

Reply to “Comment on ‘Energy-level shift of quantum systems via the scalar electric Aharonov-Bohm effect’ ”

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We respond to the points raised in the preceding Comment [Phys. Rev. A **111**, 066203 (2025)] on our recent work [Phys. Rev. A **107**, 042209 (2023)], which introduced a theoretical framework for the scalar electric Aharonov-Bohm effect. In this Reply, we take the opportunity to address specific criticisms, by referring to foundational aspects of gauge invariance and time-dependent quantum systems. We explain why the gauging procedure used in the Comment leads to inconsistencies and show that, when applied similarly to the well-established magnetic Aharonov-Bohm effect, it would yield demonstrably incorrect conclusions. Moreover, we highlight the consistency of our results with canonical treatments of the time-dependent Schrödinger equation, and we reaffirm the physicality of energy shifts in field-free regions, an effect observed in the magnetic and predicted in the electric Aharonov-Bohm contexts. We believe that this response contributes to a deeper understanding of the subtleties involved in gauge theory and quantum phase phenomena.

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In the preceding Comment [1] on a recent work of ours [2], the claim is made that our analysis in [2] is not correct and that there should be no shift in the energy level of the quantum system inside the Faraday sphere since the electric field inside the sphere is zero.¹

Sections I and II of the Comment give a brief overview and summary of the analysis from [2]. Section III of the Comment begins the critique of our work [2], by looking at the Hamiltonian and eigenfunctions of the Hamiltonian in two different gauges, referred to as gauge 1, an unprimed gauge, and gauge 2, a primed gauge.

In gauge 1 the Hamiltonian is $H = H_0 + V_0 \cos(\Omega t)$, which gives the Schrödinger equation in the form

$$i\hbar \frac{\partial \psi_i(\mathbf{r}, t)}{\partial t} = H \psi_i = H_0 \psi_i(\mathbf{r}, t) + V_0 \cos(\Omega t) \psi_i(\mathbf{r}, t). \quad (1)$$

Both the Comment [1] and our original work [2] agree that the solution to (1) is

$$\psi_i(\mathbf{r}, t) = \Psi_i(\mathbf{r}) \exp\left(-\frac{iE_i t}{\hbar} - i\varphi(t)\right) \quad \text{with} \\ \varphi(t) = \frac{e}{\hbar} \int V(t) dt = \frac{eV_0}{\hbar\Omega} \sin \Omega t. \quad (2)$$

Next the Comment [1] performs a gauge transformation in which the scalar potential is transformed to zero. The gauge function is taken as $\lambda(t) = \frac{V_0}{\Omega} \sin(\Omega t)$, which gives

$$V' = V(t) - \partial_t \lambda(t) = V_0 \cos(\Omega t) - V_0 \cos(\Omega t) = 0. \quad (3)$$

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¹Although the Comment [1] focuses on [2], it also claims to apply to our work in [3]. Our rebuttal here shows why [1] is incorrect in its criticism of both works.

The gauge transformation in (3) does not change the vector potential, since $\nabla\lambda(t) = 0$ and $\mathbf{A} = 0$, which yields $\mathbf{A}' = \mathbf{A} + \nabla\lambda(t) = 0$. The gauge in (3) is called gauge 2 or the primed gauge in the Comment [1]. The time-dependent Schrödinger equation in gauge 2 becomes

$$i\hbar \frac{\partial \psi'_i(\mathbf{r}, t)}{\partial t} = H_0 \psi'_i(\mathbf{r}, t). \quad (4)$$

Now the solution to (4), with the scalar potential gauge away, is just $\psi'_i(\mathbf{r}, t) = \Psi_i(\mathbf{r}) \exp(-\frac{iE_i t}{\hbar})$. However, one also needs to perform the gauge transformation of the wave function, which is $\psi_i(\mathbf{r}, t) = \exp(-i\frac{e}{\hbar}\lambda) \psi'_i(\mathbf{r}, t)$. Combining all this then gives

$$\begin{aligned} & \exp\left(-i\frac{e}{\hbar}\lambda\right) \psi'_i(\mathbf{r}, t) \\ & \rightarrow \exp\left(-i\frac{eV_0}{\hbar\Omega} \sin(\Omega t)\right) \Psi_i(\mathbf{r}) \exp\left(-\frac{iE_i t}{\hbar}\right), \end{aligned} \quad (5)$$

which is just the result from (2). The Comment claims to be able to gauge away the energy (i.e., that the energy is a gauge artifact), whereas we show that the Aharonov-Bohm phase appears either as the time integral of the scalar potential as in (2) or via the gauge transformation of the wave function as in (5).

Another way to show that the claims of [1] are wrong is to apply the methods of [1] to the standard magnetic Aharonov-Bohm effect and show that one could gauge away the magnetic Aharonov-Bohm effect. This is at odds with the fact that the magnetic Aharonov-Bohm effect has been experimentally observed, with the conclusion being that the gauging procedure used in the Comment [1] is wrong.

In the standard magnetic Aharonov-Bohm effect, the vector potential associated with a solenoid of radius $\rho = R$ and magnetic field B_0 is

$$\begin{aligned} \mathbf{A}_{\text{in}} &= \frac{B_0 \rho}{2} \hat{\phi} \quad \text{for } \rho < R, \\ \mathbf{A}_{\text{out}} &= \frac{B_0 R^2}{2\rho} \hat{\phi} \quad \text{for } \rho \geq R. \end{aligned} \quad (6)$$

The Aharonov-Bohm phase for a particle of charge e that circles around the solenoid is

$$\begin{aligned} \alpha &= \frac{e}{\hbar} \oint \mathbf{A}_{\text{out}} \cdot d\mathbf{r} = \frac{eB_0 R^2}{2} \hbar \int_0^{2\pi} \frac{1}{\rho} \hat{\phi} \cdot \rho d\phi \hat{\phi} \\ &= \frac{e}{\hbar} B_0 R^2 \pi = \frac{e}{\hbar} \Phi, \end{aligned} \quad (7)$$

where $\Phi = B_0 \pi R^2$ is the magnetic flux of the solenoid. Equation (7) is the standard result for the magnetic Aharonov-Bohm phase.

As in the electric case above, one can gauge away the outside vector potential with the non-single-valued gauge function $\lambda = \frac{B_0 R^2}{2} \phi$, which gives $\nabla\lambda = \frac{B_0 R^2}{2\rho} \hat{\phi}$. Thus, the outside vector potential is transformed away $\mathbf{A}'_{\text{out}} = \mathbf{A}_{\text{out}} - \nabla\lambda = 0$. Following the procedure of Comment [1] the

Aharonov-Bohm phase would be zero: $\alpha = \frac{e}{\hbar} \oint \mathbf{A}'_{\text{out}} \cdot d\mathbf{r} = 0$. However, as in the electric case, along with the gauge transformation of the gauge potentials, one has to transform the wave function using λ , i.e., using (5),

$$\begin{aligned} \psi'_i(\mathbf{r}, t) &= \exp\left(i\frac{e}{\hbar}\lambda\right) \psi_i(\mathbf{r}, t) \rightarrow \exp\left(i\frac{e}{\hbar} \frac{B_0 R^2 2\pi}{2}\right) \psi_i(\mathbf{r}, t) \\ &= \exp\left(i\frac{e}{\hbar} B_0 R^2 \pi\right) \psi_i(\mathbf{r}, t), \end{aligned} \quad (8)$$

where, since the charge e encircles the solenoid, we have taken ϕ to go from 0 to 2π , so that $\lambda = \frac{B_0 R^2}{2} 2\pi$. One can see that the phase factor in (8) matches the phase factor in (7). In the case of (7) the phase factor comes from the line integral of the outside vector gauge potential, whereas for (8) the phase factor comes from the gauge transformation applied to the wave function. Exactly the same thing occurs for the version of the electric Aharonov-Bohm discussed in our work [2]: In gauge 1 the phase factor comes from the time integral of the scalar potential, while in gauge 2 the phase factor comes from the gauge transformation of the wave function.

Another error in the Comment [1] is that the author defines the energy eigenstates as $E_i = -(\partial_t S + eV) = (\partial_t S' + eV')$ [see Eqs. (12)–(14) of the Comment]. Although the author does not define S and S' , presumably $S = -E_i t$ and $S' = -E_i t + e\lambda(t)$. Combining this with $V = 0$ and $V' = V_0 \cos(\Omega t)$ and using $\lambda(t) = \frac{V_0}{\Omega} \sin(\Omega t)$ then yields the same constant result for both $-(\partial_t S + eV)$ and $(\partial_t S' + eV')$, namely, both equal E_i , implying a constant energy eigenstate E_i in both cases. This is wrong since, in general, a time-dependent system will not have a constant energy eigenvalue. A discussion of this fact can be found in Sec. 10.1.2 of [4]. Above Eq. 10.9 the following is stated: “If the Hamiltonian changes with time, then the eigenfunctions and eigenvalues are themselves [time dependent].”²

Reference [4] states that the standard phase factor for the time-dependent case is generalized from the time-independent case as

$$\theta_i = -\frac{1}{\hbar} \int_0^t E_i dt' = -\frac{E_i t}{\hbar} \rightarrow \theta_i(t) = -\frac{1}{\hbar} \int_0^t E_i(t') dt'. \quad (9)$$

The factor θ_i to the left of the arrow is the standard phase factor for a constant energy eigenvalue, while the factor $\theta_i(t)$ is the generalization for the time-dependent situation. The factor $\theta_i(t)$ matches the time-dependent phase factor that can be read off of Eqs. (9) and (10) of our work [2],

$$-\frac{1}{\hbar} \int_0^t [E_i + eV(t')] dt'. \quad (10)$$

The term $E_i(t')$ from the right-hand side of (9) becomes $E_i + eV(t')$ in (10). Our analysis of the time-dependent Schrödinger equation [given in Eqs. (1)–(13) of Ref. [2]]

²The Hamiltonian considered in [2], and written down in (1) above, is clearly time dependent.

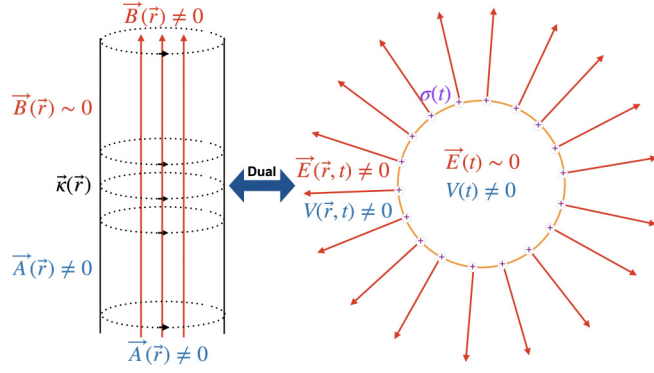


FIG. 1. Comparison of the magnetic Aharonov-Bohm effect with the dual electric Aharonov-Bohm effect. The magnetic Aharonov-Bohm effect has an azimuthal, spatially varying surface current density $\kappa(\mathbf{r})$, as compared to the electric Aharonov-Bohm effect, which has a time-varying surface charge density $\sigma(t)$.

gives a time-dependent phase factor that agrees with the general statements of Ref. [4], while the analysis in the Comment [1] leads to a time-independent energy eigenvalue in conflict with the analysis of time-dependent systems given in [4].

Next the Comment implies there should not be a shift in energy levels since the electric field inside the Faraday sphere is zero. This is incorrect, as can be seen by again looking at the standard magnetic Aharonov-Bohm effect for a quantum ring, i.e., a charged particle confined to move on a ring of radius R_0 . For a quantum ring threaded by a magnetic flux, there is a shift in energy levels despite the particle on the quantum ring being in a completely field-free region. As shown in Sec. 1.5.3 of [5] and Sec. 10.2.3, Eq. 10.74, of [4], a charged particle confined to a ring of radius R_0 , which is threaded by vector potential from an infinite solenoid of radius $R < R_0$ (i.e., a vector potential in cylindrical coordinates of the form $\mathbf{A} = \frac{\Phi}{2\pi\rho}\hat{\phi}$ for $\rho > R$, with $\Phi = B_0\pi R^2$ the magnitude of the magnetic flux of the solenoid), the energy eigenvalues are shifted from $E_n = \frac{\hbar^2 n^2}{2mR_0^2}$ without the magnetic flux to $E_n = \frac{\hbar^2}{2mR_0^2}(n - \frac{e\Phi}{2\pi\hbar})^2$ with the magnetic flux. The parameter m is the mass of the charged particle and n is an integer. For this setup, the fields $\mathbf{B}$ and $\mathbf{E}$ are zero outside the solenoid, i.e., for $\rho > R$. The particle is confined to a ring of radius R_0 in a region where all fields are zero, but the energy levels are shifted: $\frac{\hbar^2 n^2}{2mR^2} \rightarrow \frac{\hbar^2}{2mR^2}(n - \frac{e\Phi}{2\pi\hbar})^2$. This is exactly parallel to the construction of the electric Aharonov-Bohm effect in [2] where there is a nonzero scalar potential both inside [where $V(t) = V_0 \cos(\Omega t)$] and outside [where $V(r, t) = \frac{V_0 r_0}{r} \cos(\Omega t)$, where r_0 is the radius of the Faraday sphere].

Figure 1 shows that the standard magnetic Aharonov-Bohm setup is dual to the electric Aharonov-Bohm setup proposed in our work [2]. (i) The magnetic Aharonov-Bohm setup has a nonzero vector potential both inside and outside the cylinder, while the electric Aharonov-Bohm setup has a nonzero scalar potential both inside and outside the sphere. (ii) The magnetic field for the magnetic Aharonov-Bohm setup is nonzero only inside the cylinder, while the

electric field for the electric Aharonov-Bohm setup is nonzero only outside the sphere. (iii) The magnetic Aharonov-Bohm setup has a spatially varying current density $\kappa(\mathbf{r})$, while the electric Aharonov-Bohm setup has a time-varying surface charge density $\sigma(t)$. This spatial versus time distinction makes sense since the vector potential and magnetic field are the space and space-space components of the four-vector potential $A_\mu = (V, \mathbf{A})$ and Faraday field strength tensor $F_{\mu\nu}$, respectively, while the scalar potential and electric field are the time component of the four-vector potential and the Faraday field strength tensor, respectively.

In Sec. III B of the preceding Comment the author draws a distinction between the gauge-invariant phase $\frac{e}{\hbar} \oint \mathbf{A} \cdot d\mathbf{l}$ with the phase we use in our work [2] $\frac{e}{\hbar} \int V(t)dt$. In fact, as noted in the original work of Aharonov and Bohm [6], the full covariant expression for the Aharonov-Bohm phase is $\frac{e}{\hbar} \oint A_\mu dx^\mu = \frac{e}{\hbar} \oint (V dt - \mathbf{A} \cdot d\mathbf{l})$ (relative to [6], we use V instead of ϕ for the scalar potential, and we have set $c = 1$). This expression is at the bottom of the left-hand column of p. 2 of Ref. [6]. This shows that the phase $\frac{e}{\hbar} \int V(t)dt$ has the same standing as $\frac{e}{\hbar} \oint \mathbf{A} \cdot d\mathbf{l}$, since it is part of the full expression for the Aharonov-Bohm phase.

Finally, the Comment claims the following in the penultimate paragraph (the added emphasis is ours): “A possible way to resolve this puzzle is to assume that there is *one true gauge* in which the potentials accurately represent the [actual physical] state . . . , although it cannot be measured . . .” and “then the adding of the time-varying voltage indeed results in the change of the physical state inside the [shell], which is represented by the potentials in the *true gauge*.” These statements violate the gauge principle. There is no “true gauge.” Any gauge is as good as any other *in principle*. Some gauges may be easier to work with or do calculations in, but there is no such thing as a true gauge.

In summary, we have shown that Gao’s Comment [1] incorrectly analyzes the results of our work [2,3] on several points.

(i) We have shown in the first part of this Reply that the Comment incorrectly applied the gauging procedure to our electric Aharonov-Bohm setup in [2]. Additionally, we have shown that applying the gauging procedure of the Comment [1] to the magnetic Aharonov-Bohm effect would lead to the incorrect conclusion that one could gauge away the experimentally verified magnetic Aharonov-Bohm effect.

(ii) In the Comment Gao finds a constant energy eigenvalue for a time-dependent Hamiltonian [Eqs. (1) and (2) of [2]], which seems to go against the general analysis of the time-dependent Schrödinger equation given in Ref. [4].

(iii) One *can* have an energy shift even for systems that are completely in a field-free region. This can be seen directly in the similar siltation of the magnetic Aharonov-Bohm effect with a quantum ring where there is an energy shift even though the particle on the ring moves in a field-free region. The scalar electric Aharonov-Bohm setup in [2] is also an example of an energy shift despite the system moving in a field-free region.

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- [1] S. Gao, preceding Comment, [Phys. Rev. A **111**, 066203 \(2025\)](#).
- [2] R. Y. Chiao, H. Hart, M. Scheibner, J. Sharping, N. A. Inan, D. A. Singleton, and M. E. Tobar, [Phys. Rev. A **107**, 042209 \(2023\)](#).
- [3] R. Y. Chiao, N. A. Inan, M. Scheibner, J. Sharping, D. A. Singleton, and M. E. Tobar, [Phys. Rev. D **109**, 064073 \(2024\)](#).
- [4] D. J. Griffith, *Introduction to Quantum Mechanics*, 2nd ed. (Pearson Prentice Hall, Upper Saddle River, 2005).
- [5] D. Tong, [arXiv:1606.06687](#).
- [6] Y. Aharonov and D. Bohm, [Phys. Rev. **115**, 485 \(1959\)](#).