

Separation of quantum time evolution into holonomic and dynamical parts

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The issue of separating Schrödinger-type quantum time evolution into a product of holonomic and dynamical parts in the nonadiabatic non-Abelian case is examined. We identify all special cases in which this kind of separation is possible, and we prove that separability is a gauge-invariant property of quantum time evolution. The general analysis is implemented in a three-level system with Λ -type coupling structures. The typical case where the holonomic and dynamical parts do not separate is illustrated by means of two noncommuting Λ -type Hamiltonians.

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I. INTRODUCTION

Experimental tests of various forms of geometric phase [1–3] and quantum holonomy [4–6] rely on the ability to separate it from the dynamical part of the quantum time evolution [7–12]. While such a product form always exists in the Abelian geometric phase cases [1,2] and in the case of adiabatic non-Abelian quantum holonomy [4], the holonomic and dynamical parts do not generally separate for nonadiabatic time evolution of subspaces of Hilbert space [5], a setting that has gained considerable attention in the context of holonomic quantum computation [13–23]. Motivated by the recent analyses in Refs. [24–27], we undertake a detailed examination of the issue of separation for nonadiabatic Schrödinger-type quantum time evolution of subspaces.

The theory of quantum holonomy in the nonadiabatic non-Abelian case was first developed by Anandan [5]. Specifically, it was established that the Schrödinger evolution of a subspace of Hilbert space is governed by a time-ordered exponential of a sum of a matrix-valued vector potential and a matrix representation of the Hamiltonian, generating the holonomic and dynamical parts of the time evolution, respectively. The generators are typically noncommuting matrices, which implies that the quantum time evolution cannot, in general, be separated into a product of a holonomic and a dynamical part, thereby limiting the experimental accessibility of quantum holonomy.

Although separation into holonomic and dynamical parts typically does not occur, it remains pertinent to identify the

special cases in which it does occur for nonadiabatic time evolution of subspaces. These cases are of particular interest for experimental tests of quantum holonomies in nonadiabatic evolution and are relevant for holonomic quantum computation. To this end, we revisit the theory of Schrödinger-type quantum time evolution of subspaces to analyze the issue of separation systematically.

The paper is organized as follows. We commence in Sec. II by presenting an operator-based approach to the Schrödinger evolution of subspaces and we show how it underlies the theory developed in Ref. [5]. Notably, the analysis is carried out for noncyclic time evolution throughout. We identify all cases in which separation of quantum time evolution into holonomic and dynamical parts does occur. We further prove that the separability is a gauge-invariant property of quantum time evolution. In Sec. III, the general analysis is illustrated by a series of examples, all based on the three-level Λ configuration. Finally, a brief summary of the paper and an outlook delineating potential directions for further study is provided in Sec. IV.

II. SCHRÖDINGER EVOLUTION OF SUBSPACES

A. The restricted Schrödinger equation

Consider a closed quantum system associated with an N -dimensional (N finite) Hilbert space $\mathcal{H} = \mathcal{V}_M(t) \oplus \mathcal{V}_{N-M}(t)$ decomposed into two orthogonal subspaces $\mathcal{V}_M(t)$ and $\mathcal{V}_{N-M}(t)$ of fixed dimensions M and $N - M$, respectively. We assume that $C: t \in [0, \tau] \mapsto \mathcal{V}_M(t)$ is a continuous curve in the Grassmannian $\mathcal{G}(N; M)$, i.e., the space of M -dimensional subspaces of $\mathcal{H}$, and introduce an orthonormal M -frame $\mathcal{S}(t) \equiv \{|\psi_j(t)\rangle\}_{j=1}^M$ spanning $\mathcal{V}_M(t)$ at every instant t , where each $|\psi_j(t)\rangle$ is a solution of the time-dependent Schrödinger equation (we put $\hbar = 1$):

$$i\dot{|\psi_j(t)\rangle} = -iH(t)|\psi_j(t)\rangle, \quad j = 1, \dots, M \quad (1)$$

with $H(t)$ denoting the Hamiltonian operator of the system. The time evolution operator $U(t, 0): \mathcal{V}_M(0) \rightarrow \mathcal{V}_M(t)$ may be

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written as

$$U(t, 0) = |\psi_j(t)\rangle\langle\psi_j(0)|. \quad (2)$$

Here and onwards, unless otherwise stated, we use Einstein's summation convention, according to which repeated indices are implicitly summed. The operator $U(t, 0)$ is a partial isometry, i.e., $U^\dagger(t, 0)U(t, 0) = |\psi_j(0)\rangle\langle\psi_j(0)| \equiv P_M(0)$ and $U(t, 0)U^\dagger(t, 0) = |\psi_j(t)\rangle\langle\psi_j(t)| \equiv P_M(t)$ are projectors on the subspaces $\mathcal{V}_M(0)$ and $\mathcal{V}_M(t)$, respectively. The special case where $P_M(\tau) = P_M(0)$ defines cyclic evolution, i.e., when $\mathcal{V}_M(\tau)$ and $\mathcal{V}_M(0)$ coincide.

We proceed by emphasizing the general fact that all quantum holonomy experiments start from a physical setup that defines a Hamiltonian $H(t)$ and by preparing the initial projector $P_M(0)$ on $\mathcal{V}_M(0)$. The Schrödinger equation for the projector

$$\dot{P}_M(t) = i[P_M(t), H(t)] \quad (3)$$

generates a continuous curve $\tilde{C}: t \in [0, \tau] \mapsto P_M(t)$ in the space of rank- M projectors, being isomorphic to the curve C in $\mathcal{G}(N; M)$ introduced above. Thus, the experimental setup determines $H(t)$ and $P_M(t)$.

The essence of the preceding observation is captured by the differential equation [28,29]

$$\dot{U}(t, 0) = [\dot{P}_M(t) - iP_M(t)H(t)P_M(t)]U(t, 0), \quad (4)$$

where $P_M(t)H(t)P_M(t)$ is the Hamiltonian operator restricted to the instantaneous subspace $\mathcal{V}_M(t)$. We refer to Eq. (4) as the restricted Schrödinger equation (RSE).

B. Derivation of the time evolution matrix

We establish that Anandan's theory [5] follows from the RSE. This requires the introduction of an orthonormal M -frame $\mathcal{L}(t) \equiv \{|\varphi_j(t)\rangle\}_{j=1}^M$ consisting of vectors that span $\mathcal{V}_M(t)$ at every instant t , but generally do not satisfy Eq. (1), and obey the “in-phase” condition [30]

$$\mathbb{O}(0, \tau) > 0, \quad (5)$$

where $\mathbb{O}(0, \tau)$ is the $M \times M$ overlap matrix with matrix elements $\mathbb{O}_{jk}(0, \tau) = \langle\varphi_j(0)|\varphi_k(\tau)\rangle$. For cyclic evolution $\mathcal{V}_M(\tau) = \mathcal{V}_M(0)$, Eq. (5) reduces to the standard condition $|\varphi_j(\tau)\rangle = |\varphi_j(0)\rangle$, in the case of which $\mathbb{O}(0, \tau) = \mathbb{I}_M > 0$, $\mathbb{I}_M$ being the $M \times M$ identity matrix. In the language of fiber bundles, $\mathcal{L}(t)$ corresponds to a local section [31] that serves as a reference used for a gauge-covariant description of the change $\mathcal{S}(0) \mapsto \mathcal{S}(\tau)$ generated by the Hamiltonian $H(t)$.

The two M -frames $\mathcal{S}(t)$ and $\mathcal{L}(t)$ spanning $\mathcal{V}_M(t)$ are related by the operator $W(t) \equiv |\psi_j(t)\rangle\langle\varphi_j(t)|$, which is a partial isometry as $W^\dagger(t)W(t) = W(t)W^\dagger(t) = P_M(t)$. In the following, we assume that $|\varphi_j(0)\rangle = |\psi_j(0)\rangle$, corresponding to the initial condition $W(0) = P_M(0)$. We may write

$$|\psi_k(t)\rangle = |\varphi_j(t)\rangle\mathbb{W}_{jk}(t), \quad (6)$$

where $\mathbb{W}_{jk}(t) \equiv \langle\varphi_j(t)|W(t)|\varphi_k(t)\rangle = \langle\varphi_j(t)|\psi_k(t)\rangle$ are matrix elements of the unitary $M \times M$ transformation matrix $\mathbb{W}(t)$, which connects $\mathcal{L}(t)$ and $\mathcal{S}(t)$. Equations (2) and (6),

together with $\mathbb{W}_{jk}(0) = \delta_{jk}$, yield

$$\begin{aligned} U(t, 0) &= P_M(t)U(t, 0) \\ &= P_M(t)|\varphi_j(t)\rangle\langle\varphi_k(0)|\mathbb{W}_{jk}(t). \end{aligned} \quad (7)$$

By inserting this expression into the RSE, we find

$$\begin{aligned} P_M(t)|\dot{\varphi}_j(t)\rangle\langle\varphi_k(0)|\mathbb{W}_{jk}(t) + |\varphi_j(t)\rangle\langle\varphi_k(0)|\dot{\mathbb{W}}_{jk}(t) \\ = -iP_M(t)H(t)|\varphi_j(t)\rangle\langle\varphi_k(0)|\mathbb{W}_{jk}(t), \end{aligned} \quad (8)$$

where the term $\dot{P}_M(t)U(t, 0)$ has been canceled on both sides. Further, upon substitution of $P_M(t) = |\varphi_l(t)\rangle\langle\varphi_l(t)|$ and relabeling in the second term on the left-hand side, we have

$$\begin{aligned} |\varphi_l(t)\rangle\langle\varphi_k(0)|\langle\varphi_l(t)|\dot{\varphi}_j(t)\rangle\mathbb{W}_{jk}(t) \\ + |\varphi_l(t)\rangle\langle\varphi_k(0)|\dot{\mathbb{W}}_{lk}(t) \\ = -i|\varphi_l(t)\rangle\langle\varphi_k(0)|\langle\varphi_l(t)|H(t)|\varphi_j(t)\rangle\mathbb{W}_{jk}(t). \end{aligned} \quad (9)$$

By using the linear independence of the operators $|\varphi_l(t)\rangle\langle\varphi_k(0)|$ and that $\langle\varphi_l(t)|\dot{\varphi}_j(t)\rangle = -\langle\dot{\varphi}_l(t)|\varphi_j(t)\rangle$, as well as rearranging terms, we obtain the Anandan equation (AE):

$$\dot{\mathbb{W}}(t) = [\mathbb{A}(t) + \mathbb{K}(t)]\mathbb{W}(t). \quad (10)$$

Here, $\mathbb{A}(t)$ and $\mathbb{K}(t)$ are anti-Hermitian $M \times M$ matrices with matrix elements

$$\begin{aligned} \mathbb{A}_{jk}(t) &\equiv \langle\dot{\varphi}_j(t)|\varphi_k(t)\rangle, \\ \mathbb{K}_{jk}(t) &\equiv -i\langle\varphi_j(t)|H(t)|\varphi_k(t)\rangle. \end{aligned} \quad (11)$$

The formal solution of the AE is given by a time-ordered exponential:

$$\mathbb{W}(\tau) = \mathcal{T}e^{\int_0^\tau [\mathbb{A}(t) + \mathbb{K}(t)]dt}, \quad (12)$$

which typically does not separate into a product of holonomic $\mathcal{T}e^{\int_0^\tau \mathbb{A}(t)dt}$ and dynamical $\mathcal{T}e^{\int_0^\tau \mathbb{K}(t)dt}$ parts [5,24,32–34], as $\mathbb{A}(t)$ and $\mathbb{K}(t')$ generally do not commute for arbitrary $t, t' \in [0, \tau]$. Physically, the difference between $\mathcal{T}e^{\int_0^\tau \mathbb{A}(t)dt}$ and $\mathcal{T}e^{\int_0^\tau \mathbb{K}(t)dt}$ is that, whereas the former is a time nonlocal path-dependent quantity [35], the latter is time local; i.e., it is given by the time integral of the Hamiltonian evaluated at time t projected onto the subspace at the same time t [36].

The use of an M -frame $\mathcal{L}(t)$ satisfying the in-phase condition in Eq. (5) ensures that the time evolution must not be cyclic. To make this point precise, we note that

$$\begin{aligned} \mathbb{U}_{jk}(\tau, 0) &\equiv \langle\psi_j(0)|U(\tau, 0)|\psi_k(0)\rangle = \langle\psi_j(0)|\psi_k(\tau)\rangle \\ &= \langle\varphi_j(0)|\varphi_l(\tau)\rangle\mathbb{W}_{lk}(\tau) \\ &= \mathbb{O}_{jl}(0, \tau)\mathbb{W}_{lk}(\tau), \end{aligned} \quad (13)$$

where $\mathbb{U}(\tau, 0)$ is the matrix representation of the time evolution operator $U(\tau, 0)$ with respect to $\mathcal{S}(0)$ [37]. The matrices $\mathbb{O}(0, \tau)$ and $\mathbb{W}(\tau)$ are the positive and unitary parts, respectively, of $\mathbb{U}(\tau, 0)$. Thus, $\mathbb{U}(\tau, 0)$ is unitary if and only if $\mathbb{O}(0, \tau) = \mathbb{I}_M$, which, as noted above, corresponds to cyclic evolution. By combining Eqs. (12) and (13), we obtain the time evolution matrix that governs the Schrödinger evolution of the subspace $\mathcal{V}_M(t)$:

$$\mathbb{U}(\tau, 0) = \mathbb{O}(0, \tau)\mathcal{T}e^{\int_0^\tau [\mathbb{A}(t) + \mathbb{K}(t)]dt}, \quad (14)$$

which reduces to Anandan's expression [5] for cyclic evolution. Equation (14) shows that the issue of separation applies to any quantum evolution, irrespective of whether it is cyclic or noncyclic.

C. The issue of separation

Although the preceding analysis establishes that quantum time evolution does not separate into a product of a holonomic and a dynamical part, in general, it remains pertinent to determine the special cases in which such a separation does exist for nonadiabatic time evolution of subspaces of dimension $M \geq 2$. The requirement for the solution of the AE in Eq. (12) to separate is that

$$[\mathbb{A}(t), \mathbb{K}(t')] = 0_M, \quad \forall t, t' \in [0, \tau], \quad (15)$$

0_M being the $M \times M$ zero matrix. This is a highly restrictive condition in the sense that it will typically not be satisfied for a randomly chosen Hamiltonian and subspace. To make this point precise, we identify the only three distinct special cases for the requirement in Eq. (15) to hold: (i) when $\mathcal{V}_M(t) = \mathcal{V}_M(0)$, $\forall t \in [0, \tau]$, in which case just the dynamical part can be nontrivial, as the vectors defining the M -frame $\mathcal{L}(t)$ can be chosen time-independent; (ii) when $H(t)$ and $\mathcal{L}(t)$ are such that $\mathbb{K}(t) = 0_M$, $\forall t \in [0, \tau]$, leaving a nontrivial holonomic part only; (iii) when $H(t)$ and $\mathcal{L}(t)$ are such that $\mathbb{A}(t) = -iA_j(t)\mathbb{P}_j(0)$ and $\mathbb{K}(t) = -iK_j(t)\mathbb{P}_j(0)$, $\mathbb{P}_j(0)$ being mutually orthogonal projection matrices and $A_j(t)$, $K_j(t)$, $j = 1, \dots, M$, are real-valued functions that are simultaneously nonzero for at least part of the time interval $[0, \tau]$, and $A_{j'}(t) \neq A_{j \neq j'}(t)$ [$K_{k'}(t) \neq K_{k \neq k'}(t)$] for at least one j' (k'). Cases (i) and (ii) have been utilized, e.g., in Ref. [38] and in Refs. [13, 18–20], respectively, and case (iii) is reminiscent of the idea of unconventional geometric quantum computation [39], which has been considered in the non-Abelian context in Ref. [24]. The typical case, which we label (iv), is when the holonomic and dynamical parts do not separate. In Sec. III, cases (i)–(iv) are illustrated in a three-level system.

Whether a given solution $\mathbb{W}(\tau)$ separates or not is a gauge-invariant property of the time evolution. To prove this assertion, consider a change of M -frame $\mathcal{L}(t) \mapsto \tilde{\mathcal{L}}(t)$ under which

$$|\varphi_k(t)\rangle \mapsto |\bar{\varphi}_k(t)\rangle = |\varphi_j(t)\rangle \mathbb{R}_{jk}(t) \quad (16)$$

for some unitary $M \times M$ matrix $\mathbb{R}(t)$ smoothly dependent on time, $\mathbb{W}(\tau) \mapsto \bar{\mathbb{W}}(\tau)$, and $\mathbb{O}(0, \tau) \mapsto \bar{\mathbb{O}}(0, \tau)$, so that

$$\begin{aligned} \mathbb{U}(\tau, 0) &= \mathbb{O}(0, \tau) \mathbb{W}(\tau) \\ &\mapsto \bar{\mathbb{U}}(\tau, 0) = \bar{\mathbb{O}}(0, \tau) \bar{\mathbb{W}}(\tau). \end{aligned} \quad (17)$$

Now, note that

$$\bar{\mathbb{U}}(\tau, 0) = \mathbb{R}^\dagger(0) \mathbb{U}(\tau, 0) \mathbb{R}(0), \quad (18)$$

since $\mathcal{S}(0)$ is required to coincide with $\mathcal{L}(0)$. We further have from Eq. (16) that $\bar{\mathbb{O}}(0, \tau) = \mathbb{R}^\dagger(0) \mathbb{O}(0, \tau) \mathbb{R}(0) > 0$. Together with Eq. (5), this implies $\mathbb{R}(\tau) = \mathbb{R}(0)$ and thus

$$\bar{\mathbb{O}}(0, \tau) = \mathbb{R}^\dagger(0) \mathbb{O}(0, \tau) \mathbb{R}(0). \quad (19)$$

By combining Eqs. (18) and (19) with Eq. (17), we find

$$\mathbb{R}^\dagger(0) \mathbb{U}(\tau, 0) \mathbb{R}(0) = \mathbb{R}^\dagger(0) \mathbb{O}(0, \tau) \mathbb{R}(0) \bar{\mathbb{W}}(\tau). \quad (20)$$

Since $\mathbb{O}(0, \tau)$ is assumed to be strictly positive, it follows that $\mathbb{O}^{-1}(0, \tau)$ exists. Thus, $[\mathbb{R}^\dagger(0) \mathbb{O}(0, \tau) \mathbb{R}(0)]^{-1} = \mathbb{R}^\dagger(0) \mathbb{O}^{-1}(0, \tau) \mathbb{R}(0)$ is well defined and can be used to invert Eq. (20), yielding

$$\begin{aligned} \bar{\mathbb{W}}(\tau) &= \mathbb{R}^\dagger(0) \mathbb{O}^{-1}(0, \tau) \mathbb{R}(0) \mathbb{R}^\dagger(0) \mathbb{O}(0, \tau) \mathbb{W}(\tau) \mathbb{R}(0) \\ &= \mathbb{R}^\dagger(0) \mathbb{W}(\tau) \mathbb{R}(0). \end{aligned} \quad (21)$$

In the case where $\mathbb{W}(\tau)$ separates, we therefore have

$$\begin{aligned} &\mathcal{T} e^{\int_0^\tau \mathbb{A}(t) dt} \mathcal{T} e^{\int_0^\tau \mathbb{K}(t) dt} \\ &\mapsto \mathbb{R}^\dagger(0) \mathcal{T} e^{\int_0^\tau \mathbb{A}(t) dt} \mathbb{R}(0) \mathbb{R}^\dagger(0) \mathcal{T} e^{\int_0^\tau \mathbb{K}(t) dt} \mathbb{R}(0), \end{aligned} \quad (22)$$

which is a product of the holonomic part $\mathbb{R}^\dagger(0) \mathcal{T} e^{\int_0^\tau \mathbb{A}(t) dt} \mathbb{R}(0)$ and the dynamical part $\mathbb{R}^\dagger(0) \mathcal{T} e^{\int_0^\tau \mathbb{K}(t) dt} \mathbb{R}(0)$ in the transformed frame $\tilde{\mathcal{L}}(t)$. This shows that if $\mathbb{W}(\tau)$ separates in one gauge, it will do so in all gauges. In particular, it follows that if $\mathbb{W}(\tau)$ does not separate in a given gauge, it will not separate in any gauge. Consequently, cases (i)–(iii) above constitute the only special cases in which separation occurs.

Note that the eigenvalues of $\mathbb{R}^\dagger(0) \mathcal{T} e^{\int_0^\tau \mathbb{A}(t) dt} \mathbb{R}(0)$ coincide with those of $\mathcal{T} e^{\int_0^\tau \mathbb{A}(t) dt}$, meaning that the eigenvalues of the holonomic part are gauge invariant. Since the eigenvalues can be obtained by reconstructing the unitary time evolution matrix using quantum state tomography [14–17, 21], the spectrum of the first factor in Eq. (22) is the experimental signature of the holonomy associated with the loop C in all gauges.

It is instructive to consider the separation issue for cyclic evolution. In this case, there exists a set of vectors $\mathcal{S}(0) = \mathcal{L}(0) = \{|\mu_j\rangle\}_{j=1}^M$ spanning $\mathcal{V}_M(\tau) = \mathcal{V}_M(0)$ and satisfying [repeated indices are not summed in Eqs. (23)–(26)]

$$U(\tau, 0)|\mu_j\rangle = e^{i\chi_j}|\mu_j\rangle, \quad (23)$$

i.e., $|\mu_j\rangle$ and $e^{i\chi_j}$ are eigenvectors and eigenvalues of $U(\tau, 0)$. We can construct $\mathcal{L}(t)$ as

$$|\varphi_j(t)\rangle = e^{-if_j(t)} U(t, 0)|\mu_j\rangle, \quad (24)$$

with real-valued functions $f_j(t)$ such that $f_j(\tau) = \chi_j$ and $f_j(0) = 0$ [40]. This guarantees the cyclic condition as the accumulated phases precisely cancel the eigenphases of $U(\tau, 0)$, yielding $|\varphi_j(\tau)\rangle = |\varphi_j(0)\rangle = |\mu_j\rangle$ and hence $\mathbb{O}(0, \tau) = \mathbb{I}_M$. Equation (24) allows us to compute $\mathbb{K}(t)$ and $\mathbb{A}(t)$ in the cyclic case as

$$\mathbb{K}_{jk}(t) = -i\langle\mu_j|U^\dagger(t, 0)H(t)U(t, 0)|\mu_k\rangle e^{if_j(t)-f_k(t)} \quad (25)$$

and

$$\begin{aligned} \mathbb{A}_{jk}(t) &= if_j(t)\delta_{jk}e^{if_j(t)-f_k(t)} \\ &+ i\langle\mu_j|U^\dagger(t, 0)H(t)U(t, 0)|\mu_k\rangle e^{if_j(t)-f_k(t)}, \end{aligned} \quad (26)$$

where we have used Eq. (11) and that $\dot{U}^\dagger(t, 0) = iU^\dagger(t, 0)H(t)$. By comparing Eqs. (25) and (26), we find

$$\begin{aligned} \mathbb{A}(t) &= i \text{diag}\{\dot{f}_1(t), \dots, \dot{f}_M(t)\} - \mathbb{K}(t) \\ &\equiv \mathbb{J}(t) - \mathbb{K}(t), \end{aligned} \quad (27)$$

which, combined with Eq. (15), implies the criterion

$$[\mathbb{J}(t), \mathbb{K}(t')] = [\mathbb{K}(t), \mathbb{K}(t')], \quad \forall t, t' \in [0, \tau], \quad (28)$$

for when the holonomic and dynamical parts separate. The typical case (iv) occurs when $[\mathbb{J}(t), \mathbb{K}(t')] \neq [\mathbb{K}(t), \mathbb{K}(t')]$

for at least some $t, t' \in [0, \tau]$. We recognize cases (i)–(iii) as the following three special cases where Eq. (28) holds: (i) $\mathbb{J}(t) = \mathbb{K}(t)$, $\forall t \in [0, \tau]$, which implies $\mathbb{A}(t) = 0_M$ by Eq. (27); (ii) $\mathbb{K}(t) = 0_M$, $\forall t \in [0, \tau]$; (iii) $\mathbb{A}(t) = -i \text{diag}\{A_1(t), \dots, A_M(t)\} \equiv \mathbb{A}_d(t)$ and $\mathbb{K}(t) = -i \text{diag}\{K_1(t), \dots, K_M(t)\} \equiv \mathbb{K}_d(t)$, which implies $\mathbb{J}(t) = \mathbb{A}_d(t) + \mathbb{K}_d(t)$. Further, note that

$$\mathbb{U}(\tau, 0) = e^{\int_0^\tau \mathbb{J}(t) dt} = \text{diag}\{e^{i\chi_1}, \dots, e^{i\chi_M}\} \quad (29)$$

in all cases (i)–(iv). We thus find $\mathbb{U}(\tau, 0) = e^{\int_0^\tau \mathbb{K}(t) dt}$ in case (i), $\mathbb{U}(\tau, 0) = e^{\int_0^\tau \mathbb{A}_d(t) dt}$ in case (ii), and $\mathbb{U}(\tau, 0) = e^{\int_0^\tau \mathbb{A}_d(t) dt} e^{\int_0^\tau \mathbb{K}_d(t) dt} = e^{\int_0^\tau \mathbb{A}_d(t) dt} e^{\int_0^\tau \mathbb{K}_d(t) dt}$ in case (iii). Note that no time ordering is needed in any of these expressions for $\mathbb{U}(\tau, 0)$.

III. PHYSICAL EXAMPLES

To illustrate cases (i)–(iv) identified in Sec. II C, we consider the “minimal” physical setting consisting of a three-level Λ configuration, in which two transitions $|p\rangle \leftrightarrow |r\rangle$ and $|q\rangle \leftrightarrow |r\rangle$, with (p, q, r) a given permutation of some fixed orthogonal states with labels $(1, 2, 3)$, are simultaneously driven by a pair of square-shaped laser pulses both with a real-valued envelope:

$$\Omega(t) = \begin{cases} \Omega_0, & 0 \leq t \leq \tau, \\ 0, & \text{otherwise.} \end{cases} \quad (30)$$

By using the rotating-wave approximation in the appropriate rotating frame, the Hamiltonian during the pulse becomes time independent and takes the form

$$H = \Omega_0(|r\rangle\langle b| + |b\rangle\langle r|) + 2\delta|r\rangle\langle r|, \quad (31)$$

where $|b\rangle = \omega_p^*|p\rangle + \omega_q^*|q\rangle$ is the Morris-Shore bright state [41], $\omega_p, \omega_q \in \mathbb{C}$ are time-independent laser parameters satisfying $|\omega_p|^2 + |\omega_q|^2 = 1$, and we have assumed the two pulses have the same detuning δ . The setup is shown in Fig. 1. Diagonalization of H yields

$$H = \mathcal{E}_1|v_1\rangle\langle v_1| + \mathcal{E}_2|v_2\rangle\langle v_2|, \quad (32)$$

with eigenvectors

$$\begin{aligned} |v_1\rangle &= \cos \frac{\gamma}{2} |r\rangle + \sin \frac{\gamma}{2} |b\rangle, \\ |v_2\rangle &= -\sin \frac{\gamma}{2} |r\rangle + \cos \frac{\gamma}{2} |b\rangle, \end{aligned} \quad (33)$$

and corresponding energy eigenvalues

$$\begin{aligned} \mathcal{E}_1 &= \delta + \sqrt{\delta^2 + \Omega_0^2} \equiv \delta + \dot{\phi}_t, \\ \mathcal{E}_2 &= \delta - \sqrt{\delta^2 + \Omega_0^2} \equiv \delta - \dot{\phi}_t. \end{aligned} \quad (34)$$

Here, we have introduced $\tan \gamma \equiv \Omega_0/\delta$ and the precession angle $\phi_t \equiv \sqrt{\delta^2 + \Omega_0^2} t$. We note in particular that zero detuning corresponds to $\gamma = \pm \frac{\pi}{2}$. The third eigenvector $|v_0\rangle = -\omega_q|p\rangle + \omega_p|q\rangle$ with $\mathcal{E}_0 = 0$ is the Morris-Shore dark state [41], which is decoupled from the dynamics [41].

We demonstrate how the special cases (i)–(iii) are obtained in the above described Λ system by choosing the initial M -frame $\mathcal{S}(0) = \mathcal{L}(0)$ in three different ways.

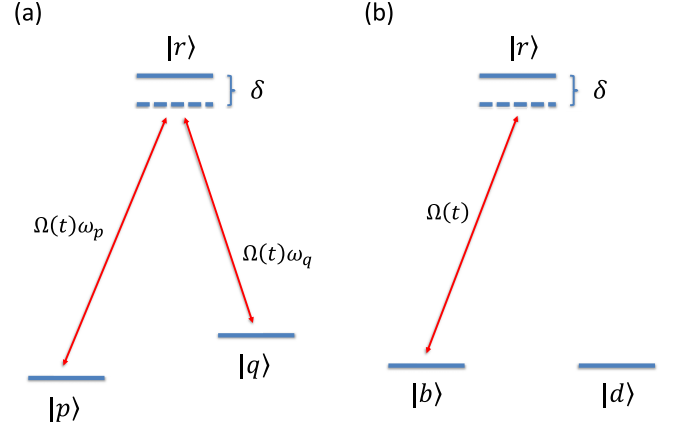


FIG. 1. (a) Three-level Λ configuration with two transitions $|p\rangle \leftrightarrow |r\rangle$ and $|q\rangle \leftrightarrow |r\rangle$, with (p, q, r) being a given permutation of some fixed orthogonal states with labels $(1, 2, 3)$. The two transitions are simultaneously driven by a pair of laser pulses both with real-valued envelope $\Omega(t)$, the same detuning δ , and time-independent relative amplitude ω_p/ω_q . (b) The system expressed in the Morris-Shore basis [41] with $|b\rangle = \omega_p^*|p\rangle + \omega_q^*|q\rangle$ and $|d\rangle = -\omega_q|p\rangle + \omega_p|q\rangle$.

To illustrate the typical case (iv), we consider a sequence of two Λ systems with different choices of (p, q, r) so as to correspond to two noncommuting Λ Hamiltonians.

A. Case (i)

Let $\mathcal{S}(0) = \mathcal{L}(0) = \{|v_1\rangle, |v_2\rangle\}$, which implies $\dot{P}_2(t) = \frac{d}{dt}(\hat{1} - |d\rangle\langle d|) = 0$, since $|d\rangle\langle d|$ does not evolve under influence of H in Eq. (31). This allows us to take the fixed vectors $|\varphi_1(t)\rangle = |\varphi_1(0)\rangle = |v_1\rangle$ and $|\varphi_2(t)\rangle = |\varphi_2(0)\rangle = |v_2\rangle$ to correspond to the local section with the overlap matrix $\mathbb{O}(0, \tau) = \mathbb{I}_2$. One therefore obtains

$$\mathbb{K}(t) = -i \begin{pmatrix} \mathcal{E}_1 & 0 \\ 0 & \mathcal{E}_2 \end{pmatrix} \quad (35)$$

and $\mathbb{A}(t) = 0_2$, yielding the purely dynamical time evolution matrix [38]

$$\mathbb{U}(\tau, 0) = e^{\int_0^\tau \mathbb{K}(t) dt} = e^{-i\delta\tau} e^{-i\phi_\tau \mathbb{Z}}, \quad (36)$$

where $\mathbb{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

B. Case (ii)

Now let $\mathcal{S}(0) = \mathcal{L}(0) = \{|d\rangle, |b\rangle\}$, which implies $\dot{P}_2(t) \neq 0$, potentially giving rise to a nontrivial holonomy. We choose $\mathcal{L}(t) = \{|\varphi_1(t)\rangle = |\varphi_1(0)\rangle = |d\rangle, |\varphi_2(t)\rangle = e^{-i \arg(b) e^{-iHt}} |b\rangle\}$ to correspond to the local section with the overlap matrix

$$\mathbb{O}(0, t) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \sin^2 \gamma \sin^2 \phi_t} \end{pmatrix} \geq 0. \quad (37)$$

Note that $\mathbb{O}(0, t) > 0$, $\forall t \in [0, \tau]$, if $\sin \gamma \neq \pm 1$, which corresponds to nonzero detuning.

From $|\varphi_1(t)\rangle = |d\rangle$ it follows that $\mathbb{K}_{11}(t) = \mathbb{K}_{12}(t) = \mathbb{K}_{21}(t) = 0$, and $|\varphi_2(t)\rangle$ implies $\mathbb{K}_{22}(t) =$

$-i\langle\varphi_2(t)|H|\varphi_2(t)\rangle = -i\langle b|e^{iHt}He^{-iHt}|b\rangle = -i\langle b|H|b\rangle = 0$.
We further have

$$\mathbb{A}(t) = i \begin{pmatrix} 0 & 0 \\ 0 & \frac{d}{dt} \arg\langle b|e^{-iHt}|b\rangle \end{pmatrix}, \quad (38)$$

yielding, together with $\mathbb{K}(t) = 0_2$, the solution of the AE:

$$\mathbb{W}(\tau) = e^{\int_0^\tau \mathbb{A}(t)dt} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i \arg\langle b|e^{-iH\tau}|b\rangle} \end{pmatrix}, \quad (39)$$

where we have used that time ordering is redundant, since $[\mathbb{A}(t), \mathbb{A}(t')] = 0_2$, $\forall t, t' \in [0, \tau]$. Thus, the time evolution matrix becomes

$$\mathbb{U}(\tau, 0) = \mathbb{O}(0, \tau) \mathbb{W}(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \sin^2 \gamma \sin^2 \phi_\tau} e^{i \arg\langle b|e^{-iH\tau}|b\rangle} \end{pmatrix}, \quad (40)$$

where $\arg\langle b|e^{-iH\tau}|b\rangle = -\delta\tau + \arg(\cos \phi_\tau + i \cos \gamma \sin \phi_\tau)$. Cyclic evolution ($\mathbb{O}(0, \tau) = \mathbb{I}_2$) corresponds to $\phi_\tau = \pi$ and $\delta\tau = \pi\delta/\sqrt{\delta^2 + \Omega_0^2} = \pi \cos \gamma$, which provides a realization of the nontrivial holonomy [18,19]:

$$\mathbb{U}(\tau, 0) = \mathbb{W}(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi(1+\cos \gamma)} \end{pmatrix}. \quad (41)$$

Note that $\mathbb{U}(\tau, 0)$ is diagonal in the dark-bright basis but, in general, off-diagonal in the fixed basis $\{|p\rangle, |q\rangle\}$. This displays the non-Abelian nature of the holonomy: two noncommuting time evolution matrices can be realized by applying two consecutive pulse-pairs with different choices of laser parameters ω_p and ω_q [13].

C. Case (iii)

Here, we make use of the energy eigenvectors by choosing $S(0) = \mathcal{L}(0) = \{|v_0\rangle = |d\rangle, \cos \frac{\xi}{2}|v_1\rangle + \sin \frac{\xi}{2}|v_2\rangle\}$, $\xi \in [0, \pi]$, which implies $\dot{P}_2(t) \neq 0$ so as to allow for a nontrivial holonomy. The choice $\mathcal{L}(t) = \{|\varphi_1(t)\rangle = |\varphi_1(0)\rangle = |d\rangle, |\varphi_2(t)\rangle = e^{-i \arg\langle \psi_2(0)|\psi_2(t)\rangle} |\psi_2(t)\rangle\}$, with $|\psi_2(t)\rangle = e^{-iHt}|\psi_2(0)\rangle = e^{-i\mathcal{E}t} \cos \frac{\xi}{2}|v_1\rangle + e^{-i\mathcal{E}t} \sin \frac{\xi}{2}|v_2\rangle$, corresponds to a local section, since

$$\mathbb{O}(0, t) = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \sin^2 \xi \sin^2 \phi_t} \end{pmatrix} \geq 0. \quad (42)$$

The inequality becomes strict for $\xi \neq \pm \frac{\pi}{2}$.

We can now calculate $\mathbb{A}(t)$ and $\mathbb{K}(t)$ by using $\mathcal{L}(t)$. One finds

$$\mathbb{A}(t) = i \begin{pmatrix} 0 & 0 \\ 0 & \frac{d}{dt} \arg\langle \psi_2(0)|\psi_2(t)\rangle + \delta + \dot{\phi}_t \cos \xi \end{pmatrix} \quad (43)$$

and

$$\mathbb{K}(t) = -i \begin{pmatrix} 0 & 0 \\ 0 & \delta + \dot{\phi}_t \cos \xi \end{pmatrix}. \quad (44)$$

Conceptually, this corresponds to the idea of unconventional geometric quantum computation [39] in the holonomic setting [24], in which $\mathbb{K}(t) = -g(t)\mathbb{A}(t)$, here with

$$g(t) = \frac{\delta + \dot{\phi}_t \cos \xi}{\frac{d}{dt} \arg\langle \psi_2(0)|\psi_2(t)\rangle + \delta + \dot{\phi}_t \cos \xi}. \quad (45)$$

Equations (43) and (44) show that $[\mathbb{A}(t), \mathbb{K}(t')] = 0_2$, $\forall t, t' \in [0, \tau]$. Thus, the solution of the AE separates:

$$\begin{aligned} \mathbb{W}(\tau) &= e^{\int_0^\tau \mathbb{A}(t)dt} e^{\int_0^\tau \mathbb{K}(t)dt} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & e^{i[\arg(\cos \phi_\tau - i \cos \xi \sin \phi_\tau) + \phi_\tau \cos \xi]} \end{pmatrix} \\ &\quad \times \begin{pmatrix} 1 & 0 \\ 0 & e^{-i(\delta\tau + \phi_\tau \cos \xi)} \end{pmatrix}, \end{aligned} \quad (46)$$

where time ordering is not needed, since $[\mathbb{A}(t), \mathbb{A}(t')] = [\mathbb{K}(t), \mathbb{K}(t')] = 0_2$, $\forall t, t' \in [0, \tau]$, and we have used that $\arg\langle \psi_2(0)|\psi_2(\tau)\rangle = -\delta\tau + \arg(\cos \phi_\tau - i \cos \xi \sin \phi_\tau)$.

Note that we recover the example given in Sec. III B when $\xi = \pi - \gamma$. This can be seen by noting that

$$\begin{aligned} |\psi_2(0)\rangle &= \cos\left(\frac{\pi - \gamma}{2}\right)|v_1\rangle + \sin\left(\frac{\pi - \gamma}{2}\right)|v_2\rangle \\ &= \sin\frac{\gamma}{2}|v_1\rangle + \cos\frac{\gamma}{2}|v_2\rangle = |b\rangle \end{aligned} \quad (47)$$

and

$$\delta + \dot{\phi}_t \cos(\pi - \gamma) = \delta - \dot{\phi}_t \cos \gamma = 0, \quad (48)$$

yielding $\mathbb{A}_{22}(t) = i \frac{d}{dt} \arg\langle b|e^{-iHt}|b\rangle$ and $\mathbb{K}_{22}(t) = 0$ in Eqs. (43) and (44), respectively. Furthermore, note that $\xi = 0$ ($\xi = \pi$) correspond to $S(0) = \mathcal{L}(0) = \{|d\rangle, |v_1\rangle\}$ ($S(0) = \mathcal{L}(0) = \{|d\rangle, |v_2\rangle\}$). Just as in the example given in Sec. III A, these two initial frames give rise to purely dynamical time evolution. In this sense, the setups discussed in Secs. III A and III B are special cases of the setting discussed here, obtained by selecting the value of ξ appropriately.

It is instructive to consider the cyclic case, in which $\mathbb{U}(\tau, 0)$ coincides with $\mathbb{W}(\tau)$. As is evident from Eq. (42), cyclic evolution corresponds to $\phi_\tau = \pi$ (the already mentioned cases $\xi = 0$ and $\xi = \pi$ also trivially correspond to cyclic evolution, since $\mathbb{O}(0, t) = \mathbb{I}_2$, $\forall t \in [0, \tau]$), which implies $\delta\tau = \pi \cos \gamma$. Upon insertion into Eq. (46), this results in the time evolution matrix

$$\mathbb{U}(\tau, 0) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi(1-\cos \xi)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi(\cos \gamma + \cos \xi)} \end{pmatrix}, \quad (49)$$

which again is consistent with the example given in Sec. III B, as it reduces to Eq. (41) for the special choice of $\xi = \pi - \gamma$. Conversely, for a given ξ , the product form displayed in Eq. (49) makes it possible to realize experimentally a pure holonomy,

$$\mathbb{U}(\tau, 0) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi(1-\cos \xi)} \end{pmatrix}, \quad (50)$$

by independently tuning γ until it coincides with $\pi - \xi$.

D. Case (iv)

To illustrate the typical case where the holonomic and dynamical parts do not separate, we consider two resonant ($\delta = 0$) Λ -type Hamiltonians, $H(t) = \Omega(t)H_0$ and $\tilde{H}(t) = \tilde{\Omega}(t)\tilde{H}_0$, such that $[H_0, \tilde{H}_0] \neq 0$. To achieve noncommutativity, we associate H_0 with a coupling structure consisting of the transitions $\{|p\rangle, |q\rangle\} \leftrightarrow |r\rangle$, and we associate $\tilde{H}_0$ similarly

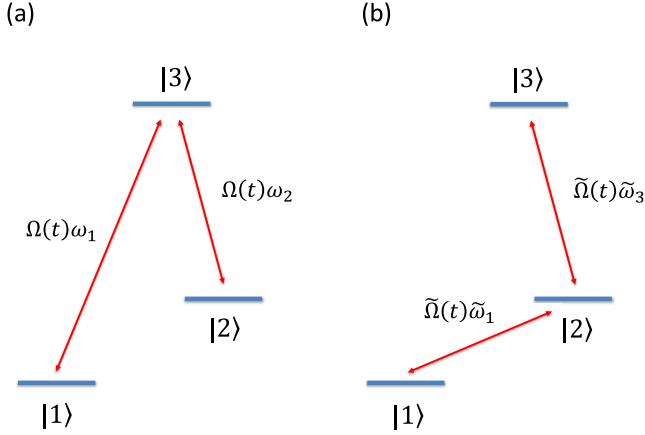


FIG. 2. Two different resonant ($\delta = 0$) Λ -type coupling structures applied sequentially: (a) a pair of laser pulses simultaneously driving the transitions $|1\rangle \leftrightarrow |3\rangle$ and $|2\rangle \leftrightarrow |3\rangle$ for $t \in [0, t_1]$, followed by (b) a pair of laser pulses simultaneously driving the transitions $|1\rangle \leftrightarrow |2\rangle$ and $|3\rangle \leftrightarrow |2\rangle$ for $t \in [t_1, \tau]$. The resulting holonomic and dynamical parts do not separate on $[0, \tau]$ due to the noncommuting nature of the Hamiltonians associated with the coupling structures.

with $\{|\tilde{p}\rangle, |\tilde{q}\rangle\} \leftrightarrow |\tilde{r}\rangle$, with (p, q, r) and $(\tilde{p}, \tilde{q}, \tilde{r})$ being different permutations of $(1, 2, 3)$. Note that, by using resonant pulses, any shape of the envelope functions $\Omega(t)$ and $\tilde{\Omega}(t)$ can be used. We focus on the cyclic case.

For notational simplicity, we make the specific choices $(p, q, r) = (1, 2, 3)$ and $(\tilde{p}, \tilde{q}, \tilde{r}) = (1, 3, 2)$, i.e.,

$$\begin{aligned} H_0 &= |3\rangle\langle b| + |b\rangle\langle 3|, \\ \tilde{H}_0 &= |2\rangle\langle \tilde{b}| + |\tilde{b}\rangle\langle 2|, \end{aligned} \quad (51)$$

with $|b\rangle = \omega_1^*|1\rangle + \omega_2^*|2\rangle$ and $|\tilde{b}\rangle = \tilde{\omega}_1^*|1\rangle + \tilde{\omega}_3^*|3\rangle$, applied sequentially on time intervals $[0, t_1]$ and $[t_1, \tau]$, respectively. The setup is illustrated in Fig. 2. By assuming π pulses ($\int_0^{t_1} \Omega(t)dt = \int_{t_1}^\tau \tilde{\Omega}(t)dt = \pi$), we obtain the combined unitary operator [42]

$$\begin{aligned} \tilde{V}(\tau, t_1)V(t_1, 0) &= e^{-i\pi\tilde{H}_0}e^{-i\pi H_0} \\ &= (-|2\rangle\langle 2| + |\tilde{d}\rangle\langle \tilde{d}| - |\tilde{b}\rangle\langle \tilde{b}|) \\ &\quad \times (-|3\rangle\langle 3| + |d\rangle\langle d| - |b\rangle\langle b|), \end{aligned} \quad (52)$$

with $|d\rangle = -\omega_2|1\rangle + \omega_1|2\rangle$ and $|\tilde{d}\rangle = -\tilde{\omega}_3|1\rangle + \tilde{\omega}_1|3\rangle$. Here, one can consider various combinations of laser parameters $\{\omega_1, \omega_2, \tilde{\omega}_1, \tilde{\omega}_3\}$ and find all sorts of different “subspace unitaries” by factoring out one of the eigenvectors of the combined unitary operator, say $|\mu_3\rangle$, so as to implement cyclic evolution $\mathcal{V}_2(\tau) = \mathcal{V}_2(0) = \text{Span}\{|\mu_1\rangle, |\mu_2\rangle\}$. The resulting time evolution operator restricted to the subspace $\mathcal{V}_2(0)$ is the partial isometry

$$\begin{aligned} U(\tau, 0) &= \tilde{V}(\tau, t_1)V(t_1, 0)(\hat{1} - |\mu_3\rangle\langle \mu_3|) \\ &= e^{i\chi_1}|\mu_1\rangle\langle \mu_1| + e^{i\chi_2}|\mu_2\rangle\langle \mu_2|. \end{aligned} \quad (53)$$

To analyze the issue of separation by purely analytical means in this generic case is, however, surprisingly difficult. Therefore, we focus on one very specific parameter choice that yields a nontrivial time evolution (i.e., $[H_0, \tilde{H}_0] \neq 0$), for

which $\mathbb{A}(t)$ and $\mathbb{K}(t)$ can be calculated analytically and shown to be noncommuting.

To this end, we take $\omega_1 = -\omega_2 = \tilde{\omega}_1 = -\tilde{\omega}_3 = \frac{1}{\sqrt{2}}$, which yields

$$\tilde{V}(\tau, t_1)V(t_1, 0) = -|1\rangle\langle 3| + |3\rangle\langle 2| - |2\rangle\langle 1|, \quad (54)$$

with eigenvalues $e^{i\chi_j} = (e^{i2\pi/3})^{j-1} \equiv \eta^{j-1}$ and corresponding eigenvectors

$$|\mu_j\rangle = \frac{1}{\sqrt{3}}(-\eta^{j-1}|1\rangle + |2\rangle + \eta^{2(j-1)}|3\rangle), \quad (55)$$

$j = 1, 2$, and 3 . We choose $\mathcal{S}(0) = \mathcal{L}(0) = \{|\mu_1\rangle, |\mu_2\rangle\}$, i.e.,

$$U(\tau, 0) = |\mu_1\rangle\langle \mu_1| + e^{i2\pi/3}|\mu_2\rangle\langle \mu_2|, \quad (56)$$

with $|\mu_1\rangle = \frac{1}{\sqrt{3}}(-|1\rangle + |2\rangle + |3\rangle)$ and $|\mu_2\rangle = \frac{1}{\sqrt{3}}(-\eta|1\rangle + |2\rangle + \eta^2|3\rangle)$.

One can note that $\mathcal{V}_2(0) = \text{Span}\{|\mu_1\rangle, |\mu_2\rangle\}$ does not coincide with the subspaces $\text{Span}\{|1\rangle, |2\rangle\}$ and $\text{Span}\{|1\rangle, |3\rangle\}$. The evolution of these two latter subspaces, governed by $V(t, 0)$ and $\tilde{V}(t, t_1)$, respectively, are associated with a trivial dynamical part, and thereby pure holonomy [13]. Thus, the evolution $\mathcal{V}_2(0) \rightarrow \mathcal{V}_2(t)$ acquires nontrivial holonomic and dynamical contributions. These contributions are not expected to separate after completing the loop C in the Grassmannian $\mathcal{G}(3; 2)$ at $t = \tau$.

To demonstrate this latter point explicitly, we calculate the corresponding $\mathbb{A}(t)$ and $\mathbb{K}(t')$ and show that they indeed do not commute for general $t, t' \in [0, \tau]$. First, $V(t_1, 0)$ is generated by the Hamiltonian

$$\begin{aligned} H(t) &= \Omega(t)\frac{1}{\sqrt{2}}(|3\rangle\langle 1| - |3\rangle\langle 2| + \text{H.c.}) \\ &\equiv \Omega(t)H_0, \end{aligned} \quad (57)$$

and $\tilde{V}(\tau, t_1)$ is generated by the Hamiltonian

$$\begin{aligned} \tilde{H}(t) &= \tilde{\Omega}(t)\frac{1}{\sqrt{2}}(|2\rangle\langle 1| - |2\rangle\langle 3| + \text{H.c.}) \\ &\equiv \tilde{\Omega}(t)\tilde{H}_0, \end{aligned} \quad (58)$$

with the real-valued pulse envelopes $\Omega(t)$ and $\tilde{\Omega}(t)$ being nonzero on $[0, t_1]$ and $[t_1, \tau]$, respectively, such that $\int_0^{t_1} \Omega(t)dt = \int_{t_1}^\tau \tilde{\Omega}(t)dt = \pi$. We verify that $[H_0, \tilde{H}_0] = \frac{1}{2}(-|1\rangle\langle 2| + |1\rangle\langle 3| - |2\rangle\langle 3| - \text{H.c.}) \neq 0$. We have

$$\begin{aligned} V(t, 0) &= e^{-ia_t H_0} \\ &= \frac{1}{2}(|1\rangle + |2\rangle)(\langle 1| + \langle 2|) \\ &\quad + e^{-ia_t}|v_+\rangle\langle v_+| + e^{ia_t}|v_-\rangle\langle v_-|, \end{aligned} \quad (59)$$

with $|v_\pm\rangle = \frac{1}{2}(|1\rangle - |2\rangle) \pm \frac{1}{\sqrt{2}}|3\rangle$ and $a_t = \int_0^t \Omega(s)ds$; similarly

$$\begin{aligned} \tilde{V}(t, t_1) &= e^{-i\tilde{a}_t \tilde{H}_0} \\ &= \frac{1}{2}(|1\rangle + |3\rangle)(\langle 1| + \langle 3|) \\ &\quad + e^{-i\tilde{a}_t}|\tilde{v}_+\rangle\langle \tilde{v}_+| + e^{i\tilde{a}_t}|\tilde{v}_-\rangle\langle \tilde{v}_-|, \end{aligned} \quad (60)$$

with $|\tilde{v}_\pm\rangle = \frac{1}{2}(|1\rangle - |3\rangle) \pm \frac{1}{\sqrt{2}}|2\rangle$ and $\tilde{a}_t = \int_{t_1}^t \tilde{\Omega}(s)ds$.

The local section $\mathcal{L}(t)$ is chosen as

$$|\varphi_1(t)\rangle = U(t, 0)|\mu_1\rangle, \quad |\varphi_2(t)\rangle = e^{-if(t)}U(t, 0)|\mu_2\rangle \quad (61)$$

with $f(t)$ being any real-valued function such that $f(\tau) = \frac{2\pi}{3}$ and $f(0) = 0$. Here, $U(t, 0) = V_{\text{tot}}(t, 0)(\hat{1} - |\mu_3\rangle\langle\mu_3|)$, where

$$V_{\text{tot}}(t, 0) = \begin{cases} V(t, 0), & t \in [0, t_1], \\ \tilde{V}(t, t_1)V(t_1, 0), & t \in [t_1, \tau], \end{cases} \quad (62)$$

is the time evolution operator on the full Hilbert space that coincides with the combined unitary $\tilde{V}(\tau, t_1)V(t_1, 0)$ in Eq. (52) at $t = \tau$. Indeed, one may verify that $|\varphi_j(\tau)\rangle = |\varphi_j(0)\rangle = |\mu_j\rangle$, $j = 1$ and 2 , ensuring that $\mathbb{O}(0, \tau) = \mathbb{I}_2$ (cyclic evolution).

Let us first consider $\mathbb{K}(t)$. We have

$$\mathbb{K}_{jk}(t) = \begin{cases} -i\langle\varphi_j(t)|H(t)|\varphi_k(t)\rangle = -i\Omega(t)\langle\mu_j(t)|H_0|\mu_k(t)\rangle, & t \in [0, t_1], \\ -i\langle\varphi_j(t)|\tilde{H}(t)|\varphi_k(t)\rangle = -i\tilde{\Omega}(t)\langle\mu_j(t)|V^\dagger(t_1, 0)\tilde{H}_0V(t_1, 0)|\mu_k(t)\rangle, & t \in [t_1, \tau], \end{cases} \quad (63)$$

where we have defined $|\mu_1(t)\rangle \equiv |\mu_1\rangle$ and $|\mu_2(t)\rangle \equiv e^{-if(t)} \times |\mu_2\rangle$ and used that $[H_0, V(t, 0)] = [\tilde{H}_0, \tilde{V}(t, t_1)] = 0$. By evaluating the matrix elements, we find

$$\mathbb{K}(t) = \begin{cases} i\frac{\Omega(t)}{3\sqrt{2}} \begin{pmatrix} 4 & \eta^2 e^{-if(t)} \\ \eta e^{if(t)} & -2 \end{pmatrix}, & t \in [0, t_1], \\ i\frac{\tilde{\Omega}(t)}{3\sqrt{2}} \begin{pmatrix} 4 & \eta^2 e^{-if(t)} \\ \eta e^{if(t)} & -2 \end{pmatrix}, & t \in [t_1, \tau], \end{cases} \quad (64)$$

where we have used that $a_{t_1} = \pi$ and the identity $1 + \eta + \eta^2 = 0$.

Next, we find $\mathbb{A}(t)$ by using $f_1(t) = 0$ and $f_2(t) = f(t)$ in Eq. (27), yielding $\mathbb{J}(t) = i \text{diag}\{0, f(t)\}$, i.e.,

$$\mathbb{A}(t) = i\dot{f}(t)\mathbb{P}_2 - \mathbb{K}(t), \quad (65)$$

where $\mathbb{P}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. This implies, for arbitrary $t, t' \in [0, \tau]$, that

$$[\mathbb{A}(t), \mathbb{K}(t')] = i\dot{f}(t)[\mathbb{P}_2, \mathbb{K}(t')] - [\mathbb{K}(t), \mathbb{K}(t')], \quad (66)$$

with the terms on the right-hand side generally nonvanishing and noncanceling. To illustrate this latter point, it is sufficient to consider $t' = t \in [0, t_1]$ for some t at which $\Omega(t) \neq 0$. For this case, we obtain

$$\begin{aligned} [\mathbb{A}(t), \mathbb{K}(t)] &= i\dot{f}(t)[\mathbb{P}_2, \mathbb{K}(t)] \\ &= -\frac{\dot{f}(t)\Omega(t)}{3\sqrt{2}} \begin{pmatrix} 0 & -\eta^2 e^{-if(t)} \\ \eta e^{if(t)} & 0 \end{pmatrix}, \end{aligned} \quad (67)$$

which, by choosing the function $f(t)$ so that $\dot{f}(t) \neq 0$, does not vanish. We thus conclude that the unitaries $\mathcal{T}e^{\int_0^\tau \mathbb{A}(t)dt}$ and $\mathcal{T}e^{\int_0^\tau \mathbb{K}(t)dt}$ do not separate in the time evolution

$$\begin{aligned} \mathbb{U}(\tau, 0) &= \mathcal{T}e^{\int_0^\tau [\mathbb{A}(t) + \mathbb{K}(t)]dt} \\ &= e^{i\int_0^\tau \dot{f}(t)dt} \mathbb{P}_2 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\pi/3} \end{pmatrix}, \end{aligned} \quad (68)$$

where we have used Eq. (65) and $\int_0^\tau \dot{f}(t)dt = f(\tau) - f(0) = \frac{2\pi}{3}$.

To see the physical significance of the above, recall that the gauge-invariant eigenvalues of $\mathcal{T}e^{\int_0^\tau \mathbb{A}(t)dt}$, in the special

case where the dynamical part can independently be made trivial, are found by performing quantum process tomography to reconstruct the time evolution matrix $\mathbb{U}(\tau, 0)$ experimentally. As a consequence of the gauge-invariant nonseparability displayed in Eq. (68), on the other hand, the holonomic part cannot be extracted from $\mathbb{U}(\tau, 0)$ by independently tuning the dynamical part. Thus, the nonadiabatic quantum holonomy in this system setup cannot be measured.

IV. CONCLUSIONS

Separation of nonadiabatic Schrödinger-type quantum time evolution of subspaces into holonomic and dynamical parts has been studied. We have established that such separation only occurs in highly restrictive situations. The special cases in which separation does manifest have been identified and found to be essentially covered by existing schemes in the literature. We have proved that separability is a gauge-invariant property of quantum time evolution. All qualitatively different special cases have been exemplified within a single Λ system setup. The typical case, in which the holonomic and dynamical parts do not separate, has been illustrated by sequentially applying two noncommuting Λ -type Hamiltonians.

While our work completes the analysis of the Schrödinger evolution of subspaces, the issue of separation appears in other contexts involving quantum holonomies. These include, e.g., the Uhlmann holonomy [43,44] of mixed quantum states and the relation between direct and iterative holonomy associated with discrete sequences of subspaces of dimension $M \geq 2$ [45,46].

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DATA AVAILABILITY

There are no publicly available research data or software supporting this manuscript. Requests for further information or data should be sent to the authors.

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