

Fault-tolerant encoding of logical qudits in spin systems

Sumin Lim ^{*}

*Graduate School of Quantum Science and Technology, KAIST, Daejeon 34141, Republic of Korea
and CAESR, The Clarendon Laboratory, Department of Physics, University of Oxford,
Parks Road, Oxford OX1 3PU, United Kingdom*



(Received 11 November 2025; revised 14 February 2026; accepted 26 March 2026; published 16 April 2026)

Universal quantum computers require fault-tolerant logical qudits, as qudits naturally align with the simulation of multilevel physical systems. Here we present a general framework and working examples for encoding fault-tolerant logical qudits in finite-dimensional spin systems. We construct distance-3 codewords, distance-5 codewords, and general $(2t + 1)$ -distance codes that can be implemented using a single physical qudit or a small number of coupled qudits for higher distances while requiring a Hilbert-space dimension significantly smaller than conventional constructions based on multiple logical qubits. Logical operations and error-correction protocols can be implemented with polynomial scaling in the number of elementary operations. We further discuss schematic designs for physical implementation and required single-gate fidelities, which are compatible with current spin qudit platforms. This strategy provides a resource-efficient path toward realizing fault-tolerant logical qudits in finite multilevel physical systems.

DOI: [10.1103/n28j-r8bg](https://doi.org/10.1103/n28j-r8bg)

I. INTRODUCTION

Quantum technologies, including quantum information processing [1–5], quantum simulation [6–8], quantum sensing [9–11], and quantum communication [12], have attracted significant interest across diverse research communities due to their potential to outperform classical approaches. Among these, quantum simulation provides a powerful framework for studying complex physical systems by directly emulating their intrinsic quantum properties, as originally proposed by Feynman [6]. While qubit-based architectures can represent multilevel quantum states, they typically require additional physical qubits and increased circuit depth. A more natural and resource-efficient approach is to directly exploit qudits, i.e., quantum systems with intrinsic multilevel structure. Recent theoretical and experimental studies [13–15] have explored various kinds of qudit-based quantum simulations, including applications to molecular dynamics [16], neutrino oscillations [17], and disorder-induced superfluidity [18], highlighting the potential of quantum multilevel simulators. Beyond quantum simulation, qudit-based quantum information processing [19–22] and quantum communication [23–25] inherently offer enhanced information capacity and improved computational or communication efficiency, thereby motivating the development of dedicated qudit-processing schemes.

It is widely recognized that we are currently in the noisy intermediate-scale quantum era [26]. In this context, it is

essential to develop strategies for encoding logical qudits using physical qudits [27–33], analogous to the encoding of logical qubits in physical qubits [34–37]. Although the Gottesman-Kitaev-Preskill (GKP) code [38] intrinsically implies the encoding of logical qudits, it relies on bosonic systems [28,39], i.e., an infinite-dimensional Hilbert space, along with additional state-normalization procedures and corresponding experimental overhead. An alternative approach is to exploit intrinsically finite-dimensional multilevel systems, such as spin qudits.

Spin systems, which were proposed and demonstrated as candidate embodiments in the early stages of the quantum computing era [2,4,5], can also provide a robust multilevel platform. Electron spins offer fast gate operations and interqubit interactions across various platforms, while nuclear spins are known to be one of the most stable quantum systems and are promising candidates for fault-tolerant quantum memory. In recent years, extensive investigations have been performed to exploit spin qudit systems as quantum memory, in both experimental [40–42] and theoretical [19,43–50] studies. This approach is not limited to spins in solid-state systems [51–53], but can also be extended to encodings in angular momentum space in trapped and laser-cooled atoms or molecules [54,55], which can have higher total angular momentum [56,57] than typical nuclear spin systems, as well as to photonic platforms exploiting orbital angular momentum degrees of freedom [58,59].

In this study, inspired by various qudit-based codes [19,27,43,44] including cat codes [60,61] and binomial codes [29,62,63], we propose a method for encoding a logical qudit using either a single larger spin qudit or multiple spin qudits, with only polynomial scaling in Hilbert-space dimension and gate complexity. We first present working examples of logical qutrit and ququart encodings against first-order Z errors (phase errors) and their generalization to arbitrary

^{*}Contact author: suminlim@kaist.ac.kr

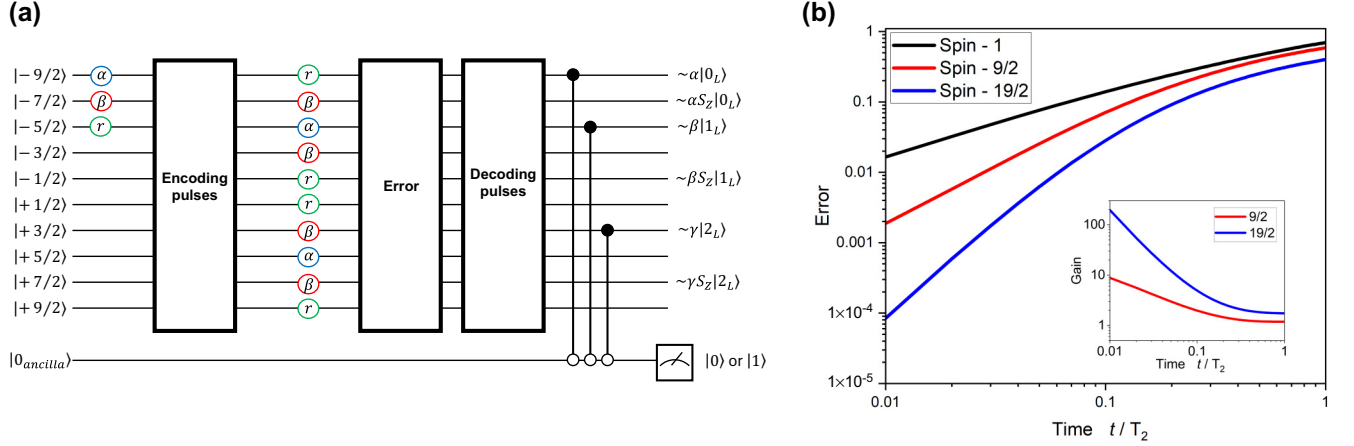


FIG. 1. (a) Brief schematic of the qutrit Z-error-correction code (see the text for details). (b) Plot of the qutrit fidelity as a function of time with the distance-1 (black line), distance-3 (red line), and distance-5 (blue line) quantum error correction codes. The inset shows the gain from fault-tolerant encoding for distance-3 (red) and distance-5 (blue) encoding.

d -dimensional qudits. We next extend the codewords to enable correction of Pauli X -, Y -, and Z -error correction and describe how the design can be generalized to d -dimensional qudits. Additionally, this strategy can be expanded to second- and higher-order error corrections, and we provide brief steps for its extension. Depending on the system, multiple coupled spin qudits can be entangled to form a fault-tolerant logical qudit, with efficient encoding-decoding schemes, which we illustrate with working examples. Finally, we discuss a schematic design for the physical implementation, fault-tolerant logical qudit operations, and required single gate fidelities. The results clearly demonstrate that direct encoding of logical qudits provides an advantage in quantum resource efficiency.

II. Z-ERROR-CORRECTION CODE FOR QUTRITS, QUQUARTS, AND GENERALIZATION

In this section we outline a general formalism for constructing logical qudits with arbitrary dimension d and codeword distance $2t + 1$. The logical codewords are constructed as linear combinations of symmetric catlike superpositions within the Z basis, with coefficients carefully chosen to satisfy the qudit Knill-Laflamme (KL) conditions up to order $2t$. For distance-3 Z -error correction, satisfying the first-order KL constraint requires at least two independent cat components per logical state, leading to a minimal Hilbert-space dimension scaling of approximately $4d$, corresponding to a total spin quantum number $S = 2d - \frac{3}{2}$. This constraint expands to Hilbert-space scaling of approximately $12d$ when correcting all Pauli X, Y , and Z errors. Increasing the code distance from $2t + 1$ to $2(t + 1) + 1$ requires additional independent basis components to correct higher-order errors, doubling the Hilbert-space dimension at each step. The detailed construction with working examples follows.

We first present Z -error-correcting codewords for qutrits and ququarts as working examples and provide a method for generalizing them to higher-dimensional qudits. In solid-state spin systems, when an external magnetic field is applied along the Z axis to quantize individual m_S and m_I levels, perturbations along the Z axis often act as a dominant source of

decoherence, i.e., the T_2 relaxation time limits the overall coherence. Therefore, error-correction schemes that focus on Z error can effectively enhance the performance of the system. In this case, logical qutrit codewords that counteract dephasing can be designed using a spin- $\frac{9}{2}$ system

$$\begin{aligned} |0_L\rangle &= \sqrt{\frac{10}{20}}(|-\frac{5}{2}\rangle + |+\frac{5}{2}\rangle), \\ |1_L\rangle &= \sqrt{\frac{6}{20}}(|-\frac{3}{2}\rangle + |+\frac{3}{2}\rangle) + \sqrt{\frac{4}{20}}(|-\frac{7}{2}\rangle + |+\frac{7}{2}\rangle), \\ |2_L\rangle &= \sqrt{\frac{7}{20}}(|-\frac{1}{2}\rangle + |+\frac{1}{2}\rangle) + \sqrt{\frac{3}{20}}(|-\frac{9}{2}\rangle + |+\frac{9}{2}\rangle), \end{aligned} \quad (1)$$

where $|m_I\rangle$ corresponds to Z -basis eigenstates. The three codewords presented above satisfy the qudit version of KL criteria [36], as shown by

$$\begin{aligned} \langle i_L | E_a^\dagger E_b | j_L \rangle &= 0 \quad \text{for } i \neq j, \\ \langle i_L | E_a^\dagger E_b | i_L \rangle - \langle j_L | E_a^\dagger E_b | j_L \rangle &= 0, \end{aligned} \quad (2)$$

where $|i_L\rangle$ and $|j_L\rangle$ correspond to codewords $|0_L\rangle, |1_L\rangle, |2_L\rangle, \dots$ and E_a and E_b represent the target error operators. For this Z -error example, the relevant operators are I and S_Z . In Eq. (1), all original codewords and error codewords are mutually orthogonal and form a pure error-correction code. Therefore, both encoding and decoding can be implemented using simple pulse sequences. A brief schematic diagram of the overall process is shown in Fig. 1(a), and the details of the encoding and decoding pulses are provided in Appendix A. Without loss of generality, the original qutrit state can be prepared as $|\psi\rangle = \alpha|-\frac{9}{2}\rangle + \beta|-\frac{7}{2}\rangle + \gamma|-\frac{5}{2}\rangle$. This state can be encoded into the logical qutrit state $|\psi_{\text{enc}}\rangle = \alpha|0_L\rangle + \beta|1_L\rangle + \gamma|2_L\rangle$, using the encoding pulses. During the storage time, Z error (i.e., phase error) may occur on the system, and the resulting state can be written as

$$\begin{aligned} E|\psi_{\text{enc}}\rangle &= \sqrt{1 - \epsilon}(\alpha|0_L\rangle + \beta|1_L\rangle + \gamma|2_L\rangle) \\ &\quad + (i\epsilon_Z)(\alpha S_Z|0_L\rangle + \beta S_Z|1_L\rangle + \gamma S_Z|2_L\rangle) \\ &\quad + \left(\frac{(i\epsilon_Z)^2}{2!}\right)(\alpha S_Z^2|0_L\rangle + \beta S_Z^2|1_L\rangle + \gamma S_Z^2|2_L\rangle) \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{(i\epsilon_Z)^3}{3!} \right) (\alpha S_Z^3 |0_L\rangle + \beta S_Z^3 |1_L\rangle + \gamma S_Z^3 |2_L\rangle) \\
 & + \dots, \quad (3)
 \end{aligned}$$

where ϵ_Z corresponds to the error probability, i.e., $\frac{t}{T_{\text{relax}}}$. As in general quantum error-correction protocols, we can assume that only the first and second terms in Eq. (3) dominate the state for short times $t \ll T_{\text{relax}}$. Since all six terms $|0_L\rangle$, $|1_L\rangle$, $|2_L\rangle$, $S_Z|0_L\rangle$, $S_Z|1_L\rangle$, and $S_Z|2_L\rangle$ are mutually orthogonal, the decoding pulses and conditional excitation of the ancilla qubit evolve the state to $|\psi\rangle = A_0(\alpha|-\frac{9}{2}\rangle + \beta|-\frac{5}{2}\rangle + \gamma|+\frac{3}{2}\rangle)|1_{\text{ancilla}}\rangle + A_1(\alpha|-\frac{7}{2}\rangle + \beta|-\frac{1}{2}\rangle + \gamma|+\frac{7}{2}\rangle)|0_{\text{ancilla}}\rangle$. The projective measurement on the ancilla qubit then restores the original state, regardless of whether the outcome is 0 or 1.

This design can also be applied to ququarts or high-dimensional logical qudits. The encoding for a ququart can be done using

$$\begin{aligned}
 |0_L\rangle &= \sqrt{\frac{14}{28}}(|-\frac{7}{2}\rangle + |+\frac{7}{2}\rangle), \\
 |1_L\rangle &= \sqrt{\frac{8}{28}}(|-\frac{5}{2}\rangle + |+\frac{5}{2}\rangle) + \sqrt{\frac{6}{28}}(|-\frac{9}{2}\rangle + |+\frac{9}{2}\rangle), \\
 |2_L\rangle &= \sqrt{\frac{9}{28}}(|-\frac{3}{2}\rangle + |+\frac{3}{2}\rangle) + \sqrt{\frac{5}{28}}(|-\frac{11}{2}\rangle + |+\frac{11}{2}\rangle), \\
 |3_L\rangle &= \sqrt{\frac{10}{28}}(|-\frac{1}{2}\rangle + |+\frac{1}{2}\rangle) + \sqrt{\frac{4}{28}}(|-\frac{13}{2}\rangle + |+\frac{13}{2}\rangle). \quad (4)
 \end{aligned}$$

Using the same strategy, a d -dimensional logical qudit can be encoded with the codewords $|0_L\rangle$, $|1_L\rangle$, $|2_L\rangle$, $\dots$, $|(d-1)_L\rangle$ and coefficients as

$$\begin{aligned}
 |0_L\rangle &= \sqrt{\frac{1}{2}}[|-(S/2 + \frac{1}{4})\rangle + |+(S/2 + \frac{1}{4})\rangle], \\
 |i_L\rangle &= a_i[|-(S/2 + \frac{1}{4} - i)\rangle + |+(S/2 + \frac{1}{4} - i)\rangle] \\
 &+ b_i[|-(S/2 + \frac{1}{4} + i)\rangle + |+(S/2 + \frac{1}{4} + i)\rangle], \quad (5)
 \end{aligned}$$

where $S = 2d - \frac{3}{2}$ and

$$a_i = \sqrt{\frac{2d-1+i}{8d-4}}, \quad b_i = \sqrt{\frac{2d-1-i}{8d-4}}. \quad (6)$$

These coefficients a_i and b_i are defined to satisfy the KL criteria (see Appendix B).

Another way to generalize the encoding scheme, beyond increasing the qudit dimension d , is to increase the distance of the error-correcting code. A higher-distance code allows the correction of more general or higher-order errors, for instance, up to second-order errors using distance-5 codes. We provide a logical qutrit encoding that has distance 5, which can be realized using a spin $S = \frac{19}{2}$ system,

$$\begin{aligned}
 |0_L\rangle &= \sqrt{\frac{5}{16}}(|-\frac{5}{2}\rangle + |+\frac{5}{2}\rangle) + \sqrt{\frac{3}{16}}(|-\frac{15}{2}\rangle + |+\frac{15}{2}\rangle), \\
 |1_L\rangle &= \sqrt{\frac{5423}{42400}}(|-\frac{1}{2}\rangle + |+\frac{1}{2}\rangle) + \sqrt{\frac{7203}{42400}}(|-\frac{9}{2}\rangle + |+\frac{9}{2}\rangle) \\
 &\times \sqrt{\frac{6517}{42400}}(|-\frac{11}{2}\rangle + |+\frac{11}{2}\rangle) \\
 &+ \sqrt{\frac{2057}{42400}}(|-\frac{19}{2}\rangle + |+\frac{19}{2}\rangle),
 \end{aligned}$$

$$\begin{aligned}
 |2_L\rangle &= \sqrt{\frac{3294}{22800}}(|-\frac{3}{2}\rangle + |+\frac{3}{2}\rangle) + \sqrt{\frac{3749}{22800}}(|-\frac{7}{2}\rangle + |+\frac{7}{2}\rangle) \\
 &\times \sqrt{\frac{2771}{22800}}(|-\frac{13}{2}\rangle + |+\frac{13}{2}\rangle) \\
 &+ \sqrt{\frac{1586}{22800}}(|-\frac{17}{2}\rangle + |+\frac{17}{2}\rangle). \quad (7)
 \end{aligned}$$

The general form of the distance-5 code and its coefficients for arbitrary d are provided in Appendix B, along with guidelines for designing codes with larger codeword distance $2t+1$.

In Fig. 1(b) we present the ideal performance of the logical qutrit encoding using spins $\frac{9}{2}$ [Eq. (1)] and $\frac{19}{2}$ [Eq. (7)] as a function of the ratio between the quantum error correction scheme cycle time and T_2 relaxation time. The error $E = 1 - \mathcal{F}^2$ is obtained from natural dephasing using the Lindblad master equation. The inset of Fig. 1(b) shows the gain from the encoding, defined as $E_{\text{uncorrected}}/E_{\text{corrected}}$, providing evidence that the residual error decreases through larger qudit Hilbert spaces.

III. X-, Y-, AND Z-ERROR-CORRECTION CODE FOR QUTRITS AND GENERALIZATION

Next we briefly discuss logical qudit encoding that can compensate for all Pauli X, Y , and Z errors, enabling fully fault-tolerant circuits. The encoding can be designed based on a similar approach, but with additional spacing between codewords to maintain orthogonality even in the presence of X and Y errors. The simplest case of a fault-tolerant qudit, i.e., a qutrit, can be encoded using the following codewords in a spin $S = \frac{29}{2}$ system:

$$\begin{aligned}
 |0_L\rangle &= \sqrt{\frac{10}{20}}(|-\frac{15}{2}\rangle + |+\frac{15}{2}\rangle), \\
 |1_L\rangle &= \sqrt{\frac{6}{20}}(|-\frac{9}{2}\rangle + |+\frac{9}{2}\rangle) + \sqrt{\frac{4}{20}}(|-\frac{21}{2}\rangle + |+\frac{21}{2}\rangle), \\
 |2_L\rangle &= \sqrt{\frac{7}{20}}(|-\frac{3}{2}\rangle + |+\frac{3}{2}\rangle) + \sqrt{\frac{3}{20}}(|-\frac{27}{2}\rangle + |+\frac{27}{2}\rangle). \quad (8)
 \end{aligned}$$

This encoding shares the same set of coefficients as the Z -error-correction encoding given in Eq. (1), but with a threefold larger Hilbert-space dimension to provide additional spacing between codewords. This encoding strategy can also be generalized to d -dimensional qudits, as shown in

$$\begin{aligned}
 |0_L\rangle &= \sqrt{\frac{1}{2}}[| - 3(S/2 + \frac{1}{4}) \rangle + | + 3(S/2 + \frac{1}{4}) \rangle], \\
 |i_L\rangle &= a_i\{| - [3(S/2 + \frac{1}{4}) - 3i] \rangle + | + [3(S/2 + \frac{1}{4}) - 3i] \rangle\} \\
 &+ b_i\{| - [3(S/2 + \frac{1}{4}) + 3i] \rangle + | + [3(S/2 + \frac{1}{4}) + 3i] \rangle\}, \quad (9)
 \end{aligned}$$

with $S = 6d - \frac{7}{2}$ and the same set of a_i and b_i coefficients as in Eq. (6).

While the designs above account for first-order errors, they can also be extended to second- and higher-order errors. This can be achieved by including additional Hilbert-space dimensions and modifying the coefficients in the logical qudit definition. As a working example, logical qutrit encoding that can correct spin-operator errors up to second order can be

implemented using a spin $S = \frac{99}{2}$, as shown by

$$\begin{aligned}
 |0_L\rangle &= \sqrt{\frac{5}{16}}(|-\frac{25}{2}\rangle + |+\frac{25}{2}\rangle) + \sqrt{\frac{3}{16}}(|-\frac{75}{2}\rangle + |+\frac{75}{2}\rangle), \\
 |1_L\rangle &= \sqrt{\frac{5423}{42400}}(|-\frac{5}{2}\rangle + |+\frac{5}{2}\rangle) + \sqrt{\frac{7203}{42400}}(|-\frac{45}{2}\rangle + |+\frac{45}{2}\rangle) \\
 &\quad \times \sqrt{\frac{6517}{42400}}(|-\frac{55}{2}\rangle + |+\frac{55}{2}\rangle) \\
 &\quad + \sqrt{\frac{2057}{42400}}(|-\frac{95}{2}\rangle + |+\frac{95}{2}\rangle), \\
 |2_L\rangle &= \sqrt{\frac{3294}{22800}}(|-\frac{15}{2}\rangle + |+\frac{15}{2}\rangle) \\
 &\quad + \sqrt{\frac{3749}{22800}}(|-\frac{35}{2}\rangle + |+\frac{35}{2}\rangle) \\
 &\quad \times \sqrt{\frac{2771}{22800}}(|-\frac{65}{2}\rangle + |+\frac{65}{2}\rangle) \\
 &\quad + \sqrt{\frac{1586}{22800}}(|-\frac{85}{2}\rangle + |+\frac{85}{2}\rangle). \tag{10}
 \end{aligned}$$

Similar to first-order error-correction codewords, this design also shares the same coefficients as second-order Z-error codewords [i.e., Eq. (7)], but with a fivefold increase in Hilbert-space dimension and spacing between basis states. In general, we find that once the $(2t+1)$ -distance Z-error-correction codewords are specified, codewords for all X-, Y-, and Z-error corrections with the same distance can

be directly obtained by increasing all basis indices by a factor of $2t+1$ in the Z basis (see Appendix B).

Depending on the quantum platform, entangled spin qudits can also be employed to construct highly coherent logical qudits. Here we provide several examples, such as a three-spin- $\frac{9}{2}$ -qudit code that can encode a fault-tolerant qutrit,

$$\begin{aligned}
 |0_L\rangle &= \sqrt{\frac{10}{20}}(|-\frac{5}{2}\rangle_{A,B,C} + |+\frac{5}{2}\rangle_{A,B,C}), \\
 |1_L\rangle &= \sqrt{\frac{6}{20}}(|-\frac{3}{2}\rangle_{A,B,C} + |+\frac{3}{2}\rangle_{A,B,C}) \\
 &\quad + \sqrt{\frac{4}{20}}(|-\frac{7}{2}\rangle_{A,B,C} + |+\frac{7}{2}\rangle_{A,B,C}), \\
 |2_L\rangle &= \sqrt{\frac{7}{20}}(|-\frac{1}{2}\rangle_{A,B,C} + |+\frac{1}{2}\rangle_{A,B,C}) \\
 &\quad + \sqrt{\frac{3}{20}}(|-\frac{9}{2}\rangle_{A,B,C} + |+\frac{9}{2}\rangle_{A,B,C}), \tag{11}
 \end{aligned}$$

where A , B , and C denote the three qudits and $|m_I\rangle_{A,B,C}$ corresponds to the entangled product state $|m_I\rangle_A \otimes |m_I\rangle_B \otimes |m_I\rangle_C$. The multiqutrit encoding and decoding pulse sequences can be designed in the same manner as proposed in Ref. [50]. The above logical qutrit codewords with three $S = \frac{9}{2}$ systems have the advantage of being implementable using nuclear spins in solid-state platforms, which are among the most coherent quantum systems.

This multispin qudit code can also be extended to second-order correction, as shown in the encoding with five spins $\frac{19}{2}$ by

$$\begin{aligned}
 |0_L\rangle &= \sqrt{\frac{5}{16}}(|-\frac{5}{2}\rangle_{A,B,C,D,E} + |+\frac{5}{2}\rangle_{A,B,C,D,E}) + \sqrt{\frac{3}{16}}(|-\frac{15}{2}\rangle_{A,B,C,D,E} + |+\frac{15}{2}\rangle_{A,B,C,D,E}), \\
 |1_L\rangle &= \sqrt{\frac{5423}{42400}}(|-\frac{1}{2}\rangle_{A,B,C,D,E} + |+\frac{1}{2}\rangle_{A,B,C,D,E}) + \sqrt{\frac{7203}{42400}}(|-\frac{9}{2}\rangle_{A,B,C,D,E} + |+\frac{9}{2}\rangle_{A,B,C,D,E}) \\
 &\quad \times \sqrt{\frac{6517}{42400}}(|-\frac{11}{2}\rangle_{A,B,C,D,E} + |+\frac{11}{2}\rangle_{A,B,C,D,E}) + \sqrt{\frac{2057}{42400}}(|-\frac{19}{2}\rangle_{A,B,C,D,E} + |+\frac{19}{2}\rangle_{A,B,C,D,E}), \\
 |2_L\rangle &= \sqrt{\frac{3294}{22800}}(|-\frac{3}{2}\rangle_{A,B,C,D,E} + |+\frac{3}{2}\rangle_{A,B,C,D,E}) + \sqrt{\frac{3749}{22800}}(|-\frac{7}{2}\rangle_{A,B,C,D,E} + |+\frac{7}{2}\rangle_{A,B,C,D,E}) \\
 &\quad \times \sqrt{\frac{2771}{22800}}(|-\frac{13}{2}\rangle_{A,B,C,D,E} + |+\frac{13}{2}\rangle_{A,B,C,D,E}) + \sqrt{\frac{1586}{22800}}(|-\frac{17}{2}\rangle_{A,B,C,D,E} + |+\frac{17}{2}\rangle_{A,B,C,D,E}), \tag{12}
 \end{aligned}$$

where the notation $|m_I\rangle_{A,B,C,D,E}$ denotes the tensor product state of five-spin qudits, as in the previous case.

IV. DISCUSSION OF DIMENSIONS, PRACTICAL IMPLEMENTATIONS, AND LOGICAL QUDIT OPERATIONS

Building on the encoding schemes introduced in Secs. II and III, this section discusses the required Hilbert-space dimensionality, practical considerations for physical implementation, and logical qudit gate operations. In addition, we identify the minimum single-qudit gate fidelities required for fault-tolerant encoding to yield a net advantage.

The conventional definition of a code distance $2t+1$ in quantum error correction is often interpreted as the ability to correct t simultaneous and independent errors in logical qubit encodings. This definition relies on two characteristic properties of qubits: first, that the square of qubit Pauli operators is

the identity, and second, that logical qubit encoding requires multiple physical qubits. These assumptions, however, are often not valid in qudit systems. A logical unit can be encoded within a single qudit, and the square of error operators is not necessarily the identity.

The first and most well-known example illustrating this distinction is the GKP code [38]. The GKP code requires a bosonic Hilbert space [29,64], as implied by the definition of the generalized d -dimensional Pauli operators

$$\begin{aligned}
 X|j\rangle &= |j+1(\text{mod } d)\rangle, \\
 Z|j\rangle &= w^j|j\rangle, \tag{13}
 \end{aligned}$$

where d denotes the qudit dimension embedded as a periodic subset of an infinite-dimensional Hilbert space and w is a phase factor. In this framework, higher-order physical errors in the system can be expressed as combinations of generalized

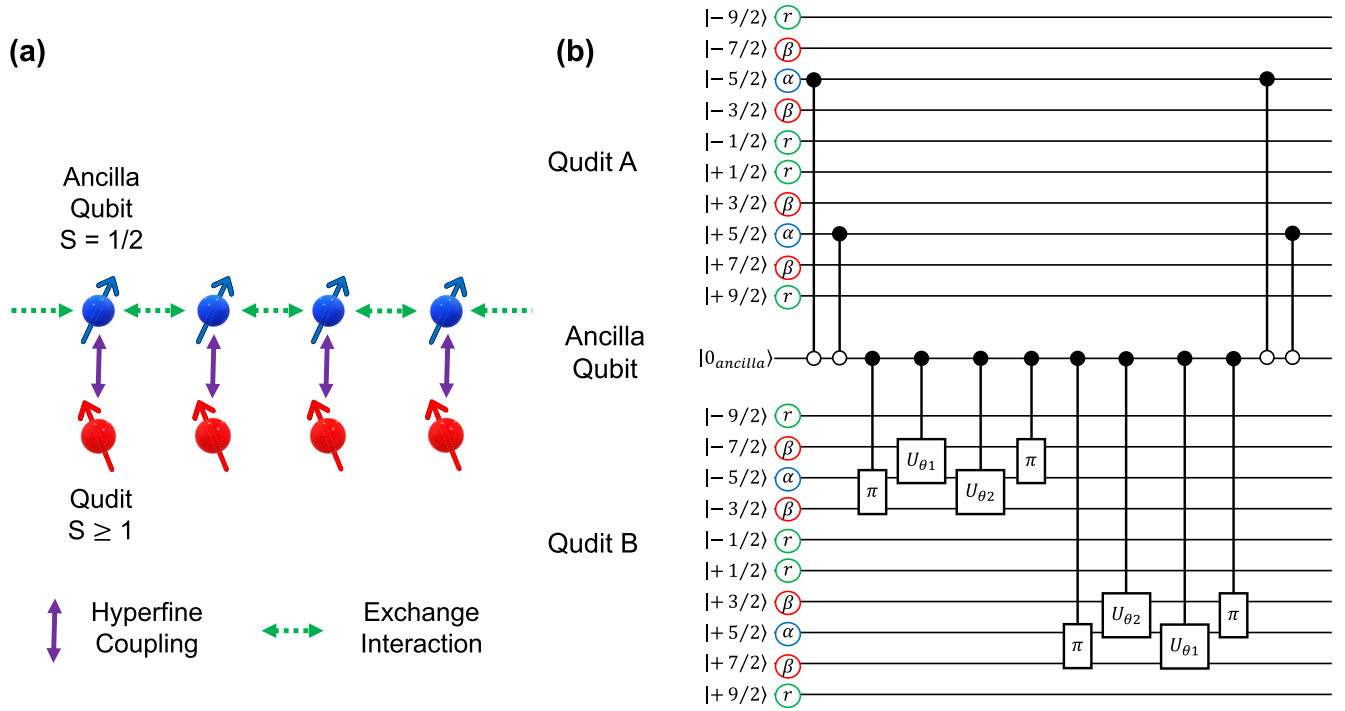


FIG. 2. (a) Schematic illustration of a scalable multiqubit architecture. (b) Example of a logical two-qubit gate (cnot) implemented via an ancilla qubit as a mediator. Unitary rotations U_{θ_i} are around the y axis, with $\cos(\theta_i) = \sqrt{\frac{2}{5}}, \sqrt{\frac{3}{5}}$ for $i = 1, 2$, respectively (see the text for details).

X and Z operators up to order t and are therefore correctable using a codeword with distance $2t + 1$.

While this mathematical structure is also well defined for finite-dimensional qudits, its physical implementation requires translational invariance (i.e., periodic boundary conditions) in Hilbert space. In particular, unknown physical errors in finite-dimensional systems, such as spin qudits, do not generate cyclic shift operations across the Hilbert-space boundary. This issue is mitigated in the case of harmonic oscillators by encoding into infinite-dimensional eigenstates with finite squeezing, which effectively preserves translational symmetry. This mechanism, however, is not available in intrinsically finite-dimensional systems.

Indeed, this consideration motivated the alternative encodings described in the previous sections. A similar perspective on codeword distance can be applied to the spin Hamiltonian of the system. In solid-state spin systems, the dominant Hamiltonian terms governing the system can often be written as

$$H = g\mu_B B \cdot S + g_I \mu_B B \cdot I + S \cdot A \cdot I + S \cdot D \cdot S + I \cdot Q \cdot I + (\text{higher order terms}), \quad (14)$$

where I and S represent the nuclear and electronic spin operators, respectively. From left to right, the terms describe the electronic and nuclear spin Zeeman splittings, the hyperfine interaction, the second-order crystal field splitting, and the nuclear quadrupole interaction. While additional higher-order terms (such as crystal field splitting represented in terms of Stevens operators) may arise from the crystalline symmetry and environmental effects, these operators can generally be

decomposed into combinations of I and S operators. These terms can act as qudit control operations under applied engineered perturbations [51] and as error sources under unknown perturbations [50]. Higher-order errors generated by I and S operators up to order of approximately t can therefore be corrected using a logical qudit encoding with a distance of $2t + 1$.

Next we discuss the resources required for practical quantum computation based on these logical qudits. These include the number of physical units (qubits or qudits), the number of elementary gate operations required to decode and correct a single logical qudit, and the implementation of logical qudit gates. As illustrated in Fig. 2(a), one possible architecture consists of each qudit being coupled to the nearest ancilla qubit. This ancilla serves both as a resource for error detection and as a mediator of interqudit interactions.

Under this connectivity condition, a distance-3 logical qudit code requires on the order of approximately d^2 elementary operations for decoding, error correction, and logical single- and two-qubit gates. The decoding and correction procedures are discussed in Sec. II and Appendix A. Logical single-qubit operations can be implemented as unitary transformations between codewords. Since all original and error codewords are mutually orthogonal in this encoding, the corresponding unitary transformations can be explicitly constructed and implemented with at most approximately d^2 elementary operations. Logical two-qubit gates can be implemented via an ancilla qubit, as depicted in Fig. 2(b). The sequence shown corresponds to a controlled-not (cnot) operation from $| 0_L \rangle_A$ to $| 0_L \rangle_B \leftrightarrow | 1_L \rangle_B$. Importantly, this operation does not depend on the presence of Z error on control qudit A, allowing

error correction to be performed either before or after the two-qudit operation. For the case of X -, Y -, and Z -error-correction encoding of a spin- $\frac{29}{2}$ system [Eq. (8)], the same operation can be achieved by adding nearest-level controlled-not operations from qudit A to the ancilla qubit. Specifically, these correspond to cnot operations conditioned on the levels $|-\frac{17}{2}\rangle$, $|-\frac{15}{2}\rangle$, $|-\frac{13}{2}\rangle$, $|+\frac{13}{2}\rangle$, $|+\frac{15}{2}\rangle$, and $|+\frac{17}{2}\rangle$ of qudit A . Other controlled operations, such as $C_i\text{NOT}_{j,k}$ with different indices i , j , and k , can be implemented in the same manner within approximately d^2 elementary operations. Since a universal gate set for the qudits consists of arbitrary single-qudit gates between adjacent levels and two-qudit entangling operations such as $C_i\text{NOT}_{j,k}$ between arbitrary levels [65,66], all logical operations can be constructed with polynomial complexity scaling in d .

This type of connectivity between spin qudits can be experimentally implemented in state-of-the-art platforms, including hybrid nuclear-electronic spin systems, where the system Hamiltonian can be described by an extension of Eq. (14), with an additional exchange interaction between electron spins $JS_1 \cdot S_2$. Controlled transitions between nuclear spin qudits and hyperfine-coupled electronic spin qubits have been experimentally demonstrated in paramagnetic impurities in semiconductor systems [40,41] as well as in molecular spin qudits [67,68]. For interqudit connectivity, electron spins coupled via exchange interactions can serve as mediators of entanglement between two nuclear spins, as originally proposed theoretically in the Kane model [69] and realized experimentally in semiconductor quantum-dot spin systems [70]. While such implementations require multiple electron spins to mediate interactions, this additional overhead scales only linearly with the number of elementary operations required for logical two-qudit gates. Regarding the addressing of the individual spin sublevels, while first-order Zeeman terms in Eq. (14) lead to uniform energy-level splittings, additional perturbative terms, such as hyperfine interactions, crystal field effects, and nuclear quadrupole interactions, provide spectral addressability of the sublevels [40,51]. Fault-tolerant encoding of a qutrit can be implemented in several nuclear-spin environments, such as Bi dopants in Si systems, and higher-dimensional qudits require access to larger Hilbert spaces. In such cases, the qudit space can be realized through the chemical design of giant molecular magnets [52,71,72] or by using several coupled nuclear spin systems [53,73]. Additional candidate platforms using similar spin Hamiltonians include trapped and laser-cooled ions or molecules (which possess additional total angular momentum contributions from motional degrees of freedom) [56,57] and photonic systems utilizing orbital angular momentum states [58,59].

The features discussed above indicate that logical qudit operations can be implemented with polynomial complexity in the qudit dimension d while maintaining similar or lower overhead compared to conventional mapping from logical qubit encodings. Importantly, this approach retains the advantage of requiring fewer physical units to be controlled. This resource efficiency can be further quantified by comparing the total Hilbert-space dimension required for logical qudit construction, as discussed in Appendix C (see Fig. 6). The total Hilbert-space dimension provides an upper bound on the physical state space accessed during decoding and

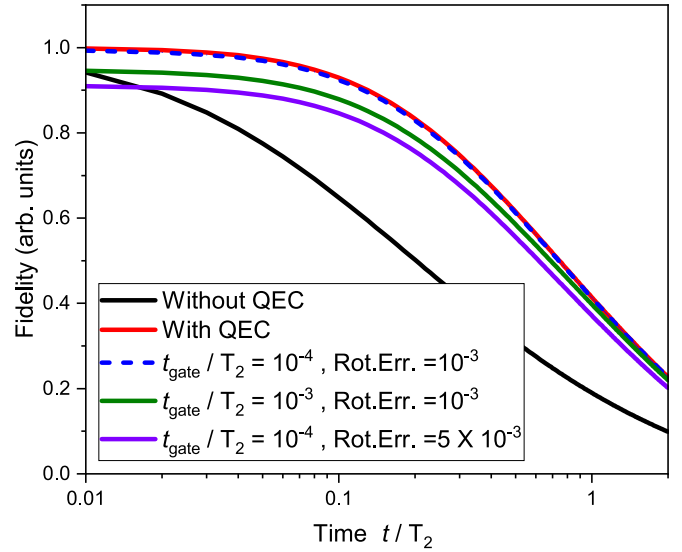


FIG. 3. Qutrit fidelity as a function of time without (black solid line) and with (red solid line) error correction. The effects of decoherence during the decoding sequence and imperfect gate rotation angle errors are also simulated, shown as green and purple solid lines and a blue dashed line (see the text for details).

is directly relevant to experimental complexity of physical implementation. Regarding gate complexity in information processing, there has been extensive work on the decomposing of qudit-based gates into qubit-based gates [74]. However, several recent studies suggest that qudit-to-qubit mapping typically requires extra resources, such as extra entangling gates in Hamiltonian exponentiation [13,75] or increased circuit depth in Grover's algorithm implementations [76]. These results further support the conclusion that the direct use of logical qudits can offer practical advantages in experimental algorithm implementation.

In Fig. 3 we present a numerical estimate of the overall fidelity of the logical qutrit code defined in Eq. (1). The time evolution of a superposed qutrit state $[|\psi\rangle = \frac{1}{\sqrt{3}}(|0_L\rangle + |1_L\rangle + |2_L\rangle)]$, encoded in a nuclear spin- $\frac{9}{2}$ qudit coupled to an electron spin- $\frac{1}{2}$ ancilla qubit, is simulated with a typical spin-spin relaxation time T_2 . While more specific microscopic models of spin decoherence exist for various experimental environments [77], we model decoherence using a Lindblad master equation, assuming fluctuations of the longitudinal magnetic field B_z , as the dominant decoherence source. This model corresponds to the conventional case of well-isolated spin qudit memories in the solid state. It is typically observed in the decay of electronic and nuclear spins of defects in semiconductors [40] or in a molecular spin qudits [47], which exhibit millisecond-scale coherence for nuclear spins and tens of microseconds for electronic spins in a typical X-band (approximately 0.3 T) electron paramagnetic resonance (EPR) setup.

A comparison of the fidelities with (red solid line) and without (black solid line) the error-correction protocol confirms the effectiveness of the encoding in the regime $t \ll T_2$. We further test a model that accounts for single-qudit gate fidelity and decoherence during the encoding and decoding

process, in order to identify practical conditions to achieve a net advantage with the error-correction protocol. In the simulation, we have assumed that only the $\delta m_l = 1$ transition is allowed in the system (which could be further improved if the system allows $\delta m_l = 2$ transitions), and gate infidelity is defined as the rotation angle and phase error of these elementary gates. As shown by comparisons across different gate fidelities and single gate duration parameters, such as single gate fidelity of 99.5% (purple solid line) and single gate duration $t_{\text{gate}}/T_2 = 10^{-3}$ (green solid line), we find that the optimal protocol requires a single gate fidelity above 99.9% and single gate duration shorter than 10^{-4} times T_2 (blue dashed line). In conventional EPR and NMR setups, the single gate duration condition has already been achieved [40], and state-of-the-art spin qubit devices report single gate fidelities up to two to three nines [78,79]. Therefore, we believe that logical qudit encoding is readily implementable and can offer an advantage over conventional qubit-to-qudit mapping schemes.

ACKNOWLEDGMENTS

We thank Arzhang Ardavan for helpful discussions. This project was supported by the European Union's Horizon 2020 research and innovation program under Grant Agreements No. 862893 (FATMOLS) and No. 863098 (SPRING) and the National Research Foundation of Korea under Grant No. RS-2023-00256050.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the authors upon reasonable request.

APPENDIX A: PULSE SEQUENCES FOR ENCODING AND DECODING LOGICAL QUDITS

Figure 4 shows the complete encoding pulse sequence for the qutrit Z-error-correction code. Without loss of generality, the initial state can be written as $|\psi\rangle = \alpha|-\frac{9}{2}\rangle + \beta|-\frac{7}{2}\rangle + \gamma|-\frac{5}{2}\rangle$.

Through the encoding sequence, the general starting state is transformed into $|\psi\rangle = \alpha|0_L\rangle + \beta|1_L\rangle + \gamma|2_L\rangle$.

Figure 5 shows the complete decoding pulse sequence. The state after Z error on the encoded state can be written as

$$E|\psi_{\text{enc}}\rangle = \sqrt{1 - \epsilon}(\alpha|0_L\rangle + \beta|1_L\rangle + \gamma|2_L\rangle) + (i\epsilon_Z)(\alpha S_Z|0_L\rangle + \beta S_Z|1_L\rangle + \gamma S_Z|2_L\rangle) + \dots, \quad (\text{A1})$$

where the ellipsis denotes higher-order terms. After applying the decoding pulses, the state is transformed into $|\psi\rangle = A_0(\alpha|-\frac{9}{2}\rangle + \beta|-\frac{5}{2}\rangle + \gamma|+\frac{3}{2}\rangle) + A_1(\alpha|-\frac{7}{2}\rangle + \beta|-\frac{1}{2}\rangle + \gamma|+\frac{7}{2}\rangle)$.

APPENDIX B: COEFFICIENTS FOR LOGICAL QUDIT ENCODING CODEWORDS

In this Appendix we present the detailed derivation of the logical qudit codeword construction, including the formalism of the Z-basis combinations and their coefficients, and the generalization to arbitrary qudit dimension d and codeword distance $2t + 1$. As a working example, we consider the first-order (i.e., distance-3) Z-error-correction code, for which each logical qudit state can be written as

$$|0_L\rangle = \sqrt{\frac{1}{2}}[|-(S/2 + \frac{1}{4})\rangle + |(S/2 + \frac{1}{4})\rangle], \\ |i_L\rangle = a_i[|-(S/2 + \frac{1}{4} - i)\rangle + |(S/2 + \frac{1}{4} - i)\rangle] + b_i[|-(S/2 + \frac{1}{4} + i)\rangle + |(S/2 + \frac{1}{4} + i)\rangle], \quad (\text{B1})$$

where $S = 2d - \frac{3}{2}$ and $i = 1, 2, \dots, d - 1$. The coefficients a_i and b_i must be carefully chosen to satisfy the KL criteria defined in Eq. (2). For the first-order Z-error case, the nontrivial conditions in the KL criteria can be summarized as the three conditions $\langle i_L | i_L \rangle = 1$ (normalization), $\langle i_L | S_Z | i_L \rangle = \langle j_L | S_Z | j_L \rangle$, and $\langle i_L | S_Z^2 | i_L \rangle = \langle j_L | S_Z^2 | j_L \rangle$ for $i, j = 0, 1, 2$. It follows straightforwardly that the coefficients b_i are determined from a_i using the normalization condition. The first-order projection conditions are automatically satisfied by symmetry. The second-order projection conditions constrain a_i coefficients through the relation $\langle 0_L | S_Z^2 | 0_L \rangle = \langle i_L | S_Z^2 | i_L \rangle$.

Encoding pulses

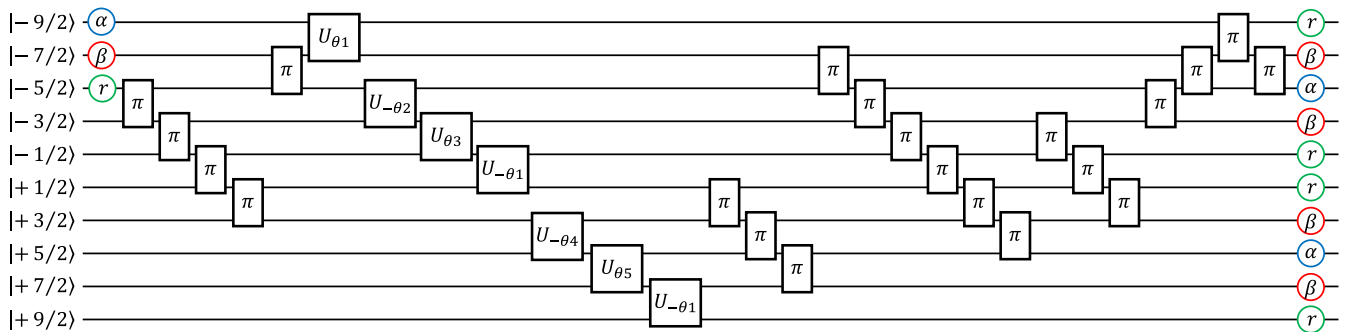


FIG. 4. Gates used in the encoding pulses sequence of the qutrit Z-error-correction code. Unitary rotations U_{θ_i} are around the y axis, with $\cos(\theta_i) = \sqrt{\frac{1}{2}}, \sqrt{\frac{3}{10}}, \sqrt{\frac{3}{7}}, \sqrt{\frac{7}{20}}, \sqrt{\frac{7}{13}}$ for $i = 1, 2, 3, 4, 5$, respectively.

Decoding pulses

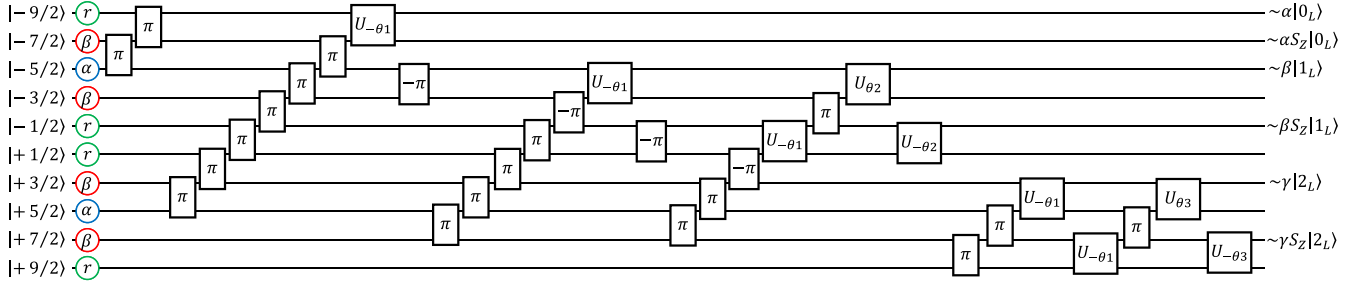


FIG. 5. Gates used in the decoding pulse sequence of the qutrit Z-error-correction code. Unitary rotations U_{θ_i} are around the y axis, with $\cos(\theta_i) = \sqrt{\frac{1}{2}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{10}}$ for $i = 1, 2, 3$, respectively.

Therefore, choosing the total angular momentum $S = 2d - \frac{3}{2}$ yields a valid solution, as shown in Eq. (6).

To construct the distance-5 code, we double the number of S_Z -basis states by duplicating the distribution used in the distance-3 code and placing it in the higher- S_Z subspace. For example, the distance-5 code for a qutrit retains the codeword

distribution from $|\frac{1}{2}\rangle \cdots |\frac{9}{2}\rangle$ and adds a similar set spanning $|\frac{11}{2}\rangle \cdots |\frac{19}{2}\rangle$.

In this distance-5 case, the codewords must satisfy the KL conditions $\langle i_L | i_L \rangle = 1$ and $\langle i_L | S_Z^n | i_L \rangle = \langle j_L | S_Z^n | j_L \rangle$ for $n = 1, 2, 3, 4$. The general form of the distance-5 code for a d -dimensional qudit can be written as

$$\begin{aligned}
 |0_L\rangle &= \sqrt{\frac{5}{16}} \left[|-(S/4 + \frac{1}{8})\rangle + |(S/4 + \frac{1}{8})\rangle \right] + \sqrt{\frac{3}{16}} \left[|-(3S/4 + \frac{3}{8})\rangle + |(3S/4 + \frac{3}{8})\rangle \right], \\
 |i_L\rangle &= a_i \left[|-(2d - \frac{1}{2} - i)\rangle + |(2d - \frac{1}{2} - i)\rangle \right] + b_i \left[|-(2d - \frac{3}{2} + i)\rangle + |(2d - \frac{3}{2} + i)\rangle \right] \\
 &\quad + c_i \left[|-(i - \frac{1}{2})\rangle + |(i - \frac{1}{2})\rangle \right] + d_i \left[|-(4d - \frac{3}{2} - i)\rangle + |(4d - \frac{3}{2} - i)\rangle \right],
 \end{aligned} \tag{B2}$$

where $S = 4d - \frac{5}{2}$ and $i = 1, 2, \dots, d - 1$. The coefficients satisfying the above KL criteria are given by

$$\begin{aligned}
 a_i &= \sqrt{\frac{8d + 2i - 5}{32d - 16}} \sqrt{\frac{66d^3 + 2d^2i - 100d^2 - 14di^2 + 12di + 47d + 2i^3 + 4i^2 - 5i - 7}{96d^3 + 32d^2i - 160d^2 - 16di^2 - 16di + 84d + 8i^2 - 14}}, \\
 b_i &= \sqrt{\frac{8d - 2i - 3}{32d - 16}} \sqrt{\frac{66d^3 + 2d^2i - 100d^2 - 14di^2 + 12di + 47d + 2i^3 + 4i^2 - 5i - 7}{96d^3 + 32d^2i - 160d^2 - 16di^2 - 16di + 84d + 8i^2 - 14}}, \\
 c_i &= \sqrt{\frac{12d - 2i - 5}{32d - 16}} \sqrt{\frac{36d^3 + 30d^2i - 60d^2 - 2di^2 - 28di + 37d - 2i^3 + 4i^2 + 5i - 7}{96d^3 + 32d^2i - 160d^2 - 16di^2 - 16di + 84d + 8i^2 - 14}}, \\
 d_i &= \sqrt{\frac{4d + 2i - 3}{32d - 16}} \sqrt{\frac{36d^3 + 30d^2i - 60d^2 - 2di^2 - 28di + 37d - 2i^3 + 4i^2 + 5i - 7}{96d^3 + 32d^2i - 160d^2 - 16di^2 - 16di + 84d + 8i^2 - 14}}.
 \end{aligned} \tag{B3}$$

Although there are five sets of conditions in the KL criteria, the odd-order error projection terms ($\langle i_L | S_Z | i_L \rangle = \langle j_L | S_Z | j_L \rangle$ and $\langle i_L | S_Z^3 | i_L \rangle = \langle j_L | S_Z^3 | j_L \rangle$) are automatically satisfied by symmetry. Therefore, it is sufficient to verify only the remaining even-order conditions $\langle i_L | i_L \rangle = 1$, $\langle i_L | S_Z^2 | i_L \rangle = \langle j_L | S_Z^2 | j_L \rangle$, and $\langle i_L | S_Z^4 | i_L \rangle = \langle j_L | S_Z^4 | j_L \rangle$. As a result, the above codeword design with four free parameters satisfies all KL criteria.

It is worth noting that the distance-5 code for a qutrit, as well as for general d -dimensional qudits, is not unique. As a simple example, we present an alternative set of coefficients

$$\begin{aligned}
 |0_L\rangle &= \sqrt{\frac{3}{10}} \left(|-\frac{5}{2}\rangle + |+\frac{5}{2}\rangle \right) + \sqrt{\frac{1}{5}} \left(|-\frac{15}{2}\rangle + |+\frac{15}{2}\rangle \right), \\
 |1_L\rangle &= \sqrt{\frac{1152}{9225}} \left(|-\frac{1}{2}\rangle + |+\frac{1}{2}\rangle \right) + \sqrt{\frac{133}{1025}} \left(|-\frac{9}{2}\rangle + |+\frac{9}{2}\rangle \right) + \sqrt{\frac{399}{2050}} \left(|-\frac{11}{2}\rangle + |+\frac{11}{2}\rangle \right) + \sqrt{\frac{468}{9225}} \left(|-\frac{19}{2}\rangle + |+\frac{19}{2}\rangle \right), \\
 |2_L\rangle &= \sqrt{\frac{1081}{7700}} \left(|-\frac{3}{2}\rangle + |+\frac{3}{2}\rangle \right) + \sqrt{\frac{252}{1650}} \left(|-\frac{7}{2}\rangle + |+\frac{7}{2}\rangle \right) + \sqrt{\frac{441}{3300}} \left(|-\frac{13}{2}\rangle + |+\frac{13}{2}\rangle \right) + \sqrt{\frac{282}{3850}} \left(|-\frac{17}{2}\rangle + |+\frac{17}{2}\rangle \right).
 \end{aligned} \tag{B4}$$

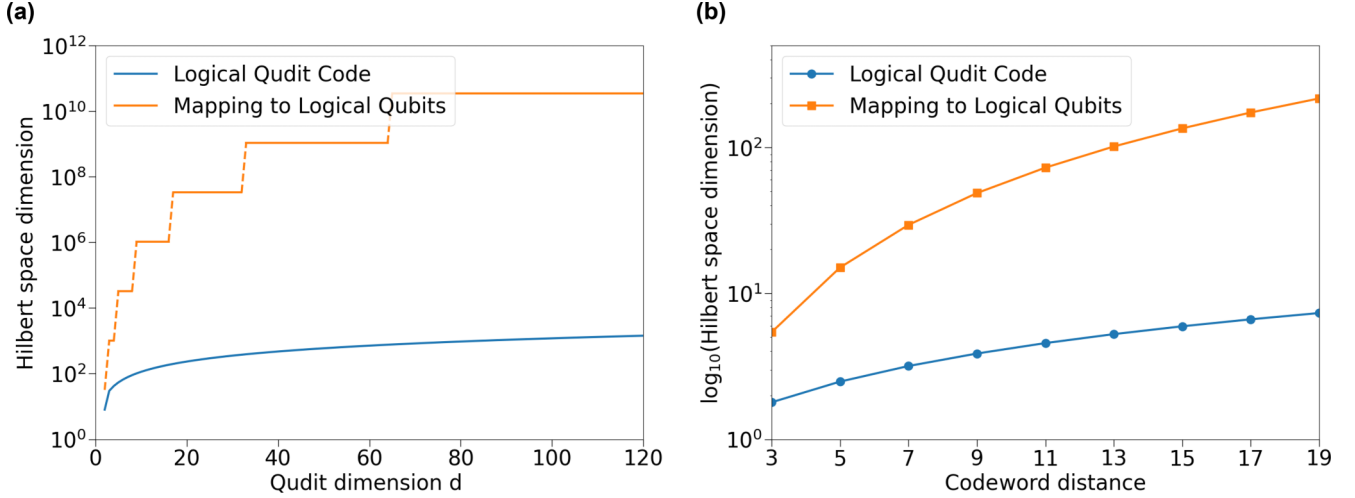


FIG. 6. Required Hilbert space for logical qudit encoding in this work (blue) and conventional mapping into multiple logical qubits (orange), according to (a) qudit dimension d and (b) codeword distance.

For a general codeword distance of $2t + 1$ with $t \geq 3$, similar expansion, which doubles the encoding S_Z basis in the same manner as the distance-3 to distance-5 construction, provides sufficient Hilbert space and degrees of freedom to satisfy the KL criteria, i.e., $\langle i_L | S_Z^n | i_L \rangle = \langle j_L | S_Z^n | j_L \rangle$ for $n = 0, 1, 2, \dots, 2t$, by appropriately tailoring the coefficients. As in the above case, for odd t , the condition $\langle i_L | S_Z^t | i_L \rangle = \langle j_L | S_Z^t | j_L \rangle$ is trivially satisfied from symmetry. Therefore, the remaining nontrivial KL conditions correspond to even orders of t . Based on this iterative doubling of the codeword structure, an arbitrary $|i_L\rangle$ consists of a linear combination of $2^t Z$ -basis states, with the same number of independent coefficients. Pairing the coefficients t times across 2^t basis states provide sufficient degrees of freedom to construct arbitrary $|i_L\rangle$ satisfying $\langle i_L | S_Z^n | i_L \rangle = \langle 0_L | S_Z^n | 0_L \rangle$ for $n = 0, 2, \dots, 2t$. We emphasize that this systematic procedure provides a sufficient condition for constructing valid codewords. Further investigation is required to identify the mathematically minimal Hilbert-space dimension that enables fault-tolerant encoding.

We have presented a general formalism for extending the Z -error-correcting code to full X -, Y -, and Z -error correction. This is achieved by increasing the basis dimension by a factor of $2t + 1$ (i.e., a threefold increase from the $\frac{9}{2}$ code to the $\frac{29}{2}$ code and a fivefold increase from the $\frac{19}{2}$ code to the $\frac{99}{2}$) or by increasing the number of qudits by a factor of $2t + 1$ (i.e., the three- $\frac{9}{2}$ -qudit code or five- $\frac{19}{2}$ -qudit code). In both approaches, the same set of coefficients used for the Z -error correction remains valid for constructing the corresponding distance- $2t + 1$ X -, Y -, and Z -error-correcting code.

As a working example, we briefly describe why the codewords in Eq. (1) can also be extended to correct X , Y , and Z errors. Equation (1) satisfies the conditions $\langle i_L | | i_L \rangle = 1$, $\langle i_L | S_Z | i_L \rangle = \langle j_L | S_Z | j_L \rangle$, and $\langle i_L | S_Z^2 | i_L \rangle = \langle j_L | S_Z^2 | j_L \rangle$ for $i, j = 0, 1, 2$. Therefore, the threefold expanded codewords in Eq. (8) automatically satisfy the above three conditions, since the expansion corresponds to a linear scaling of the S_Z basis. The orthogonality conditions in the KL criteria ($\langle i_L | E_a E_b | j_L \rangle = 0$) are also fulfilled due to the $(2t + 1)$ -fold

spacing between the basis states, for any E_a and E_b corresponding to the X , Y , or Z operator. Thus, the remaining nontrivial conditions that must be verified are $\langle i_L | S_X^2 | i_L \rangle = \langle j_L | S_X^2 | j_L \rangle$ and $\langle i_L | S_Y^2 | i_L \rangle = \langle j_L | S_Y^2 | j_L \rangle$.

In the case of S_X^2 projection, since S_X can be expressed as $S_X = \frac{1}{2}(S_+ + S_-)$, the equation can be rewritten as $\langle i_L | (S_+ + S_-)(S_+ + S_-) | i_L \rangle = \langle j_L | (S_+ + S_-)(S_+ + S_-) | j_L \rangle$.

Due to the spacing between the S_Z -basis states, the only remaining nontrivial terms are $\langle i_L | (S_+ S_- + S_- S_+) | i_L \rangle = \langle j_L | (S_+ S_- + S_- S_+) | j_L \rangle$.

Using the relations for the spin ladder operators, we obtain

$$\begin{aligned} S_+ |S, m\rangle &= \sqrt{S(S+1) - m(m+1)} |S, m+1\rangle, \\ S_- |S, m\rangle &= \sqrt{S(S+1) - m(m-1)} |S, m-1\rangle. \end{aligned} \quad (\text{B5})$$

If we decompose $|i_L\rangle = \sum c_m |S, m\rangle$, the above terms can be written as

$$\begin{aligned} \langle i_L | (S_+ S_- + S_- S_+) | i_L \rangle &= 2S(S+1) - 2 \sum \langle S, m | m^2 | c_m \rangle^2 |S, m\rangle \\ &= 2S(S+1) - 2 \langle i_L | S_Z^2 | i_L \rangle. \end{aligned} \quad (\text{B6})$$

Therefore, the conditions $\langle i_L | S_X^2 | i_L \rangle = \langle j_L | S_X^2 | j_L \rangle$ and $\langle i_L | S_Y^2 | i_L \rangle = \langle j_L | S_Y^2 | j_L \rangle$ also hold with the same coefficients as in Eq. (8). The same reasoning can be iteratively extended to arbitrary $(2t + 1)$ -distance codes, up to $\langle i_L | S_X^{2t} | i_L \rangle = \langle j_L | S_X^{2t} | j_L \rangle$ and $\langle i_L | S_Y^{2t} | i_L \rangle = \langle j_L | S_Y^{2t} | j_L \rangle$.

APPENDIX C: COMPARISON OF REQUIRED HILBERT-SPACE DIMENSION FOR ENCODING

In Fig. 6 we present the required Hilbert space for two strategies to realize a logical qudit: the logical qudit encoding scheme proposed in this work and the conventional construction of a logical qudit from multiple logical qubits. Figure 6(a) shows the scaling of Hilbert-space dimension with qudit dimension d for both approaches. As a working example, we consider the distance-3 logical qudit encoding described in Eq. (9). For the conventional strategy, we assume that the

logical qudit is constructed from multiple logical qubits, each encoded using a distance-3 surface code. Figure 6(b) shows the scaling of the Hilbert-space dimension of logical qutrits as a function of codeword distance for the two strategies.

As shown in both comparisons, encoding of logical qudits introduced in this work requires orders of magnitude smaller total Hilbert-space dimension, implying that fault-tolerant encoding can be performed in a more resource-efficient manner than the conventional approach. The total Hilbert-space dimension provides an upper bound on the physical error sub-

space that must be addressed during the error correction and is relevant to experimental complexity that may arise in practical implementations. While qubit-based quantum error-correction schemes nowadays face rapidly increasing decoding complexity with system size and significant efforts have been devoted to developing compact and efficient decoding schemes [80], such approaches generally include trade-offs between decoding scaling and error threshold. From this perspective, reducing the required Hilbert space offers a complementary and physically meaningful advantage.

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