}'$, because it enables spin singlets initially defined on one vertex to traverse and extend across the link $\langle \mathbf{r}, \mathbf{r}' \rangle$. This should be contrasted with the effect of similar spin-exchange interactions within one vertex that commute with the local $SU(N)$ Gauss's laws. Notably, the presence of $SU(N)$ spin-exchange interactions was not considered in the seminal paper [43], defining the main source of $SU(N)$ gauge-symmetry breaking of that scheme [the $U(1)$ Gauss's law is not affected by this, however [42]]. The key goal of our present work is to include such leading gauge-breaking terms and develop a scheme to suppress and control these processes.

D. Gauge protection through external field

As an initial strategy for the mitigation of the aforementioned processes, we propose to utilize an external field to guarantee that the color composition of each vertex does not change. The field is intended to introduce a different energy offset for each colored excitation. Any variation of the color composition on a vertex will bring the state out of resonance, suppressing it energetically.

We formalize this idea by adding the following gauge-protection term, with strength H , to the Hamiltonian,

$$\hat{H}_{\text{prot}} = H \sum_{\alpha} \gamma_{\alpha}^{\mathbf{r}} \left(\hat{n}_{\mathbf{r},m}^{\alpha} + \sum_{\mathbf{k}} \sum_{\mathbf{v}=\pm\mathbf{k}} \hat{n}_{\mathbf{r},\mathbf{v}}^{\alpha} \right). \quad (13)$$

This corresponds to a shift by $H\gamma_{\alpha}^{\mathbf{r}}$ of all fermions with color α , matter and rishon, on vertex $\mathbf{r}$. By choosing

$$|(\gamma_{\alpha}^{\mathbf{r}} - \gamma_{\alpha}^{\mathbf{r}'} - (\gamma_{\beta}^{\mathbf{r}} - \gamma_{\beta}^{\mathbf{r}'}))| > 0 \quad \text{for } \alpha \neq \beta \quad (14)$$

on neighboring vertices $\mathbf{r}$ and $\mathbf{r} + \mathbf{k}$, spin-exchange processes within rishon links can be suppressed: the effect of exchanging colors α and β on vertices $\mathbf{r}$, $\mathbf{r} + \mathbf{k}$ by applying the corresponding term $\propto \hat{c}_{\mathbf{r},\mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r}+\mathbf{k},-\mathbf{k}}^{\beta\dagger} \hat{c}_{\mathbf{r},\mathbf{k}}^{\beta} \hat{c}_{\mathbf{r}+\mathbf{k},-\mathbf{k}}^{\alpha}$ in Eq. (12), leads to a change of energy $\Delta H_{\text{prot}} = H(\gamma_{\beta}^{\mathbf{r}} - \gamma_{\beta}^{\mathbf{r}'} - (\gamma_{\alpha}^{\mathbf{r}} - \gamma_{\alpha}^{\mathbf{r}'}))$ for $\alpha \neq \beta$. Hence for $|H| \gg |V_{\text{ex}}|$ color-changing spin exchange is strongly suppressed. In Sec. V we will discuss a concrete experimental realization of this gauge-protection term.

Although the gauge-protection term above suppresses changes of the color configurations, it does not fully protect the $SU(N)$ Gauss's law in Eq. (5). To understand this, we consider an $SU(2)$ subgroup of $SU(N)$ obtained by focusing on two colors $\alpha \neq \beta$, and derive the selection rules associated with the spin-exchange interaction (12). We can label states in the $SU(2)$ subgroup by their total nuclear spin $F^{\mathbf{r}}$ and their spin projection $m_F^{\mathbf{r}} = -F^{\mathbf{r}}, \dots, F^{\mathbf{r}}$ quantum numbers, defined separately on each vertex $\mathbf{r}$ and $\mathbf{r}'$. In the gauge-invariant subspace, $F^{\mathbf{r}} = F^{\mathbf{r}'} = 0$ since total color singlets on the vertices also realize singlets of the $SU(2)$ subgroup. The linear gauge protection term [65] of Eq. (13) only suppresses m_F -changing processes ($\Delta m_F^{\mathbf{r}} = \pm 1$, $\Delta m_F^{\mathbf{r}} + \Delta m_F^{\mathbf{r}'} = 0$), illustrated in the top and bottom rows of Fig. 2.

In order to understand which other ($m_F^{\mathbf{r}} = m_F^{\mathbf{r}'} = 0$) states the $F_{\text{tot}} = 0$ singlets $|F^{\mathbf{r}} = 0; m_F^{\mathbf{r}} = 0\rangle \otimes$

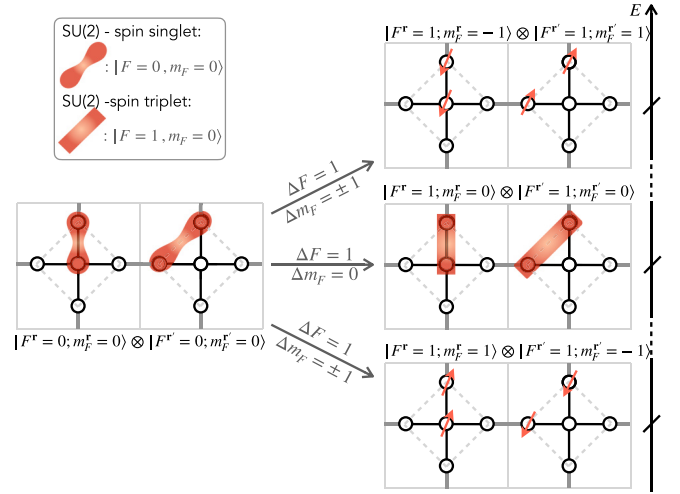


FIG. 2. Selection rules. The selection rules for a system of two neighboring vertices. The spin-exchange interaction V_{ex} couples the gauge-invariant states with $|F^{\mathbf{r}} = 0; m_F^{\mathbf{r}} = 0\rangle \otimes |F^{\mathbf{r}'} = 0; m_F^{\mathbf{r}'} = 0\rangle$ to gauge-breaking states $|F^{\mathbf{r}} = 1; m_F^{\mathbf{r}} = 0\rangle \otimes |F^{\mathbf{r}'} = 1; m_F^{\mathbf{r}'} = 0\rangle$ (represented by the red squares) and $|F^{\mathbf{r}} = 1; m_F^{\mathbf{r}} = \pm 1\rangle \otimes |F^{\mathbf{r}'} = 1; m_F^{\mathbf{r}'} = \mp 1\rangle$ (indicated by aligned spins). The introduction of an external, vertex-dependent field, Eq. (13), allows us to energetically separate the states with $m_F = \pm 1$ from those with $m_F = \pm 0$ but does not protect from oscillations into the manifolds of states with $|F^{\mathbf{r}} = 1; m_F^{\mathbf{r}} = 0\rangle \otimes |F^{\mathbf{r}'} = 1; m_F^{\mathbf{r}'} = 0\rangle$.

$|F^{\mathbf{r}} = 0; m_F^{\mathbf{r}} = 0\rangle$ (Fig. 2 left) are coupled by $\hat{T}_{\alpha\beta}^{(1)\otimes(1)} = V_{\text{ex}} \hat{c}_{\mathbf{r},\mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r}+\mathbf{k},-\mathbf{k}}^{\beta\dagger} \hat{c}_{\mathbf{r},\mathbf{k}}^{\beta} \hat{c}_{\mathbf{r}+\mathbf{k},-\mathbf{k}}^{\alpha}$, we note that the spin-exchange interaction is globally $SU(N)$ -invariant; i.e., it conserves the total $SU(N)$ spin on both vertices $\mathbf{r}$, $\mathbf{r}'$ combined and, by extension, the same is true for the $SU(2)$ subgroup we consider. In principle, this symmetry consideration allows couplings to states with $F^{\mathbf{r}} = F^{\mathbf{r}'} = F$, which can combine to $F_{\text{tot}} = 0$. However, since $\hat{T}_{\alpha\beta}^{(1)\otimes(1)}$ acts as a rank-1 tensor operator on each vertex ($\mathbf{r}$ or $\mathbf{r}'$), the Wigner-Eckart theorem implies a further selection rule (besides $\Delta m_F^{\text{tot}} = 0$),

$$\Delta F^{\mathbf{r}}, \Delta F^{\mathbf{r}'} = 0, 1. \quad (15)$$

Hence, the only gauge-noninvariant states reached by V_{ex} are $|F^{\mathbf{r}} = 1; m_F^{\mathbf{r}} = 0\rangle \otimes |F^{\mathbf{r}'} = 1; m_F^{\mathbf{r}'} = 0\rangle$, illustrated in the center row of Fig. 2. In Sec. V we will show for a realistic experimental setting that coupling matrix elements to the triplet states $|F^{\mathbf{r}} = 1; m_F^{\mathbf{r}} = 0\rangle \otimes |F^{\mathbf{r}'} = 1; m_F^{\mathbf{r}'} = 0\rangle$ are perturbative in $t^2/(\delta - V)$, in contrast to the $\Delta m_F = \pm 1$ transitions which are of order V_{ex} and thus require strong gauge protection, $|H| > |V_{\text{ex}}|$. This, in turn, allows one to design an efficient gauge protection mechanism based on weak spin-exchange processes $\propto J \sim t^2/U$, which we introduce below.

IV. MINIMAL MODEL FOR NON-ABELIAN LATTICE GAUGE THEORY

So far we focused on obtaining the gauge-invariant hopping terms, however, to confirm that our model captures inherently non-Abelian features, we must ensure that the scheme does not map to an Abelian $U(1)$ or $\mathbb{Z}_2$ LGT, or simpler models, e.g., because the model has too many constraints

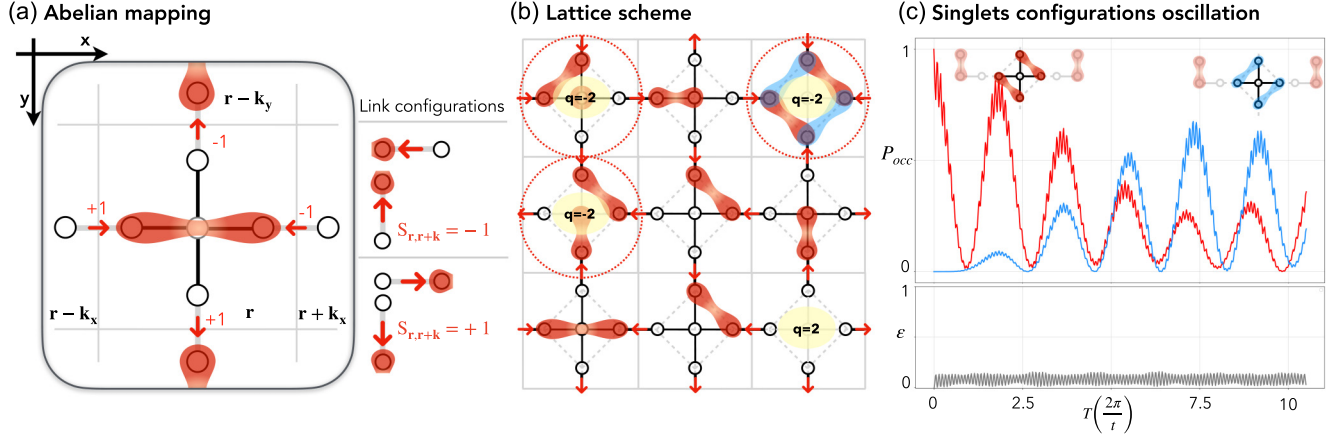


FIG. 3. Mapping of $U(2)$ to $U(1)$ LGT and minimal non-Abelian setup: $U(2)$ with one rishon per link. (a) The mapping that identifies the links' state through the operator $\hat{S}_{\mathbf{r},\mathbf{r}+\mathbf{k}}$. The mapping to a $U(1)$ theory is valid for models with one rishon per link and up to one singlet configuration per vertex. The values of the link variable are determined by the position of the occupied rishon site with respect to the link, as illustrated. The link can be described through a binary variable $S_{\mathbf{r},\mathbf{r}+\mathbf{k}} = \pm 1$. (b) An example of lattice initialization for a non-Abelian simulation: at least one vertex is occupied by more than one singlet. The vertices circled in red are filled with more than one singlet and characterize the non-Abelian nature of the scheme. In the top-right vertex we show in blue an additional gauge-invariant singlet covering including components orthogonal to the red covering. (c) This shows the exact diagonalization results of the model discussed in the main text. Top: During the time evolution, the system, which is initialized in the state colored in red, evolves into the state in blue, with different singlet covering. Since the two states are not orthogonal, in the plot we show the evolution of the orthonormalized states (i.e., the blue state is projected in the space orthogonal to the red one). Three different frequencies of oscillation are noticeable: the fastest oscillations are due to the occupation of states with a wrong number of rishon excitation, the slowest ones represent the effective evolution of the state from the red to the blue singlet covering, while the intermediate oscillations appear when the system is in a gauge-invariant state with some matter excitation in the central vertex (e.g., when one particle or an entire singlet resides in the matter site). Bottom: The gray plot shows the occupation ε of states with the wrong rishon-number and indeed matches the frequency of the fastest oscillations in the top plot.

such that the dynamics becomes effectively Abelian. Indeed, we find that $U(2)$ one-singlet-per-vertex ($\hat{N}_{\mathbf{r}} = 2$) models can be trivially mapped onto a $U(1)$ QLM, as we derive in this section. In addition, our analysis allows us to determine the inherently non-Abelian sectors of one-rishon models, which can be targeted experimentally by the choice of the initial state.

To identify the Abelian component in $U(2)$ one-rishon models, we first establish a one-to-one correspondence to a $U(1)$ QLM. If such a mapping can be applied, this implies that the spin states of the particles are irrelevant in the underlying non-Abelian LGT. As a result, the $U(2)$ LGT reduces to a $U(1)$ LGT with background charges. In order to find the simplest non-Abelian instance of one-rishon ($\mathcal{N} = 1$) models, we subsequently extend to *multiple-singlet-per-vertex* ($\hat{N}_{\mathbf{r}} > 2$) models—i.e., higher particle fillings at the vertices—and consider different initializations, until we reach a minimal model with components that are beyond Abelian LGTs.

A. Mapping to an Abelian model

The mapping to a $U(1)$ LGT that we attempt to make defines an orientation on each link establishing a one-to-one correspondence to a $U(1)$ electric field $\hat{E}_{\mathbf{r},\mathbf{r}'} = \sum_{\alpha} \hat{E}_{\mathbf{r},\mathbf{r}'}^{\alpha}$, as defined in Eq. (3). In Fig. 3(a), we illustrate the fundamental elements of the mapping: we consider a vertex filled with up to one singlet and we impose the constraint of one rishon per link. The mapping consists in assigning a specific orientation to the links between two vertices $\langle \mathbf{r}, \mathbf{r}' \rangle$ depending on the occupation of the corresponding rishon

sites. In particular, we choose the “orientation” of the link to point towards the occupied site, and we define the operator $\hat{S}_{\mathbf{r},\mathbf{r}+\mathbf{k}} = \text{sgn}(\hat{E}_{\mathbf{r},\mathbf{r}'}^{\alpha})$, with $\mathbf{r}' = \mathbf{r} + \mathbf{k}$ as before. Therefore, when the right rishon of a horizontal link is occupied, the link variable will accordingly be positive ($\hat{S}_{\mathbf{r},\mathbf{r}+\mathbf{k}} = +1$) while the link variable will be negative when the left rishon is occupied ($\hat{S}_{\mathbf{r},\mathbf{r}+\mathbf{k}} = -1$); see Fig. 3(a). The same applies to a vertical link: when the down rishon is occupied the link variable is positive, while it is negative when the upper rishon is occupied.

In this model the $U(1)$ Gauss's law is defined as $\hat{G}_{\mathbf{r}} = \sum_{\mathbf{k}} (\hat{E}_{\mathbf{r},\mathbf{r}+\mathbf{k}} - \hat{E}_{\mathbf{r}-\mathbf{k},\mathbf{r}}) - \hat{\psi}_{\mathbf{r}}^{\dagger} \hat{\psi}_{\mathbf{r}} - q_{\mathbf{r}}^0$ where the local background charge $q_{\mathbf{r}}^0$ is assigned accordingly in each vertex. In the example of Fig. 3(a), the background charge is $q_{\mathbf{r}}^0 = 0$, while the charge would be $q_{\mathbf{r}}^0 = 2$ if the vertex were empty. Since a $U(1)$ constraint is present [i.e., the singlets cannot hop from one vertex to the other; see Eq. (6)], the background charge is also conserved. In Appendix A we show that this way a one-to-one correspondence between *one-singlet-per-vertex* and the $U(1)$ QLM is established.

B. Non-Abelian model

The minimal non-Abelian model is obtained by initializing the lattice with at least one vertex having more than one singlet. To be precise, we consider vertices with four particles, including matter and rishon particles, ($\hat{N}_{\mathbf{r}} = 4$); For these configurations Gauss's law implies that states formed by different (nonorthogonal) singlet coverings constitute gauge-invariant states; see Fig. 3(b). These states cannot be mapped to a

$U(1)$ QLM, because the particle's position does not uniquely determine its color-state, hence these configurations are necessary to construct models beyond Abelian LGTs. Accordingly, the purpose of the present geometry is to isolate the relevant local microscopic processes in a controlled setting, not to reproduce all features of a full higher-dimensional gauge-theory dynamics.

Next, to confirm that the addition of these four-rishon/matter configurations enables to probe features inherent to non-Abelian LGTs in our proposed Hamiltonian (see Sec. III), we show dynamical coupling between the different singlet configurations on such a vertex. To this end, we employ exact diagonalization on a minimal model and show that the time evolution under the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_t$ [see Eq. (7) and Eq. (8)] leads to oscillations between the two gauge-invariant singlet coverings on a vertex with four rishon/matter particles. The minimal model is constituted by one central vertex and we additionally consider the two neighboring vertices (e.g., left and right), which fulfill an ancillary role and are both occupied by one singlet; see Fig. 3(c). This situation defines a simple instance of a model with minimal non-Abelian dynamics, where the auxiliary sites are introduced to promote matter hopping, and where the remaining two neighboring vertices (above and below) are disregarded for simplicity.

Here we assume on-site (extended) Hubbard interactions of strength $U = 2\delta$ ($V = 2\delta$) on all rishon sites (rishon links), which energetically protects the rishon number per link; see Eq. (7). The tunneling term $\hat{H}_t$ of particles within a vertex is a weak perturbation, and here we choose $t = \delta/10$. An additional on-site Hubbard interaction U is included on the matter site; this term helps to enhance the oscillation between the two singlet configurations, and we introduce a stronger on-site interaction on the unpaired rishons to avoid double occupancy in those sites and promote the singlet-dynamics in the central vertex.

To probe the coupling between the two singlet configurations, we first construct a set of orthonormal states $\{|\varphi_1\rangle, |\varphi_2\rangle\}$, where $|\varphi_1\rangle$ is the initial singlet state [see Fig. 3(c)] (red singlet covering) while $|\varphi_2\rangle$ represents the normalized projection of the second configuration of singlets (blue singlet covering) on the orthonormal space. We initialize the system in the state $|\varphi_1\rangle$ and compute the occupation of states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ under time evolution with Hamiltonian $\hat{H}$ with time T ; see Fig. 3(c). By construction of the Hamiltonian (in the absence of the spin-exchange interactions between neighboring vertices discussed in Sec. III), the Gauss-law constraints Eq. (5) are conserved throughout the dynamics. In the present three-vertex setting, these constraints are directly reflected in the total-spin quantum numbers of the local vertex configurations. For this reason, the corresponding spin observables provide a direct diagnostic of whether the evolution remains confined to the gauge-invariant sector or develops gauge-violating admixtures. Since these quantities are conserved by construction in the regime considered here, we do not include them in the shown plots. Moreover we observe that the correct rishon number constraint is satisfied to a good approximation, as can be seen in the plot of the gauge-breaking states occupation ε in Fig. 3(c). The errors ε in the rishon number constraint are controlled by the perturbation

t/δ , while they decrease for weaker perturbation t , as does the effective tunneling amplitude t_{eff} .

The total evolution of the system is characterized by a superposition of multiple oscillations with three different characteristic frequencies: indeed, the system couples to different intermediate states which are not represented in Fig. 3(c). For example, the matter excitation on the right (or left) auxiliary vertex can perturbatively hop to the central vertex resulting in a gauge-invariant state that respects the rishon number constraint: the occupation of these states explains the oscillation of the system with intermediate frequency. The fastest oscillations correspond to a micromotion associated with the virtual occupation of the off-resonant states with the wrong rishon number and match the profile of ε . Finally, the evolution associated with the slowest frequency shows the oscillation between the red and blue singlet coverings which happens while the values of the link variables, shown as arrows in Fig. 3(b), do not change. The scale of the time evolution suggests that, with appropriate parameter tuning, this oscillation can be directly accessed in experiments. Hence, for a given link configuration, there exist at least two different spin configurations participating in the dynamics, so that the mapping to the $U(1)$ QLM is no longer bijective. In other words, the local Hilbert-space structure of our scheme is richer than that of a spin-1/2 $U(1)$ model.

V. EXPERIMENTAL IMPLEMENTATION 1: AELAS IN OPTICAL LATTICES

To realize the microscopic model described in Sec. III in an analog quantum simulation experiment, we propose two different schemes utilizing ultracold $SU(N)$ fermions [43,76,78,79]. In this section we focus on $U(2)$ LGTs with rishon number $\mathcal{N} = 1$ per link and fermionic spin-1/2 particles; in particular we establish a gauge protection mechanism necessary to maintain gauge invariance in the presence of strong exchange interactions, Eq. (12), present in AELAs. In Sec. VI we propose a periodically driven Rydberg scheme which allows the gauge-invariant implementation of more complicated theories beyond one-rishon per link models.

Implementing toy models of non-Abelian LGTs in $(2+1)\text{D}$ constitutes one of the main goals in the quantum simulation of LGTs. Recently there have been numerous proposals to build schemes for the simulation of QLMs and non-Abelian gauge theories [43,44,57,80–82]: indeed, $SU(N)$ Hubbard models are generally suited to implement the microscopic model (7); however, they require the engineering of additional strategies to fully ensure the protection of the gauge symmetry, as discussed in Sec. III. Previous works such as Refs. [43,44] have tackled these challenges and suggested experimental proposals for engineering gauge-invariant Hamiltonians. In Banerjee *et al.* [43] a model similar to Eq. (7) has been considered, obtaining a scheme for the implementation of $U(N)$ and $SU(N)$ theories, but some terms of the Hamiltonian that are naturally present and give rise to problematic symmetry-breaking interactions have not been included in the calculations. On the other hand, Surace *et al.* [44] studied in great detail their proposed model, including imperfections of the underlying tight-binding Wannier states, but did not achieve full gauge invariance of the system. In our

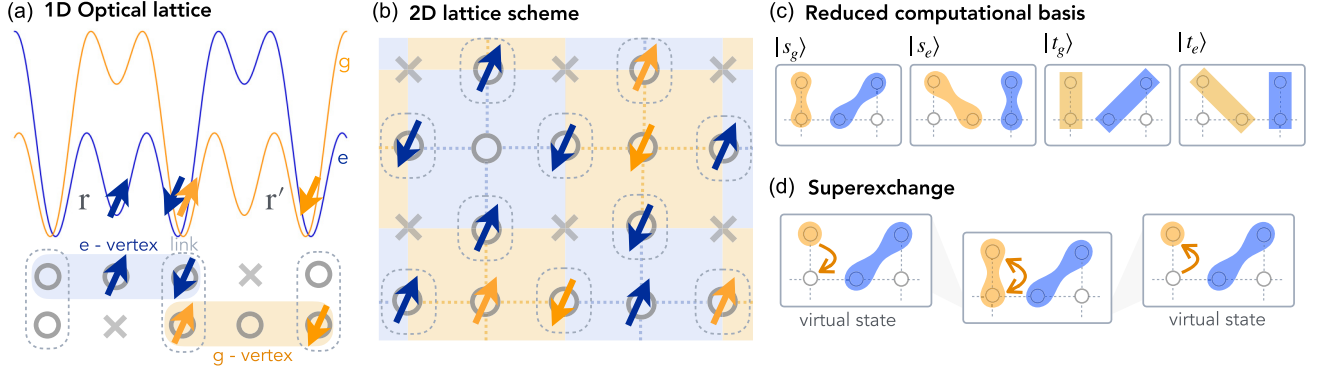


FIG. 4. AELA scheme with its reduced basis and superexchange mechanism. (a) One-dimensional cut of a minimal building block: the orange line represents the optical lattice potential of atoms in the ground state (g) and the blue one of the atoms in the excited clock state (e). Along one dimension, each vertex is given by a group of three wells that are of alternate (g and e) nature, and it is initialized to only contain spin singlets. The links are identified by the rishon sites, where the g and e wells overlap and where we realize interactions between atoms of the neighboring vertices. (b) Two-dimensional generalization of the lattice scheme: each vertex is identified with blue or orange and composed of one matter site and four adjacent rishon sites, for a total of five wells. The hopping of particles can happen only within the same vertex, and it is shown with blue and orange dotted lines. (c) The four $m_F^{\text{tot}} = 0$ states that dominate the dynamics in the regime protected through the Zeeman splitting mechanism. (d) The superexchange (here shown in the g vertex) yields a correction to the total energy of the central state: the states in the left and right boxes are perturbatively occupied. Analogously, superexchange processes also occur in the e vertices but have not been depicted here.

work, we attempt to consider all the relevant symmetry breaking interactions in the system and to find a set of strategies that leads to a full protection of the gauge symmetry. This work provides a systematic investigation of the role of exchange terms in the loss of gauge invariance: we focus on understanding the microscopic origin of these gauge-breaking terms and assessing their impact on the stability and scalability of the simulation. This allows us to identify conditions under which such processes become significant, and to propose strategies to control their suppression.

As we discussed in Sec. III and explicitly show below, spin-exchange interactions in the form of Eq. (12) naturally arise in AELAs between neighboring vertices with a strength that is on the order of the largest energy scale in the system. To achieve the simulation of large-scale non-Abelian LGTs, we propose to implement a twofold gauge protection scheme to efficiently suppress gauge-breaking terms present in AELA $SU(N)$ Hubbard models. First, we implement linear gauge protection [65] realized by an external magnetic field that allows us to suppress oscillations to the states $|F^r = 1; m_F^r = \pm 1\rangle \otimes |F^{r'} = 1; m_F^{r'} = \mp 1\rangle$ (see Fig. 2). Second, we derive the strength of the remaining gauge-breaking terms, and we find that on-site Hubbard interactions on the matter sites are sufficient to fully protect the non-Abelian gauge constraints.

Our starting point is the setup proposed in previous works that utilizes a rishon formulation to implement Abelian $U(1)$ [42] and non-Abelian $U(N)$ LGTs [43,44]. In particular, we consider ultracold fermionic AELAs distinguished through their electronic ground $|g\rangle$ (metastable $|e\rangle$) state and nuclear spin $|m_F = \pm \frac{1}{2}\rangle$ for $U(2)$ models; we note that our scheme can be generalized to $U(N)$ models. Using tune-out optical lattices for the $|g\rangle$ and $|e\rangle$ states, we engineer an optical superlattice with $|g\rangle$ ($|e\rangle$) atoms on sublattice A (sublattice B) of a square lattice in $(2 + 1)$ dimensions.

In Fig. 4(a) we illustrate a one-dimensional cut of a minimal building block: the two lattices have the same periodicity but are shifted relative to each other in space: the $|g\rangle$ and $|e\rangle$ lattice sites overlap only on the rishon sites allowing us to effectively generate the interactions between neighboring vertices shown in Fig. 1(d), without tunneling between them; we refer to Ref. [42] for an explicit calculation of the Wannier functions. Similarly, on the two-dimensional square lattice the neighboring vertices overlap on the rishon sites. Hence, each block is composed of five wells: a central matter site surrounded by four rishon sites, as illustrated in Fig. 4(b). The total tight-binding discrete Hamiltonian of this system reads [42,44,76]

$$\hat{H} = \hat{H}_g + \hat{H}_e + \hat{H}_{gg} + \hat{H}_{ee} + \hat{H}_{eg}. \quad (16)$$

Here $\hat{H}_\epsilon$ ($\epsilon = g, e$) is the single-particle Hamiltonian in vertex ϵ that includes the chemical potential terms, proportional to δ^ϵ in Eq. (7), and the intravertex hopping proportional to t^ϵ in Eq. (8). In general, t^ϵ and δ^ϵ can be tuned independently in this setting for $\epsilon = g, e$ states. $\hat{H}_{\epsilon\epsilon}$ is the on-site Hubbard interaction, proportional to $U_{\epsilon\epsilon}$, that takes the role of the on-site Hubbard interaction proportional to U in Eq. (7) and that will allow us to additionally implement a superexchange-based gauge protection mechanism. Finally, $\hat{H}_{eg} = \hat{H}_V + \hat{H}_{\text{ex}}$ describes the interaction between the g and e atoms in every rishon site and it is composed of two terms: the first one, $\hat{H}_V$, is an extended Hubbard term that arises when a g and e atom occupy the same rishon site and we identify it with the extended Hubbard term $\propto V$ in Eq. (7). The second term, that we indicate with $\hat{H}_{\text{ex}}$, is defined as in Eq. (12) and is proportional to V_{ex} : it represents a spin-exchange interaction on the rishon sites, where the two lattices are spatially overlapping, and introduces gauge-breaking couplings, as explained in Sec. III. For further discussion on the origin of the interaction terms

$U_{\epsilon\epsilon}$, V , and V_{ex} we refer to Appendix B. In the next paragraphs we discuss our strategies for realizing gauge protection.

A. Protection of gauge invariance

After defining the general structure of the scheme at the level of the microscopic Hamiltonian, the goal is now to suppress gauge-breaking processes that are inevitable in the above described scheme due to the spin-exchange interaction of strength V_{ex} [see Eq. (12)]: this term introduces an energy splitting between the triplet and singlet states of two atoms located on the same rishon site and leads to the expansion of the spin singlets across a link, as discussed in Sec. III.

To corroborate this picture, we study the microscopic Hamiltonian on a minimal building block comprising two neighboring vertices [Fig. 4(a)]. For a faithful quantum simulation, it is necessary to maintain gauge invariance, i.e., to remain in the total spin singlet sector $|F^{\mathbf{r}} = 0; m_F^{\mathbf{r}} = 0\rangle$ at each vertex $\mathbf{r}$, and suppress leakage into the triplet sector $|F^{\mathbf{r}} = 1; m_F^{\mathbf{r}} = 0, \pm 1\rangle$, as already explained in Sec. III and shown in Fig. 2. To this end, we propose a gauge protection scheme constituted by

(1) Linear Zeeman shifts suppressing singlet/triplet oscillation between neighboring vertices $\mathbf{r}, \mathbf{r}'$, i.e., between states $|F^{\mathbf{r}} = 0; m_F^{\mathbf{r}} = 0\rangle \otimes |F^{\mathbf{r}'} = 0; m_F^{\mathbf{r}'} = 0\rangle$ and $|F^{\mathbf{r}} = 1; m_F^{\mathbf{r}} = \pm 1\rangle \otimes |F^{\mathbf{r}'} = 1; m_F^{\mathbf{r}'} = \mp 1\rangle$ with $m_F^{\mathbf{r}} = -m_F^{\mathbf{r}'}$,

(2) Superexchange processes within a vertex enabling protection [41] against couplings between the $|F^{\mathbf{r}} = 0; m_F^{\mathbf{r}} = 0\rangle \otimes |F^{\mathbf{r}'} = 0; m_F^{\mathbf{r}'} = 0\rangle$ and $|F^{\mathbf{r}} = 1; m_F^{\mathbf{r}} = 0\rangle \otimes |F^{\mathbf{r}'} = 1; m_F^{\mathbf{r}'} = 0\rangle$ manifold.

The first protection mechanism is conveniently obtained by applying an external magnetic field B : the atomic states $\epsilon = g, e$ are affected by Zeeman shifts $\Delta_Z = g_e m_F \mu_B B$, where g_e is the corresponding Landé factor, m_F the nuclear spin quantum number encoding color (spin), and μ_B is the Bohr magneton. Due to the nonzero difference $\delta g = g_e - g_g$ between the Landé factors of the $|g\rangle$ and $|e\rangle$ states, neighboring vertices experience different Zeeman shifts. This realizes the desired general gauge-protection term introduced in Sec. III, Eq. (13), with the identification $H = \mu_B B$ and

$$\gamma_{\alpha}^{\mathbf{r}} = m_F^{\alpha} \begin{cases} g_g & \mathbf{r} \in g \text{ vertices} \\ g_e & \mathbf{r} \in e \text{ vertices,} \end{cases} \quad (17)$$

where m_F^{α} is the nuclear magnetic state corresponding to color α . For $\alpha \equiv m_F^{\alpha}$ it follows that $\gamma_{\alpha}^{\mathbf{r}} - \gamma_{\alpha'}^{\mathbf{r}'} = \pm \alpha \delta g \neq \gamma_{\beta}^{\mathbf{r}} - \gamma_{\beta'}^{\mathbf{r}'}$ unless $\alpha = \beta$ with $\delta g = g_g - g_e$, i.e., condition (14) is satisfied. Hence, when $|\delta g \mu_B B| \gg |V_{\text{ex}}|$, linear gauge protection is achieved and m_F -changing spin-exchange processes are suppressed [76] for strong enough magnetic fields. Additionally, we assume to remain far away from Feshbach resonances, for simplicity, since near such Feshbach resonances modifications of the interaction strengths have to be accounted for [83–85].

We note that the Zeeman splitting introduced here cannot protect against a class of processes where Gauss's law is violated on multiple vertices such that the contributions of the Zeeman energy sum up to zero and hence are coupled resonantly via a higher-order process. These weak higher-order processes are not captured by the minimal two vertex model

considered. However, they are suppressed by the superexchange protection mechanism, introduced in the following paragraph. Alternatively, the higher-order resonances can be efficiently suppressed by the introduction of a magnetic field gradient [86,87].

As explained in Sec. III, the linear Zeeman term cannot protect from oscillations between the $|F^{\mathbf{r}} = 0; m_F^{\mathbf{r}} = 0\rangle \otimes |F^{\mathbf{r}'} = 0; m_F^{\mathbf{r}'} = 0\rangle$ and $|F^{\mathbf{r}} = 1; m_F^{\mathbf{r}} = 0\rangle \otimes |F^{\mathbf{r}'} = 1; m_F^{\mathbf{r}'} = 0\rangle$ states, and we therefore need an additional strategy. To understand the origin of this problem, we consider the minimal model consisting of two vertices and explicitly derive how the effective matter tunneling from Eq. (11) introduces perturbative gauge-breaking exchange interactions of type (2) proportional to V_{ex} . Concretely, we consider the configurations with one singlet on each vertex and those with one triplet on each vertex and calculate the effective couplings between these states.

We find that the low-energy effective dynamics of the minimal model in the zero total magnetization sector ($m_F^{\text{tot}} = 0$) can be described through a four-dimensional Hilbert space spanned by the basis states [see Fig. 4(c)]

$$\{|s_g\rangle, |s_e\rangle, |t_g\rangle, |t_e\rangle\}, \quad (18)$$

where the subscript g (e) indicates that the matter site in the vertex g (e) is occupied and where $|s\rangle$ are the singlets states $|F^{\mathbf{r}} = 0; m_F^{\mathbf{r}} = 0\rangle \otimes |F^{\mathbf{r}'} = 0; m_F^{\mathbf{r}'} = 0\rangle$, while $|t\rangle$ are the triplets states $|F^{\mathbf{r}} = 1; m_F^{\mathbf{r}} = 0\rangle \otimes |F^{\mathbf{r}'} = 1; m_F^{\mathbf{r}'} = 0\rangle$. Using the computational basis (18) and assuming that the microscopic hopping $t = t^g = t^e$ and the chemical potential $\delta = \delta^g = \delta^e$ have the same values for g and e vertices, we can describe the system through the Hamiltonian

$$\hat{H}_{\text{eff}} = \begin{pmatrix} \delta + \delta_{\text{eff}} & t_{\text{eff}} & \lambda & \lambda \\ t_{\text{eff}} & \delta + \delta_{\text{eff}} & \lambda & \lambda \\ \lambda & \lambda & \delta + \delta_{\text{eff}} & t_{\text{eff}} \\ \lambda & \lambda & t_{\text{eff}} & \delta + \delta_{\text{eff}} \end{pmatrix}. \quad (19)$$

From perturbation theory calculations (see more details in the Appendix B), we obtain the corrections to the chemical potential of the matter site $\delta_{\text{eff}} = O(t^2/\delta)$, the effective hopping within the singlets and triplets sectors t_{eff} similar to Eq. (11), and the coupling coefficient λ between the singlets and the triplets

$$\lambda = -\frac{t^2}{2(\delta - V)} \left(\frac{V_{\text{ex}}}{\delta - V + V_{\text{ex}}} \right); \quad (20)$$

see Table II in the appendixes for full expressions of t_{eff} and δ_{eff} .

In the limit $|t| \ll \delta \ll V$, the coupling between singlets and triplets is much weaker than the gauge-invariant hopping, $|\lambda| \ll t_{\text{eff}}$, and appears as a perturbation to the effective gauge-invariant model. This is because in this limit $\lambda \sim t^2/V$ requires virtual two-rishon states whereas $t_{\text{eff}} \sim t^2/\delta$ includes a contribution from a virtual zero-rishon state. In any case, λ leads to a resonant coupling between the triplet and singlet sectors.

Up to this point, the analysis did not include the contribution of superexchange processes within each vertex, which can introduce a splitting between singlet and triplet states and thus serve as a gauge-protection mechanism. Moreover,

we assumed the values of δ and t to be the same on the g and e vertices. In the following, we distinguish between the on-site Hubbard interactions U_{gg} (U_{ee}) on the rishon and matter sites of g (e) vertices, which yield intravertex superexchange interactions but cannot be tuned independently in a realistic experimental setting. In our scheme, we propose to use staggered potentials on the matter sites of the g and e vertices, i.e., $\delta^g \neq \delta^e$, such that the different contributions of the superexchange interaction are balanced and we obtain resonant gauge-protected dynamics.

In Fig. 4(d) we illustrate the intravertex superexchange processes in the g vertex. We distinguish the processes in the g and e vertices by introducing a state-dependent chemical potential δ^ϵ ($\epsilon = g, e$) on the matter site. The effective Hamiltonian describing the resulting superexchange [88,89] is given by

$$\hat{H}_J = \sum_{\mathbf{r}, \mathbf{v}=\pm\mathbf{k}} \frac{J_\epsilon}{2} \sum_{\alpha \neq \beta} (\hat{c}_{\mathbf{r},\mathbf{v}}^{\alpha\dagger} \hat{c}_{\mathbf{r},\mathbf{v}}^\beta \hat{\psi}_{\mathbf{r}}^{\beta\dagger} \hat{\psi}_{\mathbf{r}}^\alpha - \hat{n}_{\mathbf{r},m}^\alpha \hat{n}_{\mathbf{r},\mathbf{v}}^\beta), \quad (21)$$

where $\hat{c}_{\mathbf{r},\mathbf{v}}^\alpha$ ($\hat{\psi}_{\mathbf{r}}^\alpha$) is the fermionic operator of the rishon (matter) site and

$$J_\epsilon = \left(\frac{2t^2}{U_{\epsilon\epsilon} + \delta^\epsilon} + \frac{2t^2}{U_{\epsilon\epsilon} - \delta^\epsilon} \right). \quad (22)$$

This effective interaction can also be reformulated in terms of spin operators yielding the standard superexchange Hamiltonian

$$\hat{H}_J = \sum_{\mathbf{r}, \mathbf{v}=\pm\mathbf{k}} J_\epsilon \left(\hat{\mathbf{F}}_{\mathbf{r},m} \cdot \hat{\mathbf{F}}_{\mathbf{r},\mathbf{v}} - \frac{1}{4} \hat{n}_{\mathbf{r},m} \hat{n}_{\mathbf{r},\mathbf{v}} \right). \quad (23)$$

This interaction introduces an energy shift between the triplet and the singlet manifold, lowering singlet energy by $-J_\epsilon$. If $U_{gg} = U_{ee}$, these energy contributions would immediately allow us to separate the singlets and triplets into two

TABLE I. Scattering lengths for ^{173}Yb in units of the Bohr radius a_0 [90]. The Hubbard interaction strength $U_X \propto a_X$ depends on the two-particle scattering length with $X = gg, ee, eg^+, eg^-$. The absolute strength is determined by the Wannier function overlap between g and e sites. In this paper we define $V = (U_{eg}^+ + U_{eg}^-)/2$ and $V_{\text{ex}} = (U_{eg}^+ - U_{eg}^-)/2$.

a_{gg}	a_{ee}	a_{eg}^-	a_{eg}^+
199.4 a_0	306.2 a_0	219.7 a_0	1800 a_0

well-defined energy sectors, as required for gauge protection, but typically the atomic scattering interactions are different for g and e states; see Table I. For example, in the fermionic isotope ^{173}Yb , the superexchange energy scale is smaller for e atoms than for g atoms.

To mitigate the difference due to the unequal scattering lengths on g and e vertices, we propose to tune the chemical potentials on the matter sites of the two vertices δ^g and δ^e independently in such a way that the overall energy splitting between the two singlet configurations $|s_g\rangle$ and $|s_e\rangle$ vanishes, $\Delta^s = 0$. Moreover, their values will be chosen such that a controlled energy separation $|\Delta^{s-t}| \gg \lambda$ from the gauge-breaking triplet states $|t_g\rangle$ and $|t_e\rangle$ is obtained. To this end, we repeat the analysis of the effective Hamiltonian in the basis (18) for $\delta^g \neq \delta^e$ and we obtain a modified version of Eq. (19), where the coefficients now depend on $\delta^g - \delta^e$.

We write the local chemical potential δ^ϵ in the microscopic Hamiltonian (16) as

$$\delta^\epsilon = \delta + \tilde{\delta}^\epsilon, \quad \tilde{\delta}^g = -\tilde{\delta}^e, \quad \delta = \frac{1}{2}(\delta^e + \delta^g), \quad (24)$$

where the small difference $\delta^e - \delta^g = \tilde{\delta}^e - \tilde{\delta}^g$ is expressed in terms of the small state-dependent detuning $|\tilde{\delta}^\epsilon| \ll |\delta|$; note that $|\delta| \gg t$ in order to conserve the correct rishon number per link; see Sec. III. The effective Hamiltonian (19) then takes the form

$$\hat{H}_{\text{eff}} = \begin{pmatrix} \delta^g + \delta_{\text{eff}}^g - J_g & t_{\text{eff}} & \lambda_{gg} & \lambda_{ge} \\ t_{\text{eff}} & \delta^e + \delta_{\text{eff}}^e - J_e & \lambda_{ge} & \lambda_{ee} \\ \lambda_{gg} & \lambda_{ge} & \delta^g + \delta_{\text{eff}}^g & t_{\text{eff}} \\ \lambda_{ge} & \lambda_{ee} & t_{\text{eff}} & \delta^e + \delta_{\text{eff}}^e \end{pmatrix}, \quad (25)$$

where $|\delta_{\text{eff}}^\epsilon(t^\epsilon, \delta^\epsilon)| \ll \delta$ is a perturbative, dispersive energy shift $O(t^2/\delta)$ and $t_{\text{eff}} = t_{\text{eff}}(t^\epsilon, \delta^\epsilon)$ as well as $J_{\epsilon'} = J_{\epsilon'}(t^\epsilon, \delta^\epsilon)$ and the off-diagonal couplings $O(\lambda)$ depend on all microscopic parameters; explicit expressions are provided in Appendix B.

The energy splitting between the two singlet states in Eq. (25) is given by $\Delta^s = (\delta^g + \delta_{\text{eff}}^g - J_g) - (\delta^e + \delta_{\text{eff}}^e - J_e)$. In order to obtain resonant, gauge-invariant dynamics we now determine δ^g and δ^e such that $\Delta^s = 0$. To this end, we set $t = t^e = t^g$, fix $\delta^e = 10t$, and find suitable parameters δ^g and V such that $\Delta^s = 0$; see Fig. 5(a). Here we keep $V_{\text{ex}}/V = 790/1010$, $U_{gg}/V = 199/1010$ and $U_{ee}/V = 306/1010$ constant, at values corresponding to ^{173}Yb ; see Table I. Moreover, we assume the g and e optical potentials to be

similar, in order to have equal g - e Wannier orbitals and avoid their explicit calculation. For the points of (δ^g, V) for which $\Delta^s = 0$, we then compute $|\Delta^{s-t}/\lambda_{\epsilon, \epsilon'}|$ for all combinations $\epsilon, \epsilon' = g, e$ to estimate the efficiency of the superexchange gauge protection. Our results are shown in Fig. 5(b) where we observe that the atomic interactions in ^{173}Yb naturally place the model in a gauge protected regime. The value of $\min |\Delta^{s-t}/\lambda_{\epsilon, \epsilon'}|$, minimized over all combinations $\epsilon, \epsilon' = g, e$, increases for growing values of V/t . The dashed line shows $\min |\Delta^{s-t}/\lambda_{\epsilon, \epsilon'}| = 6.6$ obtained for the reference values $V/t = 505$ and $\delta^e/t = 10$, which we use for the following time evolution; see Figs. 5(c) and 5(d).

To validate our proposed gauge protection mechanism in a more realistic setting, we perform exact time evolution

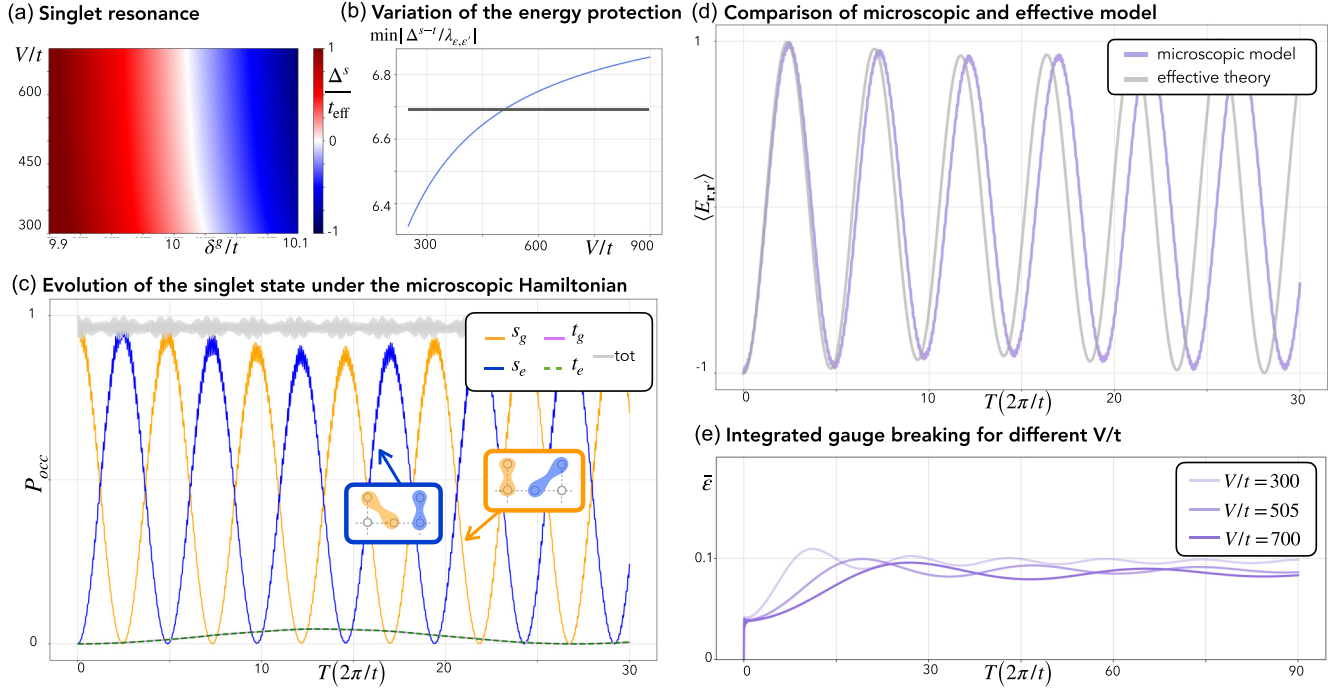


FIG. 5. Analysis of the superexchange-induced protection and time evolution of the full microscopic model. (a) The plot shows that for a fixed choice of $\delta^e = 10t$ and varying values of V/t (but we fix V_{ex}/V , U_{gg}/V and U_{ee}/V) we obtain the resonance of the singlets states, $\Delta^s = 0$, for $|\delta^s/t| \approx 10$. (b) We show the ratio between the protection energy Δ^{s-t} (i.e., the minimal energy difference between the triplets and singlets sector) and the singlet-triplet coupling coefficients $\lambda_{e,e'}$ for all combinations $e, e' = g, e$ varying values of V/t (and fixed V_{ex}/V , U_{gg}/V and U_{ee}/V). The gray thick line corresponds to $V/t = 505$ that we use in the following analysis. (c) We apply exact diagonalization to obtain the dynamics of an initial gauge-invariant state $|s_g\rangle$ (inset orange box) under the full microscopic model, i.e., including the gauge-breaking exchange interactions and the gauge protection strategies. The orange (blue) line denotes the population of the gauge-invariant state $|s_g\rangle$ ($|s_e\rangle$), whereas the magenta (green) curve shows the population of the gauge-violating triplet state $|t_g\rangle$ ($|t_e\rangle$). The gray curve shows the total occupation in the subspace spanned by $|s_g\rangle$, $|s_e\rangle$, $|t_g\rangle$, and $|t_e\rangle$; the remaining population is constituted by, e.g., high energy states with doublons. (d) We compare the electric field $\langle \hat{E}_{r,r'} \rangle$ in the microscopic (purple line) and effective (gray line) model, showing the emergence of an effective dynamics. (e) We calculate the integrated gauge violation $\bar{\epsilon}$ on long times for three values of V/t and show that the gauge protection slightly improves when increasing it.

of the minimal toy model described in Fig. 4(c) using the microscopic Hamiltonian (16). We consider a system of two neighboring vertices, we set $t = t^e = t^s$ and $\delta^e = 10t$. We use $V_{\text{ex}}/V = 790/1010$, $U_{\text{gg}}/V = 199/1010$ and $U_{\text{ee}}/V = 306/1010$ to be proportional to the scattering lengths in Table I and choose the value of δ^s determined perturbatively above to yield $\Delta^s = 0$. The microscopic model is based on the Hamiltonian (16) and additionally includes the Zeeman splitting, Eqs. (14) and (17). We use $\mu_B B \delta g = 3V$ to guarantee sufficient protection. For instance, by choosing a value of $V \sim 10$ kHz, one obtains an estimated magnetic field strength of approximately $B = 110$ G. To ensure the preservation of gauge invariance in the simulation, we could chose to operate in a regime where $t \sim V/300$, which would corresponds to $t \sim 30$ Hz. While larger tunneling rates (e.g., $t \sim 100$ Hz) are in principle accessible, such values would clearly reduce the degree of protection against gauge-violating processes, therefore the choice of parameters must take in consideration both arguments. Additional experimental considerations arise due to the presence of Feshbach resonances, which occur at specific critical values of the magnetic field. In proximity to these resonances, the scattering length is significantly modified, whereas operating sufficiently far from the resonance leads only to a modest renormalization. For the isotope ^{173}Yb , the

relevant Feshbach resonance lies around $B = 50$ G [84,85], while for ^{171}Yb the resonance is found at a much higher field, around $B = 1300$ G [83]. These estimates also highlight a more general point: the robustness of the protected subspace under this mechanism should be understood as a necessary theoretical ingredient for scaling to larger systems, rather than as a sufficient demonstration of experimental scalability. In practice, the latter also depends on coherence times, control precision, and on maintaining the relevant energy protections in more extended systems. More generally, improving the protection requires reducing the bare tunneling relative to the interaction and detuning scales, which in turn slows down the effective dynamics. The optimal parameter regime is therefore set by a compromise between the suppression of gauge-violating processes and the accessibility of the corresponding dynamical timescales. While the corresponding effective timescales may reach the 100 ms to 1 s range and thus make long-time dynamics increasingly demanding, they are not outside the range in which experimentally relevant signatures of non-Abelian gauge dynamics may be accessed.

The system is initialized in the state $|s_g\rangle$ [see Fig. 5(c) (inset)], and we calculate the occupation of the low-energy states $\{|s_g\rangle, |s_e\rangle, |t_g\rangle, |t_e\rangle\}$ from the full time-evolved state. Our results are shown in Fig. 5(c). We observe that within this

subspace, the singlet and triplet manifolds are well decoupled due to the introduction of the superexchange gauge protection. Hence, our simulation confirms that the system remains in a controlled regime with limited gauge breaking, consistent with the perturbative calculations.

We further compare the dynamics obtained from the microscopic time evolution to our effective model (25) and find excellent agreement; see Fig. 5(d). For the parameters used in this simulation, the resulting characteristic timescale of the effective model is on the order of five tunneling times $2\pi/t$, realistic for current optical lattice experiments. We emphasize that the simulation includes the intrinsic gauge-breaking errors due to direct Hubbard exchange interactions V_{ex} . Therefore, our proposed gauge protection mechanism using linear Zeeman shifts and superexchange interactions allows us to still obtain an effective gauge-invariant $U(2)$ LGT with fermionic matter, as we demonstrate numerically and analytically.

To further study the efficiency of our gauge protection mechanism, we tune the parameter V/t . Thus, we directly control the strength of gauge-breaking terms by increasing V , and consequently also the other interaction parameters, with respect to t and δ^e that remain unchanged (δ^g is updated to maintain the singlet resonance as explained above). In Fig. 5(e) we show the long-time dynamics of the integrated gauge violation $\bar{\varepsilon}(T) = \frac{1}{T} \int \varepsilon(\tilde{T}) d\tilde{T}$ [75], where $\varepsilon(T)$ is the instantaneous gauge violation at time T defined as the probability of occupation of gauge-breaking states normalized by its infinite-temperature value. Our results confirm that the gauge protection can be controlled by V/t , although $\bar{\varepsilon}$ does not change much for the chosen parameters when V is doubled, in agreement with the expectations of Fig. 5(b). Additionally, Fig. 5(e) shows that when the ratio V/t is lowered, the gauge protection persists. In Fig. 8 in Appendix B we show an example of dynamics without the superexchange protection mechanism. The results obtained in this analysis on a single building block confirm that the intrinsic gauge protection mechanism is a promising starting point to achieve gauge-invariant dynamics in large-scale quantum simulations. Our scheme can be naturally implemented in an AELA setup since it relies on an external magnetic field, staggered chemical potentials, and on-site Hubbard interactions protecting it against both intrinsic and external gauge-breaking terms.

B. Limitations

The scheme that we proposed offers promising results for the simulation of non-Abelian $U(2)$ LGTs, but it is important to recognize few inherent limitations that may impact its broader applicability and accuracy. These limitations arise mainly from the occurrence of atomic losses: while this phenomenon can actually appear during any two-body interaction, it is more pronounced for the e - e interactions in the e optical lattice. Atomic losses can clearly influence the stability of the scheme and impact the gauge invariance of the simulations, and generally affect the precision and scalability of the model in scenarios with larger fillings and higher atomic occupancies. For this reason, we propose to limit the particle occupation in the e lattice to a maximum of one singlet (therefore two atoms) on each e vertex. In Appendix C we check

the occurrence of losses due to the e - e and e - g interactions on experimentally relevant timescales and find that this is acceptable for the range of parameters that we choose for our simulations. Clearly, the constraint on the number of particles in the e vertices generates a limitation for the implementation of more complex and interesting theories: to access a wider range of simulations, we therefore propose a second scheme, in the next section, which is based on Rydberg interactions and can ideally sustain $U(N)$ theories with higher rishon-number occupation.

VI. EXPERIMENTAL IMPLEMENTATION 2: RYDBERG PLATFORM

Next, we focus on models with more than one rishon per link $\mathcal{N} > 1$, i.e., non-Abelian $U(N)$ LGTs with a richer and more complex representation of the gauge group. When referring to the scheme as digital-analog hybrid implementation, we mean a combination of analog time evolution with a digitized interaction gate: we propose to use a combination of a two-dimensional square lattice and optical tweezer arrays to engineer a periodically driven Floquet scheme [91–94], designed to yield the desired gauge-invariant dynamics for AELAs. Specifically, the tweezers are employed to “cookie cut” some of the sites in the lattice, in order to avoid the hopping of particles within neighboring vertices; see Figs. 6(a) and 6(b) [95,96].

In order to obtain a fully scalable scheme, in terms of color index N and number of rishons per link $\mathcal{N}$, we pursue a different strategy from before. Our goal is to implement a generalization of the idealized $SU(N)$ invariant Hamiltonian, Eqs. (7) and (8), fine-tuned as much as possible to be $SU(N)$ -invariant. To this end, we propose to realize a generalized gauge-invariant multi-rishon interaction $\propto V$ using a Rydberg scheme that completely avoids gauge-symmetry breaking spin-exchange interactions V_{ex} . Indeed, this second scheme works only with AELAs in their electronic ground state $|g\rangle$ with spatially nonoverlapping rishon sites on each link, which furthermore completely eliminates the problem of e - e losses, contributing to its full scalability. Since deviations from the perfect $SU(N)$ invariance of the new interactions $\propto V$ cannot be ruled out completely, we assume that residual g - g superexchange interactions similar to those described in the previous scheme will serve as a gauge-protection mechanism. The main goal of this section is to introduce an alternative scheme for implementing non-Abelian QLMs. This section is intended as an order-of-magnitude feasibility outlook for a Rydberg-based implementation, rather than as a fully benchmarked experimental proposal. A quantitative assessment of the performance in a full-scale experiment and of platform-specific effects, including imperfect blockade, differential trapping, and spin-dependent interactions, lies beyond the scope of the present work and is left for future investigation.

A. $SU(N)$ -invariant Dicke interaction

To obtain the interaction and the local addressability between neighboring vertices [see Eq. (7)] necessary to achieve the gauge-invariant dynamics (see Sec. III), we propose to use a sequence of fast excitations to the Rydberg states of the

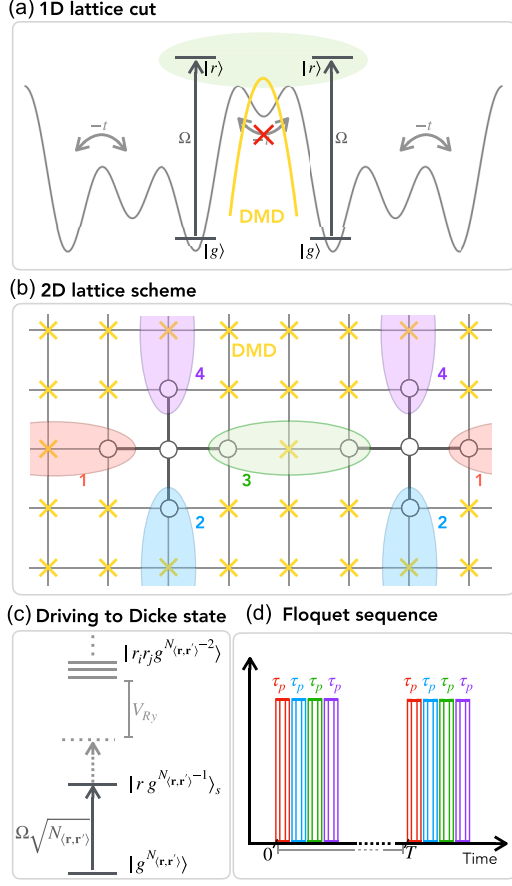


FIG. 6. Digital-analog hybrid Rydberg scheme. (a) A one-dimensional cut of the lattice setup: atoms in the rishon sites of a link are excited to the Rydberg states with Rabi frequency Ω , they interact in a Rydberg-blockade regime, and are subsequently deexcited, while the hopping between different vertices is eliminated through optical potentials, e.g., using DMDs [96]. (b) In the two-dimensional lattice scheme we identify with different colors the four links of each vertex: we propose to excite the rishon sites of these links through four separated pulses to ensure that there are no interactions between rishon excitations of different links. (c) A set of $N_{(\mathbf{r}, \mathbf{r}')}$ atoms in the blockade regime undergoes a collective Rabi oscillation with frequency $\Omega\sqrt{N_{(\mathbf{r}, \mathbf{r}')}}$, leading to a coupling between the ground state $|g^{N_{(\mathbf{r}, \mathbf{r}')}}\rangle$ and the symmetric Dicke state $|r g^{N_{(\mathbf{r}, \mathbf{r}')-1}}\rangle_s$. Our scheme utilizes the corresponding collective AC-Stark shifts $\propto \Omega\sqrt{N_{(\mathbf{r}, \mathbf{r}')}}$. (d) We show the Floquet sequence of period T used in the scheme: initially, the four different pulses of duration τ_p act separately on the different links, as in (b), then the system evolves under its free Hamiltonian for the rest of the period T .

atoms in neighboring rishon sites. The goal of this scheme is to achieve an interaction term in the Hamiltonian that takes the role of V in Eq. (7) and depends on the total rishon number $\hat{N}_{(\mathbf{r}, \mathbf{r}')}$ on a link $(\mathbf{r}, \mathbf{r}')$ in a nonlinear way:

$$\hat{H}_\Omega = \Omega \sum_{(\mathbf{r}, \mathbf{r}')} f(\hat{N}_{(\mathbf{r}, \mathbf{r}')}). \quad (26)$$

This term, proportional to Ω , plays a fundamental role for the scheme, serving two goals: it enforces the rishon-number conservation since it contributes an energy that is nonlinear in

$\hat{N}_{(\mathbf{r}, \mathbf{r}')}$ and therefore implements the constraint $\hat{N}_{(\mathbf{r}, \mathbf{r}')} = \mathcal{N}$ on all links separately for large enough Ω . Moreover, this allows one to obtain effective perturbative dynamics induced by a microscopic intravertex hopping t , as explained in Sec. III; nonlinearity is required to obtain a nonzero perturbative hopping coefficient.

The Hamiltonian that we aim to implement in the scheme therefore comprises the following terms:

$$\hat{H} = \hat{H}_t + \hat{H}_\delta + \hat{H}_\Omega, \quad (27)$$

with $\hat{H}_t$ defined as in Eq. (8), $\hat{H}_\Omega$ as in Eq. (26) and $\hat{H}_\delta = \delta \sum_\alpha \sum_{\mathbf{r}} \hat{n}_{\mathbf{r}, m}^\alpha$; in the following analysis we will assume $U_{gg} = 0$ for simplicity. This ideal Hamiltonian respects the gauge invariance requirements: indeed, the intravertex hopping $\hat{H}_t$ and the potential staggering $\hat{H}_\delta$ do not affect the gauge symmetry, while $\hat{H}_\Omega$ exclusively depends on the total number of rishon excitations of a link and not on the color of the particles, allowing one to couple neighboring vertices while preserving the $SU(N)$ invariance on both vertices independently.

The main challenge in realising the interaction $f(\hat{N}_{(\mathbf{r}, \mathbf{r}')})$ in a cold-atom platform. We start considering a simple isolated link, and we assume the distance between the rishons of such a link to be smaller than the Rydberg blockade radius: the general idea is to resonantly couple the ground state of the atomic system to the symmetric Dicke state with exactly one Rydberg excitation shared by all the atoms of the link. In particular, we consider a system of $N_{(\mathbf{r}, \mathbf{r}')}$ atoms in the two rishon sites of a link, whose ground states $|g\rangle_i$ are resonantly coupled to Rydberg states $|r\rangle_i$ with Rabi frequency Ω_i , mediating van der Waals interactions, where $i = 1, \dots, N_{(\mathbf{r}, \mathbf{r}')}$ labels the atoms. The strong Rydberg blockade (due to interactions $\propto C_6$) between two excited atoms prevents the simultaneous excitation of multiple particles to the states $|r_i r_j g^{N_{(\mathbf{r}, \mathbf{r}')-2}}\rangle$, $\forall i \neq j$ [see Fig. 6(c)]. If we assume $\Omega_i \equiv \Omega \forall i$, the system oscillates between the ground state $|g^{N_{(\mathbf{r}, \mathbf{r}')}}\rangle$ and the resonant symmetric Dicke state $|r g^{N_{(\mathbf{r}, \mathbf{r}')-1}}\rangle_s \equiv \frac{1}{\sqrt{N_{(\mathbf{r}, \mathbf{r}')}}} \sum_i |r_i g^{N_{(\mathbf{r}, \mathbf{r}')-1}}\rangle$ in an effective collective Rabi oscillation with frequency $\Omega^* = \Omega\sqrt{N_{(\mathbf{r}, \mathbf{r}')}}$ [97,98]. This reduced system can be diagonalized in the basis $\{|g^{N_{(\mathbf{r}, \mathbf{r}')}}\rangle, |r g^{N_{(\mathbf{r}, \mathbf{r}')-1}}\rangle_s\}$, and the instant eigenvalues are $E_\pm = \pm \Omega\sqrt{N_{(\mathbf{r}, \mathbf{r}')}}$ with relative eigenvectors $|\varphi_\pm\rangle = \frac{1}{\sqrt{2}}(|g^{N_{(\mathbf{r}, \mathbf{r}')}}\rangle \pm |r g^{N_{(\mathbf{r}, \mathbf{r}')-1}}\rangle_s)$. In our scheme, we want to exploit the nonlinear dependence of these eigenenergies on $N_{(\mathbf{r}, \mathbf{r}'})$ to implement the interaction $f(\hat{N}_{(\mathbf{r}, \mathbf{r}')})$.

To maintain gauge invariance, spin-dependent interactions between vertices have to be suppressed; see Sec. III. While the amplitude of scattering interactions between AELAs in the electronic ground state are independent of the nuclear spin, this is in general not true for the pairwise interactions between atoms excited to the Rydberg states [99]. This implies that the coefficients C_6^{ij} of the van der Waals interactions arising for the states $|r_i r_j g^{N_{(\mathbf{r}, \mathbf{r}')-2}}\rangle$ depend on the spins of the excited atoms (i.e., if the atom of index i has spin α and the atom j has spin β , then $C_6^{ij} \equiv C_6^{\alpha\beta}$). Moreover, atoms in different spin states, once excited to their Rydberg states, are subjected to different hyperfine energy splitting [99,100] and the corresponding Rabi frequencies also depend on the spin, i.e., $\Omega_i \equiv \Omega_\alpha$ for the atom of index i with spin α . To ensure that the collective Rabi dynamics takes place, and to achieve the

desired spin-independent interactions, our scheme is designed to bypass these issues: the Rydberg atoms on the links interact in a blockade regime, which requires that all $C_6^{\alpha\beta}$ are sufficiently large, such that their precise values become irrelevant for the effective interaction. Moreover, Rydberg states must be excited with a beam of multicolor lasers, which can be correctly tuned in order to achieve the same Rabi frequency for all spins, $\Omega_\alpha \equiv \Omega$, $\forall \alpha$ [99,101,102] (see more details in Appendix D).

B. Floquet sequence

To include the coherent dynamics within the vertices and to ensure that the collective Rabi oscillations are correctly applied on each link, we propose to utilize the following Floquet sequence: (1) On each link, Rydberg states are quickly excited, held briefly, and deexcited for a total time τ_p . We assume large enough Rabi frequencies, on a MHz scale, such that τ_p can be on a μs scale—much shorter than a typical ms scale of a tunneling time in the optical lattice. (2) In order to avoid interactions between Rydberg atoms in different links, we distribute the entire square lattice into four inequivalent types of links, as shown in Fig. 6(b). This is done in such a way that a link of a given type only has links on other types of neighbours. Individual local Rydberg excitation and deexcitation pulses are then applied sequentially for each type of link. (3) After these four pulses, the system undergoes free dynamics described by the Hamiltonian

$$\hat{H}_{\text{free}} = -t \sum_{\alpha} \sum_{\mathbf{r}, \mathbf{k}} \sum_{\mathbf{v}=\pm\mathbf{k}} (\hat{c}_{\mathbf{r},\mathbf{v}}^{\alpha\dagger} \hat{\psi}_{\mathbf{r}}^{\alpha} + \text{H.c.}) + \delta \sum_{\alpha} \sum_{\mathbf{r}} \hat{n}_{\mathbf{r},m}^{\alpha}, \quad (28)$$

and the entire cycle is repeated after a determined Floquet period $T \gg \tau_p$; see Fig. 6(d). This strategy addresses two main problems that would otherwise arise: firstly, the short duration of the excitation and the following deexcitation from the Rydberg level allow ones to reduce the probability of significant losses in the Rydberg-excited states since the experimental parameters and τ_p can be fine-tuned accordingly to achieve this; secondly, the application of four separated pulses allows one to exclude the interaction between rishon sites that do not belong to the same link and further protects against losses, as the Rydberg atoms are significantly distant from each other.

Each link $\langle \mathbf{r}, \mathbf{r}' \rangle$ can now be described as the ideal system introduced in the previous paragraph. The pulse applied on the link must guarantee a controlled evolution of the system. Therefore, we need to implement a fast but adiabatic process that allows the evolution from the ground state $|g^{N_{(\mathbf{r},\mathbf{r}')}}\rangle$ to its antisymmetric superposition $|\varphi_{-}\rangle$ with the Dicke state, which contributes with an energy equivalent to $-\Omega\sqrt{N_{(\mathbf{r},\mathbf{r}')}}$. Here we do not include a deep analysis of the adiabatic evolution but assume that the dynamical phase can be neglected or controlled through a spin echo sequence [103–106]: we therefore approximate the time evolution operator of the pulse on the link $\langle \mathbf{r}, \mathbf{r}' \rangle$ with $\hat{U}_{(\mathbf{r},\mathbf{r}')} = e^{i\Omega\sqrt{N_{(\mathbf{r},\mathbf{r}')}}\tau_p} |g^{N_{(\mathbf{r},\mathbf{r}')}}\rangle \langle g^{N_{(\mathbf{r},\mathbf{r}')}}|$; see more details in Appendix E.

We can define the total period of the Floquet sequence as $T = N_p \tau_p$, with $N_p \gg 1$ and the time-evolution operator

over the Floquet period as $\hat{U}(T) = \hat{U}_{\text{free}} \hat{U}_4 \hat{U}_3 \hat{U}_2 \hat{U}_1$, where $\hat{U}_{\text{free}} = e^{-i\hat{H}_{\text{free}}(N_p-4)\tau_p} \approx e^{-i\hat{H}_{\text{free}}T}$ [104,107,108]. In the limit of $\tau_p \ll T \ll 1/t$, with t the hopping coefficient of the free Hamiltonian, we use the Trotter decomposition method to obtain the effective Hamiltonian of the periodically driven system $\hat{U}(T) \approx e^{-i\hat{H}_{\text{Floq}}T}$ up to errors $O(\tau_p^2)$, obtaining

$$\begin{aligned} \hat{H}_{\text{Floq}} \approx & -t \sum_{\alpha} \sum_{\mathbf{r}, \mathbf{k}} \sum_{\mathbf{v}=\pm\mathbf{k}} (\hat{c}_{\mathbf{r},\mathbf{v}}^{\alpha\dagger} \hat{\psi}_{\mathbf{r}}^{\alpha} + \text{H.c.}) \\ & + \delta \sum_{\alpha} \sum_{\mathbf{r}} \hat{n}_{\mathbf{r},m}^{\alpha} - \frac{\Omega}{N_p} \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} \sqrt{\hat{N}_{(\mathbf{r},\mathbf{r}')}}, \end{aligned} \quad (29)$$

where $\hat{N}_{(\mathbf{r},\mathbf{r}')} = \sum_{\alpha\beta} \hat{n}_{\mathbf{r},\mathbf{k}}^{\alpha} + \hat{n}_{\mathbf{r}',-\mathbf{k}}^{\beta}$ is the total number of atoms on the rishon sites of the link $\langle \mathbf{r}, \mathbf{r}' \rangle$. This effective Hamiltonian has the same structure as the ideal one of Eq. (27) and therefore can be used to implement different types of non-Abelian dynamics. The emerging term, $\hat{H}_{\Omega}$, plays the same role as the Hubbard (U) and extended Hubbard (V) interactions in Eq. (7) with the difference that it is proportional to the square root of the number of rishons. Despite the different form, this term still fulfils the general requirement of nonlinear proportionality to $N_{(\mathbf{r},\mathbf{r}')}$, necessary to enforce the correct rishon number $\mathcal{N}$ constraint, as explained in the previous paragraph.

C. Effective theory

From Eq. (29) we can determine the effective hopping of the gauge-invariant perturbative dynamics for a fixed rishon number $\mathcal{N}$,

$$\begin{aligned} t_{\text{eff}} = & \frac{t^2}{\delta + (\sqrt{\mathcal{N}} - \sqrt{\mathcal{N}+1}) \frac{\Omega}{N_p}} \\ & - \frac{t^2}{\delta + (\sqrt{\mathcal{N}} - 1 - \sqrt{\mathcal{N}}) \frac{\Omega}{N_p}}. \end{aligned} \quad (30)$$

The coefficient is therefore influenced by the ratio $\frac{\Omega}{N_p\delta}$, which governs the scale of the dynamics. The effective hopping is calculated as previously done in Eq. (11), with the only difference that in the latter the rishon number was fixed to one, while here we assume it to be $\mathcal{N}$. Typically, we assume Ω to be on the order of $O(\text{MHz})$ or $O(10 \text{ MHz})$, while t is on the order of $O(\text{kHz})$: by choosing the values of N_p and δ while respecting all the necessary requirements for the correctness of the scheme, we gain flexibility in tuning the effective hopping strength. This method is particularly efficient for small $\mathcal{N}$, leading to an effective dynamics observable on the timescales of the experiment: e.g., the strength of the effective hopping can be tuned to values between $|t_{\text{eff}}| \approx |t|/10$ to $|t_{\text{eff}}| \approx |t|/100$, assuming $\mathcal{N} = 3$ or $\mathcal{N} = 4$, $N_p \approx 100$ and $\delta \approx O(\text{kHz})$ or $O(10 \text{ kHz})$.

Few comments on the effective implementation and the possible limitations of this platform are necessary. Atoms excited in their Rydberg states are inevitably affected by losses, through our Floquet scheme we aim to strongly limit the time that atoms spend in the excited states: if we assume the duration of the excitation pulses τ_p to be limited to a maximum of a fraction of a microsecond [109], and we estimate the theoretical lifetime of the atoms in the Rydberg

states to be of the order of $1 \times 10^2 \mu\text{s}$, we can expect that a sufficient number of Floquet cycles can be performed before observing experimentally relevant atomic losses. To provide a rough quantitative estimate of Rydberg-state losses, we assume that only one atom per link is resonantly excited to the Rydberg state at any given time, corresponding to two links per vertex. Approximating the excitation as a resonant pulse, the time-averaged Rydberg population per addressed atom is $\langle n_R \rangle \simeq 1/2$. We estimate an example quantity for the experimental regimes that could be implemented. Using the following set of parameters regime: $\Omega = 1 \text{ MHz}$, $t = 1 \text{ kHz}$, $\delta = 5 \text{ kHz}$, $N_p = 40$, dwell time $\tau_p = T/N_p$ and a constrained rishon number $\mathcal{N} = 3$ we obtain $t_{\text{eff}} \approx 1/4$. Considering a Floquet period $T \approx \frac{1}{50} \frac{1}{t}$, and a Rydberg lifetime of $\tau_{\text{life}} \approx 100 \mu\text{s}$, the probability of loss per vertex and per period can be estimated as $p_{\text{loss}} \approx 2 \langle n_R \rangle \frac{\tau_p}{\tau_{\text{life}}} \frac{t_p}{T} \sim 8 \times 10^{-4}$. If we consider four active vertices (that can correspond to a plaquette) evolved over time $1/t_{\text{eff}}$, the total survival probability is therefore $P_{\text{surv}} \approx (1 - p_{\text{loss}})^{4.450} \approx 0.9$, and reaches 0.8 when evolving for $2/t_{\text{eff}}$. In an actual experiment it is not fundamental to achieve a full “zero-loss” trajectory, therefore the approximate duration of the impulse and of the Floquet period could be longer.

Additionally, since generally optical traps result in different trapping potentials for the ground and Rydberg states we expect heating and higher-band excitations; we note, however, that our proposal is based on AELAs, which offer the chance to realize magic traps for Rydberg atoms [110,111], improving the reliability of the simulations.

VII. SUMMARY AND OUTLOOK

Simulating non-Abelian LGTs with dynamical matter remains a central goal of contemporary quantum simulators [16,18,19,22]. In this article we have analyzed a general approach to simulate $(2+1)\text{D}$ non-Abelian $U(N)$ LGTs using cold atoms in optical lattices. We started considering a scheme previously proposed in the literature [43], which we showed to be limited by gauge-symmetry breaking interactions. However, we demonstrated how the latter can be overcome by exploiting various protection terms in the microscopic Hamiltonian. Specifically, we have developed a comprehensive model that allows for the gauge-invariant hopping of matter while maintaining the system’s gauge symmetry. The target non-Abelian LGT emerges as an effective theory within a protected region of the Hilbert space, where different energy terms enforce the separation between physical sectors—with the correct rishon number and color charge—and unphysical ones. We discussed a minimal experimentally relevant initialization of the lattice to achieve quantum simulations with genuinely non-Abelian character.

After outlining the general framework, we have proposed two specific experimental implementations of our scheme. The first makes use of AELAs in their atomic ground and metastable clock states and is closest to existing experiments [112,113], but subject to several fundamental limitations. Among them are strong losses occurring when two metastable-excited clock state atoms occupy the same lattice site [76]. This leads to probably the most restrictive condition, namely, that two clock-state atoms cannot occupy

the same site, which motivated us to study our first scheme only for cases with one rishon per link. Likewise, for this first scheme, it is advisable to work with only one singlet per clock-state vertex in order to limit undesired losses; but as we have demonstrated, this still allows for genuinely non-Abelian simulations if more than one singlet is allowed on ground-state vertices. For similar reasons, we also restricted our detailed analysis of the AELA scheme to the $U(2)$ gauge group, although in principle larger groups $U(N)$ with $N = 3, 4, \dots$ could be experimentally realized. The closeness of the first proposed scheme to realistic experimental setups and the built-in gauge-protection scheme we describe in this article nevertheless, makes it highly promising for performing large-scale quantum simulation studies of a restricted class of non-Abelian quantum link models. Unless the strong collisional losses, which have only been measured experimentally in the recent years [76], can somehow be overcome, we conclude that the scheme originally introduced by Banerjee *et al.* [43], on which our first proposal is also based, cannot be used to scale up the realization of more complex quantum link models with tunable gauge-invariant couplings and more than one rishon per link [18]. For this reason, in the last part of our article, we devised a digital-analog hybrid scheme that implements the required gauge-invariant couplings through Rydberg interactions in the blockade regime [97]. This allows one to work only with ground-state atoms. As we have demonstrated, by fine-tuning all Rabi couplings to the Rydberg states corresponding to different colors—realized through nuclear spin states—a fully scalable scheme for simulating non-Abelian quantum link models is obtained. We expect that gauge-symmetry breaking interactions are significantly weaker in this setup than in the first AELA scheme we analyzed, but a detailed study of the atomic losses in the excited states and the general stability of the scheme is necessary for a full experimental implementation and will be devoted to future work.

Looking forward, we can conclude that our two proposed implementations serve distinct but complementary roles in the roadmap toward scalable non-Abelian quantum simulation. The first (AELA) one provides the most direct path to near-term experiments and basic demonstrations of non-Abelian dynamics, while the hybrid (Rydberg) approach offers a promising route toward more tunable couplings and more complex geometries, paving the way for a first experimental study of a large-scale non-Abelian quantum link model [69]. At the same time, the results presented here should be interpreted as a controlled analysis of a minimal local building block, whose underlying mechanisms are expected to remain relevant in more elaborate future realizations.

A next major step that will be necessary to explore the rich phase diagram of such models, in particular in the deconfined regime, is the implementation of strong plaquette terms [45,73,114]. In the schemes proposed here so far, only perturbatively small plaquette terms can be expected [72], and we believe that further extensions of the digital-analog hybrid scheme we introduced are most promising for reaching this task. On the theoretical side, only a few quantitative results exist on the detailed structure of the equilibrium phase diagrams of the resulting non-Abelian quantum link models [115]. In contrast to the field-theoretical models studied in particle

physics, the quantum link models discussed here are subject to strong lattice and discretization effects due to the finite dimension of the link Hilbert space, which requires advanced numerical methods or quantum simulation platforms to reach a full quantitative description. Finally, quantum simulators offer new perspectives, such as studies of far-from-equilibrium dynamics [26,116–118].

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G.D.P. conducted the analytical and numerical simulations, G.D.P., L.H., and F.G. conceived the conceptual ideas, with

input from M.A. on the experimental realization, and J.C.H. on the theoretical scheme. L.H. and F.G. supervised; all authors contributed to writing the manuscript.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: MAPPING TO ABELIAN MODELS

In the main text, we have introduced a scheme to identify a discrete spin variable of the links given the position of the particles in the lattice. Here we show some examples of configurations that map to an Abelian theory. The first one has only one singlet per vertex and no matter excitations, as illustrated in Fig. 7(a). Compared to the general discussion in the main text, here the central *matter* site is absent. Doublons are not permitted on any site, meaning the singlets at each vertex can be represented by an equivalent state consisting of two hard-core bosons. As a result, the atomic spin becomes irrelevant to the scheme's description. We identify the vertices initialized without any atom with a background charge $q_r^0 = 2$ in order to fulfill the Gauss's laws, $\sum_{i=x,y} (\hat{E}_{r,r+k_i} - \hat{E}_{r-k_i,r}) - \hat{\psi}_r^\dagger \hat{\psi}_r = 2$. Because of the local $U(1)$ symmetry of the system, the charge remains conserved as a static background charge. The next case, shown in Fig. 7(b), includes matter excitations in the vertices. If all vertices are initialized to have only one or zero singlets per vertex, the entire lattice configuration can effectively be translated into an Abelian theory fully characterized by the spins on the links. Another possible configuration allows two singlets on a vertex but we enforce a double occupancy on the matter site in that vertex [Fig. 7(c)]. The vertices with two singlets would be associated with a background charge $q_r^0 = -2$ as $\sum_{i=x,y} (\hat{E}_{r,r+k_i} - \hat{E}_{r-k_i,r}) - \hat{\psi}_r^\dagger \hat{\psi}_r = -2$. The last configuration, shown in Fig. 7(d), has the constraint of one or zero singlets per vertex and allows for double occupancy on the matter site. In this case, only the empty vertices will have a background charge $q_r^0 = 2$.

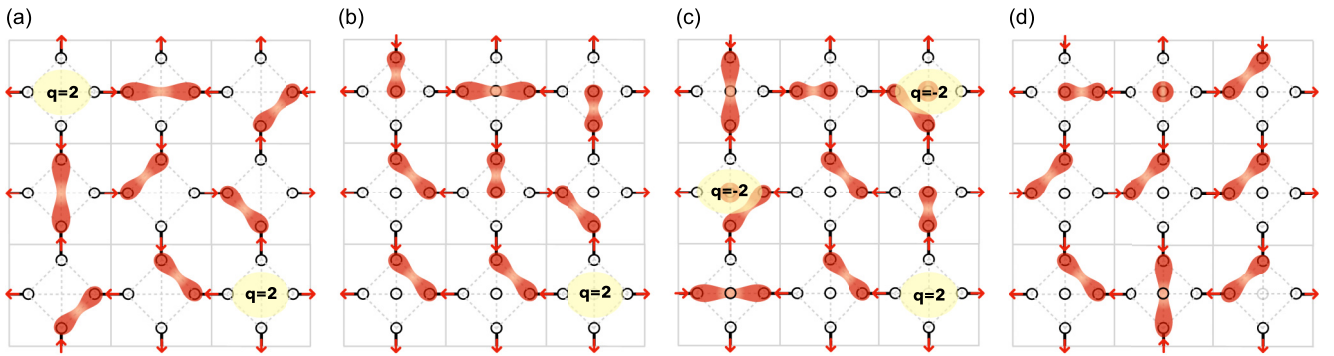
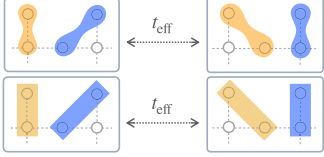
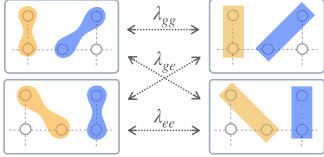
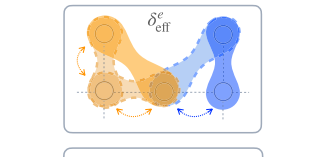
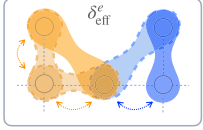
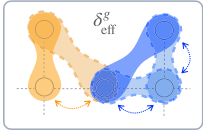


FIG. 7. Abelian configurations. Examples of simple configurations that map to an Abelian theory, i.e., configurations where the locations of the fermions as well as their spin state can be uniquely determined from the link variables (red arrows). (a) The first scheme is characterized by the absence of matter sites, each vertex has one or zero singlets that correspond to a background charge of $q = 0$ or $q = 2$, respectively. (b) In this case we allow for matter excitations, and each vertex contains one or zero singlets (background charge of $q = 0$ or $q = 2$). (c) Here each vertex can have either one or two singlets, but in the latter case one of the two singlets is constrained to the matter site and thus corresponds to a background charge of $q = -2$. (d) Here we allow one or zero singlets per vertex including matter excitations with double occupancy.

TABLE II. Effective model of the two-vertex building block. We show the terms that contribute to the effective model in Eq. (19). The coefficients of the effective singlet hopping (t_{eff}) and the coupling between singlet and triplet sectors (λ_{ge}) are calculated with corrected second-order matrix element expression in perturbation theory, since δ^g can be different from δ^e . For reference, the values here represented are those used in Eq. (25), while for Eq. (19) it is sufficient to impose $t^e = t^g = t$ and $\delta^e = \delta^g = \delta$.

Total contribution	Effective theory term
$t_{\text{eff}} = -\frac{t^g t^e}{2} \sum_{\epsilon=g,e} \left[\frac{1}{\delta^\epsilon} + \frac{1}{2} \left(\frac{1}{V-\delta^\epsilon} + \frac{1}{V-V_{\text{ex}}-\delta^\epsilon} \right) \right]$	
$\lambda_{\epsilon\epsilon} = -\frac{1}{2} \left(\frac{(t^\epsilon)^2}{\delta^\epsilon - V} - \frac{(t^\epsilon)^2}{\delta^\epsilon - V + V_{\text{ex}}} \right)$	
$\lambda_{ge} = -\frac{t^g t^e}{4} \sum_{\epsilon=g,e} \left(\frac{1}{\delta^\epsilon - V} - \frac{1}{\delta^\epsilon - V + V_{\text{ex}}} \right)$	
$\delta_{\text{eff}}^e = -2 \frac{(t^g)^2}{\delta^g} - \frac{(t^e)^2}{2} \left(\frac{1}{V-\delta^e} + \frac{1}{V-V_{\text{ex}}-\delta^e} \right)$	
$\delta_{\text{eff}}^g = -2 \frac{(t^e)^2}{\delta^e} - \frac{(t^g)^2}{2} \left(\frac{1}{V-\delta^g} + \frac{1}{V-V_{\text{ex}}-\delta^g} \right)$	

APPENDIX B: TIGHT BINDING HAMILTONIAN AND EFFECTIVE THEORY

In the main text, we introduced the tight binding discrete Hamiltonian as Eq. (16). Here we specify the individual components in detail. The single particle terms are defined as

$$\hat{H}_\epsilon = \sum_{\alpha} \sum_{\mathbf{r}_\epsilon, \mathbf{k}} \sum_{\mathbf{v}=\pm\mathbf{k}} \left[-t^\epsilon (\hat{\psi}_{\mathbf{r}_\epsilon}^{\alpha\dagger} \hat{c}_{\mathbf{r}_\epsilon, \mathbf{v}}^\alpha + \text{H.c.}) + \delta^\epsilon \hat{\psi}_{\mathbf{r}_\epsilon}^{\alpha\dagger} \hat{\psi}_{\mathbf{r}_\epsilon}^\alpha \right], \quad (\text{B1})$$

while the interaction terms read

$$\begin{aligned} \hat{H}_{\epsilon\epsilon} &= \sum_{\alpha < \beta} \sum_{\mathbf{r}_\epsilon, \mathbf{k}} \sum_{\mathbf{v}=0, \pm\mathbf{k}} U_{\epsilon\epsilon} \hat{n}_{\mathbf{r}_\epsilon, \mathbf{v}}^\alpha \hat{n}_{\mathbf{r}_\epsilon, \mathbf{v}}^\beta, \\ \hat{H}_{ge} &= \sum_{\alpha, \beta} \sum_{(\mathbf{r}_g, \mathbf{r}_e)} (V \hat{c}_{\mathbf{r}_g, \mathbf{k}}^\alpha \hat{c}_{\mathbf{r}_g, \mathbf{k}}^\beta \hat{c}_{\mathbf{r}_e, -\mathbf{k}}^{\beta\dagger} \hat{c}_{\mathbf{r}_e, -\mathbf{k}}^\alpha \\ &\quad + V_{\text{ex}} \hat{c}_{\mathbf{r}_g, -\mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r}_e, \mathbf{k}}^{\beta\dagger} \hat{c}_{\mathbf{r}_g, \mathbf{k}}^\beta \hat{c}_{\mathbf{r}_e, \mathbf{k}}^\alpha), \end{aligned} \quad (\text{B2})$$

where we use $\hat{c}_{\mathbf{r}_\epsilon, \mathbf{v}}^\alpha$ ($\hat{c}_{\mathbf{r}_\epsilon, \mathbf{v}}^{\alpha\dagger}$) as generic lattice fermionic operators that create or annihilate an atom on a site described by a Wannier function centered at position $\mathbf{r}_\epsilon$ ($\epsilon = g, e$) and with spin α ; further we denote by $\hat{n}_{\mathbf{r}_\epsilon, \mathbf{v}=0}^\alpha = \hat{\psi}_{\mathbf{r}_\epsilon}^{\alpha\dagger} \hat{\psi}_{\mathbf{r}_\epsilon}^\alpha$ the matter density on vertex $\mathbf{r}_\epsilon$. The strength of the two-body contact interactions V , V_{ex} , U_{gg} , and U_{ee} depend on the spatial shape and overlap of the Wannier functions of the lattice sites and are proportional to the interaction parameters $g_X = \frac{4\pi\hbar^2 a_X}{m}$, with m the atomic mass and a_X the four scattering lengths of the interaction channels, where $X = gg, ee, eg^+, eg^-$, and where $V = (U_{eg}^+ + U_{eg}^-)/2$ and $V_{\text{ex}} = (U_{eg}^+ - U_{eg}^-)/2$ are combination of the symmetric and antisymmetric channel strengths [42, 76].

Now we assume the first gauge-protection mechanism, i.e., the Zeeman protection, to be already present and from this microscopic Hamiltonian we can calculate the coefficients that characterize the perturbation theory, shown in Table II, where we did not include the superexchange-induced terms, as they are already shown in the main text. We limit the perturbative processes to those on the vertices included in the toy-model simulation: t_{eff} is the coupling coefficient between the two singlet (triplet) states $|s_g\rangle$ and $|s_e\rangle$ ($|t_g\rangle$ and $|t_e\rangle$) while $\lambda_{\epsilon\epsilon'}$ are the couplings between the singlet and triplet sectors and $\delta_{\text{eff}}^\epsilon$ are the corrections to the total energy δ^ϵ . Since t_{eff} and λ_{ge} couple states with different energy (at zeroth order), we used the symmetric second-order matrix element expression.

In general, we can expect $\lambda_{gg} \approx \lambda_{ee} \approx \lambda_{ge}$: the error that would come from assuming them equal is negligible for the effective dynamics. For example, if we consider $t^e = t^g = t$ and $\delta = \delta^e = 10t$, the difference between the coupling λ_{gg} and λ_{ee} can be approximated to 0 when considering $\tilde{\delta}^g \ll \delta$:

$$\begin{aligned} \lambda_{gg} - \lambda_{ee} &= -\frac{t^2}{2(\delta + \tilde{\delta}^g - V)} \left(\frac{V_{\text{ex}}}{\delta + \tilde{\delta}^g - V + V_{\text{ex}}} \right) \\ &\quad + \frac{t^2}{2(\delta - V)} \left(\frac{V_{\text{ex}}}{\delta - V + V_{\text{ex}}} \right) \\ &\approx \frac{t^2 V_{\text{ex}}}{2(\delta - V)(\delta - V + V_{\text{ex}})} \\ &\quad \times \left(\frac{\tilde{\delta}^g}{\delta - V} + \frac{\tilde{\delta}^g}{\delta - V + V_{\text{ex}}} \right) \ll \lambda_{gg}, \lambda_{ee}. \end{aligned} \quad (\text{B4})$$

A similar comparison can be done with λ_{ge} . In Sec. V we have mentioned how the spin dependence of the interactions between particles on the links introduces an oscillation between states in the singlet and triplet states with $m_F = 0$.

From the microscopic model and considering the representation of these states in the Fock's basis, we now provide a more physical intuition for the nonzero coupling λ (see Table II) between the triplet and singlet sector. Only as a reference, we first write the general form of the state with a singlet on both neighboring vertices, $|F^r = 0; m_F^r = 0\rangle \otimes |F^{r'} = 0; m_F^{r'} = 0\rangle$:

$$\begin{aligned} |s\rangle &= |s\rangle_g \otimes |s\rangle_e = \frac{|\uparrow\downarrow\rangle_g - |\downarrow\uparrow\rangle_g}{\sqrt{2}} \otimes \frac{|\uparrow\downarrow\rangle_e - |\downarrow\uparrow\rangle_e}{\sqrt{2}} \\ &= \frac{1}{2}(|\uparrow\downarrow\rangle_g |\uparrow\downarrow\rangle_e - |\uparrow\downarrow\rangle_g |\downarrow\uparrow\rangle_e \\ &\quad - |\downarrow\uparrow\rangle_g |\uparrow\downarrow\rangle_e + |\downarrow\uparrow\rangle_g |\downarrow\uparrow\rangle_e), \end{aligned} \quad (\text{B5})$$

and a state with spin triplets on both neighboring vertices, $|F^r = 1; m_F^r = 0\rangle \otimes |F^{r'} = 1; m_F^{r'} = 0\rangle$:

$$\begin{aligned} |t\rangle &= |t\rangle_g \otimes |t\rangle_e = \frac{|\uparrow\downarrow\rangle_g + |\downarrow\uparrow\rangle_g}{\sqrt{2}} \otimes \frac{|\uparrow\downarrow\rangle_e + |\downarrow\uparrow\rangle_e}{\sqrt{2}} \\ &= \frac{1}{2}(|\uparrow\downarrow\rangle_g |\uparrow\downarrow\rangle_e + |\uparrow\downarrow\rangle_g |\downarrow\uparrow\rangle_e \\ &\quad + |\downarrow\uparrow\rangle_g |\uparrow\downarrow\rangle_e + |\downarrow\uparrow\rangle_g |\downarrow\uparrow\rangle_e), \end{aligned} \quad (\text{B6})$$

where we colored the spins that interact when both occupy the rishon sites, using blue for spin with the same state $m_F = \pm 1/2$, and purple for those with opposite m_F . Indeed, the contribution of the atoms interacting in the rishon sites depends on their specific spin configuration: the term proportional to V_{ex} in Eq. (B3) yields a spin exchange between two neighboring vertices when the interaction in the rishon site happens between two atoms with opposite $m_F = \pm 1/2$ [purple in Eq. (B5) and Eq. (B6)]; in that case, the Zeeman mechanism that we introduced energetically protects from the occupation of gauge-breaking states and the exchange interaction is therefore suppressed. On the other hand, when the interaction in the rishon site happens between atoms with the same quantum number m_F [blue in Eq. (B5) and Eq. (B6)], the term proportional to V_{ex} in Eq. (B3) can be written as

$$\begin{aligned} &V_{\text{ex}} \sum_{\alpha, \beta} (\hat{c}_{\mathbf{r}_g, -\mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r}_e, \mathbf{k}}^{\beta\dagger} \hat{c}_{\mathbf{r}_g, -\mathbf{k}}^{\alpha} \hat{c}_{\mathbf{r}_e, \mathbf{k}}^{\beta}) \delta_{\beta}^{\alpha} \\ &= V_{\text{ex}} \sum_{\alpha} (\hat{c}_{\mathbf{r}_g, -\mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r}_e, \mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r}_g, -\mathbf{k}}^{\alpha} \hat{c}_{\mathbf{r}_e, \mathbf{k}}^{\alpha}) \\ &= -V_{\text{ex}} \sum_{\alpha} (\hat{c}_{\mathbf{r}_g, -\mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r}_g, -\mathbf{k}}^{\alpha} \hat{c}_{\mathbf{r}_e, \mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r}_e, \mathbf{k}}^{\alpha}) \\ &= -V_{\text{ex}} \sum_{\alpha} \hat{n}_{\mathbf{r}_g, -\mathbf{k}}^{\alpha} \hat{n}_{\mathbf{r}_e, \mathbf{k}}^{\alpha}, \end{aligned} \quad (\text{B7})$$

where we considered only the contribution on a single rishon. This leads to a reformulated version of Eq. (B3) that we can

express as follows [78,119,120]:

$$\begin{aligned} \hat{H}_{ge} &= \sum_{(\mathbf{r}_g, \mathbf{r}_e)} \left[V \sum_{\alpha \neq \beta} (\hat{c}_{\mathbf{r}_g, \mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r}_g, \mathbf{k}}^{\alpha} \hat{c}_{\mathbf{r}_e, -\mathbf{k}}^{\beta\dagger} \hat{c}_{\mathbf{r}_e, -\mathbf{k}}^{\beta}) \right. \\ &\quad + (V - V_{\text{ex}}) \sum_{\alpha} (\hat{c}_{\mathbf{r}_g, \mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r}_g, \mathbf{k}}^{\alpha} \hat{c}_{\mathbf{r}_e, -\mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r}_e, -\mathbf{k}}^{\alpha}) \\ &\quad \left. + V_{\text{ex}} \sum_{\alpha \neq \beta} (\hat{c}_{\mathbf{r}_g, -\mathbf{k}}^{\alpha\dagger} \hat{c}_{\mathbf{r}_e, \mathbf{k}}^{\beta\dagger} \hat{c}_{\mathbf{r}_g, -\mathbf{k}}^{\beta} \hat{c}_{\mathbf{r}_e, \mathbf{k}}^{\alpha}) \right], \end{aligned} \quad (\text{B8})$$

where the term in the last line is effectively suppressed by the Zeeman protection. The term in the second line is proportional to $(V - V_{\text{ex}}) = U_{eg}^-$, meaning that the atoms interact via the antisymmetric interaction channel. The different energy of the intermediate states leads to a phase oscillation between singlet and triplet states.

In the limit $|t| \ll |\delta|$ we can define the two quantities $t_{\text{eff}}^{\uparrow\downarrow} = -\frac{t^2}{\delta} + \frac{t^2}{(\delta - V)}$ and $t_{\text{eff}}^{\uparrow\uparrow} = -\frac{t^2}{\delta} + \frac{t^2}{(\delta - V + V_{\text{ex}})}$, which represent the perturbative hopping of the states components that interact in the rishon sites with spins of same m_F ($t_{\text{eff}}^{\uparrow\uparrow}$) or opposite m_F ($t_{\text{eff}}^{\uparrow\downarrow}$). The imbalance in the tunneling amplitudes causes a dephasing during the time evolution that leads to oscillations between the singlet state $|s\rangle$ and the state with a triplet on each vertex $|t\rangle$. We can also use these quantities to define the coupling λ between these two sectors as $\lambda = (t_{\text{eff}}^{\uparrow\uparrow} - t_{\text{eff}}^{\uparrow\downarrow})/2$. This analysis of Fig. 8, which does not include the superexchange interactions, shows that these couplings λ quickly lead to accumulation of gauge-breaking errors as the four states of the basis $\{|s_g\rangle, |s_e\rangle, |t_g\rangle, |t_e\rangle\}$ are on resonance, the gauge symmetry is easily broken by the evolution of the system. As shown in Fig. 5, the superexchange couplings lead to gauge protection, overcoming the effect of λ .

APPENDIX C: ATOMIC LOSSES

The atomic two-body interactions can lead to losses in the system. When atoms interact with each other in the same lattice site, it is possible for some of them to be lost during the simulation. This phenomenon can affect any two-body interaction, but it is particularly pronounced in the case of e - e ones. Here we want to estimate the contribution of such events to the scheme. We start by considering the on-site pair lifetime, τ_{a_s} , where a_s indicates the scattering length of a generic two-body interaction, which we can define through the two-body loss rate coefficient β_{a_s} using the relation [90,121]

$$(\tau_{a_s})^{-1} = \beta_{a_s} \int d\mathbf{r} |\mathbf{w}(\mathbf{r})|^4. \quad (\text{C1})$$

We use this formula to calculate the loss rate γ_{ee} , with $\beta_{ee} = 2.2 \times 10^{-11} \text{ cm}^3 \text{ s}^{-1}$ [90]:

$$\gamma_{ee} = P_{ee} (\tau_{a_{ee}})^{-1}, \quad (\text{C2})$$

where we indicate with P_{ee} the probability of double occupancy on the same e lattice site. Assuming that $t_e \ll \delta \ll U_{ee}$, we can estimate from the perturbative processes shown in Fig. 9(a)

$$P_{ee} \approx 2 \left(\frac{t_e}{U_{ee}} \right)^2, \quad (\text{C3})$$

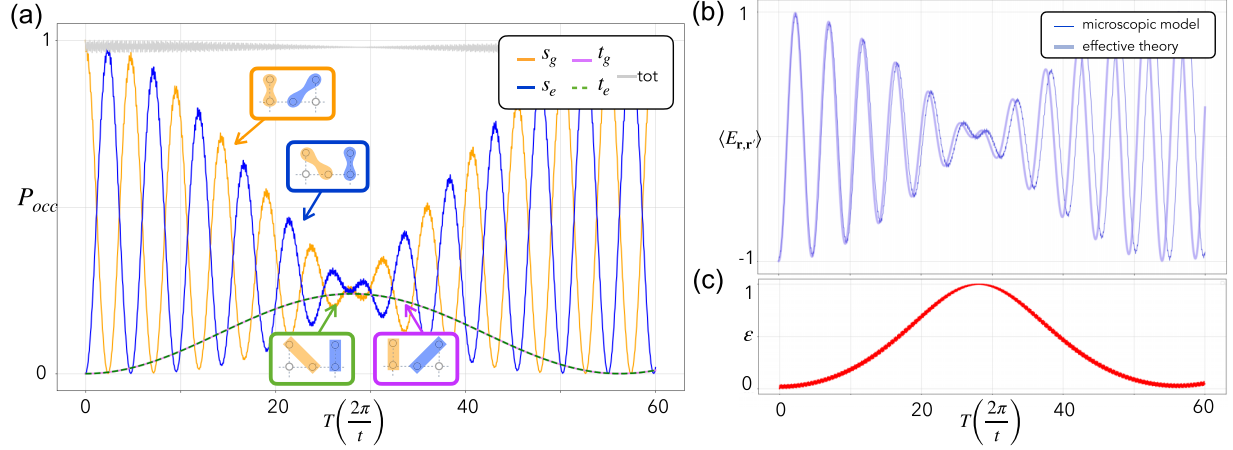


FIG. 8. Time evolution of the system without superexchange protection. The plots show the time evolution of the two-vertex toy model presented in the main text: in this case the Zeeman-splitting mechanism is implemented and it protects from the occupation of triplet states $|F^r = 1; m_F^r = \pm 1\rangle \otimes |F^{r'} = 1; m_F^{r'} = \mp 1\rangle$, while we do not include the on-site Hubbard interactions; therefore there is no superexchange-induced protection. (a) The first plots shows the evolution of populations in the initial state into a superposition of the singlet states (in orange and blue) and the triplet states (in green and purple). (b) The second plot shows the evolution of the electric field in the microscopic (thin line) and effective (thick line) theory: it has a peculiar beat, which corresponds with the peak of the gauge violation ε , shown in red in (c).

and we obtain

$$\gamma_{ee} = P_{ee}(\tau_{a_{ee}})^{-1} = \frac{t_e^2}{U_{ee}^2} \beta_{ee} \int |w|^4. \quad (C4)$$

To avoid numerically evaluating the integrals with the Wannier functions we assume the e lattice to be fairly deep, and we use the harmonic approximation to calculate it. In this case, we consider the approximation introduced in [89,122] where the lattice structure is approximately described by the function

$$W(\mathbf{r}) = \sum_{\mu} W_{\mu} \sin^2(x_{\mu}), \quad (C5)$$

where the index μ is referred to the spacial coordinates $\mu = x, y, z$. The variables W_{μ} indicate the depth of the lattice in the three possible directions and are assumed to be large enough

to allow a correct application of the approximation. The three-dimensional lattice can be treated as a two-dimensional one if the depth of one of the three variables is much bigger than the others, basically eliminating any kind of interaction and hopping in that direction. First, the interaction and the hopping coefficients are calculated as

$$t_{\mu} = - \int d\mathbf{r} w_{\mu}^*(\mathbf{r}) \left[-\frac{\hbar^2}{2m} \nabla^2 + W_{\mu} \right] w_{\alpha}(\mathbf{r} - \mathbf{r}), \quad (C6)$$

$$U_{\alpha\alpha} = g_{\alpha\alpha} \int d\mathbf{r} |w_{\alpha}(\mathbf{r})|^4, \quad (C7)$$

where we use the usual definition

$$g_{\alpha\alpha} = \frac{4\pi \hbar^2 a_{\alpha\alpha}}{m}. \quad (C8)$$

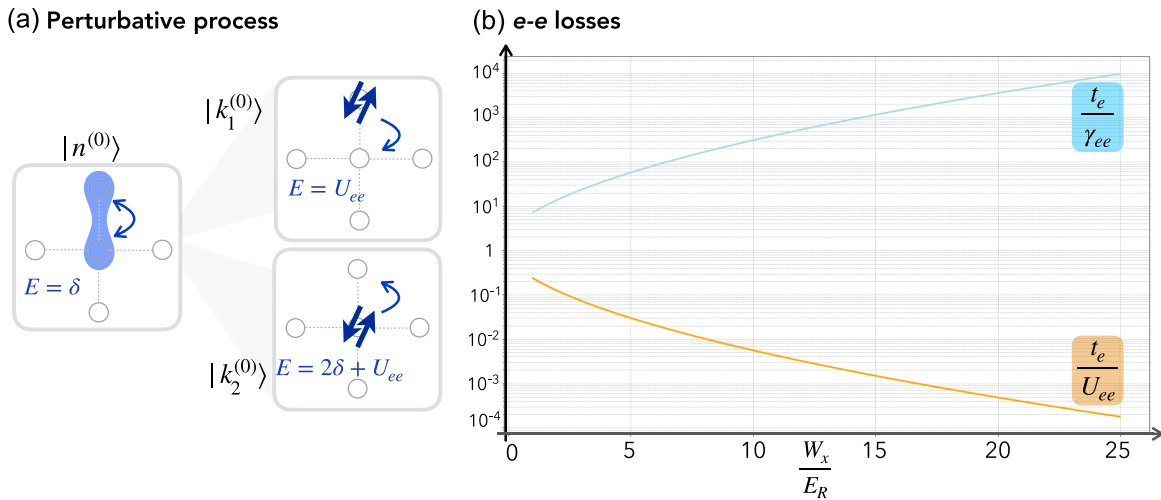


FIG. 9. Analysis for the e-e losses. (a) The superexchange process that contributes to a correction of the total energy of the initial singlet configuration $|n^{(0)}\rangle$: the two virtual states $|k_1^{(0)}\rangle$ and $|k_2^{(0)}\rangle$ are characterized by a two-body interaction that can lead to losses. (b) The ratio t_e/γ_{ee} (blue) with respect to t_e/U_{ee} (orange) for the different values of lattice depth.

Moreover, we can use Eq. (C7) and rewrite Eq. (C4) as follows:

$$\gamma_{ee} = \frac{t_e^2}{U_{ee}} \frac{\beta_{ee}}{g_{ee}}. \quad (\text{C9})$$

To apply the approximation we assume the Wannier functions to be expressed in a Gaussian-like form around the lattice site they refer to, and we obtain [89]

$$t_{e,\mu} \approx \left(\frac{4}{\sqrt{\pi}} \right) E_R \left(\frac{W_\mu}{E_R} \right)^{\frac{3}{4}} e^{-2\left(\frac{W_\mu}{E_R}\right)^{\frac{1}{2}}}, \quad (\text{C10})$$

$$U_{ee} \approx \left(\frac{8}{\pi} \right) \left(\frac{\pi a_{ee}}{a} \right) E_R \left(\frac{W_x}{E_R} \frac{W_y}{E_R} \frac{W_z}{E_R} \right)^{\frac{1}{4}}, \quad (\text{C11})$$

where E_R is the recoil energy that depends on the lattice momentum

$$E_R = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2a^2 m}. \quad (\text{C12})$$

For our calculations, we assume that $W_x/E_R = W_y/E_R \ll W_z/E_R$. Moreover, we focus on calculating the hopping in the x (or y) direction, as ($t_x = t_y \gg t_z$). Using these approximations, we substitute (C10) and (C11) in (C9) and we obtain

$$\begin{aligned} \gamma_{ee} &= \frac{\frac{16}{\pi} E_R \left(\frac{W_x}{E_R} \right)^{\frac{3}{2}} e^{-4\left(\frac{W_x}{E_R}\right)^{\frac{1}{2}}}}{\frac{8}{\pi} \left(\frac{\pi a_{ee}}{a} \right) E_R \left(\frac{W_x}{E_R} \frac{W_y}{E_R} \frac{W_z}{E_R} \right)^{\frac{1}{4}}} \frac{\beta_{ee} m}{4\pi \hbar^2 a_{ee}} \\ &= \frac{\pi}{4} \frac{W_x/E_R}{(W_z/E_R)^{\frac{1}{4}}} e^{-4\left(\frac{W_x}{E_R}\right)^{\frac{1}{2}}} \frac{\beta_{ee}}{a a_{ee}^2}, \end{aligned} \quad (\text{C13})$$

which is correctly expressed in Hz. We calculate the ratio γ_{ee}/t_e to be able to estimate the average number of hopping processes that can happen before observing a loss

$$\frac{\gamma_{ee}}{t_e} = \frac{t_e}{U_{ee}} \frac{\beta_{ee}}{g_{ee}} = \frac{\sqrt{\pi}}{8} \left(\frac{W_x/E_R}{(W_z/E_R)} \right)^{\frac{1}{4}} e^{-2\left(\frac{W_x}{E_R}\right)^{\frac{1}{2}}} \frac{\beta_{ee}}{a a_{ee}^2} \frac{1}{E_R}. \quad (\text{C14})$$

From the same calculations we obtain the ratio t_e/U_{ee}

$$\frac{t_e}{U_{ee}} = \frac{1}{2\sqrt{\pi}} \frac{a}{a_{ee}} \left(\frac{W_x/E_R}{(W_z/E_R)} \right)^{\frac{1}{4}} e^{-2\left(\frac{W_x}{E_R}\right)^{\frac{1}{2}}}. \quad (\text{C15})$$

We plot these different ratios to find a specific range of the variable W_x/E_R for which the losses remain controlled. We have used a typical value for $a \approx 300$ nm, and we set $W_z/E_R = 150$. The harmonic approximation fails to correctly reproduce the hopping amplitude for too small values of the lattice depth, and we assume it to be reliable for values $W/E_R > 5$. In Fig. 9(b) we observe that, when considering a lattice with ratio $t_e/U_{ee} \approx 1/100$ to $1/300$ (so when W/E_R has values between 7 and 9), the ratio of particle decay in units of the hopping coefficient is set to be in a range that goes from $1/100$ to almost $1/300$, meaning that the decay is not particularly relevant. This particular range of t_e/U_{ee} has been chosen because it is similar to the values used in the various analysis in the main text, where we considered $t/\delta \approx 1/10$ and $U_{ee}/\delta \approx 15$. From similar approximated calculation and considerations on the symmetry of the lattice, and using $\beta_{ge} = 3.9 \times 10^{-13} \text{ cm}^3 \text{ s}^{-1}$ [90], we estimate that the corresponding value $\gamma_{ge}/t \approx 10^{-2} \gamma_{ee}/t_e$ while $t_e/U_{ee} \approx 3 t_e/U_{ee}$ making

the g - e losses even less relevant. Because of these results, we assume the scheme to be reliable and experimentally implementable.

APPENDIX D: RYDBERG MULTICOLOR DRIVE

In the main text, we assumed that $N_{(r,r')}$ atoms in the ris-hon sites of a link are excited to the Rydberg states in a process that results in a collective Rabi oscillation between the ground state $|g^{N_{(r,r')}}\rangle$ and the symmetric Dicke-state $|r g_F^{N_{(r,r')} - 1}\rangle$ with a frequency $\Omega^* = \sqrt{N_{(r,r')}} \Omega$. The general requirement for the Dicke dynamics to be valid is that each atom must be excited to the Rydberg state with Rabi frequency Ω but, in our case, the excitation cannot be achieved using a monochromatic laser: the Rydberg states are split into different hyperfine levels, resulting in the excited states of atoms with different color degree of freedom to be out of resonance; see Fig. 10(a). We therefore propose to use of multi-chromatic lasers that allow one to achieve excitations of the atoms in the ground state to the correct Rydberg states. The lasers' frequencies and intensities must be accordingly tuned, in order to achieve the same Rabi frequency Ω . For clarity, we calculate the collective Rabi frequency for an easy example of a link with two atoms of different colors [e.g., the blue and red atoms in Fig. 10(a)]. We start defining the basis used, $\{|gg\rangle, |rg\rangle, |gr\rangle, |rr\rangle\}$, where g is the ground state and r is the Rydberg excited one and we assume the atoms to be separated by distance R . The Hamiltonian of the system is then

$$\hat{H} = \begin{pmatrix} 0 & \Omega_b & \Omega_r & 0 \\ \Omega_b & 0 & 0 & \Omega_r \\ \Omega_r & 0 & 0 & \Omega_b \\ 0 & \Omega_r & \Omega_b & C_6/R^6 \end{pmatrix}. \quad (\text{D1})$$

The states $|rg\rangle, |gr\rangle$ can be rotated to

$$|\varphi_{\pm}\rangle = \frac{\Omega_r}{\sqrt{\Omega_r^2 + \Omega_b^2}} |rg\rangle \pm \frac{\Omega_b}{\sqrt{\Omega_r^2 + \Omega_b^2}} |gr\rangle. \quad (\text{D2})$$

Since $|\varphi_{-}\rangle$ is a “dark state,” the Hamiltonian can be reduced using the basis $\{|gg\rangle, |\varphi_{+}\rangle, |rr\rangle\}$ yielding

$$\hat{H} = \begin{pmatrix} 0 & \Omega^* & 0 \\ \Omega^* & 0 & \Omega^* \\ 0 & \Omega^* & C_6/R^6 \end{pmatrix}, \quad (\text{D3})$$

where $\Omega^* = \frac{2\Omega_r\Omega_b}{\sqrt{\Omega_r^2 + \Omega_b^2}}$ which is therefore dependent on the color of the particles in the link. Once the Rabi frequencies are tuned to be all equal to Ω for each one of the lasers, Ω^* reduces to $\sqrt{2}\Omega$ and the color independence is achieved. The same process can be repeated for more atoms in each link.

APPENDIX E: ADIABATIC SWEEP

The adiabatic process used for the Rydberg scheme in the main text could be achieved through a fast Rydberg dressing sweep [see Fig. 10(b)]: We define the Hamiltonian of the dressing to the Rydberg state as a function of a time-dependent energy offset $\Delta(t)$. The instantaneous eigenvalues are $E_{\pm}(t) = (-\Delta(t) \pm \sqrt{\Delta(t)^2 + 4\Omega^{*2}})/2$ with relative eigenvectors (nonnormalized) $|\varphi_{+}\rangle = |g^{N_{(r,r')}}\rangle +$

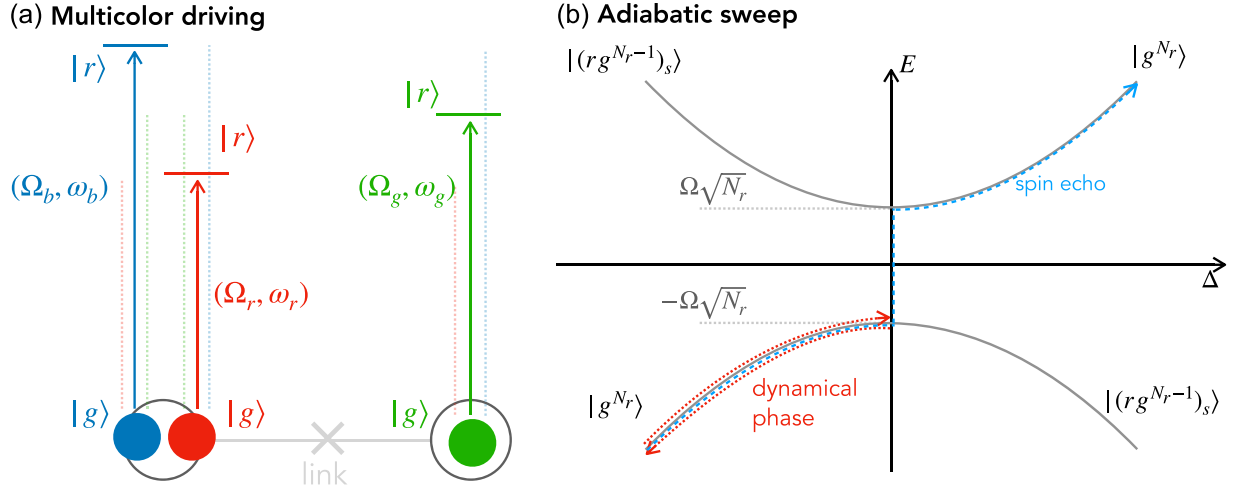


FIG. 10. Rydberg scheme. (a) The Rydberg-excited states of atoms with different nuclear spin experience an hyperfine splitting that separate the energy levels. For this reason, we propose to use a multicolor-laser scheme to excite all the atoms in the link at the same time. Each laser must be tuned to yield the same Rabi frequency Ω . Here we show the example of a link with three atoms of different color. (b) We show the energy of the two eigenstates of an Hamiltonian describing a Rydberg dressing scheme for different values of the energy offset Δ , to pick up the phase necessary to our scheme we assume to undergo an adiabatic evolution to the eigenenergy of the Hamiltonian with $\Delta = 0$, hold the system in such state for a certain period, and then evolve the state back to the original ground state (red line). Alternatively, it is possible to use a spin echo procedure (blue line).

$\frac{E_{\pm}}{\Omega^*} |r g^{N_{(r,r')-1}}\rangle_s$ and $|\varphi_{-}\rangle = \frac{\Delta(t)+E_{-}}{\Omega^*} |g^{N_{(r,r')}}\rangle + |r g^{N_{(r,r')-1}}\rangle_s$. When $\Delta(t) = 0$ we obtain the energy $E_{-} = \Omega^* = \sqrt{N_r}\Omega$ and the eigenstate is the antisymmetric superposition $|\varphi_{-}\rangle = |A\rangle = (|g^{N_{(r,r')}}\rangle - |r g^{N_{(r,r')-1}}\rangle_s)/\sqrt{2}$.

The total sequence comprises three steps: first, the adiabatic evolution from $|g^{N_{(r,r')}}\rangle$ to $|A\rangle$ happens over the time τ_s and is associated to an evolution operator $\hat{U}_{\text{fwd}}$, afterwards the Hamiltonian is not changed; then the state is held to remain in the eigenstate with energy $-\sqrt{N_{(r,r')}}\Omega$ ($\hat{U}_{\text{hold}}$) for a time $\tau_h \gg \tau_s$; and finally it is evolved back through the same adiabatic path ($\hat{U}_{\text{bwd}}$). The adiabatic evolution leads to the appearance of a dynamical $\theta = -\int_0^{\tau_s} E(t')dt'$ and Berry γ phase, but the latter does not contribute, since the contributions from the forward and backward evolution sum up to zero. Therefore, we define the dynamical phase of the forward process as θ_f , that of the backward process as θ_b and the total operator of the sweep as

$$\hat{U}_p = \hat{U}_{\text{bwd}}\hat{U}_{\text{hold}}\hat{U}_{\text{fwd}} = e^{i\theta_b} e^{i\theta_f} e^{i\sqrt{N_{(r,r')}}\Omega\tau_h} |g^{N_{(r,r')}}\rangle \langle g^{(r,r')}|. \quad (\text{E1})$$

In our analysis we assume that the contribution of the dynamical phases can be neglected or canceled if using a spin-echo procedure; see Fig. 10(b). For reference we show its analytical

form assuming that the energy offset of the dressing varies as $\Delta(t) = -\Delta_0 + vt$ with $v = \Delta_0/\tau_s$ and $\Delta_0 > 0$

$$\theta_f = -\int_0^{\tau_s} E_{-}(t')dt' = -\int_0^{\tau_s} \frac{-\Delta(t')}{2} - \frac{\sqrt{\Delta(t')^2 + 4\Omega^{*2}}}{2} dt'. \quad (\text{E2})$$

An incomplete short evolution is not sufficient to guarantee the desired $\sqrt{N_{(r,r')}}$ term; indeed, we calculate the contribution of the dynamical phase on a short time $\tilde{\tau} \ll \tau_s$, therefore assuming that the state does not completely evolve to $|A\rangle$ [i.e., we assume the red dashed path in Fig. 10(b) to advance much less before turning back]

$$-\int_0^{\tilde{\tau}} E_{-}(t')dt' \approx -\left[\frac{\Delta_0}{12}\left(\frac{\tilde{\tau}}{\tau_s}\right)^2 - \frac{N_{(r,r')}\Omega^2}{2\Delta_0}\right]\tilde{\tau}. \quad (\text{E3})$$

The contribution of this integral is linear in $N_{(r,r')}$, not sufficient to guarantee the wanted gauge invariance. Holding the system in the state $|A\rangle$ for a time τ_h , during which the Hamiltonian remains unchanged, also contributes to the tunability of the effective interaction resulting from the full Floquet scheme.

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