

Generation of perfectly achromatic optical vortices using a compensated tandem twisted nematic cell

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In this work, we theoretically investigate a tandem crossed twisted nematic liquid crystal architecture for the generation of broadband (“white”) optical vortices. Operating in transmission, the system consists of two twisted nematic cells cascaded such that the liquid crystal director at the output of the first cell is strictly orthogonal to the director at the input of the second. Unlike conventional q -plates, this crossed configuration preserves the handedness of incident circularly polarized light. By rotating the entire polarization state by a target angle, the architecture imparts a corresponding spatial phase shift directly to the circular polarization. To evaluate its performance under realistic physical conditions, we develop a rigorous Jones matrix model. We systematically analyze the impact of deviations from the ideal waveguiding regime, manufacturing thickness mismatches, oblique incidence, and structural misalignments. By utilizing the modal overlap integral, we establish strict analytical figures of merit that provide a universal lower bound for the global mode purity. To mitigate the inherent amplitude and phase distortions, we propose three distinct operational strategies, ranging from parameter optimization to active phase correction. We demonstrate that active compensation effectively decouples the amplitude transmission efficiency from fabrication thickness tolerances, ensuring high optical throughput and superior phase fidelity. Ultimately, the direct structural compatibility of this architecture with emerging in-plane switching technologies establishes the tandem twisted nematic cell as a highly versatile blueprint for next-generation, fully dynamic broadband spatial light modulators.

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I. INTRODUCTION

Optical vortices are a fascinating class of structured light beams characterized by a helical phase front, described by the term $\exp(il\phi)$, where l is an integer known as the topological charge and ϕ is the azimuthal angle. This helical phase results in a phase singularity on the beam axis, leading to a characteristic donut-shaped intensity profile. A landmark discovery by Allen *et al.* in 1992 revealed that such beams carry a well-defined orbital angular momentum (OAM) of $l\hbar$ per photon, distinct from the spin angular momentum associated with polarization [1–4]. This property enables the transfer of torque to microscopic particles, fueling a vast range of applications, including advanced optical tweezers for manipulating and rotating particles [5–10], and broadband superresolution imaging where achromatic depletion beams enable multicolor stimulated emission depletion microscopy [11,12], increasing channel capacity in optical communications through mode-division multiplexing [13,14], and pushing the boundaries of quantum information science with a high-dimensional state space [15].

The utility of these applications is significantly enhanced when they can be performed with broadband light, which has spurred intense interest in “white light” or achromatic optical vortices [16–20]. A truly achromatic vortex generator must ideally preserve the key topological properties across a broad spectral range. This implies not only topological achromaticity (a constant integer charge l) but also device achromaticity. For a device, this means providing wavelength-independent phase modulation and, crucially, maintaining a uniform amplitude response without introducing chromatic power losses. While the diffractive nature of light means that a beam’s size will inevitably scale with wavelength, the generating element itself should ideally offer a spatially and spectrally uniform transformation.

However, generating high-quality white vortices remains a significant challenge, as most conventional generation methods are inherently chromatic. Techniques based on diffractive optical elements, such as spiral phase plates and forked gratings, introduce a phase shift proportional to the optical path difference, which is fundamentally dependent on wavelength [16,21]. While ubiquitous, standard liquid crystal spatial light modulators (SLMs) operate on the same principle and thus suffer from the same chromatic limitations [19,22,23]. Alternative approaches using dielectric metasurfaces have demonstrated broadband capabilities [24–26], yet they often face challenges related to fabrication complexity and static response profiles [27].

The most promising route to tunable achromaticity has been through devices that impart a geometric (Pancharatnam-Berry) phase [28]. Q -plates, typically fabricated from spatially

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patterned liquid crystals, are the leading candidates in this category [29–32]. They function as spatially variant half-wave plates that convert the spin of circularly polarized light into OAM. While the geometric phase itself is achromatic, the underlying half-wave plate condition is not, constraining broadband performance. Numerous strategies have been proposed to create achromatic half-wave plates, for instance, by stacking multiple retarders [33,34] or using Bragg reflection in helical liquid crystal structures [35,36].

Building on these principles, significant progress has been achieved by utilizing layers of twisted nematics to synthesize highly achromatic broadband retarders [33]. Specifically, multitwist liquid crystal polymer networks and self-compensating double-layer reverse-twist architectures have demonstrated exceptional ultrabroadband vortex generation in both transmission [37] and reflection [38,39]. However, while these patterned polymer architectures provide outstanding achromaticity, they inherently function as completely static optical components. Their spatial phase profiles and twist geometries are permanently fixed during fabrication via complex dual-substrate photoalignment or polymerized chiral doping. Consequently, real-time spatial phase reconfiguration is highly challenging. Another approach to achieve a wavelength-independent phase relies on using a standard, chromatic q -plate followed by a polarization filtering system [40]. This method successfully achieves a perfectly achromatic phase response, but at a significant cost: it introduces a strong, nonuniform amplitude modulation, as unwanted polarization components at off-design wavelengths are filtered out, leading to chromatic power loss.

In this work, we introduce and analyze a fundamentally new approach to address the critical need for a fully dynamic, transmissive, broadband spatial light modulator. Our method is based on a device architecture, previously known only for its application as a broadband polarization rotator, which consists of a tandem of two crossed twisted nematic (TN) liquid crystal cells [41]. To the best of our knowledge, the potential of this tandem cell architecture as a high-performance vortex generator has not been previously explored. Unlike static polymer films, this configuration can be effectively employed for dynamic electro-optic control. By maintaining a uniform rubbing alignment on the two static internal boundaries and allowing for dynamic in-plane rotation of the liquid crystal director on the active external boundaries (using two one-side zero-azimuth anchoring in-plane switching (OZ-IPS) cells [42–45]), this theoretical framework lays the foundation for next-generation SLMs. We demonstrate that, with a proposed active compensation scheme, this tandem system can be engineered to provide not only a perfectly achromatic phase response but also a fully achromatic amplitude transmission, thereby avoiding the typical trade-off between phase achromaticity and transmission efficiency often encountered in broadband geometric phase optics. Herein, we develop a comprehensive theoretical model for this device, analyze the sources of non-idealities, and present three distinct compensation strategies that establish the tandem TN cell as a superior and highly versatile platform for generating high-fidelity white optical vortices.

II. THEORETICAL MODEL

A. Light propagation in a single twisted nematic cell

Let us consider a single liquid crystal cell with a planar alignment of the director. We will choose the Cartesian coordinate system such that the z axis is orthogonal to the two parallel substrates of the cell. The standard-basis vectors of the coordinate system we denote as $\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\}$. The orientation of the director of the liquid crystal can be defined by a unit vector which we denote as $\hat{\mathbf{e}}$. We assume that the alignment of the director is planar, therefore the vector $\hat{\mathbf{e}}$ can be defined by a twist angle ϕ , which is an angle between axis x and director orientation:

$$\hat{\mathbf{e}} = \hat{\mathbf{e}}_x \cos \phi + \hat{\mathbf{e}}_y \sin \phi. \quad (1)$$

We model the propagation of light along the z axis through the liquid crystal by treating it as a birefringent medium. The model is built upon two principal refractive indices: the ordinary index, n_o , and the extraordinary index, n_e , which are assumed to be constant throughout the medium.

The local orientation of the director determines the optical axes for a light ray. Specifically, the extraordinary wave is polarized parallel to the director, while the ordinary wave is polarized orthogonally to it.

This framework is well suited for configurations where the director varies slowly. However, it can also effectively model systems with sharp interfaces by considering them as a stack of discrete, uniform layers. Thus, for a single homogeneous layer of thickness h , the light propagation can be described by the following equation:

$$\begin{aligned} \mathbf{E}(z+h) &= (\mathbf{E}(z) \cdot \hat{\mathbf{e}}) \hat{\mathbf{e}} \exp(in_e kh) \\ &+ [\mathbf{E}(z) - (\mathbf{E}(z) \cdot \hat{\mathbf{e}}) \hat{\mathbf{e}}] \exp(in_o kh), \end{aligned} \quad (2)$$

where $\mathbf{E}$ is a vector of electric field, k is a wave number in vacuum. For the subsequent analysis, it is convenient to use the Jones formalism. Within this framework, transmission through the layer h is described by the Jones matrix $T^E(h)$:

$$\mathbf{E}(z+h) = T_h^E(h) \mathbf{E}(z), \quad (3)$$

$$\begin{aligned} T^E(h) &= \exp(i\bar{k}h) \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\gamma h) \right. \\ &\quad \left. + i \begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix} \sin(\gamma h) \right], \end{aligned} \quad (4)$$

$$\bar{k} = \frac{n_e + n_o}{2} k, \quad (5)$$

$$\gamma = \frac{\Delta n}{2} k = \frac{n_e - n_o}{2} k, \quad (6)$$

where T^E is a Jones transmission matrix, $\bar{k}$ is a mean wave number in the liquid crystal, and γ is an anisotropy parameter.

For the matrix formulation in Eq. (3), we express the vector $\mathbf{E}$ as a column vector of its components in the Cartesian basis.

To better analyze the evolution of the polarization state, it is convenient to switch from the linear (Cartesian) basis to the circular polarization basis. We introduce a new Jones vector, $\mathcal{E}$, which is related to the original vector $\mathbf{E}$ through a basis

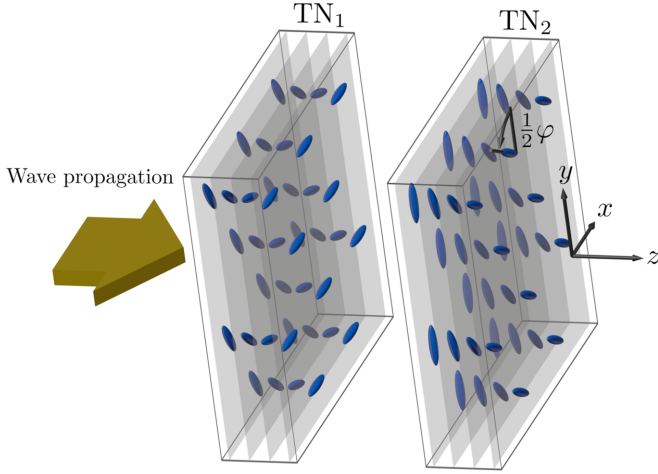


FIG. 1. Configuration of the tandem TN cell system. Two TN cells, TN_1 and TN_2 , are cascaded. The director at the output of TN_1 is aligned with the x axis, while the director at the input of TN_2 is aligned with the y axis, creating an orthogonal interface.

transformation matrix U . Concurrently, we separate out the common phase factor that corresponds to the average optical path through the liquid crystal. The resulting transformation is given by

$$\mathcal{E}(z) = \exp(-i\bar{k}z)U\mathbf{E}(z), \quad (7)$$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \quad (8)$$

With the Jones vector $\mathcal{E}$ now expressed in the circular basis, its components $\mathcal{E}_1$ and $\mathcal{E}_2$ correspond to the complex amplitudes of the right-hand circular (RHC) and left-hand circular

(LHC) polarizations, respectively. The transmission through the system is described by the Jones matrix in this basis, $T^\mathcal{E}$, given by

$$T^\mathcal{E} = \exp(-i\bar{k}z)UT^EU^{-1}, \quad (9)$$

$$T^\mathcal{E}(h) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\gamma h) + i \begin{pmatrix} 0 & \exp(i2\phi) \\ \exp(-i2\phi) & 0 \end{pmatrix} \sin(\gamma h). \quad (10)$$

The matrix $T^\mathcal{E}$ describes light propagation through a uniform layer. To generalize this for a nonuniform liquid crystal, where the director orientation varies continuously along the z axis, we now consider an infinitely thin layer of thickness dz . By taking the limit of the transfer matrix $T^\mathcal{E}(h)$ as $h \rightarrow 0$, we can obtain a differential equation that governs the continuous evolution of the light's polarization state:

$$\frac{d}{dz}\mathcal{E}(z) = i\gamma \begin{pmatrix} 0 & \exp(i2\phi) \\ \exp(-i2\phi) & 0 \end{pmatrix} \mathcal{E}(z). \quad (11)$$

While in the general case, Eq. (11) does not have a simple analytical solution due to the noncommuting nature of the corresponding matrix in the equation, a closed-form solution can be found for the important case of a linear director twist, $\phi(z) = \phi_0 + qz$. This configuration corresponds to a uniformly twisted nematic or cholesteric structure.

The standard procedure for this case involves reducing the system of two first-order coupled differential equations to a single equivalent second-order linear homogeneous equation with constant coefficients. Since this is a well-established method for solving this class of problems, we will omit the detailed derivation and present the final analytical expression for the system's Jones matrix $T_{TN}^\mathcal{E}(h)$:

$$T_{TN}^\mathcal{E}(h) = \begin{pmatrix} \exp(i\phi_d) [\cos(\gamma'h) - i\frac{q}{\gamma'} \sin(\gamma'h)] & i\frac{\gamma}{\gamma'} \exp(i2\bar{\phi}) \sin(\gamma'h) \\ i\frac{\gamma}{\gamma'} \exp(-i2\bar{\phi}) \sin(\gamma'h) & \exp(-i\phi_d) [\cos(\gamma'h) + i\frac{q}{\gamma'} \sin(\gamma'h)] \end{pmatrix}, \quad (12)$$

where h is a layer thickness, q is a twist wave number ($q = h^{-1}[\phi(z+h) - \phi(z)]$), $\bar{\phi}$ is a mean value of the director's orientation angle $\bar{\phi} = \frac{1}{2}[\phi(z) + \phi(z+h)] = \phi(z) + \frac{1}{2}qh$, ϕ_d is full angle of rotation of the director in the layer

$$\phi_d = \phi(z+h) - \phi(z) = qh, \quad (13)$$

$$\gamma' = \sqrt{\gamma^2 + q^2}. \quad (14)$$

B. The ideal tandem cell as an achromatic rotator

The tandem twisted nematic liquid crystal cell is a known device, previously investigated for its properties as a broadband polarization rotator [41]. In this work, we explore an application for this system: its potential for generating optical vortices, particularly with broadband (white) light. To understand why this configuration is advantageous for phase modulation, one must first consider its basic mechanism. It was shown that such a tandem cell, in the first approximation, geometrically rotates any incident linear polarization by a

fixed angle, preserving the linear state. Crucially, the accumulated dynamical phase during this rotation is independent of the initial polarization angle. Since circularly polarized light can be represented as a superposition of linear states, rotating the entire polarization state as a rigid body geometrically imparts a corresponding phase shift (with opposite signs for right- and left-handed circular polarizations). This geometric phase accumulation is the core mechanism enabling the cell to function as a phase modulator.

To establish the fundamental principle mathematically, we first analyze an idealized model. We construct a system composed of two identical TN cells. The cells are assumed to have identical material properties, thickness h , and twist angle $\frac{\phi}{2}$ (yielding a total twist angle of ϕ for the combined system). They are cascaded such that the director at the output of the first cell is exactly orthogonal to the director at the input of the second, as illustrated in Fig. 1.

The key to this preliminary analysis is the assumption that the system operates deep within the Mauguin (wave-guiding)

limit. According to classical liquid crystal optics [46], the adiabaticity of light propagation in a twisted medium is governed by the Mauguin parameter $u = k \frac{\Delta n h}{2\phi_d}$, which compares the rate of phase accumulation to the rate of director twist. The magnitude of this parameter determines the ideality of the waveguiding effect: for increasingly large values ($u \gg 1$), the local polarization eigenmodes (corresponding to linear polarizations parallel and orthogonal to the local director) adiabatically track the continuously twisting structure without mutual cross-coupling.

Mathematically, this idealized wave-guiding limit implies that the twist wave number is negligibly small compared to the anisotropy parameter ($q \ll \gamma$). By applying the limits $q/\gamma' \approx 0$ and $\gamma/\gamma' \approx 1$ exclusively to the amplitude coefficients of the general propagation matrix (12), the expression simplifies significantly while leaving the oscillating phase arguments ($\gamma'h$) intact.

Using this approximation, we can explicitly derive the response of the tandem system by multiplying the simplified Jones matrices of the individual cells: $T_{\text{CTN}}^{\mathcal{E}} = T_{\text{TN2}}^{\mathcal{E}} T_{\text{TN1}}^{\mathcal{E}}$. For the first cell (TN₁), the director twists from an initial angle of $-\frac{\varphi}{2}$ to an output angle of 0 (aligned with the x axis), corresponding to a mean angle $\bar{\phi}_1 = -\frac{\varphi}{4}$ and a total layer twist $\phi_d = \frac{\varphi}{2}$. Substituting these into the simplified matrix yields

$$T_{\text{TN1}}^{\mathcal{E}} \approx \begin{pmatrix} e^{i\frac{\varphi}{2}} \cos(\gamma'h) & ie^{-i\frac{\varphi}{2}} \sin(\gamma'h) \\ ie^{i\frac{\varphi}{2}} \sin(\gamma'h) & e^{-i\frac{\varphi}{2}} \cos(\gamma'h) \end{pmatrix}. \quad (15)$$

For the second cell (TN₂), due to the orthogonal interface, the director twists from an input angle of $\frac{\pi}{2}$ (y axis) to a final angle of $\frac{\pi}{2} + \frac{\varphi}{2}$. This corresponds to a mean angle $\bar{\phi}_2 = \frac{\pi}{2} + \frac{\varphi}{4}$. The $\frac{\pi}{2}$ offset introduces a sign change in the exponential phase factors representing the mean orientation:

$$T_{\text{TN2}}^{\mathcal{E}} \approx \begin{pmatrix} e^{i\frac{\varphi}{2}} \cos(\gamma'h) & -ie^{i\frac{\varphi}{2}} \sin(\gamma'h) \\ -ie^{-i\frac{\varphi}{2}} \sin(\gamma'h) & e^{-i\frac{\varphi}{2}} \cos(\gamma'h) \end{pmatrix}. \quad (16)$$

Performing the direct matrix multiplication $T_{\text{TN2}}^{\mathcal{E}} T_{\text{TN1}}^{\mathcal{E}}$, the dynamic phase components $[\cos(\gamma'h)$ and $\sin(\gamma'h)]$ mutually cancel in the off-diagonal elements and sum to unity via the trigonometric identity $\cos^2(\gamma'h) + \sin^2(\gamma'h) = 1$ on the diagonal. This leads to the elegant, fully diagonal result:

$$T_{\text{CTN}}^{\mathcal{E}} \approx \begin{pmatrix} \exp(i\varphi) & 0 \\ 0 & \exp(-i\varphi) \end{pmatrix}. \quad (17)$$

A remarkable feature of this explicit derivation is that the wavelength-dependent dynamical phase factors entirely vanish from the resulting matrix. The transformation depends strictly on the total twist angle φ and acts purely by applying a phase shift of φ to the right-hand circular (RHC) component and $-\varphi$ to the left-hand circular (LHC) component.

This mechanism is fundamentally different from that of a standard q -plate. While a q -plate flips the helicity of circular polarizations (RHC becomes LHC and vice versa), this tandem cell preserves it: RHC light remains RHC, and LHC remains LHC, with the device solely modulating their respective phases.

Because the imparted phase modulation $\pm\varphi$ is inherently independent of wavelength, the tandem cell acts as a perfectly

achromatic phase shifter within the bounds of this idealized waveguiding approximation. Therefore, if the total twist angle $\varphi(x, y)$ is patterned to vary spatially across the beam's transverse profile, the device functions as a spatial light modulator. It can transform a circularly polarized plane wave into a high-quality optical vortex across a broad spectrum (i.e., a “white” vortex), provided that the waveguiding parameter u is sufficiently large for all relevant wavelengths.

III. ANALYSIS OF A NONIDEAL TANDEM CELL

The idealized analysis presented in Sec. II serves purely to illustrate the fundamental operating principle of the tandem architecture. The resulting transmission matrix (17) relies on the mathematical approximation of perfect adiabatic following, assuming that the light polarization ideally tracks the continuously twisting director. However, for any physically realizable device with a finite optical thickness, this waveguiding effect is never truly perfect. The actual polarization state inevitably deviates from the ideal parallel and orthogonal orientations relative to the local director. Furthermore, practical fabrication introduces inevitable thickness mismatches between the two constituent cells. Therefore, to accurately evaluate the device performance under actual polychromatic illumination, we must transition from this simplified approximation to a rigorous physical model.

In this section we develop a comprehensive analytical framework to investigate how these specific physical realities—imperfect waveguiding, and manufacturing tolerances—generate fundamental amplitude and phase distortions arising from the nonideal behavior of this specific tandem cell architecture.

A. Perturbation model and small parameters

To construct the rigorous physical model, we maintain the exact tandem configuration established in Sec. II (crossed interfaces, individual geometric twists of $\varphi/2$, and a total twist of φ). However, we now account for two primary nonidealities: finite optical thickness and imperfect matching of the constituent cell thicknesses (h_1 and h_2).

While an exact analytical expression for the total Jones matrix of this system can be derived by directly multiplying $T_{\text{TN2}}^{\mathcal{E}}(h_2)T_{\text{TN1}}^{\mathcal{E}}(h_1)$, the resulting equations are excessively cumbersome and obscure the underlying physics. A more powerful approach is to expand the exact solution in terms of small, independent dimensionless parameters that characterize these nonidealities.

First, to quantify the deviation from perfect adiabatic waveguiding, we must transition from the classical Mauguin parameter $u = \frac{1}{2}k \frac{\Delta n h}{\phi_d}$ (introduced in Sec. II) to a variable better suited for perturbative analysis. We introduce the local nonadiabaticity parameter $\alpha_{\text{TN}} = q/\gamma'$. In the strong waveguiding limit ($u \gg 1$, meaning $q \ll \gamma$), this parameter is intrinsically small and inversely proportional to u : $\alpha_{\text{TN}} \approx q/\gamma = 1/u$. Because α_{TN} is bounded within $[0, 1]$, it serves as an ideal expansion parameter.

Second, we define the thickness mismatch parameter $\beta = (h_1 - h_2)/(h_2 + h_1)$. This parameter is bounded within $[-1, 1]$ and physically characterizes the relative (fractional)

geometric thickness mismatch between the two constituent cells in a specific device instance. If both $\alpha_{\text{TN}} \rightarrow 0$ and $\beta \rightarrow 0$, the system perfectly converges to the ideal analytical limit (17).

Because the two cells have distinct thicknesses, their individual nonadiabaticity parameters ($\alpha_1 = q_1/\gamma'_1$ and $\alpha_2 = q_2/\gamma'_2$) differ. However, they are mathematically coupled via β . To formulate a pair of strictly independent variables, we introduce the mean nonadiabaticity parameter α :

$$\alpha = \frac{1}{2} \left(\frac{q_1}{\gamma'_1} + \frac{q_2}{\gamma'_2} \right). \quad (18)$$

Together, α and β constitute our fundamental pair of independent small variables for the subsequent Taylor expansion.

$$T_{\text{CTN}}^{\varepsilon} = \begin{pmatrix} \exp(i\varphi)[\cos(\Phi_d) - i\alpha \sin(\Phi_s)] & i \sin(\Phi_d) + 2\alpha \sin(\gamma'_2 h_2) \sin(\gamma'_1 h_1) \\ i \sin(\Phi_d) - 2\alpha \sin(\gamma'_2 h_2) \sin(\gamma'_1 h_1) & \exp(-i\varphi)[\cos(\Phi_d) + i\alpha \sin(\Phi_s)] \end{pmatrix}. \quad (19)$$

Here the arguments Φ_s and Φ_d compactly describe the phase properties of the tandem system. Φ_s represents the cumulative phase retardation, which is the sum of the phase differences accumulated between the wave-guided eigenmodes in each of the two cells, given by

$$\Phi_s = \gamma'_1 h_1 + \gamma'_2 h_2. \quad (20)$$

Conversely, Φ_d represents the retardation mismatch between the two cells and is defined as

$$\Phi_d = \gamma'_1 h_1 - \gamma'_2 h_2. \quad (21)$$

This parameter quantifies the degree of phase imbalance in the system.

The small dimensionless parameters, α and β , introduced previously in this section, are now intuitively linked to these phase arguments. The thickness mismatch parameter, β , is directly proportional to the phase imbalance: $\beta \approx \Phi_d/\Phi_s$. Similarly, the mean nonadiabaticity parameter α can be directly related to the total director twist φ and the cumulative retardation Φ_s as $\alpha \approx \varphi/\Phi_s$. This explicit dependence demonstrates that the ideal system behavior (corresponding to $\alpha \rightarrow 0$ and $\beta \rightarrow 0$) is approached when the cumulative retardation significantly exceeds the total director twist angle ($\Phi_s \gg \varphi$) and the retardation imbalance vanishes ($\Phi_d \rightarrow 0$), thereby explicitly highlighting both the Mauguin regime condition and the structural symmetry of the tandem architecture.

B. Transmission analysis and filtering of parasitic components

Analyzing the effect of the nonidealities reveals a key difference from the ideal case. When circularly polarized light (e.g., RHC) passes through the system, a parasitic, orthogonally polarized (LHC) component emerges in addition to the primary RHC wave. This unwanted component lacks the desired vortex-forming phase modulation and would otherwise degrade the quality of the output beam.

Fortunately, the parasitic component can be effectively filtered out using a standard polarimetric setup similar to that proposed in [40]. The optical configuration matches the input and output stages depicted in Fig. 3 (noting that for the un-

To facilitate a rigorous analysis capturing both phase and amplitude effects, we perform a perturbative expansion of the exact transmission matrix. Crucially, we expand only the amplitude coefficients, while leaving the arguments of the highly oscillatory trigonometric functions strictly unexpanded. Modifying these phase arguments perturbatively would lead to significant cumulative errors.

The complete, rigorous expressions incorporating full second-order corrections (including α^2 , β^2 , and the cross-term $\alpha\beta$) are detailed in Appendix B. For clarity in the main text, we present the resulting transmission matrix retaining the most dominant perturbational terms:

compensated analysis in this section, the compensating phase plate (CPP) is omitted). The optical train consists of an input circular polarization generator (a linear polarizer followed by an achromatic quarter-wave plate) and an output circular polarization analyzer (a second achromatic quarter-wave plate followed by a linear analyzer).

A key distinction dictates the specific configuration of the output stage. The q -plate described in [40] acts as a half-wave plate, inverting the polarization handedness [e.g., transforming right-handed circular (RHC) to left-handed circular (LHC)]. Consequently, their output stage is configured to transmit the orthogonal polarization (LHC). In contrast, our tandem TN cell operates as a polarization rotator, preserving the incident handedness (RHC remains RHC, with only a phase modulation). Therefore, the output filtering stage must be configured to transmit the same circular polarization state as generated at the input. This implies that the relative orientation of the final Analyzer with respect to the second quarter-wave plate (QWP) must be set to pass the incident helicity, whereas the q -plate scheme requires a configuration that isolates the orthogonal (opposite) helicity.

To account for the full physical dependencies including finite-thickness and mismatch corrections, we utilize the rigorous second-order transmission coefficient T_{11} derived in Appendix B [Eqs. (B13)–(B16)], which governs the principal polarized component passing through the entire optical system:

$$T_{\text{CTN}} = \exp(i\varphi)[\cos(\Phi_d) - i\alpha \sin(\Phi_s) - 2\alpha^2 \sin(\gamma'_1 h_1) \sin(\gamma'_2 h_2) - i\alpha\beta \sin(\Phi_d)]. \quad (22)$$

In the limit where $|\cos(\Phi_d)| \gg \alpha$ and $|\beta| \ll 1$, the transmission coefficient can be approximated in a form that explicitly separates its amplitude and phase components:

$$T_{\text{CTN}} \approx \cos(\Phi_d) \exp \left(i\varphi \left[1 - \frac{\sin(\Phi_s)}{\Phi_s \cos(\Phi_d)} \right] \right). \quad (23)$$

Equation (23) explicitly separates the optical response into an amplitude envelope and a phase modulation. In the ideal case of perfectly matched cells ($\beta = 0$, thus $\Phi_d = 0$), the

transmission is lossless. However, in a real device, manufacturing tolerances introduce a retardation mismatch Φ_d , leading to wavelength-dependent power losses governed primarily by the $\cos(\Phi_d)$ factor. For the regime of interest—thick cells designed to minimize phase distortion—the sensitivity to this retardation mismatch increases, as the absolute phase imbalance scales directly with the mean cell thickness h ($\Phi_d \approx \beta \Phi_s \propto h$). Consequently, the geometric mismatch typically dominates the macroscopic power loss mechanism.

The phase response of the system consists of two parts: the desired linear phase shift proportional to the total director twist angle φ , and an unwanted perturbation term. In the ideal Mauguin regime ($\alpha \rightarrow 0$, which implies $\Phi_s \rightarrow \infty$), this perturbation vanishes, and the imparted phase is exactly equal to φ . For a nonzero α , a phase distortion appears. The magnitude of this distortion is directly proportional to α , and its value oscillates as a function of wavelength with a periodicity determined by the cumulative phase retardation, Φ_s .

In the limit of small α and β , the relationship $\beta \approx \Phi_d/\Phi_s$ holds. If Φ_s is large, the phase perturbation becomes a rapidly oscillating function of wavelength. To suppress this phase distortion, one must reduce α , which can be achieved by increasing the mean cell thickness h .

However, this presents a significant practical trade-off. While increasing the cell thickness improves the phase fidelity, it simultaneously makes it more challenging to maintain a small absolute thickness difference ($h_1 - h_2$). A larger absolute difference leads to a larger Φ_d , which in turn increases the chromatic power losses from the amplitude modulation term. Thus, a manufacturer would face a compromise: improving the phase response at the cost of worsening the amplitude response.

C. Figures of merit for nonachromaticity

As established in the preceding analysis, the exact amplitude and phase errors of the tandem cell are characterized by complex spectral oscillations driven by the cumulative phase retardation Φ_s . Because Φ_s depends on both the wavelength and the cell thickness, these oscillating interference terms can shift significantly with even minor changes in the physical parameters. To quantitatively evaluate the device's true broadband performance without being constrained by the exact positions of these oscillating peaks, we introduce specific figures of merit (FoMs) for achromatic fidelity. The physical rationale behind these FoMs is to construct a robust, smooth, slowly varying envelope that strictly bounds these oscillations from above. Consequently, a FoM serves as a guaranteed worst-case limit for the device error at any given wavelength, determined entirely by the fundamental cell configuration and operational parameters.

Amplitude nonachromaticity ($\mathcal{A}$). The amplitude nonachromaticity ($\mathcal{A}$) serves as this spectral envelope for the amplitude loss. It is defined by analytically maximizing the oscillating terms (denoted by the operator $\max_{\text{local}}$) to extract the local upper bound for the fractional reduction in amplitude transmission at a given wavelength λ :

$$\mathcal{A} = \max_{\text{local}} (1 - |T(\lambda)|), \quad (24)$$

where T is the transmission coefficient of the system. By applying this local maximization, $\mathcal{A}$ becomes a smooth function of wavelength that estimates the worst-case loss magnitude for the current optical parameters, providing a clear boundary for amplitude transmission efficiency.

Phase nonachromaticity. Similarly, the phase nonachromaticity ($\mathcal{P}$) quantifies the absolute worst-case smooth envelope of the chromatic error in the generated phase shift, $\delta\varphi = \arg(T) - \varphi$. For a device generating a phase modulation within the range $[-\Delta\varphi, \Delta\varphi]$ (where $|\varphi| \leq \Delta\varphi$), the total phase nonachromaticity is formally defined by applying the same local maximization over the oscillating components:

$$\mathcal{P} = \max_{\text{local}} |\delta\varphi(\lambda, \varphi)|. \quad (25)$$

To construct a highly insightful global bound, we partition this maximum error into two distinct components:

$$\mathcal{P} = \mathcal{P}_0 + \Delta\varphi \mathcal{P}_1, \quad (26)$$

where $\mathcal{P}_0$ is defined as the *constant phase distortion bound* (representing the component of the total bound $\mathcal{P}$ that is independent of $\Delta\varphi$), and $\mathcal{P}_1$ is defined as the *angular phase distortion bound* (representing the component of $\mathcal{P}$ that depends on $\Delta\varphi$). As the subsequent analysis will demonstrate, this specific decomposition is both physically meaningful and mathematically justified. Specifically, we will show that within the regime of small perturbation parameters (α and β), the derived envelopes $\mathcal{P}_0$ and $\mathcal{P}_1$ act as fundamental system characteristics governed by the physical cell configuration and manufacturing tolerances, remaining essentially independent of the maximum modulation phase $\Delta\varphi$. Therefore, throughout the following sections, we will explicitly evaluate these individual terms to construct the complete global phase distortion bound $\mathcal{P}$.

D. Nonachromaticity of the nonideal tandem cell

The complex transmission coefficient Eq. (22) describes the optical response of a specific device instance characterized by a realized fractional thickness mismatch β and a mean nonadiabaticity parameter α . However, our objective is to derive the robust figures of merit defined in Sec. III C, which must guarantee the performance not just for one specific cell, but for any device produced under a given technological process. To achieve this, we introduce the manufacturing tolerance parameter $b \ll 1$. While β is a random variable that can be positive, negative, or zero for any individual tandem pair, its magnitude is strictly bounded by the fabrication precision: $|\beta| \leq b$. By evaluating the transmission expressions at this maximum absolute boundary b , we transition from describing a single device to establishing the absolute worst-case estimate for the entire manufactured batch.

Before deriving these bounds, we must establish the valid operational regime. As is evident from the transmission coefficient, if the cosine of the retardation mismatch ($\cos \Phi_d$) approaches zero, the amplitude vanishes and the target phase φ is completely corrupted. Therefore, while we evaluate the worst-case scenarios across all possible random mismatches $\beta \in [-b, b]$, we must strictly require that the system never crosses the transmission node. This imposes the conditions that $\Phi_d \in [0, \pi/2)$ and $\cos(\Phi_d) \gg \alpha$.

Within this valid regime, the rigorous amplitude transmission magnitude for a specific device, expanding up to the second order of small parameters α and β , is given by

$$|T_{\text{CTN}}| \approx \cos(\Phi_d) + \alpha^2 \left[\frac{1 \sin^2(\Phi_s)}{2 \cos(\Phi_d)} - \cos(\Phi_d) + \cos(\Phi_s) \right]. \quad (27)$$

To estimate the worst-case amplitude nonachromaticity ($\mathcal{A}^{(0)}$) for the specified tolerance b , we evaluate this magnitude at the maximum possible mismatch ($\Phi_d \approx b\Phi_s$). The upper bound for the transmission loss ($1 - |T_{\text{CTN}}|$) is explicitly found to be

$$\mathcal{A}^{(0)} = 2 \sin^2\left(\frac{1}{2}b\Phi_s\right) + 2\alpha^2. \quad (28)$$

The coefficient $2\alpha^2$ in this bound is mathematically robust: the absolute value of the bracketed term in Eq. (27) never exceeds 2 as long as the operational restriction $\cos(b\Phi_s) \geq \sqrt{5} - 2 \approx 0.236$ is satisfied.

The phase distortion for a specific device is

$$\delta\varphi \approx -\varphi \frac{\sin(\Phi_s)}{\Phi_s \cos(\beta\Phi_s)}. \quad (29)$$

As this distortion is directly proportional to φ , the constant phase distortion is zero in this approximation: $\mathcal{P}_0^{(0)} = 0$. To estimate the worst-case angular phase distortion ($\mathcal{P}_1^{(0)}$), we again evaluate the function at the tolerance bound b and assume the worst-case scenario for the oscillating terms within the bandwidth:

$$\mathcal{P}_1^{(0)} = \frac{1}{\Phi_s \cos(b\Phi_s)}. \quad (30)$$

The operational conditions established above for the amplitude analysis are also sufficient for this estimate to be valid.

While these figures of merit are derived and expressed in terms of the analytical parameters Φ_s , b , and α to clearly illustrate the mathematical asymptotic behavior, they can be directly translated into tangible manufacturing variables. For practical engineering design, the cumulative retardation and the mean nonadiabaticity parameter can be explicitly evaluated using the fundamental fabrication specifications: the mean cell thickness h , the material birefringence Δn , the free-space wave vector $k = 2\pi/\lambda$, and the maximum required modulation angle $\Delta\varphi$. By substituting the geometric approximations

$$\Phi_s \approx \sqrt{(\Delta n k h)^2 + (\Delta\varphi)^2}, \quad \alpha \approx \frac{\Delta\varphi}{\Phi_s} \quad (31)$$

into the FoM equations, a manufacturer can directly compute the guaranteed theoretical bounds for amplitude and phase distortions across the entire spectral band based solely on the initial physical design parameters.

To validate the derived analytical expressions and the proposed figures of merit, we performed exact numerical simulations via direct multiplication of the relevant Jones matrices for each optical element. Figure 2 presents the comparison between this exact numerical solution and our analytical approximations for a representative cell with a mean thickness of $h = 10 \mu\text{m}$ and a thickness mismatch of $b = 0.02$. Here the “second-order approximation” refers to the full analytical expression given by Eq. (22), while the “first-order

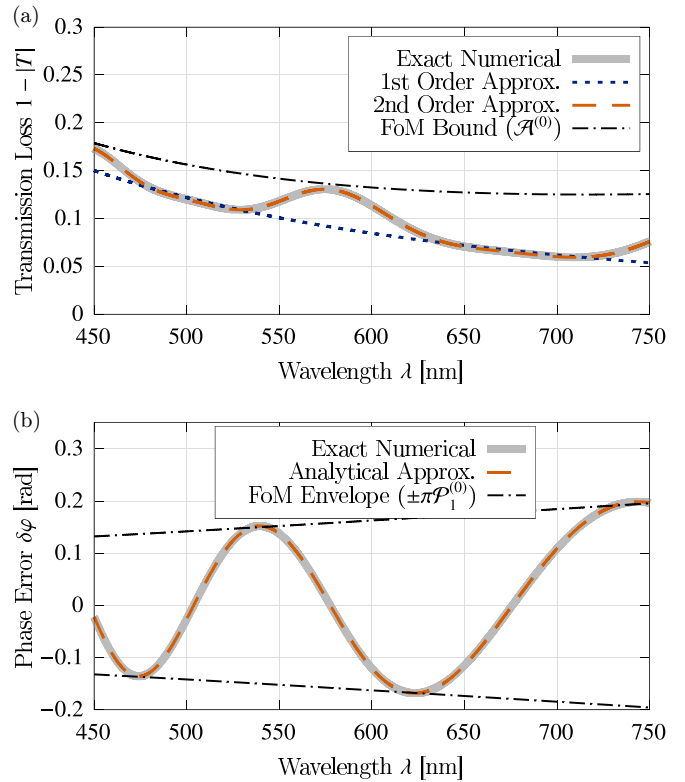


FIG. 2. Numerical validation of the theoretical model for a nonideal tandem cell. (a) Amplitude transmission loss ($1 - |T|$). (b) Phase error ($\delta\varphi$). Simulation parameters: single-cell thickness $h = 10 \mu\text{m}$, $\beta = b = 0.02$, $\Delta n = 0.2$, $\varphi = \pi$.

approximation” corresponds to the same equation with the second-order perturbation terms (those containing α^2 and $\alpha\beta$) omitted.

As shown in Fig. 2(a), the first-order approximation (dotted line) correctly captures the spectral trend of the transmission loss but exhibits a magnitude offset. In contrast, the second-order approximation (dashed line) is virtually indistinguishable from the exact numerical curve (solid gray line), confirming that retaining quadratic terms is essential for precise amplitude evaluation. The amplitude FoM ($\mathcal{A}^{(0)}$), plotted as the black dash-dotted line, correctly predicts the upper bound of these losses.

Similarly, Fig. 2(b) demonstrates that the phase error oscillates within the envelope defined by the total phase nonachromaticity evaluated for the target modulation phase $\Delta\varphi = \pi$, specifically bounded by $\mathcal{P}^{(0)}(\pi) = \pi\mathcal{P}_1^{(0)}$ (dash-dotted lines). Note that in Fig. 2(b), only a single analytical curve (Analytical Approx.) is shown, as the first- and second-order approximations are visually indistinguishable for the phase response on this scale. This confirms that our derived metrics provide a reliable worst-case estimate for the device performance.

For a more comprehensive validation, a detailed analysis is provided in the Supplemental Material [47]. This includes additional graphical comparisons for various parameter sets and a quantitative evaluation of how accurately the second-order approximation matches the exact direct Jones matrix calculations.

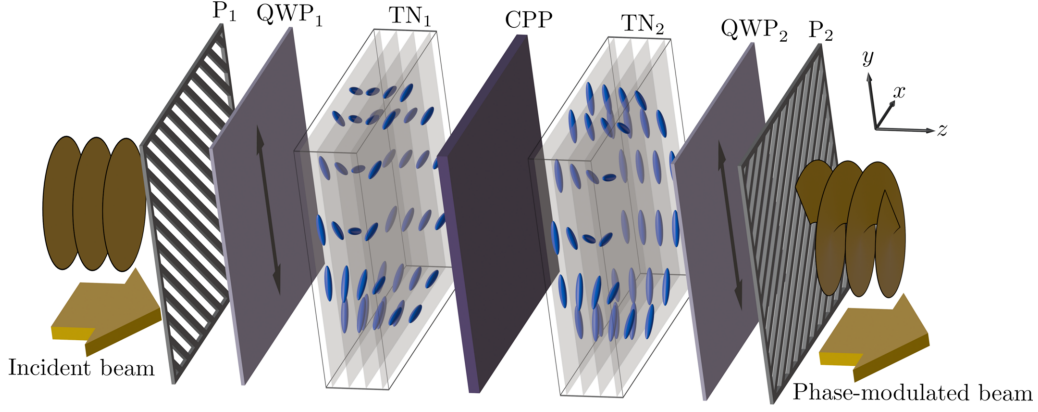


FIG. 3. Schematic of the proposed optical setup for the generation and filtering of optical vortices. The incident beam is converted to circular polarization by the input stage consisting of a linear polarizer (P_1 , polarization axis indicated by the parallel grating lines) and an achromatic quarter-wave plate (QWP_1 , slow axis indicated by the double-headed arrow). The tandem cell structure (TN_1 and TN_2) imparts the desired phase profile to the beam. A compensating phase plate (CPP) is inserted at the interface to correct the system's chromatic response: it is used to nullify manufacturing-induced amplitude modulation (Strategy A) or to suppress phase distortion (Strategy B). The output stage, comprising a second quarter-wave plate (QWP_2) and an analyzer (P_2), filters out the unwanted parasitic polarization component (the component with inverted handedness and phase), transmitting only the vortex beam.

IV. COMPENSATION STRATEGIES AND DISCUSSION

A. Mitigation of nonidealities: The compensated system model

As established in Sec. III, the nonideal tandem cell's performance is governed by a fundamental manufacturing trade-off. Minimizing the angular phase distortion ($\mathcal{P}$) requires a large cumulative retardation Φ_s . However, a large Φ_s simultaneously amplifies the system's sensitivity to manufacturing tolerances, leading to severe amplitude nonachromaticity ($\mathcal{A}$). This distortion scales with the maximum thickness inaccuracy (b) and cumulative retardation Φ_s as shown in Eq. (28). To overcome this limitation and decouple the phase modulation from the amplitude artifacts, we propose a modified device architecture. While the director orientations at the external boundaries of the tandem cell depend on the overall twist angle, the orientations at the interface between the two cells are fixed (orthogonally along the x and y axes). A tunable phase retarder (e.g., a voltage-controlled liquid crystal cell) can therefore be inserted at this interface (depicted as CPP in Fig. 3). This element introduces a controllable retardation between the x and y polarizations.

The calibration procedure is straightforward: by adjusting the voltage on the compensation cell to maximize the total light transmission through the entire optical train (including the final analyzer), one can actively nullify the effect of Φ_d . This active compensation effectively eliminates the dominant source of amplitude modulation (the $\cos \Phi_d$ term). While minor second-order effects related to the finite mean nonadiabaticity parameter persist (as detailed in Appendix C), the compensated tandem cell achieves a highly achromatic amplitude response. Its overall transformation is described by

$$T'_{\text{CTN}} = \exp(i\varphi) \cos(\Phi_d + \Phi_c) - i\alpha \exp(i\varphi) \cos(\Phi_c) \sin(\Phi_s). \quad (32)$$

This expression represents the dominant terms derived from the rigorous analysis. While Eq. (32) corresponds to the

first-order approximation in terms of the small parameters α and β , the complete derivation utilizing the second-order approximation is provided in Appendix C. The new parameter, Φ_c , which appears in this equation, represents the controllable phase retardation introduced by the compensating element.

For this derivation, the compensator was assumed to be oriented with its slow axis along the x direction (aligned with the director orientation at the output of the first TN cell) and its fast axis along the y direction. Thus, Φ_c is specifically defined as the phase lag of the x -polarized component relative to the y -polarized component. In the limit where $|\cos(\Phi_d + \Phi_c)| \gg \alpha$, we can rewrite Eq. (32) in the following form:

$$T'_{\text{CTN}} = \cos(\Phi_d + \Phi_c) \times \exp \left\{ i\varphi \left[1 - \frac{\sin(\Phi_s) \cos(\Phi_c)}{\Phi_s \cos(\Phi_d + \Phi_c)} \right] \right\}. \quad (33)$$

As established, the performance of the tandem cell is limited by both amplitude and phase distortions. With the general theoretical framework established in Eq. (33), we can now identify specific physical implementations of the compensation retardation Φ_c . We present these as a hierarchy of strategies, ranging from a rigorous active control scheme to simplified passive designs. We begin by analyzing the most comprehensive approach—active compensation—which theoretically allows for the complete elimination of manufacturing nonidealities and the achievement of perfect broadband performance.

B. Strategy A: Active compensation for full achromaticity

The first approach targets a complete, broadband compensation of the amplitude modulation. It involves using a tunable retarder as the compensating element. Through an active calibration process (e.g., maximizing light transmission), the compensator's retardation, Φ_c , is adjusted to satisfy the condition $\Phi_d + \Phi_c = 0$. This effectively nullifies the manufacturing mismatch Φ_d across the spectrum.

Evaluating this strategy using the figures of merit defined in Sec. III C, and referring to the rigorous second-order analysis in Appendix C, we find that the active compensation theoretically eliminates chromatic power loss:

$$\mathcal{A}^{(A)} = 2\alpha^2. \quad (34)$$

This result indicates that while the compensator corrects for the retardation errors (Φ_d), the fundamental scattering losses into the orthogonal polarization mode (proportional to α^2) persist. However, since $\alpha \propto 1/\Phi_s$, these losses diminish rapidly for thicker cells ($\mathcal{A}^{(A)} \propto \Phi_s^{-2}$), confirming the high efficiency of the approach.

The phase fidelity is also maintained. As in the uncompensated case, the constant phase distortion remains zero ($\mathcal{P}_0^{(A)} = 0$). However, the angular phase distortion is now determined solely by the cumulative retardation:

$$\mathcal{P}_1^{(A)} = \frac{1}{\Phi_s}. \quad (35)$$

Unlike the uncompensated case, where performance is dictated by manufacturing tolerances (via the parameter b), Strategy A effectively decouples the amplitude fidelity from fabrication errors. The residual loss arises purely from the nonadiabatic nature of light propagation in the twisted medium (the finite mean nonadiabaticity parameter). Since the nonideality parameter scales inversely with the cumulative retardation ($\alpha \approx \varphi/\Phi_s$), these residual second-order effects diminish rapidly with increasing cell thickness.

In summary, Strategy A offers a pathway to near-ideal performance where the dominant manufacturing-induced artifacts are nullified, and the ultimate fidelity is limited only by the optical thickness of the cells.

To verify the effectiveness of Strategy A, we performed numerical simulations for an optically thick tandem system ($h = 100 \mu\text{m}$) subject to a significant manufacturing error ($b = 0.05$). The results are presented in Fig. 4.

Figure 4(a) illustrates the residual amplitude nonachromaticity on a logarithmic scale. The exact numerical result (solid black line) demonstrates that the active compensation successfully nullifies the dominant manufacturing-induced modulation ($\cos(b\Phi_s)$ term), reducing losses to the level of 10^{-4} . The residual signal is bounded strictly by the theoretical limit $\mathcal{A}^{(A)} = 2\alpha^2$ (dashed orange line), confirming that the performance is limited only by the adiabaticity of the waveguide. It should be noted that these values represent the polarization conversion efficiency; static insertion losses such as Fresnel reflections are excluded from this analysis.

Simultaneously, Fig. 4(b) confirms the high phase fidelity. By enabling the use of thick cells without the penalty of amplitude modulation, the angular phase distortion is suppressed to negligible levels, remaining well within the theoretical envelope defined by $\pm\pi\mathcal{P}_1^{(A)} = \pm\pi/\Phi_s$.

C. STRATEGY B: PASSIVE COMPENSATION WITH A QUARTER-WAVE PLATE

An alternative strategy prioritizes design simplicity using a static (passive) compensator. The core of this approach is to use a nonachromatic quarter-wave plate (QWP) as the compensating element, which sets the nominal retardation to

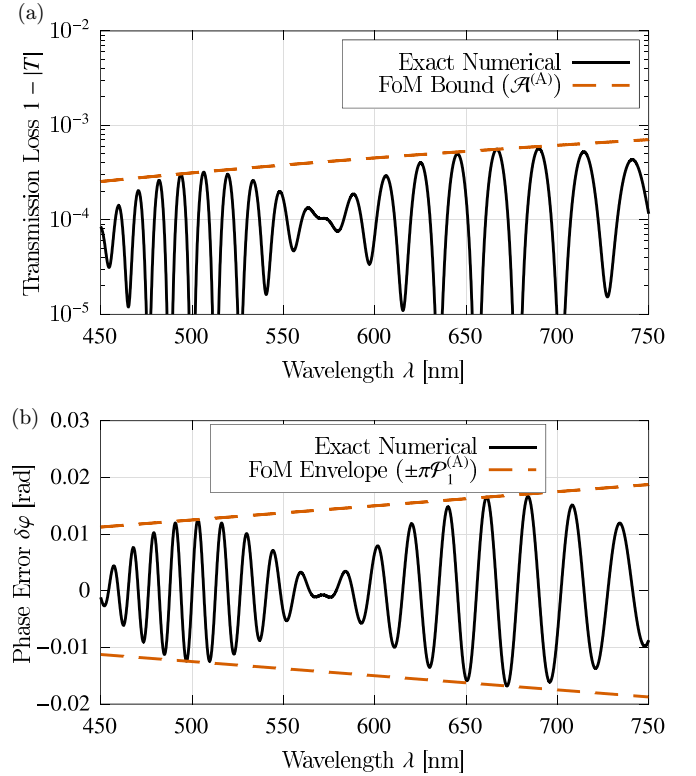


FIG. 4. Numerical verification of the active compensation strategy (Strategy A). (a) Residual amplitude transmission loss ($1 - |T|$) on a logarithmic scale. (b) Phase error ($\delta\varphi$). The simulation confirms that active tuning suppresses manufacturing artifacts, reducing errors to the fundamental limit defined by second-order Mauguin scattering ($\mathcal{A}^{(A)} = 2\alpha^2$). Simulation parameters: single-cell thickness $h = 100 \mu\text{m}$, $\beta = b = 0.05$, $\Delta n = 0.2$, $\varphi = \pi$.

$\Phi_c \approx \pi/2$ at the central angular frequency ω_0 . The primary role of the QWP is to suppress the phase perturbation term (by setting $\cos(\Phi_c) \approx 0$). This choice imposes a new design constraint: the two TN cells must be intentionally designed to be nonidentical to satisfy the dispersion-compensated amplitude condition $\Phi_2 \approx \Phi_1 + \Phi_c$ across a broad spectral range.

The device is considered quasi-achromatic because its performance is constrained by two distinct physical limitations:

(1) *Residual phase error*: The phase perturbation is only perfectly nullified exactly at ω_0 (corresponding wavelength we denote as λ_0). At other frequencies ($\omega \neq \omega_0$), $\cos(\Phi_c)$ will be small but nonzero, leading to a residual, chromatic phase error.

(2) *Amplitude distortion*: Due to manufacturing tolerances, the dispersion-compensated condition ($\Phi_2 = \Phi_1 + \Phi_c$) can never be perfectly met, resulting in a slight, wavelength-dependent amplitude modulation.

These limitations are quantified by the following figures of merit:

$$\mathcal{A}^{(B)} = 2 \sin^2 \left(\frac{1}{2} b \Phi_s \right) + \frac{1}{2} \alpha^2 + 2\alpha^2 \left| \sin \left(\frac{\pi}{2} \frac{\omega - \omega_0}{\omega_0} \right) \right|, \quad (36)$$

$$\mathcal{P}_1^{(B)} = \frac{\left| \sin \left(\frac{\pi}{2} \frac{\omega - \omega_0}{\omega_0} \right) \right|}{\Phi_s \cos(b\Phi_s)}. \quad (37)$$

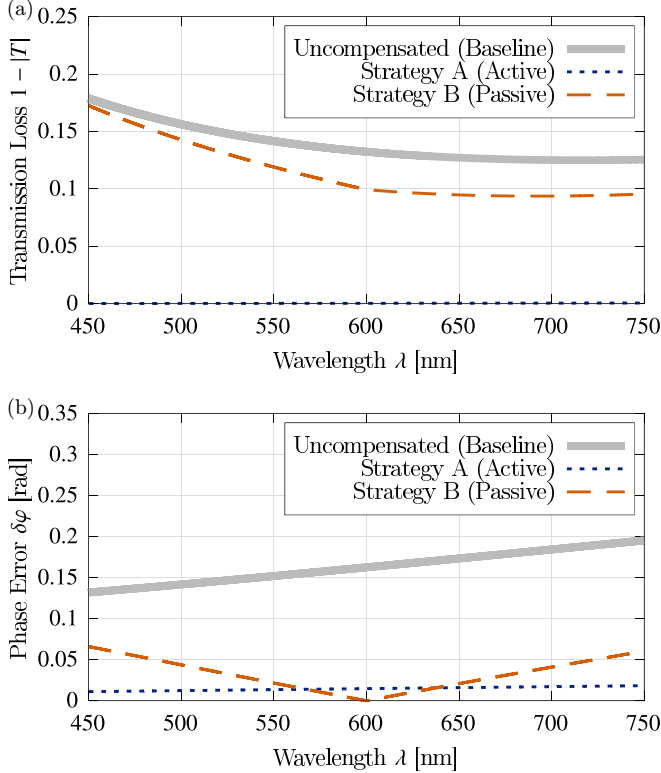


FIG. 5. Comparative analysis of the figures of merit for the proposed compensation strategies. (a) Amplitude nonachromaticity ($\mathcal{A}$). The uncompensated baseline $\mathcal{A}^{(0)}$ (solid gray) and passive strategy $\mathcal{A}^{(B)}$ (dashed line) are of comparable magnitude. The dotted line corresponds to active strategy $\mathcal{A}^{(A)}$. (b) Angular phase distortion bounds ($\mathcal{P} = \pi \mathcal{P}_1$). The solid gray line represents the uncompensated baseline ($h = 10 \mu\text{m}$, $b = 0.02$). The dashed line corresponds to passive Strategy B ($h = 10 \mu\text{m}$, $b = 0.02$, $\lambda_0 = 600 \text{ nm}$). The dotted line represents active Strategy A ($h = 100 \mu\text{m}$, $b = 0.05$), $\Delta n = 0.2$.

These bounds are valid under the established operational regime $b\Phi_s \in [0, \arctan(\frac{3}{2})]$ and $\cos(b\Phi_s) \gg \alpha$. A detailed derivation of these figures of merit is provided in Appendix D. As the distortion remains proportional to the desired phase φ , the constant phase distortion is zero: $\mathcal{P}_0^{(B)} = 0$.

Ultimately, Strategy B is effective within a spectral range where both the residual amplitude modulation and the residual phase perturbation are below an acceptable threshold.

The direct quantitative comparison of the strategies is summarized in Fig. 5.

Figure 5(a) reveals the practical limitation of passive compensation regarding amplitude fidelity. The curves for the uncompensated baseline (gray) and passive Strategy B (dashed line) overlap significantly. This indicates that for typical manufacturing tolerances ($b = 0.02$ in this simulation), the amplitude nonachromaticity is dominated by the retardation mismatch term $[\propto \sin^2(\frac{1}{2}b\Phi_s)]$, which overshadows the secondary contributions from the mean nonadiabaticity parameter or dispersion effects. Consequently, without active tuning to correct the thickness error, the passive strategy remains vulnerable to fabrication defects. In sharp contrast, active Strategy A (dotted line) reduces modulation to negligible levels. Notably, this result is achieved even with a

deliberately relaxed manufacturing tolerance ($b = 0.05$) and a thicker cell, proving that active tuning effectively decouples amplitude fidelity from fabrication defects.

Figure 5(b) highlights the trade-off in phase fidelity. Passive Strategy B exhibits a characteristic “V-shape” with a deep minimum at the central wavelength, indicating successful dispersion matching. However, the error rises at the spectral edges. Conversely, active Strategy A maintains a consistently low phase distortion across the entire spectrum. By enabling the use of optically thick cells ($100 \mu\text{m}$) without the penalty of amplitude modulation, this strategy suppresses the fundamental mean nonadiabaticity parameter α , resulting in superior broadband phase purity.

This comparison underscores a critical practical advantage: the active compensation mechanism (Strategy A) significantly relaxes manufacturing tolerances. As illustrated in Fig. 5, the active system suppresses amplitude modulation to negligible levels even with optically thick cells ($h = 100 \mu\text{m}$) and relaxed tolerances ($b = 0.05$), distinctly outperforming passive designs despite their tighter constraints ($h = 10 \mu\text{m}$, $b = 0.02$).

Ultimately, it should be emphasized that these two strategies target fundamentally different sources of error. Strategy A is primarily designed to actively compensate for amplitude losses arising from fabrication mismatches, whereas Strategy B relies on a static quarter-wave plate to passively suppress phase distortions. Consequently, the primary comparative analysis in Fig. 5 deliberately highlights each strategy within its optimal geometric regime: Strategy A leverages thick cells to inherently suppress the nonadiabatic phase errors, while Strategy B requires thinner cells to minimize its vulnerability to mismatch-induced amplitude losses. For a comprehensive side-by-side evaluation of strategies under strictly identical cell geometries, additional comparative analysis is provided in the Supplemental Material [47].

D. Strategy C: Optimized operating point for narrow-band applications

For applications limited to a relatively narrow spectral bandwidth, an alternative design approach can be employed. This strategy is applicable when the cells are manufactured with sufficient precision such that the retardation mismatch is small ($\Phi_d \ll 1$). In this regime, we can exploit the intrinsic properties of the transmission function to minimize chromatic dispersion without external compensation.

By analyzing the phase term, we observe that the most rapidly oscillating component is governed by the sinc-like function, $\sin(\Phi_s)/\Phi_s$. The derivative of this function vanishes at its local extrema, implying that in the vicinity of these points, the phase perturbation is minimally sensitive to changes in Φ_s (and thus to wavelength). This allows for a “passive” compensation strategy by engineering the cell thickness such that the cumulative retardation at the central frequency ω_0 , denoted as Φ_{s0} , coincides with a local extremum of the perturbation function.

However, this approach requires a redefinition of the device’s modulation capability. In this regime, the effective phase modulation of the optical vortex, φ_{mod} , is no longer identical to the geometric twist angle φ . It is scaled by the

perturbation factor:

$$\varphi_{\text{mod}} = \varphi \left[1 - \frac{\sin(\Phi_{s0})}{\Phi_{s0}} \right]. \quad (38)$$

To ensure sufficient modulation efficiency (specifically, to ensure a scaling factor ≥ 1 , which enables a full 2π phase stroke within the standard director rotation range), we must specifically target the local minima of the sinc function.

Evaluating this optimized strategy using our figures of merit (derivation provided in Appendix E).

The amplitude response is still governed by the thickness manufacturing tolerance b :

$$\mathcal{A}^{(C)} = \frac{1}{2}(b\Phi_s)^2 + \frac{1}{2}\alpha^2. \quad (39)$$

Unlike previous strategies, the constant phase distortion is nonzero. This arises because the optimal retardation condition (Φ_{s0}^{opt}) depends on the local twist angle φ . Since φ varies spatially across the vortex beam while the cell thickness h remains constant, the optimization condition cannot be perfectly satisfied at all points simultaneously. However, this error can be minimized by selecting an optimal design angle $\varphi^{\text{opt}} = \frac{\sqrt{3}}{2}\pi$ (see derivation in Appendix E).

For a design optimized for twist angle ($\varphi^{\text{opt}} = \frac{\sqrt{3}}{2}\pi$), the constant phase distortion is

$$\mathcal{P}_0^{(C)} = \frac{1}{16\Phi_{s0}} \frac{\Delta\omega}{\omega_0} \pi^3. \quad (40)$$

The angular phase distortion $\mathcal{P}_1^{(C)}$ remains dominated by a second-order quadratic term and a term coupling the bandwidth with the manufacturing tolerance:

$$\mathcal{P}_1^{(C)} = \frac{1}{8}\Phi_{s0} \left(\frac{\Delta\omega}{\omega_0} \right)^2 + \frac{1}{2}b\Phi_{s0} \frac{\Delta\omega}{\omega_0}. \quad (41)$$

Here $\Delta\omega/\omega_0$ represents the relative spectral bandwidth.

This strategy demonstrates that for narrow-band sources (where $\Delta\omega/\omega_0$ is small) and high-precision manufacturing (small b), exceptional phase fidelity can be achieved without any external compensation components, simply by selecting the correct optical thickness.

E. Impact of material birefringence dispersion

In the preceding derivations, the figures of merit and transmission matrices were expressed explicitly in terms of the free-space wavelength λ (via the parameter $\gamma = \frac{\pi}{\lambda}\Delta n$). However, real liquid crystal materials exhibit intrinsic birefringence dispersion, $\Delta n(\lambda)$, which typically decreases with increasing wavelength in the visible spectrum. A critical question arises regarding how this material dispersion alters the previously derived amplitude ($\mathcal{A}$) and phase ($\mathcal{P}$) nonachromaticity metrics.

Unlike conventional broadband polarization converters (which often rely on stacking different materials to balance distinct dispersion curves), the tandem TN architecture benefits from a fundamental structural symmetry. This symmetry relies on the condition that all active anisotropic elements in the optical train—the first TN cell, the compensating retarder, and the second TN cell—are fabricated from the exact same

liquid crystal material. Consequently, they all share an identical dispersion law, $\Delta n(\lambda)$.

To fully evaluate this, we must distinguish between two types of dispersive effects. First, transitioning back to the laboratory Cartesian coordinate system introduces an overall dynamic phase factor $\exp(i \sum \vec{k}_j h_j)$. This scalar term represents standard homogeneous material dispersion. It acts uniformly across the entire beam cross section—exactly like passing through an unmodulated isotropic glass plate. Because it is independent of the local modulation angle φ , it dictates only the absolute optical path length and does not influence the differential phase modulation responsible for the topological structure of the optical vortex.

The second effect concerns the actual differential polarization dynamics. Let us examine the transmission matrix of a single layer, $T_{\text{TN}}^{\mathcal{E}}(h)$, as defined in Eq. (12). Crucially, the entire wavelength dependence of this matrix is encapsulated within a single variable: the anisotropy parameter $\gamma(\lambda) = \frac{\pi}{\lambda}\Delta n(\lambda)$. The other parameters governing the matrix structure—twist rate q , thickness h , twist angle φ_d , and mean orientation $\tilde{\varphi}$ —are purely geometric and strictly independent of the wavelength.

Because the global system matrix $T_{\text{system}}^{\mathcal{E}}$ is generated via the direct multiplication of these layer matrices, it inherits this property. Thus, the normalized transmission coefficient in the circular basis is mathematically a function of a single dispersive variable: $T_{\text{system}}^{\mathcal{E}} \equiv F(\gamma, \text{geometry})$.

This structural dependence allows us to relate the exact dispersive response of the system to a hypothetical dispersion-free baseline. Let $T_{\text{system}}^{\mathcal{E}\{0\}}(\lambda)$ denote the theoretical transmission matrix of the system if the birefringence were perfectly constant across the spectrum, fixed at its value at a central design wavelength λ_0 [i.e., assuming $\Delta n(\lambda) \equiv \Delta n(\lambda_0)$].

For the real physical system exhibiting dispersion, the transmission at any arbitrary wavelength λ is exactly equivalent to the transmission of the nondispersive baseline evaluated at a shifted “effective” wavelength $\tilde{\lambda}$. By equating the γ parameters, $\frac{\pi}{\lambda}\Delta n(\lambda_0) = \frac{\pi}{\tilde{\lambda}}\Delta n(\tilde{\lambda})$, we derive this exact mapping:

$$T_{\text{system}}^{\mathcal{E}}(\lambda) = T_{\text{system}}^{\mathcal{E}\{0\}} \left(\lambda \frac{\Delta n(\lambda_0)}{\Delta n(\lambda)} \right) = T_{\text{system}}^{\mathcal{E}\{0\}}(\tilde{\lambda}). \quad (42)$$

This mapping constitutes a powerful analytical tool. The introduction of any arbitrary material dispersion law $\Delta n(\lambda)$ does not introduce new physical artifacts or complex modulation-dependent cross-polarization scattering. Mathematically, material dispersion acts merely as a nonlinear coordinate stretching (or compression) of the wavelength axis relative to the dispersion-free case, leaving the central point λ_0 invariant.

Because this nonlinear mapping applies fundamentally to the entire transmission matrix $T_{\text{system}}^{\mathcal{E}}$, it universally governs all derived optical properties, including both the exact oscillating phase and amplitude errors, and their bounding figures of merit ($\mathcal{A}$ and $\mathcal{P}$). For realistic liquid crystal materials, the dispersive shift between λ and $\tilde{\lambda}$ is quantitatively small. While warping the spectral axis horizontally shifts the exact local values of the oscillating terms at a specific wavelength, these

actual errors remain strictly confined within the smoothly varying envelopes defined by the FoMs. Thus, the overall spectral profile remains visually and functionally intact. Therefore, provided the homogeneity of the liquid crystal material is maintained throughout all components, the tandem crossed TN architecture mathematically decouples material dispersion from the underlying mechanism of the spatial phase modulation.

The preceding analysis relies on the fundamental assumption that all anisotropic elements share an identical spectral dependence $\Delta n(\lambda)$. If different materials are employed (for instance, if the compensating retarder is fabricated from a different birefringent medium), their respective phase retardations will accumulate a relative chromatic mismatch across the spectrum. While such spectral desynchronization may shift the exact spectral positions of the oscillating error terms, its impact on the bounding figures of merit remains mathematically straightforward. Within the FoM framework, any difference in material dispersion acts purely as an additional source of retardation mismatch. For the uncompensated tandem cell and the passive Strategy B, this chromatic mismatch is accommodated by simply renormalizing (increasing) the effective manufacturing tolerance parameter b to encompass the maximum dispersive deviation over the target bandwidth. For the active Strategy A, a mismatch in dispersion between the TN cells and the active compensator manifests as a spectral drift in the applied compensation phase. This is mathematically equivalent to introducing a deviation parameter $\delta\Phi_c$, the resulting impact of which is explicitly evaluated in Sec. IV G 1.

F. Global mode purity and generation fidelity

The figures of merit ($\mathcal{A}$ and $\mathcal{P}$) derived in Sec. III C establish the local, worst-case bounds for amplitude and phase distortions. However, when the tandem cell architecture is employed as a two-dimensional spatial light modulator, the optical response varies continuously across the transverse plane (x, y) . To generate a phase profile spanning $[-\Delta\varphi, \Delta\varphi]$ (where $\Delta\varphi \geq \pi$ is required for full-cycle modulation, such as integer vortex generation), the local target phase modulation $\varphi(x, y)$ varies spatially. Consequently, the local nonadiabaticity parameter $\alpha(x, y) \approx \varphi(x, y)/\Phi_s$ varies accordingly. Furthermore, the manufacturing thickness mismatch parameter $\beta(x, y)$ may fluctuate randomly across the aperture within its strict tolerance bound $[-b, b]$.

To evaluate the macroscopic quality of the generated structured light beam under these spatially varying conditions, we must define a global fidelity metric. Since the final element of the optical train is a linear analyzer, the transmitted wave is a linearly polarized scalar field. We evaluate the generation quality using the modal overlap integral [48] (mode purity) η , defined between the realistically generated complex amplitude $E_{\text{real}}(x, y)$ and the theoretically ideal amplitude $E_{\text{ideal}}(x, y)$:

$$\eta = \frac{|\iint_S E_{\text{real}} E_{\text{ideal}}^* dS|^2}{\iint_S |E_{\text{real}}|^2 dS \iint_S |E_{\text{ideal}}|^2 dS}. \quad (43)$$

For a pure phase modulator, the ideal output field differs from the incident field $E_0(x, y)$ only by the target spatial phase

profile $\varphi(x, y)$:

$$E_{\text{ideal}} = E_0(x, y) \exp[i\varphi(x, y)]. \quad (44)$$

The realistic transmitted field incorporates the local amplitude transmission modulation $T(x, y) \in [0, 1]$ and the local phase error $\delta\varphi(x, y)$:

$$E_{\text{real}} = T(x, y) E_0(x, y) \exp\{i[\varphi(x, y) + \delta\varphi(x, y)]\}. \quad (45)$$

To facilitate a rigorous analytical approximation, we express the transmission magnitude as $T(x, y) = 1 - a(x, y)$, where $a(x, y)$ represents the local fractional amplitude loss. To condense the notation, we introduce a weighted spatial average operator $\langle f \rangle$, where the weighting function is the intensity profile of the incident beam $|E_0(x, y)|^2$:

$$\langle f \rangle = \frac{\iint_S f(x, y) |E_0(x, y)|^2 dS}{\iint_S |E_0(x, y)|^2 dS}. \quad (46)$$

Substituting these fields into the overlap integral (43), the mode purity can be compactly rewritten in terms of this averaging operator:

$$\eta = \frac{|\langle (1 - a) \exp(i\delta\varphi) \rangle|^2}{\langle (1 - a)^2 \rangle}. \quad (47)$$

Performing a Taylor series expansion with respect to the small local perturbations a and $\delta\varphi$ [up to the second order, taking $\exp(i\delta\varphi) \approx 1 + i\delta\varphi - \frac{1}{2}\delta\varphi^2$], the mode purity elegantly reduces to

$$\eta \approx 1 - \text{Var}(a) - \text{Var}(\delta\varphi), \quad (48)$$

where $\text{Var}(f)$ is a variance across the beam profile $\text{Var}(f) = \langle f^2 \rangle - \langle f \rangle^2$.

Because the exact spatial distributions of the manufacturing errors $\beta(x, y)$ and the beam profile $E_0(x, y)$ are generally unknown *a priori*, calculating the exact variances is impractical. However, we can establish a strict lower bound for the global mode purity using our previously derived worst-case figures of merit. By applying Popoviciu's inequality [49] on variances for bounded probability distributions, the maximum possible variance of any variable bounded within $[m, M]$ cannot exceed $\frac{1}{4}(M - m)^2$. Knowing that $a(x, y) \in [0, \mathcal{A}]$ and $|\delta\varphi(x, y)| \leq \mathcal{P}$, the mode purity is strictly bounded by

$$\eta \geq 1 - \frac{1}{4}\mathcal{A}^2 - \mathcal{P}^2. \quad (49)$$

This inequality provides a direct, computable criterion for the overall device performance. By its mathematical nature, this formulation yields a highly conservative, lower-bound estimate for the mode purity. This conservatism arises from two primary factors. First, Popoviciu's inequality assumes the absolute worst-case variance distribution. Second, the local amplitude and phase errors depend on the local nonadiabaticity parameter $\alpha(x, y)$, which is directly proportional to the spatial modulation angle $\varphi(x, y)$. To evaluate the global figures of merit $\mathcal{A}$ and $\mathcal{P}$, one must use the maximum modulation angle required by the device's operational stroke (e.g., $|\varphi|_{\text{max}} = \pi$ for an integer vortex). Consequently, across the majority of the beam profile where the local twist is smaller than this maximum, the actual physical distortions are significantly lower than these limiting bounds.

Despite being a conservative estimate, this formulation is exceptionally general. The right-hand side of Eq. (49) serves as a universal, wavelength-dependent device characteristic determined entirely by the fundamental structural parameters: the mean thickness h , the liquid crystal birefringence Δn , the manufacturing tolerance b , and the chosen compensation strategy. This establishes a strict optical engineering framework. If an application demands a guaranteed mode purity (e.g., $\eta > 0.99$), the designer must select the hardware parameters (h , b) and the compensation scheme to satisfy this threshold. Furthermore, this explicitly resolves the requirement for a computable bandwidth: the acceptable broadband operational regime is rigorously defined as the contiguous spectral band over which this target purity threshold is continuously maintained.

To illustrate the practical utility of this framework, we can evaluate the worst-case lower bound of the mode purity for a full-cycle phase modulation ($|\varphi|_{\max} = \pi$) across the visible spectrum ($\lambda \in [450, 750]$ nm), assuming a liquid crystal birefringence of $\Delta n = 0.2$. For a standard scenario with a cell thickness $h = 10$ μm and a tolerance $b = 0.02$, the uncompensated tandem baseline yields a purity estimate that never drops below $\eta = 0.957$ over the entire band. Applying the passive Strategy B (tuned to $\lambda_0 = 600$ nm) elevates this minimum guaranteed purity to $\eta = 0.978$.

By transitioning to the active Strategy A, the amplitude fidelity becomes decoupled from manufacturing tolerances. This allows the use of significantly thicker cells (e.g., $h = 20$ μm), which inherently suppresses the nonadiabaticity parameter α . Under these conditions, the lowest value of our conservative estimate across the entire broadband spectrum reaches $\eta = 0.991$. Given the strictly bounding nature of Eq. (49), the exact theoretical mode purity—such as that evaluated via direct Jones matrix integration—will consistently exceed these calculated minimums. Thus, these values establish a rigorous baseline for the intrinsic capabilities of the device, isolating its fundamental physical limits from additional secondary hardware and operational nonidealities inevitably present in a real experimental setup.

G. Integration of secondary nonidealities into the fidelity model

In a real experimental setup, the tandem SLM is subject to multiple, simultaneous physical imperfections beyond the primary thickness mismatches. These secondary nonidealities—such as compensator drift, interface reflections, or oblique incidence—are governed by their own respective small parameters. Provided that the overall amplitude losses and phase distortions remain sufficiently small to justify the underlying Taylor expansion, the perturbation theory applies. This allows us to express the total spatial amplitude loss $a(x, y)$ and total phase distortion $\delta\varphi(x, y)$ as a linear superposition of the individual perturbation fields generated by each independent physical mechanism: $a = \sum a_i$ and $\delta\varphi = \sum \delta\varphi_i$.

Consequently, the absolute worst-case bounding figures of merit for the entire system can be obtained by simple algebraic addition: $\mathcal{A}_{\text{total}} \leq \sum \mathcal{A}_i$ and $\mathcal{P}_{\text{total}} \leq \sum \mathcal{P}_i$. This direct summation is mathematically valid and corresponds physically to a perfect constructive interference (or perfect positive correlation) of all defects across the entire beam profile. While

this provides a guaranteed, albeit severely pessimistic, lower bound for the global mode purity η , a more nuanced picture is provided by the variance-based approximation derived earlier: $\eta \approx 1 - \text{Var}(a) - \text{Var}(\delta\varphi)$.

When multiple small physical mechanisms contribute simultaneously, their respective variances combine according to the standard rules of probability theory. For instance, for two amplitude loss components a_1 and a_2 , the total variance is

$$\text{Var}(a) = \text{Var}(a_1) + \text{Var}(a_2) + 2r_a\sqrt{\text{Var}(a_1)\text{Var}(a_2)}, \quad (50)$$

where $r_a \in [-1, 1]$ represents the Pearson correlation coefficient between the two perturbation fields. A completely analogous addition rule applies to the phase variance $\text{Var}(\delta\varphi)$.

This statistical framework highlights two bounding regimes for evaluating cumulative errors. The absolute worst-case scenario ($r = 1$) scales as the square of the sum of the standard deviations, strictly mirroring the direct addition of the figures of merit. However, if the disparate hardware defects are assumed to be statistically independent ($r = 0$), the interference cross-terms vanish, and the variances sum linearly: $\text{Var}_{\text{total}} \approx \sum \text{Var}_i$, yielding a significantly higher realistic purity estimate.

Because the exact correlation coefficients depend intimately on the specific spatial intensity distribution of the incident beam and the exact alignment of the setup, calculating the true interference terms a priori is impractical. Therefore, in the following subsections, we will not extensively theorize about these cross-correlations or advocate for a single final combined value. Instead, our goal is simply to evaluate the isolated physical mechanisms and establish the individual, independent bounds $(\mathcal{A}_i, \mathcal{P}_i)$ for each secondary defect. These individual metrics provide a versatile foundation for estimating the overall error budget, accommodating both conservative worst-case summations and more nuanced statistical evaluations depending on the specific experimental constraints.

1. Active compensator drift and stability

A critical practical requirement for the active compensation approach (Strategy A) is the stability of the tunable retarder. As established, active compensation minimizes the intrinsic amplitude nonachromaticity ($\mathcal{A}^{(A)}$) to near zero, thereby allowing the use of optically thick cells to minimize the phase distortion ($\mathcal{P}^{(A)}$). The core of this strategy relies on satisfying the ideal calibration condition, $\Phi_d + \Phi_{c(\text{ideal})} = 0$.

However, in a practical setup, the compensation cannot be maintained perfectly. We model this by introducing a deviation parameter $\delta\Phi_c$, representing drift due to thermal fluctuations, voltage instability, or calibration inaccuracies, such that the actual retardation is $\Phi_c = \Phi_{c(\text{ideal})} + \delta\Phi_c$. For Strategy A to remain viable, this deviation must be actively controlled within a strict tolerance bound, $\delta\Phi_c \in [-\Delta\Phi_c, \Delta\Phi_c]$.

To find the corresponding FoMs generated by this nonideality, we express the transmission magnitude $|T'_{\text{CTN}}|$ and the phase error $\delta\varphi$ as functions of $\delta\Phi_c$. Utilizing the rigorous transmission coefficient derived in Appendix C [Eq. (C5)], we evaluate the deviations under the condition $\Phi_d + \Phi_{c(\text{ideal})} = 0$

and maximize the error over all other parameters. The resulting bounding metrics for the compensator drift are

$$\mathcal{A}^{(\delta\Phi_c)} = \frac{1}{2} \Delta\Phi_c^2, \quad (51)$$

$$\mathcal{P}_1^{(\delta\Phi_c)} = \frac{\Delta\Phi_c}{\Phi_s}, \quad \mathcal{P}_0^{(\delta\Phi_c)} = 0. \quad (52)$$

These formulas provide a direct, computable criterion for the required hardware stability. They reveal that a small drift $\Delta\Phi_c$ contributes quadratically to the amplitude loss, but linearly to the phase error, with the latter being heavily suppressed by the large cumulative retardation Φ_s of the thick cells. Consequently, $\mathcal{A}^{(\delta\Phi_c)}$ and $\mathcal{P}^{(\delta\Phi_c)}$ can simply be added to the overall error budget to determine the true mode purity of the operating device.

2. Interface reflections

While the theoretical mode purity of the actively compensated tandem cell (Strategy A) is fundamentally unlimited and can be arbitrarily improved by simply increasing the optical thickness of the cells, real-world implementations are constrained by boundary effects. The standard Jones calculus employed in our theoretical model assumes perfect transmission at all boundaries and inherently does not account for Fresnel reflections.

In a physical device, reflections at the refractive index mismatches decrease the intensity of the target transmitted mode and generate parasitic counterpropagating or multiply-reflected waves. A practical optical SLM setup contains numerous such interfaces (glass substrates, ITO electrodes, alignment layers, etc.). However, high-quality optical assemblies minimize internal reflections by utilizing index-matching adhesives (cementing) or index-matched fluids. Assuming a fully cemented internal structure where all refractive indices are well-matched, interface reflections are minimized.

To evaluate the impact of the remaining outer interfaces, we can approximate the local transmitted field as $E_{\text{out}} \approx E_{\text{ideal}}[(1 - r') + r'' \exp(i\Phi_{\text{FP}})]$. Here, r' represents the total baseline amplitude reduction (forward transmission loss), and r'' represents the cumulative amplitude of the parasitic waves arising from internal reflections ($r'' \leq r'$). The highly sensitive propagation phase Φ_{FP} governs the interference.

This introduces two distinct amplitude effects: a static baseline depletion and a dynamic Fabry-Perot interference ripple. Accordingly, the amplitude nonachromaticity is bounded by $\mathcal{A}^{(R)} = r' + r''$, while the phase distortion envelope is bounded by $\mathcal{P}^{(R)} = r''$. For a single effective external boundary with an intensity reflection coefficient r , we can establish the worst-case upper bounds by setting $r' \approx r$ and $r'' \approx r$:

$$\mathcal{A}^{(R)} = 2r, \quad \mathcal{P}^{(R)} = r. \quad (53)$$

To determine the actual mode degradation, we must evaluate the combined variance of the local amplitude loss a and phase error $\delta\varphi$, which are coupled through the interference phase Φ_{FP} . Mathematically, because the baseline depletion r' acts as a spatial constant, it identically vanishes when calculating the variance $\text{Var}(a)$. Thus, the combined variance depends

entirely on the interference ripple: $\text{Var}(a) + \text{Var}(\delta\varphi) = r'^2 - r'^2[(\cos(\Phi_{\text{FP}}))^2 + (\sin(\Phi_{\text{FP}}))^2]$. Using the worst-case substitution $r'' \approx r$, this yields a strict and elegant upper bound for the total purity degradation:

$$\text{Var}(a) + \text{Var}(\delta\varphi) \leq r^2. \quad (54)$$

Physically, the fact that the baseline depletion r' drops out of this purity bound is fully consistent with the definition of the mode purity η Eq. (43). Because η is explicitly normalized by the total transmitted intensity of the resulting beam, a uniform loss merely reduces the absolute beam power without degrading its spatial topological structure.

For a bare, uncoated air-glass interface, the reflection coefficient is typically $r \approx 0.04$ (4%). If left untreated, this introduces a proportional 4% amplitude and phase ripple. However, because the global mode purity degradation is bounded by the combined variance, this untreated boundary reduces the overall purity bound by only a small fraction, approximately $r^2 \approx 0.16\%$. Although this baseline degradation is already relatively minor, standard broadband antireflection (AR) coatings routinely reduce the reflection to $r < 0.005$ (0.5%), suppressing the combined variance to the order of 2.5×10^{-5} and rendering its impact on mode purity strictly negligible. Furthermore, these external boundary reflections can be completely eliminated by utilizing index-matched prism coupling for beam entry and exit, thereby removing this specific source of hardware distortion from the overall SLM error budget.

Finally, this combined variance formalism elegantly scales to systems with multiple unoptimized internal interfaces. One does not need to compute complex multilayer interference paths. The cumulative parasitic ripple amplitude is strictly bounded by the total scattering loss of the system. To establish a robust worst-case upper bound for such a multi-interface system, one can simply substitute the parameter r in Eqs. (53) and (54) with the effective cumulative reflection $r \approx \frac{1}{2} \sum r_i$, where r_i are the individual intensity reflection coefficients. This seamlessly incorporates all multiple reflections into the global error budget without overinflating the theoretical limits.

3. Angular deviation and oblique incidence

In practical liquid crystal spatial light modulators, the incident beam is rarely perfectly collimated. Oblique incidence—angular deviation of the light beam from the cell normal—introduces parasitic birefringence effects and phase retardation errors. Because these angular dependencies are a well-known limitation of LC devices, evaluating the robustness of the tandem CTN configuration against small incidence angles θ_i is crucial for real-world applications.

To quantify this effect, we conducted a rigorous paraxial analysis using the extended Jones matrix method [50] and perturbation theory, fully detailed in Appendix F. For small incidence angles ($\theta_i \ll 1$), we introduce a dimensionless small parameter χ that governs the cumulative phase perturbation over the cell thickness h :

$$\chi = hk \frac{n_e - n_o}{n_o n_e} \theta_i^2. \quad (55)$$

Our analytical derivation demonstrates that, in the first approximation, the primary effect of oblique incidence is a

modification of the effective refractive indices for the two orthogonal eigenmodes operating in the Mauguin regime (the modes rotating alongside the local director). This results in an additional phase accumulation for each of these captured modes as they propagate through the tandem structure.

As shown in Appendix F, the spatially varying part of this additional phase shift—which directly distorts the target geometric phase pattern φ —is strictly proportional to χ . This deterministic phase distortion yields the following linear bound for the phase figure of merit:

$$\mathcal{P}^{(\text{ang})} = \frac{n_e + n_o}{4n_o} \chi. \quad (56)$$

Conversely, regarding the amplitude figure of merit ($\mathcal{A}^{(\text{ang})}$), this purely phase-accumulating perturbation does not generate any linear amplitude ripple. Physically, when the perturbed modes are projected onto the target circular polarization state at the exit analyzer, the amplitude variation depends on the cosine of the induced phase differences. Consequently, the amplitude error strictly begins with quadratic terms of the small parameters (e.g., χ^2 , $\chi\alpha$), meaning that no linear term proportional solely to χ exists within this approximation. The tandem configuration thus preserves the spatial uniformity of the amplitude intensity against small off-axis deviations to the first order.

4. Spatial misalignment of the tandem cells

Beyond the intrinsic material and waveguiding nonidealities, the physical assembly of the tandem architecture introduces the potential for a relative transverse spatial shift (misalignment) between the two constituent TN cells. Because a comprehensive theoretical and numerical analysis of this effect is extensive, the complete derivation is relegated to Sec. S3 of the Supplemental Material [47]. Here, we summarize the fundamental results and integrate them into our global fidelity model.

Our analysis reveals that the primary mechanism degrading the mode purity under spatial misalignment is the splitting of the structural disclination lines. In an ideally aligned system, the disclination lines of both cells perfectly coincide. When the cells are transversely shifted, these lines separate, creating narrow mismatched gaps. Inside these gaps, the propagating wave experiences an abrupt, uncompensated π phase jump.

Crucially, we demonstrated that for small perturbations, this topological degradation acts as an independent multiplicative loss channel. The total mode purity factorizes as

$$\eta \approx \eta_0 \eta_{\text{ma}}, \quad (57)$$

where η_0 represents the intrinsic mode purity of the perfectly aligned device (evaluated previously), and η_{ma} encapsulates the contribution solely from the misalignment. For an arbitrary incident intensity profile $I(x, y) = |E_0(x, y)|^2$, this topological loss is universally estimated by integrating the intensity strictly over the area of the mismatched disclination gaps, S_d :

$$\eta_{\text{ma}} \approx |1 - \varsigma_d|^2, \quad \text{where} \quad \varsigma_d \approx 2 \frac{\iint_{S_d} I(x, y) dS}{\iint_S I(x, y) dS}. \quad (58)$$

For the specific case of generating an optical vortex of topological charge l , we analytically determined the total gap area S_d generated by a transverse displacement vector of magnitude H and azimuthal angle ϑ_H :

$$S_d \approx \rho_0 H \sum_{m=0}^{l-1} |\sin(\vartheta_{dm} - \vartheta_H)|, \quad (59)$$

where ρ_0 is the beam radius, and ϑ_{dm} denotes the fixed azimuthal angles of the disclination rays:

$$\vartheta_{dm} = \frac{1 + 2m}{l} \pi, \quad m \in \{0, 1, \dots, l-1\}. \quad (60)$$

A key physical consequence of this geometric relationship is that for small displacements ($H \ll \rho_0$), the area S_d , and thus the distortion parameter ς_d , scales linearly with H . Therefore, unlike many phase perturbations that scale quadratically, the topological mode degradation induced by spatial misalignment decreases strictly linearly as the alignment precision improves.

V. FUTURE OUTLOOK AND APPLICATIONS

A. Static vortex generators and fabrication challenges

This architecture holds significant promise for fabricating static, patterned vortex generators using techniques such as photo-alignment [51–53]. A single nematic layer typically allows for a director rotation range restricted to -90° to $+90^\circ$ (bounded by disclination lines). Consequently, the induced phase shift, which in the Mauguin regime tracks the twist angle, is limited to the same range (π). This is insufficient for continuous full-cycle (2π) phase control required for integer vortex generation.

Our tandem structure inherently overcomes this limitation. By summing the rotations of two orthogonal cells, the system extends the available twist range to -180° to $+180^\circ$. This provides the full 2π phase modulation coverage. The primary challenge in fabricating such a device lies in the precise registration of the two patterned substrates. Specifically, the disclination lines—where the director twist jumps between the extreme values ($+180^\circ$ and -180°)—exist in both cells. To ensure a high-quality phase profile without artifacts, these disclination lines in the first and second cells must be perfectly aligned spatially.

B. Towards a full complex-amplitude SLM

Building upon the static concept, the fundamental principles of the compensated tandem architecture offer a highly versatile toolkit for next-generation dynamic spatial light modulators. Crucially, the physical structure of a single TN cell in our model—featuring a fixed uniform alignment on one substrate and a variable in-plane director angle on the other—maps exactly onto emerging in-plane switching technologies, specifically the one-side zero-anchoring in-plane switching (OZ-IPS) configuration [42–45]. This structural isomorphism implies that OZ-IPS matrices inherently provide the hardware foundation required to construct the active layers of our tandem device, paving the way for a universal platform for arbitrary, broadband light field synthesis.

The generation of such high-fidelity, structured light fields is a vibrant area of research with diverse applications ranging from exotic light-matter interactions to complex optical information processing [54–56]. The compensated tandem architecture contributes a crucial capability to this domain: the potential for fully achromatic operation.

Translating this theoretical framework into a physical device presents clear engineering pathways rather than fundamental roadblocks. While current generations of in-plane switching technologies may have stroke limitations in achieving a full 2π phase modulation within a standard two-cell tandem, this can be resolved through a scalable “tandem of tandems” architecture. By cascading four TN cells, the effective stroke is multiplied to cover the full-cycle modulation without sacrificing the required broadband achromaticity.

VI. CONCLUSION

In this work, we have developed a comprehensive theoretical framework for generating broadband structured light using a tandem crossed twisted nematic liquid crystal architecture. Moving beyond idealized adiabatic approximations, we formulated a rigorous perturbative Jones matrix model to evaluate the optical response under realistic physical conditions. We quantified the impact of fundamental nonidealities—specifically, waveguiding nonadiabaticity and manufacturing thickness variations—by expanding the transmission matrices to the second order of smallness. Furthermore, we assessed the influence of secondary macroscopic perturbations, such as spatial misalignments, oblique incidence, and boundary reflections. By introducing the modal overlap integral to define mode purity, we successfully factorized the contributions of these disparate error sources. Crucially, this analytical approach allowed us to establish strict figures of merit that provide a universal lower bound for the device’s global mode purity, evaluating the intrinsic limits of the hardware independent of the specific spatial pattern being modulated.

To mitigate these artifacts, we analyzed three distinct operational strategies. Strategy C requires no additional optical elements; it relies solely on the optimization of fabrication parameters for a specific target wavelength and the corresponding renormalization of the modulation phase. Strategy B involves the insertion of a passive static quarter-wave plate between the tandem cells, a configuration specifically targeted at improving the accuracy of the phase modulation over a broad spectrum. Strategy A employs an active tunable retarder at the cascaded interface. By adjusting the phase retardation, this strategy minimizes the manufacturing thickness mismatch between the constituent cells, thereby dramatically reducing intensity losses. This active configuration permits the use of optically thick cells, severely suppressing nonadiabatic phase errors while maintaining high amplitude throughput across the entire operational band.

Ultimately, the compensated tandem TN architecture provides a highly versatile platform for complex light field synthesis. While the derived principles offer a solid foundation for designing high-fidelity static optical components, the true potential of this architecture lies in its dynamic capabilities. The structural requirement of combining a uniform static alignment with a variable in-plane director twist aligns

perfectly with emerging electro-optic technologies, such as OZ-IPS. This direct hardware compatibility transforms the theoretical tandem cell into a practical blueprint for next-generation, fully dynamic broadband spatial light modulators.

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DATA AVAILABILITY

The source code used to generate the figures in the main text is openly available at [57]. All other data, including those supporting the Supplemental Material, are included within the article and its supplemental files, or are available from the authors upon reasonable request.

APPENDIX A: DERIVATION OF THE JONES MATRIX FOR A SINGLE TWISTED NEMATIC LAYER [EQ. (12)]

We start from the differential equation describing the evolution of the Jones vector in the circular basis, as given by Eq. (11):

$$\frac{d}{dz}\mathcal{E}(z) = i\gamma \begin{pmatrix} 0 & \exp(i2\phi) \\ \exp(-i2\phi) & 0 \end{pmatrix} \mathcal{E}(z). \quad (\text{A1})$$

For the case of a linear director twist, where $\phi(z) = \phi_0 + qz$, we can rewrite this matrix equation as a system of two coupled, first-order differential equations for the components $\mathcal{E}_1$ and $\mathcal{E}_2$:

$$\begin{aligned} \frac{d}{dz}\mathcal{E}_1 &= i\gamma \exp(i2(\phi_0 + qz))\mathcal{E}_2 \\ \frac{d}{dz}\mathcal{E}_2 &= i\gamma \exp(-i2(\phi_0 + qz))\mathcal{E}_1. \end{aligned} \quad (\text{A2})$$

The standard procedure to solve this system is to decouple these equations to obtain a single second-order homogeneous differential equation with constant coefficients. By differentiating the first system (A2) and subsequently substituting for all terms containing the variable $\mathcal{E}_2$ using the original system, we obtain the following second-order differential equation with constant coefficients:

$$\frac{d^2}{dz^2}\mathcal{E}_1 - 2qi\frac{d}{dz}\mathcal{E}_1 + \gamma^2\mathcal{E}_1 = 0. \quad (\text{A3})$$

The solution of this equation in general case is

$$\mathcal{E}_1(z) = A \exp(i(q + \gamma')z) + B \exp(i(q - \gamma')z), \quad (\text{A4})$$

where A and B are complex constants determined by the initial conditions, and γ' is defined as

$$\gamma' = \sqrt{\gamma^2 + q^2}. \quad (\text{A5})$$

The second component, $\mathcal{E}_2(z)$, can then be found by substituting the solution for $\mathcal{E}_1(z)$ and its derivative back into the first equation of the original system (A2). After performing

the substitution and simplification, we obtain

$$\mathcal{E}_2(z) = \frac{q + \gamma'}{\gamma} A \exp(-i[qz - \gamma'z + 2\phi_0]) + \frac{q - \gamma'}{\gamma} B \exp(-i[qz + \gamma'z + 2\phi_0]). \quad (\text{A6})$$

To find the specific solution for a given input polarization, we need to express the complex constants A and B in terms of the initial conditions of the field at $z = 0$, which we denote as $\mathcal{E}(0) = (\mathcal{E}_1(0), \mathcal{E}_2(0))^T$. By setting $z = 0$ in Eqs. (A4)–(A6), we obtain the following system of linear equations for A and B :

$$\begin{aligned} \mathcal{E}_1(0) &= A + B \\ \mathcal{E}_2(0) &= \frac{q + \gamma'}{\gamma} A \exp(-i2\phi_0) + \frac{q - \gamma'}{\gamma} B \exp(-i2\phi_0). \end{aligned} \quad (\text{A7})$$

Solving this system yields the expressions for A and B :

$$A = \frac{\gamma' - q}{2\gamma'} \mathcal{E}_1(0) + \frac{\gamma}{2\gamma'} \exp(i2\phi_0) \mathcal{E}_2(0), \quad (\text{A8})$$

$$B = \frac{q + \gamma'}{2\gamma'} \mathcal{E}_1(0) - \frac{\gamma}{2\gamma'} \exp(i2\phi_0) \mathcal{E}_2(0). \quad (\text{A9})$$

By substituting these expressions for A and B back into the general solutions (A4)–(A6), we can construct the Jones matrix that maps the input state $\mathcal{E}(0)$ to the output state $\mathcal{E}(h)$:

$$\mathcal{E}(h) = \begin{pmatrix} \exp(iqh) [\cos(\gamma'h) - i\frac{q}{\gamma'} \sin(\gamma'h)] & i\frac{\gamma}{\gamma'} \exp(i(2\phi_0 + qh)) \sin(\gamma'h) \\ i\frac{\gamma}{\gamma'} \exp(-i(2\phi_0 + qh)) \sin(\gamma'h) & \exp(-iqh) [\cos(\gamma'h) + i\frac{q}{\gamma'} \sin(\gamma'h)] \end{pmatrix} \mathcal{E}(0). \quad (\text{A10})$$

APPENDIX B: APPROXIMATION OF THE JONES MATRIX FOR THE NONIDEAL TANDEM CELL [EQ. (19)]

The total Jones matrix for the nonideal tandem system, T , is found by multiplying the individual Jones matrices of the first and second cells, $T_{\text{TN1}}^{\mathcal{E}}(h_1)$ and $T_{\text{TN2}}^{\mathcal{E}}(h_2)$, respectively:

$$T = T_{\text{TN2}}^{\mathcal{E}}(h_2) T_{\text{TN1}}^{\mathcal{E}}(h_1). \quad (\text{B1})$$

The parameters for each cell correspond to those in Eq. (12), but are now denoted with subscripts 1 and 2 to distinguish between the two cells. We assume the same liquid crystal material is used in both cells, so the anisotropy parameter γ is identical for both. The director twist in each cell is also set to be the same, with a total twist of φ for the system, meaning $\phi_{d1} = \phi_{d2} = \varphi/2$.

The specific orientation of the tandem system—where the director at the output of the first cell is along the x axis and the director at the input of the second cell is along the y axis—is achieved by setting the mean director angles as follows: for the first cell, we set $\bar{\phi}_1 = -\varphi/4$, and for the second cell $\bar{\phi}_2 = \pi/2 + \varphi/4$:

$$T_{\text{TN1}}^{\mathcal{E}}(h_1) = \begin{pmatrix} \exp(i\frac{\varphi}{2}) [\cos(\gamma'_1 h_1) - i\frac{q_1}{\gamma'_1} \sin(\gamma'_1 h_1)] & i\frac{\gamma}{\gamma'_1} \exp(-i\frac{\varphi}{2}) \sin(\gamma'_1 h_1) \\ i\frac{\gamma}{\gamma'_1} \exp(i\frac{\varphi}{2}) \sin(\gamma'_1 h_1) & \exp(-i\frac{\varphi}{2}) [\cos(\gamma'_1 h_1) + i\frac{q_1}{\gamma'_1} \sin(\gamma'_1 h_1)] \end{pmatrix}, \quad (\text{B2})$$

$$T_{\text{TN2}}^{\mathcal{E}}(h_2) = \begin{pmatrix} \exp(i\frac{\varphi}{2}) [\cos(\gamma'_2 h_2) - i\frac{q_2}{\gamma'_2} \sin(\gamma'_2 h_2)] & -i\frac{\gamma}{\gamma'_2} \exp(i\frac{\varphi}{2}) \sin(\gamma'_2 h_2) \\ -i\frac{\gamma}{\gamma'_2} \exp(-i\frac{\varphi}{2}) \sin(\gamma'_2 h_2) & \exp(-i\frac{\varphi}{2}) [\cos(\gamma'_2 h_2) + i\frac{q_2}{\gamma'_2} \sin(\gamma'_2 h_2)] \end{pmatrix}. \quad (\text{B3})$$

The resulting matrix T is algebraically complex. For clarity, instead of presenting the full 2×2 matrix at once, we will list each of its components individually:

$$T_{11} = \exp(i\varphi) \left\{ \left[\cos(\gamma'_2 h_2) - i\frac{q_2}{\gamma'_2} \sin(\gamma'_2 h_2) \right] \left[\cos(\gamma'_1 h_1) - i\frac{q_1}{\gamma'_1} \sin(\gamma'_1 h_1) \right] + \frac{\gamma^2}{\gamma'_2 \gamma'_1} \sin(\gamma'_2 h_2) \sin(\gamma'_1 h_1) \right\}, \quad (\text{B4})$$

$$T_{12} = i\frac{\gamma}{\gamma'_1} \left[\cos(\gamma'_2 h_2) - i\frac{q_2}{\gamma'_2} \sin(\gamma'_2 h_2) \right] \sin(\gamma'_1 h_1) - i\frac{\gamma}{\gamma'_2} \sin(\gamma'_2 h_2) \left[\cos(\gamma'_1 h_1) + i\frac{q_1}{\gamma'_1} \sin(\gamma'_1 h_1) \right], \quad (\text{B5})$$

$$T_{21} = -i\frac{\gamma}{\gamma'_2} \sin(\gamma'_2 h_2) \left[\cos(\gamma'_1 h_1) - i\frac{q_1}{\gamma'_1} \sin(\gamma'_1 h_1) \right] + i\frac{\gamma}{\gamma'_1} \left[\cos(\gamma'_2 h_2) + i\frac{q_2}{\gamma'_2} \sin(\gamma'_2 h_2) \right] \sin(\gamma'_1 h_1), \quad (\text{B6})$$

$$T_{22} = \exp(-i\varphi) \left\{ \frac{\gamma^2}{\gamma'_1 \gamma'_2} \sin(\gamma'_2 h_2) \sin(\gamma'_1 h_1) + \left[\cos(\gamma'_2 h_2) + i\frac{q_2}{\gamma'_2} \sin(\gamma'_2 h_2) \right] \left[\cos(\gamma'_1 h_1) + i\frac{q_1}{\gamma'_1} \sin(\gamma'_1 h_1) \right] \right\}. \quad (\text{B7})$$

The exact expressions for the matrix components Eqs. (B4)–(B7) are algebraically complex. To facilitate a rigorous analysis, we introduce a set of independent dimensionless parameters characterizing the system's nonidealities.

First, we define the thickness mismatch parameter β and the mean cell thickness h :

$$\beta = \frac{h_1 - h_2}{h_1 + h_2}, \quad h = \frac{h_1 + h_2}{2}. \quad (\text{B8})$$

Consequently, the individual thicknesses are expressed as $h_1 = h(1 + \beta)$ and $h_2 = h(1 - \beta)$.

Next, we consider the nonadiabaticity parameters for each cell, defined using the effective wavenumber inside the twisted medium, $\gamma' = \sqrt{\gamma^2 + q^2}$. Thus, $\alpha_1 = q_1/\gamma'_1$ and $\alpha_2 = q_2/\gamma'_2$. We introduce the mean nonadiabaticity parameter α :

$$\alpha = \frac{1}{2}(\alpha_1 + \alpha_2). \quad (\text{B9})$$

To derive the relationship between the individual parameters $\alpha_{1,2}$ and the independent variables α and β , we analyze the difference $\alpha_2 - \alpha_1$. Consistent with our overall perturbative approach, we explicitly limit our analysis to the second order of smallness with respect to the parameters α and β . Thus,

$$\alpha_2 - \alpha_1 = \frac{q_2}{\gamma'_2} - \frac{q_1}{\gamma'_1} = 2\alpha \frac{q_2\gamma'_1 - q_1\gamma'_2}{q_1\gamma'_2 + q_2\gamma'_1}. \quad (\text{B10})$$

Under this constraint (neglecting cubic terms and higher), we have

$$\alpha_2 - \alpha_1 = 2\alpha \frac{h_2^{-1} - h_1^{-1}}{h_2^{-1} + h_1^{-1}} = 2\alpha\beta. \quad (\text{B11})$$

From this relation, we derive the expressions for the individual parameters:

$$\alpha_1 = \alpha(1 - \beta), \quad \alpha_2 = \alpha(1 + \beta). \quad (\text{B12})$$

We now approximate the Jones matrix components by expanding their coefficients in powers of α and β , retaining terms up to the second order (including α^2 , β^2 , and $\alpha\beta$). The arguments of the trigonometric functions (like $\gamma'h$) are left unexpanded.

Following this methodology, the approximated matrix components are given by

$$T_{11} = \exp(i\varphi)[\cos(\gamma'_1 h_1 - \gamma'_2 h_2) - i\alpha \sin(\gamma'_2 h_2 + \gamma'_1 h_1)] \\ - \exp(i\varphi)[2\alpha^2 \sin(\gamma'_1 h_1) \sin(\gamma'_2 h_2) + i\alpha\beta \sin(\gamma'_2 h_2 - \gamma'_1 h_1)], \quad (\text{B13})$$

$$T_{12} = i(1 - \frac{1}{2}\alpha^2) \sin(\gamma'_1 h_1 - \gamma'_2 h_2) + 2\alpha \sin(\gamma'_1 h_1) \sin(\gamma'_2 h_2), \quad (\text{B14})$$

$$T_{21} = i(1 - \frac{1}{2}\alpha^2) \sin(\gamma'_1 h_1 - \gamma'_2 h_2) - 2\alpha \sin(\gamma'_2 h_2) \sin(\gamma'_1 h_1), \quad (\text{B15})$$

$$T_{22} = \exp(-i\varphi)[\cos(\gamma'_1 h_1 - \gamma'_2 h_2) + i\alpha \sin(\gamma'_2 h_2 + \gamma'_1 h_1)] \\ - \exp(-i\varphi)[2\alpha^2 \sin(\gamma'_2 h_2) \sin(\gamma'_1 h_1) - i\alpha\beta \sin(\gamma'_2 h_2 - \gamma'_1 h_1)]. \quad (\text{B16})$$

Finally, for the filtered optical system where only the circular polarization component is preserved, the total transmission coefficient T_{system} is determined primarily by T_{11} . The derived expression in Eqs. (B13)–(B16) provides the basis for the figures of merit analysis, capturing both the first-order phase distortions and the second-order amplitude modulation effects.

APPENDIX C: DERIVATION OF THE TRANSMISSION COEFFICIENT FOR THE FILTERED SYSTEM [EQ. (22)]

To derive the transmission coefficient for the complete filtered system, we first write the Jones matrix for the central tandem element, which now includes the compensating plate between the two TN cells. This matrix, T_{core} , is the product of the three individual matrices:

$$T_{\text{core}} = T_{\text{TN2}}^{\mathcal{E}}(h_2) T_C^{\mathcal{E}}(\Phi_c) T_{\text{TN1}}^{\mathcal{E}}(h_1). \quad (\text{C1})$$

The matrices $T_{\text{TN1}}^{\mathcal{E}}(h_1)$ and $T_{\text{TN2}}^{\mathcal{E}}(h_2)$ are the Jones matrices for the two twisted nematic cells, with components as derived in the previous section. The central matrix, $T_C^{\mathcal{E}}(\Phi_c)$, represents the compensating phase plate. For a linear retarder with its slow axis oriented along the x direction and a controllable retardation of Φ_c , this matrix in the circular basis is

$$T_C^{\mathcal{E}}(\Phi_c) = \begin{pmatrix} \cos(\Phi_c) & i \sin(\Phi_c) \\ i \sin(\Phi_c) & \cos(\Phi_c) \end{pmatrix}. \quad (\text{C2})$$

We can now derive the total transmission coefficient, T_{system} , for the full optical train. The first two elements, a linear polarizer and a quarter-wave plate, prepare a pure right-hand circular (RHC) input state, which is described by the Jones vector $(1, 0)^T$ in the circular basis.

Correspondingly, the final two elements (the second QWP and the analyzer) are configured to act as a filter that transmits only the RHC component of the light.

Therefore, the complex transmission coefficient for the entire system is simply the $(1, 1)$ element of the core

Jones matrix, T_{core} , defined in Eq. (C1). This simplifies the calculation significantly, as we do not need to compute the other three matrix elements. The expression

for this coefficient, $T_{\text{system}} = (T_{\text{core}})_{11}$, is obtained by performing the matrix multiplication $(T_{\text{TN2}}^{\mathcal{E}} T_{\text{C}}^{\mathcal{E}} T_{\text{TN1}}^{\mathcal{E}})_{11}$, which yields

$$T_{\text{system}} = (T_{\text{TN2}}^{\mathcal{E}})_{11} (T_{\text{C}}^{\mathcal{E}})_{11} (T_{\text{TN1}}^{\mathcal{E}})_{11} + (T_{\text{TN2}}^{\mathcal{E}})_{11} (T_{\text{C}}^{\mathcal{E}})_{12} (T_{\text{TN1}}^{\mathcal{E}})_{21} + (T_{\text{TN2}}^{\mathcal{E}})_{12} (T_{\text{C}}^{\mathcal{E}})_{21} (T_{\text{TN1}}^{\mathcal{E}})_{11} + (T_{\text{TN2}}^{\mathcal{E}})_{12} (T_{\text{C}}^{\mathcal{E}})_{22} (T_{\text{TN1}}^{\mathcal{E}})_{21}. \quad (\text{C3})$$

After substituting the corresponding elements of the matrices $T_{\text{TN1}}^{\mathcal{E}}$, $T_{\text{C}}^{\mathcal{E}}$ and $T_{\text{TN2}}^{\mathcal{E}}$ into the expression for $T_{\text{system}} = (T_{\text{core}})_{11}$, we obtain the following exact expression for the transmission coefficient:

$$\begin{aligned} T_{\text{system}} = \exp(i\varphi) & \left[\cos(\gamma'_2 h_2) - i \frac{q_2}{\gamma'_2} \sin(\gamma'_2 h_2) \right] \\ & \times \left\{ \cos(\Phi_c) \left[\cos(\gamma'_1 h_1) - i \frac{q_1}{\gamma'_1} \sin(\gamma'_1 h_1) \right] - \frac{\gamma}{\gamma'_1} \sin(\Phi_c) \sin(\gamma'_1 h_1) \right\} \\ & + \frac{\gamma}{\gamma'_2} \exp(i\varphi) \sin(\gamma'_2 h_2) \left\{ \sin(\Phi_c) \left[\cos(\gamma'_1 h_1) - i \frac{q_1}{\gamma'_1} \sin(\gamma'_1 h_1) \right] + \frac{\gamma}{\gamma'_1} \cos(\Phi_c) \sin(\gamma'_1 h_1) \right\}. \end{aligned} \quad (\text{C4})$$

This exact expression is cumbersome. We now simplify it by applying the second-order approximation methodology discussed in previous section. This involves expanding the expression and retaining only the terms that are not higher than second order in the small parameters α and β , while keeping the arguments of the trigonometric functions in their exact form.

After significant algebraic simplification, the transmission coefficient reduces to the much more compact and insightful form:

$$\begin{aligned} T_{\text{system}} = \exp(i\varphi) & [\cos(\Phi_c + \gamma'_1 h_1 - \gamma'_2 h_2) - i\alpha \cos(\Phi_c) \sin(\gamma'_2 h_2 + \gamma'_1 h_1)] \\ & + i\alpha\beta \exp(i\varphi) [\sin(\gamma'_1 h_1 - \gamma'_2 h_2) \cos(\Phi_c) + 2 \sin(\gamma'_1 h_1) \sin(\gamma'_2 h_2) \sin(\Phi_c)] \\ & + \alpha^2 \exp(i\varphi) \left[\frac{1}{2} \sin(\gamma'_1 h_1 - \gamma'_2 h_2) \sin(\Phi_c) - 2 \sin(\gamma'_1 h_1) \sin(\gamma'_2 h_2) \cos(\Phi_c) \right]. \end{aligned} \quad (\text{C5})$$

APPENDIX D: DERIVATION OF FIGURES OF MERIT FOR STRATEGY B

To strictly derive the amplitude fidelity, we refer to the exact transmission coefficient derived from the Jones matrix multiplication, denoted here as Eq. (C4) (referring to the unexpanded matrix product in Appendix C). Unlike the previous derivation for the active strategy, where Φ_c was an independent variable, here Φ_c is determined by the physical properties of the quarter-wave plate.

We assume the QWP is made of a material with similar dispersion characteristics to the liquid crystal, or that its dispersion tracks the LC linearly,

$$\Phi_c(\omega) = \frac{\pi}{2} \frac{\omega}{\omega_0}, \quad (\text{D1})$$

where ω_0 is the central design frequency at which the plate acts as a perfect $\lambda/4$ retarder ($\Phi_c(\omega_0) = \Phi_{c0} = \pi/2$).

We define the “equivalent thickness” of the compensator, h_c , as the thickness of a hypothetical LC layer that would produce the same retardation $\Phi_{c0} = \Phi_c(\omega_0) = \pi/2$ at the central frequency ω_0 :

$$h_c = \Phi_{c0} \gamma^{-1}(\omega_0) = \frac{\pi}{2} \gamma^{-1}(\omega_0). \quad (\text{D2})$$

To analyze the system as a unified structure, we redefine the thickness mismatch parameter β to include this effective thickness. The effective mismatch of the compensated system is

$$\beta = \frac{h_1 - h_2 + h_c}{h_2 + h_1}. \quad (\text{D3})$$

By design, Strategy B aims to satisfy the condition $h_1 - h_2 + h_c = 0$, which corresponds to $\beta = 0$. However, due to manufacturing inaccuracies, the realized parameter deviates from zero. The maximum thickness inaccuracy b sets the bound for this deviation: $|\beta| \leq b$.

With the redefinition of the thickness mismatch parameter β [Eq. (D3)], we must update the expressions for the geometric and optical parameters used in the perturbative expansion.

We retain the definitions for the mean nonadiabaticity parameter α and the mean cell thickness h as established in Appendix B:

$$\alpha_1 = q_1/\gamma'_1, \quad \alpha_2 = q_2/\gamma'_2, \quad (\text{D4})$$

$$\alpha = \frac{1}{2}(\alpha_1 + \alpha_2), \quad h = \frac{h_1 + h_2}{2}. \quad (\text{D5})$$

However, the expressions for the individual cell thicknesses $h_{1,2}$ must be reformulated to account for the inclusion of the compensator equivalent thickness h_c in the definition of β . Solving for h_1 and h_2 , we obtain

$$h_1 = h(1 + \beta) - \frac{1}{2}h_c, \quad h_2 = h(1 - \beta) + \frac{1}{2}h_c. \quad (\text{D6})$$

Consequently, the difference between the nonadiabaticity parameters, which scales with the physical thickness difference $(h_1 - h_2)$, is also modified. Using the relation $\alpha_2 - \alpha_1 \approx 2\alpha(h_1 - h_2)/(h_1 + h_2)$, we find

$$\alpha_2 - \alpha_1 = 2\alpha \left(\beta - \frac{1}{2} \frac{h_c}{h} \right). \quad (\text{D7})$$

This leads to the updated expressions for the individual parameters α_1 and α_2 in terms of the independent variables:

$$\alpha_1 = \alpha \left(1 - \beta + \frac{1}{2} \frac{h_c}{h} \right), \quad (\text{D8})$$

$$\alpha_2 = \alpha \left(1 + \beta - \frac{1}{2} \frac{h_c}{h} \right). \quad (\text{D9})$$

We retain the standard form of the definitions for the phase arguments, noting that their values are now governed by the

modified thicknesses derived above:

$$\Phi_d = \gamma'_1 h_1 - \gamma'_2 h_2, \quad \Phi_s = \gamma'_1 h_1 + \gamma'_2 h_2. \quad (\text{D10})$$

Substituting these redefined parameters into the exact transmission coefficient (C4) (derived from the raw Jones matrix multiplication) and performing a series expansion, we retain terms up to the second order in the small parameters α and β .

Using the linear dispersion relation for the QWP, $\Phi_c(\omega) = \frac{\pi}{2} \frac{\omega}{\omega_0}$, the transmission coefficient for Strategy B takes the following form:

$$\begin{aligned} T_{\text{system}} = \exp(i\varphi) & \left[\cos\left(\frac{\pi}{2} \frac{\omega}{\omega_0} + \Phi_d\right) - i\alpha \cos\left(\frac{\pi}{2} \frac{\omega}{\omega_0}\right) \sin(\Phi_s) \right] \\ & + i\alpha \left[\beta - \frac{1}{2} \frac{h_c}{h} \right] \exp(i\varphi) \left[\sin(\Phi_d) \cos\left(\frac{\pi}{2} \frac{\omega}{\omega_0}\right) + 2 \sin(\gamma'_1 h_1) \sin(\gamma'_2 h_2) \sin\left(\frac{\pi}{2} \frac{\omega}{\omega_0}\right) \right] \\ & + \alpha^2 \exp(i\varphi) \left[\frac{1}{2} \sin(\Phi_d) \sin\left(\frac{\pi}{2} \frac{\omega}{\omega_0}\right) - 2 \sin(\gamma'_1 h_1) \sin(\gamma'_2 h_2) \cos\left(\frac{\pi}{2} \frac{\omega}{\omega_0}\right) \right]. \end{aligned} \quad (\text{D11})$$

Although the full expression for the transmission coefficient (D11) is cumbersome, our primary goal is to derive the figures of merit. First, we determine the transmission magnitude $|T_{\text{system}}|$. Calculating the magnitude $|T| = \sqrt{\text{Re}^2 T + \text{Im}^2 T}$ (up to the second order in α , β and $\frac{h_c}{h}$) and grouping the terms, we obtain

$$\begin{aligned} |T_{\text{system}}| \approx & \cos\left(\frac{\pi}{2} \frac{\omega}{\omega_0} + \Phi_d\right) + \frac{1}{2} \alpha^2 \frac{\cos^2\left(\frac{\pi}{2} \frac{\omega}{\omega_0}\right) \sin^2(\Phi_s)}{\cos\left(\frac{\pi}{2} \frac{\omega}{\omega_0} + \Phi_d\right)} \\ & + \alpha^2 \left[\frac{1}{2} \sin(\Phi_d) \sin\left(\frac{\pi}{2} \frac{\omega}{\omega_0}\right) - 2 \sin(\gamma'_1 h_1) \sin(\gamma'_2 h_2) \cos\left(\frac{\pi}{2} \frac{\omega}{\omega_0}\right) \right]. \end{aligned} \quad (\text{D12})$$

This approximation is valid under the operational condition $\cos(\frac{\pi}{2} \frac{\omega}{\omega_0} + \Phi_d) \gg \alpha$. To simplify the analysis, we express the total residual retardation phase of the compensated system directly in terms of the mismatch parameter β :

$$\frac{\pi}{2} \frac{\omega}{\omega_0} + \Phi_d \approx \beta \Phi_s. \quad (\text{D13})$$

Thus, grouping the terms proportional to the spectral detuning sine function, the magnitude can be written as

$$\begin{aligned} |T_{\text{system}}| \approx & \cos(\beta \Phi_s) + \frac{1}{2} \alpha^2 \cos\left(\frac{\pi}{2} \frac{\omega - \omega_0}{\omega_0} - \beta \Phi_s\right) \cos\left(\frac{\pi}{2} \frac{\omega - \omega_0}{\omega_0}\right) \\ & + \alpha^2 \sin\left(\frac{\pi}{2} \frac{\omega - \omega_0}{\omega_0}\right) \left[\frac{1}{2} \frac{\sin\left(\frac{\pi}{2} \frac{\omega - \omega_0}{\omega_0}\right) \sin^2(\Phi_s)}{\cos(\beta \Phi_s)} + 2 \sin(\gamma'_1 h_1) \sin(\gamma'_2 h_2) \right]. \end{aligned} \quad (\text{D14})$$

To estimate the amplitude nonachromaticity ($\mathcal{A}$), defined as the maximum deviation $1 - |T_{\text{system}}|$, we evaluate this expression under the worst-case tolerance limit b (where $|\beta| \leq b$). The perturbation enclosed in the square brackets can be analytically bounded by expressing the internal arguments through the phase sum Φ_s , the mismatch Φ_d , and the total residual phase $\beta \Phi_s$. Mathematical optimization of this term reveals that its absolute value never exceeds 2, provided the operational condition $b \Phi_s \in [0, \arctan \frac{3}{2}]$ is satisfied. Under this practical constraint, bounding the remaining cosine product by its maximum value of 1/2, we obtain a rigorous upper envelope for the amplitude loss:

$$\mathcal{A} = 2 \sin^2\left(\frac{1}{2} b \Phi_s\right) + \frac{1}{2} \alpha^2 + 2 \alpha^2 \left| \sin\left(\frac{\pi}{2} \frac{\omega - \omega_0}{\omega_0}\right) \right|. \quad (\text{D15})$$

This formula explicitly captures the dependence on the spectral detuning $(\omega - \omega_0)$ and provides a mathematically robust upper bound for the amplitude loss without imposing further limits on the magnitude of the rapidly oscillating arguments.

The phase distortion is dominated by the imaginary term in the transmission coefficient, which is proportional to $\cos(\Phi_c)$. Assuming the QWP retardation follows a linear dispersion law $\Phi_c(\omega) = \frac{\pi}{2} \frac{\omega}{\omega_0}$, this term transforms as

$$\cos[\Phi_c(\omega)] = -\sin\left(\frac{\pi}{2} \frac{\omega - \omega_0}{\omega_0}\right). \quad (\text{D16})$$

Since the distortion is directly proportional to the desired modulation angle φ , the constant phase distortion is zero in this approximation:

$$\mathcal{P}_0 = 0. \quad (\text{D17})$$

For the angular phase distortion $\mathcal{P}_1$, we normalize the phase error by φ . Considering the worst-case scenario where the manufacturing tolerance b minimizes the denominator (amplitude term), we obtain

$$\mathcal{P}_1 = \frac{\left| \sin\left(\frac{\pi}{2} \frac{\omega - \omega_0}{\omega_0}\right) \right|}{\Phi_s |\cos(b\Phi_s)|}. \quad (\text{D18})$$

This expression demonstrates that the phase fidelity is determined by the spectral bandwidth (numerator) and the optical thickness (denominator). It represents the inevitable residual error in a passive system where the dispersion of the compensating element is utilized to balance the amplitude response, making perfect cancellation impossible over a finite frequency range.

APPENDIX E: DERIVATION OF FIGURES OF MERIT FOR STRATEGY C

To derive the performance metrics for the narrow-band optimization strategy, we start from the rigorous second-order approximation of the Jones matrix element T_{11} , derived in Appendix B [Eq. (B4)]. The transmission coefficient is

$$T_{11} = \exp(i\varphi) [\cos(\Phi_d) - 2\alpha^2 \sin(\gamma'_1 h_1) \sin(\gamma'_2 h_2)] - i \exp(i\varphi) [\alpha \sin(\Phi_s) + \alpha\beta \sin(\Phi_d)]. \quad (\text{E1})$$

We represent this complex expression in the polar form $T_{\text{CTN}} = |T_{\text{CTN}}| \exp(i\varphi) \exp(i\delta\psi)$, separating the amplitude and the phase distortion components. Assuming the phase perturbation $\delta\psi$ is small, we can approximate $\delta\psi \approx \tan(\delta\psi) = \text{Im}(T_{11}e^{-i\varphi})/\text{Re}(T_{11}e^{-i\varphi})$. This yields

$$T_{\text{CTN}} \approx |T_{\text{CTN}}| \exp(i\varphi) \times \exp\left(-i \frac{\alpha \sin(\Phi_s) + \alpha\beta \sin(\Phi_d)}{\cos(\Phi_d) - 2\alpha^2 \sin(\gamma'_1 h_1) \sin(\gamma'_2 h_2)}\right). \quad (\text{E2})$$

The amplitude magnitude $|T_{\text{CTN}}|$, calculated as the square root of the intensity up to the second order in small parameters, is given by

$$|T_{\text{CTN}}| \approx \cos(\Phi_d) + \frac{1}{2}\alpha^2 \sin^2(\Phi_s) - 2\alpha^2 \sin(\gamma'_1 h_1) \sin(\gamma'_2 h_2). \quad (\text{E3})$$

For the analysis of Strategy C, we consider the regime of small retardation mismatch ($\Phi_d \ll 1$), allowing us to expand $\cos(\Phi_d) \approx 1 - \Phi_d^2/2$ and approximate the geometric sine terms as $\sin(\gamma'_1 h_1) \sin(\gamma'_2 h_2) \approx \sin^2(\Phi_s/2)$. Under these assumptions, the expressions simplify significantly:

$$|T_{\text{CTN}}| \approx 1 - \frac{1}{2}\Phi_d^2 - 2\alpha^2 \sin^2\left(\frac{\Phi_s}{2}\right) + \frac{1}{2}\alpha^2 \sin^2(\Phi_s). \quad (\text{E4})$$

For the phase term, retaining only the dominant contributions up to the second order (neglecting terms like $\alpha\beta\Phi_d$ which are third-order), the expression reduces to a cleaner form dependent on the cumulative retardation Φ_s :

$$T_{\text{CTN}} \approx |T_{\text{CTN}}| \exp\left(i\varphi \left[1 - \frac{\sin(\Phi_s)}{\Phi_s}\right]\right). \quad (\text{E5})$$

Here we used the relation $\alpha = \varphi\Phi_s^{-1}(1 - \beta^2)^{-1}$. For narrow bandwidth applications, we assume $\frac{\Delta\omega}{\omega_0}\Phi_s \ll 1$.

Optimization of the operating point. We introduce the effective phase modulation φ_{mod} distinct from the director twist φ :

$$\varphi_{\text{mod}} = \varphi \left[1 - \frac{\sin(\Phi_{s0})}{\Phi_{s0}}\right], \quad (\text{E6})$$

where $\Phi_{s0} = \Phi_s(\omega_0)$. To minimize chromaticity, we select Φ_{s0} to be an extremum of the function $f(x) = \sin(x)/x$. Setting the derivative to zero yields the condition

$$\tan(\Phi_{s0}^{\text{opt}}) = \Phi_{s0}^{\text{opt}}. \quad (\text{E7})$$

To ensure the modulation scaling factor in Eq. (E6) is greater than unity (maximizing the achievable phase stroke), we specifically target the local minima of the sinc function. We approximate the solution as

$$\Phi_{s0}^{\text{opt}} = 2\pi n - \frac{\pi}{2} + \delta. \quad (\text{E8})$$

To find the value of δ , we can employ the fixed-point iteration method. Rearranging the transcendental Eq. (E7), we define the iteration function as

$$\delta = -\arcsin\left[\frac{\cos \delta}{2\pi n - \frac{\pi}{2} + \delta}\right]. \quad (\text{E9})$$

The first-order approximation yields: $\delta_1 \approx -(2\pi n - \pi/2)^{-1}$.

However, in a real device, the actual retardation will deviate from this optimal target due to manufacturing inaccuracy. We define the actual retardation as

$$\Phi_{s0} = \Phi_{s0}^{\text{opt}} + \Delta\Phi, \quad (\text{E10})$$

where the bracketed term is the theoretical optimum, and $\Delta\Phi$ represents the deviation due to thickness inaccuracy. In the worst-case scenario, $\Delta\Phi = \pm b\Phi_{s0}$.

Derivation of angular phase distortion ($\mathcal{P}_1$). The chromatic perturbation of the modulation phase, $\delta\varphi_{\text{mod}}(\omega)$, is defined as the deviation of the actual phase from the target φ_{mod} across the bandwidth:

$$\delta\varphi_{\text{mod}} = \varphi_{\text{mod}} \left[\frac{1 - \text{sinc}(\Phi_s(\omega))}{1 - \text{sinc}(\Phi_{s0})} - 1 \right], \quad (\text{E11})$$

where $\text{sinc}(x) = \sin(x)/x$. Substituting $\Phi_s(\omega) \approx \Phi_{s0}(1 + \frac{\omega - \omega_0}{\omega_0})$ and expanding the expression in powers of the small parameter $\epsilon = \frac{\omega - \omega_0}{\omega_0}$, we perform a Taylor series expansion around the optimized point defined in Eq. (E10).

Crucially, because we are operating near a local extremum, the first-order term proportional to ϵ vanishes for the ideal thickness. The remaining linear dependence arises solely from the thickness error $\Delta\Phi$. The dominant higher-order term is quadratic (ϵ^2). After algebraic simplification and neglecting negligible terms (assuming $\Delta\Phi \ll 1$), the phase distortion normalized by the target phase becomes

$$\frac{\delta\varphi_{\text{mod}}}{\varphi_{\text{mod}}} \approx \frac{1}{2}\Phi_{s0}^{\text{opt}} \left(\frac{\omega - \omega_0}{\omega_0}\right)^2 - \Delta\Phi \left(\frac{\omega - \omega_0}{\omega_0}\right). \quad (\text{E12})$$

Defining $\mathcal{P}_1$ as the maximum deviation over the full bandwidth $\Delta\omega$, we arrive at

$$\mathcal{P}_1 = \frac{1}{8} \Phi_{s0}^{\text{opt}} \left(\frac{\Delta\omega}{\omega_0} \right)^2 + \frac{1}{2} b \Phi_{s0}^{\text{opt}} \frac{\Delta\omega}{\omega_0}. \quad (\text{E13})$$

Derivation of constant phase distortion ($\mathcal{P}_0$). In Strategy C, the cell thickness is optimized to satisfy the extremum condition $\Phi_{s0} = \Phi_{s0}^{\text{opt}}$ at a central frequency ω_0 . However, the cumulative retardation Φ_{s0} depends not only on the cell thickness h and anisotropy γ , but also on the director twist angle φ (via the twist wave number $q = \varphi(2h)^{-1}$). Explicitly,

$$\Phi_{s0}(\varphi) = 2h\sqrt{q^2 + \gamma^2} = \sqrt{\varphi^2 + 4h^2\gamma^2}. \quad (\text{E14})$$

Since the physical thickness h is uniform, the optimization condition $\Phi_{s0}(\varphi) = \Phi_{s0}^{\text{opt}}$ can only be strictly satisfied for a specific design angle φ^{opt} . For all other angles, a deviation $\delta\Phi(\varphi)$ arises. Assuming the Mauguin regime ($2h\gamma \gg \varphi$), the first-order Taylor expansion yields

$$\delta\Phi(\varphi) \stackrel{\text{def}}{=} \Phi_{s0}(\varphi) - \Phi_{s0}(\varphi^{\text{opt}}) \approx \frac{\varphi^2 - (\varphi^{\text{opt}})^2}{2\Phi_{s0}^{\text{opt}}}. \quad (\text{E15})$$

This deviation leads to a residual phase error. We define the constant phase distortion parameter $\mathcal{P}_0$ based on the maximum contribution of this effect over the spectral bandwidth $\Delta\omega$:

$$\mathcal{P}_0 = \frac{1}{2} \frac{\Delta\omega}{\omega_0} \max_{\varphi} [|\varphi| \cdot |\delta\Phi(\varphi)|]. \quad (\text{E16})$$

Here we approximated $\varphi_{\text{mod}} \approx \varphi$. Substituting Eq. (E15) into the definition of $\mathcal{P}_0$ [Eq. (E16)] we have

$$\mathcal{P}_0 = \frac{1}{4\Phi_{s0}^{\text{opt}}} \frac{\Delta\omega}{\omega_0} \max_{\varphi \in [0, \pi]} |\varphi[\varphi^2 - (\varphi^{\text{opt}})^2]|. \quad (\text{E17})$$

To minimize the constant phase distortion $\mathcal{P}_0$, we must select the optimal design angle φ^{opt} that minimizes the maximum value of function $f = \varphi[\varphi^2 - (\varphi^{\text{opt}})^2]$ over the full modulation range $\varphi \in [0, \pi]$. This is a minimax problem. The maximum error occurs either at the local extremum of the function (φ_{peak}) or at the boundary of the domain ($\varphi = \pi$). The derivative of f vanishes at $\varphi_{\text{peak}} = \varphi^{\text{opt}}/\sqrt{3}$. The error at this point is

$$|f(\varphi_{\text{peak}})| = \frac{2}{3\sqrt{3}} (\varphi^{\text{opt}})^3, \quad (\text{E18})$$

the error at the maximum twist angle $\varphi = \pi$ is

$$|f(\pi)| = \pi|\pi^2 - (\varphi^{\text{opt}})^2|. \quad (\text{E19})$$

The optimal solution is found by equating the error magnitudes at the peak and the boundary: $|f(\varphi_{\text{peak}})| = |f(\pi)|$. Solving this equation yields the analytical solution:

$$\varphi^{\text{opt}} = \frac{\sqrt{3}}{2} \pi. \quad (\text{E20})$$

To retrieve the $\mathcal{P}_0$ we should substitute φ^{opt} [Eq. (E20)] and $\varphi = \pi$ back into the expression (E17):

$$\mathcal{P}_0 = \frac{\pi^3}{16\Phi_{s0}} \frac{\Delta\omega}{\omega_0}. \quad (\text{E21})$$

Derivation of amplitude nonachromaticity ($\mathcal{A}$). To estimate the amplitude fidelity, we evaluate the magnitude expression (E4) at the optimized operating point $\Phi_{s0}^{\text{opt}} \approx 2\pi n - \frac{\pi}{2}$ [Eq. (E8)]. Substituting the worst-case retardation mismatch $\Phi_d \approx b\Phi_{s0}$, we obtain the final estimate for the amplitude nonachromaticity:

$$\mathcal{A} = \frac{1}{2} (b\Phi_{s0})^2 + \frac{1}{2} \alpha^2. \quad (\text{E22})$$

This confirms that the amplitude loss is driven quadratically by both the manufacturing tolerance and the fundamental nonadiabaticity.

APPENDIX F: EXTENDED JONES MATRIX ANALYSIS FOR OBLIQUE INCIDENCE

In this Appendix, we describe the diffraction of a plane wave propagating through the tandem cell at a very small angle relative to the cell normal. We define the incident plane wave by a polar angle θ_i (where $\theta_i = 0$ corresponds to normal incidence) and an azimuthal angle ϕ_i . We assume that the incidence angle is small ($\theta_i \ll 1$) and introduce a corresponding dimensionless small parameter χ , which governs the cumulative oblique phase perturbation over the cell thickness h :

$$\chi = hk \frac{n_e - n_o}{n_o n_e} \theta_i^2 \ll 1. \quad (\text{F1})$$

The subsequent analytical derivation is strictly restricted to this paraxial regime, allowing us to treat the angular deviation as a small perturbation.

Under oblique incidence, the electric field acquires a longitudinal component E_z . However, for the propagation analysis using the extended Jones matrix method [50], it is sufficient to evaluate only the transverse components $\mathbf{E}_{\perp} = (E_x, E_y)^T$. For the propagation of the electromagnetic field in a birefringent medium (such as a liquid crystal where the director $\hat{\mathbf{e}}$ lies in the xy plane and its direction is constant over an infinitesimal step h), we can express the field evolution similarly to the normal incidence case:

$$\begin{aligned} \mathbf{E}_{\perp}(z+h) &= (\mathbf{E}_{\perp}(z) \cdot \hat{\mathbf{e}}) \hat{\mathbf{e}} \exp(in'_e kh) \\ &+ [\mathbf{E}_{\perp}(z) - (\mathbf{E}_{\perp}(z) \cdot \hat{\mathbf{e}}) \hat{\mathbf{e}}] \exp(in'_o kh). \end{aligned} \quad (\text{F2})$$

This equation explicitly describes the decomposition of the transverse field into two orthogonal eigenmodes propagating with different effective refractive indices. For oblique incidence, these indices become functions of both the incidence angles and the local director orientation ϕ [50]:

$$n'_o = \sqrt{n_o^2 - \sin^2 \theta_i}, \quad (\text{F3})$$

$$n'_e = \sqrt{n_e^2 - \sin^2 \theta_i \left[\frac{n_e^2}{n_o^2} \cos^2(\phi_i - \phi) + \sin^2(\phi_i - \phi) \right]}. \quad (\text{F4})$$

By projecting this field propagation onto the Cartesian axes, we obtain the localized Jones matrix:

$$\begin{aligned} T_{\theta}^E(h) &= \exp(i\bar{k}_o h) \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\gamma_{\theta} h) \right. \\ &\quad \left. + i \begin{pmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{pmatrix} \sin(\gamma_{\theta} h) \right], \end{aligned} \quad (\text{F5})$$

where the modified mean wave number and anisotropy parameter are defined as

$$\bar{k}_\theta(\theta_i, \phi_i, \phi) = \frac{n'_e + n'_0}{2}k, \quad \gamma_\theta(\theta_i, \phi_i, \phi) = \frac{n'_e - n'_0}{2}k. \quad (\text{F6})$$

Note that the parameters for normal incidence are naturally recovered as $\bar{k} = \bar{k}_\theta(0, \phi_i, \phi)$ and $\gamma = \gamma_\theta(0, \phi_i, \phi)$. To facilitate further perturbative analysis, it is convenient to introduce the parameter difference γ_- and the parameter sum γ_+ . Because the incidence angle θ_i is small, γ_- acts as a small perturbation, while γ_+ remains a highly oscillatory factor:

$$\gamma_- = \gamma_\theta - \gamma, \quad \gamma_+ = \gamma_\theta + \gamma \approx 2\gamma, \\ \gamma_- \approx \frac{k}{4} \sin^2 \theta_i \left[\frac{n_o - n_e}{n_o^2} + \left(\frac{n_e}{n_o^2} - \frac{1}{n_e} \right) \sin^2(\phi_i - \phi) \right].$$

Following the exact same approach used in the main text to transition to the circular polarization amplitudes [similar to Eq. (7)], the modified Jones matrix in the circular basis takes the form

$$T_\theta^\mathcal{E}(h) = \exp(i[\bar{k}_\theta - \bar{k}]h) \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos(\gamma_\theta h) + i \begin{pmatrix} 0 & \exp(i2\phi) \\ \exp(-i2\phi) & 0 \end{pmatrix} \sin(\gamma_\theta h) \right]. \quad (\text{F7})$$

This leads to the corresponding coupled-wave differential equation for the circular field amplitudes $\mathcal{E}(z)$:

$$\frac{d}{dz} \mathcal{E}(z) = i \left[\gamma_\theta \begin{pmatrix} 0 & e^{i2\phi} \\ e^{-i2\phi} & 0 \end{pmatrix} + (\bar{k}_\theta - \bar{k}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \mathcal{E}(z). \quad (\text{F8})$$

For normal incidence ($\theta_i = 0$), this reduces to the unperturbed differential equation (11), the exact solution of which is known to be $\mathcal{E}(z) = T_{\text{TN}}^\mathcal{E}(z, z_0) \mathcal{E}(z_0)$, where $T_{\text{TN}}^\mathcal{E}$ is defined by Eq. (12) with $h = z - z_0$.

To understand the wave behavior when the incident field deviates from the normal by a small angle, we treat Eq. (F8) as a perturbed version of Eq. (11). We can seek the solution in the form of the unperturbed solution multiplied by a slowly varying perturbation matrix:

$$\mathcal{E}(z) = W_L(z) T_{\text{TN}}^\mathcal{E}(z, z_0) \mathcal{E}(z_0) \quad (\text{F9})$$

or alternatively, multiplied from the right,

$$\mathcal{E}(z) = T_{\text{TN}}^\mathcal{E}(z, z_0) W_R(z) \mathcal{E}(z_0), \quad (\text{F10})$$

where $W_L(z)$ and $W_R(z)$ are perturbation matrices satisfying the initial condition $W_L(z_0) = I$ and $W_R(z_0) = I$ (I is a unit matrix).

First, let us consider the solution variant with $W_L(z)$. Substituting this ansatz directly into the differential equation yields the governing equation for W_L . Introducing the standard commutator notation $[A, B] = AB - BA$, we obtain

$$\frac{d}{dz} W_L = i\gamma [\sigma_\phi, W_L] + i\gamma_- \sigma_\phi W_L + i(\bar{k}_\theta - \bar{k}) W_L, \quad (\text{F11})$$

where the coupling matrix is

$$\sigma_\phi = \begin{pmatrix} 0 & \exp(i2\phi) \\ \exp(-i2\phi) & 0 \end{pmatrix}. \quad (\text{F12})$$

To physically interpret this perturbation, we perform a substitution to transition from the circular polarization amplitudes

to the amplitudes of linear polarizations rotating alongside the local director (parallel and perpendicular to the director). This transformation is explicitly represented by

$$W_L = UR[\phi(z)] \Lambda_L R[-\phi(z)] U^{-1}, \quad (\text{F13})$$

where U is the standard basis transformation matrix to the circular basis, defined previously in Eq. (8), and $R(\phi) = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$ is the spatial rotation matrix. This compact notation clearly demonstrates the multiplication of the circular basis transition by the continuous geometric rotation. Letting $q = d\phi/dz$, the differential equation for the local perturbation matrix Λ_L becomes

$$\frac{d}{dz} \Lambda_L = i\gamma [\sigma_z, \Lambda_L] - q [\sigma_y, \Lambda_L] + i\gamma_- \sigma_z \Lambda_L + i(\bar{k}_\theta - \bar{k}) \Lambda_L, \quad (\text{F14})$$

where $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The last scalar term can be easily separated by substituting $\Lambda_L = v \tilde{\Lambda}_L$, where $v(z)$ is a scalar function representing a common phase multiplier acting on the entire field. We obtain the following system:

$$\frac{d}{dz} v = i(\bar{k}_\theta - \bar{k})v, \quad (\text{F15})$$

$$\frac{d}{dz} \tilde{\Lambda}_L = i\gamma [\sigma_z, \tilde{\Lambda}_L] - q [\sigma_y, \tilde{\Lambda}_L] + i\gamma_- \sigma_z \tilde{\Lambda}_L. \quad (\text{F16})$$

To satisfy the condition $W_L(z_0) = I$ at the initial boundary, we impose the corresponding initial conditions $v(z_0) = 1$ and $\tilde{\Lambda}_L(z_0) = I$. The exact scalar phase multiplier is thus found to be

$$v(z) = \exp \left(i \int_{z_0}^z (\bar{k}_\theta(\phi) - \bar{k}) dz' \right). \quad (\text{F17})$$

Given the structural properties of the decoupled equation, we can parameterize $\tilde{\Lambda}_L$ as $\tilde{\Lambda}_L = \begin{pmatrix} l_1 & l_2 \\ -l_2^* & l_1^* \end{pmatrix}$. This yields a coupled system of linear ordinary differential equations with variable coefficients:

$$\frac{d}{dz} l_1 = 2iq \text{Im}(l_2) + i\gamma_- l_1, \quad (\text{F18})$$

$$\frac{d}{dz} l_2 = -2iq \text{Im}(l_1) + i\gamma_+ l_2, \quad (\text{F19})$$

where $\text{Im}(\dots)$ denotes the imaginary part operator, subject to the initial conditions $l_1(z_0) = 1$ and $l_2(z_0) = 0$.

To analyze this system, we can treat the equations for l_1 and l_2 as linear inhomogeneous differential equations. Their exact solutions can be written in the integral form:

$$l_1 = e^{(i \int_{z_0}^z \gamma_- dz')} \left[1 + 2i \int_{z_0}^z e^{(-i \int_{z_0}^{z'} \gamma_- dz'')} q \text{Im}(l_2) dz' \right], \quad (\text{F20})$$

$$l_2 = -2ie^{(i \int_{z_0}^z \gamma_+ dz')} \int_{z_0}^z e^{(-i \int_{z_0}^{z'} \gamma_+ dz'')} q \text{Im}(l_1) dz', \quad (\text{F21})$$

where it is implicitly understood that the state variables within the integrals, l_1 and l_2 , depend on the local integration variable z' .

Directly from these integral equations, it becomes evident how to expand the solution into an iterative series

$l_1 = \sum_{j=0}^{\infty} l_1^{(j)}$ and $l_2 = \sum_{j=0}^{\infty} l_2^{(j)}$, governed by the recursive relations

$$l_1^{(j+1)} = 2ie^{(i \int_{z_0}^z \gamma_- dz')} \int_{z_0}^z e^{(-i \int_{z_0}^{z'} \gamma_- dz'')} q \text{Im}(l_2^{(j)}) dz', \quad (\text{F22})$$

$$l_2^{(j)} = -2ie^{(i \int_{z_0}^z \gamma_+ dz')} \int_{z_0}^z e^{(-i \int_{z_0}^{z'} \gamma_+ dz'')} q \text{Im}(l_1^{(j)}) dz', \quad (\text{F23})$$

with the lowest-order term being

$$l_1^{(0)} = \exp\left(i \int_{z_0}^z \gamma_- dz'\right). \quad (\text{F24})$$

Detailed analysis reveals that the dominant term is $l_1^{(0)}$. Given the smallness of the incidence angle, the parameter difference γ_- is strictly bounded. Depending on the azimuthal orientation $(\phi_i - \phi)$, the lower bound for γ_- is approximately $-\frac{1}{4} \frac{\chi}{h} \frac{n_e}{n_o}$, while the upper bound is approximately $\frac{1}{4} \frac{\chi}{h}$. Consequently, the integral of this perturbation over the cell thickness h is bounded by the small parameter χ , meaning $l_1^{(0)}$ deviates from unity by at most:

$$|l_1^{(0)} - 1| \lesssim \frac{1}{4} \frac{n_e}{n_o} \chi. \quad (\text{F25})$$

Concurrently, evaluating the first-order cross-coupling term $l_2^{(0)}$ involves integrating over the highly oscillatory function $\gamma_+ \approx 2\gamma$. Introducing the adiabatic parameter $\alpha = q/\gamma$, the magnitude of this cross-coupling is strongly suppressed, yielding

$$|l_2^{(0)}| \sim \frac{q}{\gamma} |l_1^{(0)} - 1| \sim \frac{1}{4} \alpha \chi. \quad (\text{F26})$$

In higher orders, the integral over this highly oscillatory term persists, ensuring the rapid convergence of the series, with subsequent terms in l_1 being bounded by the elements of l_2 .

Thus, ignoring the negligibly small off-diagonal elements, the local perturbation matrix simplifies to a purely diagonal form. Combining $v(z)$ and $l_1(z)$, and explicitly indicating the asymptotic error bound in terms of our small parameters, yields

$$\Lambda_L = \begin{pmatrix} e^{(ik \int_{z_0}^z [n'_e - n_e] dz')} & 0 \\ 0 & e^{(ik \int_{z_0}^z [n'_o - n_o] dz')} \end{pmatrix} + O(\alpha \chi). \quad (\text{F27})$$

Alternatively, let us consider the right-multiplied perturbation ansatz:

$$\mathcal{E}(z) = T_{\text{TN}}^{\mathcal{E}}(z, z_0) W_R(z) \mathcal{E}(z_0). \quad (\text{F28})$$

Substituting this form into Eq. (F8) yields a structurally different governing differential equation for W_R . Due to the noncommutativity of the perturbation and the unperturbed evolution matrix $T_{\text{TN}}^{\mathcal{E}}$, the perturbation term becomes conjugated by the unperturbed solutions:

$$\begin{aligned} \frac{d}{dz} W_R &= i \left[\gamma_- (T_{\text{TN}}^{\mathcal{E}}(z, z_0))^{-1} \sigma_{\phi} T_{\text{TN}}^{\mathcal{E}}(z, z_0) \right. \\ &\quad \left. + (\bar{k}_{\theta} - \bar{k}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] W_R. \end{aligned} \quad (\text{F29})$$

To analyze this equation, we can invoke the strong waveguiding limit. By neglecting the small nonadiabatic terms

proportional to α within the unperturbed matrix $T_{\text{TN}}^{\mathcal{E}}(z, z_0)$, the conjugation effectively “unrotates” the local director axis back to its initial orientation. This mathematical property dictates that the appropriate transformation into the local frame for the right-side perturbation relies on the director orientation at the initial boundary z_0 , rather than the local coordinate z :

$$W_R = U R[\phi(z_0)] \Lambda_R R[-\phi(z_0)] U^{-1}. \quad (\text{F30})$$

Applying this transformation demonstrates that the resulting differential equation for Λ_R is mathematically equivalent to the one derived for Λ_L . Therefore, up to the evaluated accuracy bound, the right-side local perturbation matrix shares the exact same diagonal phase-accumulation form, such that $\Lambda_R = \Lambda_L + O(\alpha \chi)$.

With the perturbation matrices established, we can now construct the complete Jones matrix for the tandem configuration under oblique incidence. For a tandem cell, the unperturbed optical core is given by the product of the individual cell matrices. In the general case involving an active compensating plate Eq. (C1), the core matrix is $T_{\text{core}} = T_{\text{TN2}}^{\mathcal{E}}(h_2) T_C^{\mathcal{E}}(\Phi_c) T_{\text{TN1}}^{\mathcal{E}}(h_1)$. As demonstrated in the main text, under the ideal operational regime and neglecting secondary nonidealities, this unperturbed core simplifies to a purely diagonal matrix:

$$T_{\text{CTN}}^{\mathcal{E}} \approx \begin{pmatrix} \exp(i\varphi) & 0 \\ 0 & \exp(-i\varphi) \end{pmatrix}, \quad (\text{F31})$$

where φ is the target phase.

To evaluate the specific impact of the oblique incidence on the figures of merit ($\mathcal{A}$ and $\mathcal{P}$), we apply the derived perturbation matrices to the constituent cells. Because our objective here is to quantify the isolated contribution of the angular deviation rather than to investigate the complex interference between multiple distinct hardware nonidealities, we can effectively evaluate this specific effect by sandwiching the ideal, unperturbed diagonal core between the corresponding angular perturbations.

To achieve this elegantly, we utilize the left-multiplied perturbation ansatz for the second cell and the right-multiplied perturbation ansatz for the first cell. The total Jones matrix of the perturbed tandem cell then takes the remarkably compact form:

$$T_{\theta, \text{CTN}}^{\mathcal{E}} = (W_{L2} T_{\text{TN2}}^{\mathcal{E}}) \cdot (T_{\text{TN1}}^{\mathcal{E}} W_{R1}) = W_{L2} T_{\text{CTN}}^{\mathcal{E}} W_{R1}. \quad (\text{F32})$$

This formulation completely encapsulates the unperturbed twisted evolution within the central matrix $T_{\text{CTN}}^{\mathcal{E}}$, reducing the problem to the multiplication of the analytically derived, near-identity perturbation matrices W_{L2} and W_{R1} at the external boundaries.

To evaluate this matrix product, we explicitly substitute the local coordinate transformations for W_{L2} (evaluated at the exit boundary $z = z_2$, where $\phi = \frac{\pi}{2} + \frac{\varphi}{2}$) and W_{R1} (evaluated at the entrance boundary $z = z_1$, where $\phi = -\frac{\varphi}{2}$). The core unperturbed matrix $T_{\text{CTN}}^{\mathcal{E}}$ is flanked by the spatial rotation matrices. Due to the 90° director twist discontinuity at the center of the tandem cell, the inner product of these rotation matrices and the ideal core simplifies to the off-diagonal Pauli

matrix, governed by the identity

$$R\left(-\frac{\pi}{2} - \frac{\varphi}{2}\right)U^{-1}T_{\text{CTN}}^{\mathcal{E}}UR\left(-\frac{\varphi}{2}\right) = -\sigma_y. \quad (\text{F33})$$

This mathematically encapsulates the central self-compensation mechanism: the 90° jump effectively swaps the local ordinary and extraordinary axes between the first and second cells. Consequently, the complete perturbed Jones matrix takes the form

$$T_{\theta, \text{CTN}}^{\mathcal{E}} = -UR\left(\frac{\pi}{2} + \frac{\varphi}{2}\right)\Lambda_L\sigma_y\Lambda_R R\left(\frac{\varphi}{2}\right)U^{-1}. \quad (\text{F34})$$

In our spatial light modulation scheme, the target geometric phase is encoded strictly in the (1,1) element of the circular Jones matrix, as the orthogonal polarization state is filtered out by the output analyzer. Extracting this specific element yields the sum of two phase-accumulating paths:

$$(T_{\theta, \text{CTN}}^{\mathcal{E}})_{11} = \frac{1}{2}e^{i\varphi} \left[e^{ik(\int_{z_c}^{z_2} n_{e-} dz' + \int_{z_1}^{z_c} n_{o-} dz')} + e^{ik(\int_{z_c}^{z_2} n_{o-} dz' + \int_{z_1}^{z_c} n_{e-} dz')} \right], \quad (\text{F35})$$

where $n_{e-} = n'_e - n_e$ and $n_{o-} = n'_o - n_o$ represent the local refractive index perturbations. The integration intervals span the first cell ($[z_1, z_c]$) and the second cell ($[z_c, z_2]$).

This expression reveals a profound result: the superposition of these two cross-coupled paths generates a mean phase shift and an amplitude modulation. However, because the amplitude modulation scales with the cosine of the phase difference between the two paths, the amplitude error contains no linear terms with respect to our small parameter χ . Since the amplitude perturbation arises strictly from the quadratic terms $O(\chi^2)$ and the previously neglected nona-

diabatic cross-coupling $O(\alpha\chi)$, and we restrict our analysis to the linear perturbation regime, we do not evaluate these higher-order contributions. Consequently, to the first order of approximation, the angular deviation induces no amplitude ripple, yielding $\mathcal{A}^{(\text{ang})} = O(\chi^2 + \alpha\chi)$.

The observable effect is purely a phase perturbation, $\delta\varphi$, determined by the arithmetic mean of the two path integrals:

$$\delta\varphi = k \int_{z_1}^{z_2} \frac{n_{e-} + n_{o-}}{2} dz'. \quad (\text{F36})$$

This cumulative phase perturbation can be decomposed into a constant piston term (which affects the entire aperture uniformly and carries no spatial information) and a spatially varying term $\delta\varphi_{\text{var}}$ that depends on the local director orientation ϕ . Expanding the integrand with respect to $\theta_i \ll 1$ yields

$$\begin{aligned} \frac{n_{e-} + n_{o-}}{2} \approx & -\frac{1}{4} \sin^2 \theta_i \left[\frac{1}{n_o} + \frac{n_e}{2n_o^2} + \frac{1}{2n_e} \right. \\ & \left. + \frac{n_e}{2} \left(\frac{1}{n_o^2} - \frac{1}{n_e^2} \right) \cos(2\phi_i - 2\phi) \right]. \end{aligned} \quad (\text{F37})$$

The spatially varying component is therefore isolated as

$$\delta\varphi_{\text{var}} = -\frac{1}{4}k \sin^2 \theta_i \int_{z_1}^{z_2} \frac{n_e}{2} \left[\frac{1}{n_o^2} - \frac{1}{n_e^2} \right] \cos(2\phi_i - 2\phi) dz'. \quad (\text{F38})$$

By evaluating this integral over the entire tandem structure and maximizing over the target geometric phase φ , we arrive at the final analytical bound for the phase figure of merit associated with oblique incidence:

$$\mathcal{P}^{(\text{ang})} = \frac{n_e + n_o}{4n_o} \chi. \quad (\text{F39})$$

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