

Exact quantum speed limits

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The traditional quantum speed limits are not attainable for many physical processes, as they tend to be loose and fail to determine the exact time taken by quantum systems to evolve. To address this we derive exact quantum speed limits for the unitary dynamics of pure-state quantum systems that outperform the existing quantum speed limits. Using these exact quantum speed limits, we can precisely estimate the evolution time for two- and higher-dimensional quantum systems. Additionally, for both finite- and infinite-dimensional quantum systems, we derive an improved Mandelstam-Tamm bound for pure states and show that this bound always saturates for any unitary generated by self-inverse Hamiltonians. Furthermore, we show that our speed limits establish an upper bound on the quantum computational circuit complexity and yield a tighter quantum Cramér-Rao bound. These results will have a significant impact on our understanding of quantum physics as well as rapidly developing quantum technologies, such as quantum computing, quantum control, and quantum batteries.

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I. INTRODUCTION

Quantum mechanics imposes fundamental limitations on the rate at which quantum systems evolve over time when subjected to external fields or environments, and these limitations are often referred to as quantum speed limits (QSLs). Quantum speed limits provide a lower bound on the time required to transport an initial quantum state to a final quantum state through a physical process. The first QSL was discovered by Mandelstam and Tamm (MT) [1], and provides an operational interpretation of the widely discussed “energy-time uncertainty relation” as it sets the intrinsic timescale for a quantum system undergoing unitary evolution [2]. Several years later, another QSL was discovered by Margolus and Levitin (ML), which is distinct from the MT bound and depends on the mean of the generator of time evolution, rather than its variance [3]. Later on, Levitin and Toffoli demonstrated that a unified bound is achievable under suitable evolution [4]. This unified bound is given by

$$T \geq T_{\text{QSL}} = \max\{T^{\text{MT}}, T^{\text{ML}}\}, \quad (1)$$

where $T^{\text{MT}} = \hbar \Theta_{\text{OT}} / \Delta H$ and $T^{\text{ML}} = \hbar \Theta_{\text{OT}} / \langle H \rangle$ are the MT and ML bounds, respectively. Here, we express these bounds

in terms of the Hilbert-space angle $\Theta_{\text{OT}} \equiv \arccos(|\langle \psi_0 | \psi_T \rangle|)$ between the initial state $|\psi_0\rangle$ and the final state $|\psi_T\rangle$ (instead of twice this angle, e.g., in Ref. [5]), ΔH is the variance in the Hamiltonian, and $\langle H \rangle$ is its expectation value, both determined for the initial state of the quantum system. Subsequent developments have generalized the Mandelstam-Tamm bound to time-dependent Hamiltonians [6].

Recently it has been shown that the evolution speed of a physical system is constrained not only by the energetic cost but also by the topological structure of the underlying dynamics [7]. In Ref. [8], theoretical proposals were presented to measure quantum speed limits. Interestingly, the MT and ML bounds have recently been measured in matter wave interferometry [9]. QSLs were initially studied for the unitary dynamics of quantum systems described by pure or mixed states [1,3,5,10–19]. Since then, they have been extensively studied for various types of dynamics, including many-body dynamics [20,21], non-Hermitian dynamics [22–25], open quantum dynamics [26–33], and arbitrary quantum dynamics [25]. Previously, QSLs have been studied in the Schrödinger picture. However, recently QSLs have been studied in the Heisenberg picture, which establishes an upper bound on the rate at which the expectation values of observables change over time [34–36]. It has also been found that the celebrated Mandelstam-Tamm bound is a special case of the quantum speed limit of observables [34].

The study of QSLs not only enhances our theoretical comprehension of fundamental physics but also carries immense practical significance in the rapidly evolving field of quantum technologies, including but not limited to quantum computation [37], quantum metrology [38,39], quantum

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optimal control [40], quantum communication [41], and quantum batteries [34,42–44]. The original QSLs are based on the distinguishability of the initial and final states of a quantum system. It is important to note that QSLs have also been formulated for several other quantities of interest, such as operator flow [45], generation of quantumness [46], generation of correlations [47,48], and quantum resource production and depletion [49]. QSLs have a significant impact on understanding the limitations of various quantum many-body phenomena, such as information scrambling [50], entanglement growth [51–53], complexity growth [54], and dynamical quantum phase transition [55,56]. The concept of speed limits is not limited to quantum physics; it is also widely studied in classical systems [57,58], thermodynamics [59–61], and quantum gravity [62].

The main goal of studying QSLs is to speed up the quantum process and estimate the minimal time required to implement a given quantum operation. However, a major drawback of the QSLs obtained to date is that they barely saturate, even for unitary dynamics of a closed quantum system described by pure states. Hence, they fail to estimate the exact amount of time required to implement a given quantum operation (or time taken by a given quantum process). The same shortcoming also occurs with both the MT and ML bounds. For example, the famous MT bound only saturates for optimal Hamiltonians that transport the quantum system along a geodesic. Such optimal Hamiltonians have a specific form $H_{\text{opt}} = \hbar\omega(|\psi\rangle\langle\psi| + \text{H.c.})$ [63,64]. It is worth mentioning that all optimal Hamiltonians are self-inverse, but the converse is not always true. The MT bound can be derived in two different ways: (i) using the Robertson-Heisenberg uncertainty relation [1] and (ii) using the geometric approach given by Anandan and Aharonov [5]. Since both approaches use inequalities, the MT bound is also an inequality. Recently, a stronger speed limit [65] has been derived using the stronger uncertainty relation introduced by Maccone and Pati [66], which is tighter than the MT bound. However, this bound is also an inequality, and finding where it saturates is difficult.

In this work, we introduce the notion of exact QSLs that hold for arbitrary unitary processes in two-dimensional as well as higher-dimensional quantum systems described by pure states. We refer to it as the “exact” quantum speed limit because it precisely estimates the time required to implement any unitary process. The key to the derivation of these exact QSLs is the exact uncertainty relation proved by Hall [67]. The exact quantum speed limit for a two-dimensional system depends on the shortest path connecting the initial and final points and on the fluctuation of the nonclassical part of the Hamiltonian. For higher-dimensional systems, the exact quantum speed limit depends on the Wootters length (obtained by integrating the distinguishability metric) and on the fluctuation of the nonclassical part of the Hamiltonian. The exact speed limits coincide with the MT bound for optimal Hamiltonians because they have no classical part. Additionally, we have derived an additional QSL in the form of an inequality that applies to any finite- or infinite-dimensional quantum system undergoing unitary dynamics described by pure states. Notably, this bound is tighter than the well-known MT bound and, interestingly, for any d -dimensional quantum system it saturates for unitaries generated by self-inverse Hamiltonians.

Moreover, we find a new way to characterize the optimal Hamiltonians (which saturate the MT bound), namely, that all optimal Hamiltonians have a vanishing classical part as defined in Ref. [67]; this means they are purely nonclassical in nature.

1. Exact uncertainty relation

The uncertainty relations are considered as a cornerstone of quantum mechanics. Most of the uncertainty relations derived to date are in the form of inequalities, even for pure states, except for the uncertainty relation given by Hall [67], which is known as the exact uncertainty relation. The exact uncertainty relation for any two observables A and B , when the state of a quantum system is described by a pure state $\rho = |\psi\rangle\langle\psi|$, is given by

$$\delta_B A \Delta B^{\text{nc}} = \frac{\hbar}{2}. \quad (2)$$

Here,

$$(\delta_B A)^{-2} = \sum_k \frac{\langle a_k | \frac{i}{\hbar} [B, \rho] | a_k \rangle^2}{\langle a_k | \rho | a_k \rangle} \quad (3)$$

generalizes the quantum Fisher information for observable A (having eigenbasis $\{|a_k\rangle\}$) with regard to infinitesimal changes in the quantum state ρ generated by the operator B [67]. This therefore provides a measure of dispersion, $\delta_B A$, for A . Further, ΔB^{nc} is the uncertainty of the so-called nonclassical part of observable B in a state ρ , which is defined as $B^{\text{nc}} = B - B^{\text{cl}}$, where the classical part is given by

$$B^{\text{cl}} = \sum_k |a_k\rangle\langle a_k| \frac{\langle a_k | \frac{1}{2} \{B, \rho\} | a_k \rangle}{\langle a_k | \rho | a_k \rangle}, \quad (4)$$

and $\{, \}$ denotes the anticommutator. It is important to note that here B^{cl} is not defined in a traditional way (which would simply be the diagonal part of B in some reference basis). The classical part of B commutes with A while its nonclassical part does not. In this sense, Eq. (2) is an exact uncertainty relation for pure states. For a comprehensive understanding of this exact uncertainty relation refer to Refs. [67–69], which provide detailed insights into the mathematical formulation and practical implications of this uncertainty relation.

II. RESULTS

Now we will discuss the main results of this letter, beginning with the derivation of the exact speed limit for arbitrary unitary dynamics of two-dimensional quantum systems, as well as dynamics of d -dimensional quantum systems.

Theorem 1. For a two-dimensional quantum system, the time required to transport a given state $|\psi_0\rangle$ to a target state $|\psi_T\rangle$ via unitary dynamics generated by the Hamiltonian H_t is given by

$$T = \frac{\hbar \Theta_{0T}}{\langle\langle \Delta H_t^{\text{nc}} \rangle\rangle_T} \quad (5)$$

whenever the evolution makes $p_t \equiv \langle\langle \psi_0 | \psi_t \rangle\rangle^2$ a monotonic function. Here Θ_{0T} is the Hilbert-space angle between the

initial and final states, $|\psi_0\rangle$ and $|\psi_T\rangle$, respectively, H_t^{nc} is computed using the basis $\{|\psi_0\rangle, |\psi_0^\perp\rangle\}$, and $\langle X_t \rangle_T \equiv \frac{1}{T} \int_0^T X_t dt$ is the time average of the quantity X_t .

Proof. Let us consider a two-dimensional quantum system which is described by pure states, the time evolution of which is governed by the Hamiltonian H_t . Let us take $A = \Pi_{\psi_0} = |\psi_0\rangle\langle\psi_0|$, $B = H_t$, and $\rho_t = |\psi_t\rangle\langle\psi_t|$ (where $|\psi_t\rangle = U_t|\psi_0\rangle$) in the exact uncertainty relation Eq. (2). Then, using the definition of $\delta_{H_t} \Pi_{\psi_0}$ defined in Eq. (3), we have

$$(\delta_{H_t} \Pi_{\psi_0})^{-1} = -\frac{\frac{d}{dt} |\langle\psi_0|\psi_t\rangle|^2}{\sqrt{|\langle\psi_0|\psi_t\rangle|^2 - |\langle\psi_0|\psi_t\rangle|^4}}, \quad (6)$$

where we take the negative root by assuming p_t is monotonically decreasing. To arrive at Eq. (6), we use the fact that $|\langle\psi_0^\perp|\psi_t\rangle|^2 = 1 - |\langle\psi_0|\psi_t\rangle|^2 = 1 - p_t$, where $\{|\psi_0\rangle, |\psi_0^\perp\rangle\}$ is the relevant orthonormal basis in two-dimensional Hilbert space. Now the exact uncertainty relation (2) becomes (for a detailed proof see the Appendix)

$$\Delta H_t^{\text{nc}} = -\frac{\hbar}{2} \frac{\frac{d}{dt} p_t}{\sqrt{p_t(1-p_t)}}. \quad (7)$$

Integrating both sides of this equation then yields

$$T = \frac{\hbar \Theta_{0T}}{\langle\langle \Delta H_t^{\text{nc}} \rangle\rangle_T}. \quad (8)$$

The exact speed limit for a two-dimensional quantum system holds for both time-dependent and time-independent Hamiltonians as long as the survival probability p_t monotonically decreases during evolution. It is important to note that the survival probability p_t naturally monotonically decreases for all noncyclic unitary dynamics generated by time-independent evolution. Moreover, it is natural to demand that the survival probability p_t is monotonically decreasing for the fastest evolution. Theorem 1 also holds for d -dimensional quantum systems if the time-evolved state belongs to an effective two-dimensional Hilbert space. Since Theorem 1 is applicable for a particular type of unitary evolution of d -dimensional quantum systems, we will now derive the exact speed limit for arbitrary unitary evolution of d -dimensional quantum systems.

Theorem 2. For a d -dimensional quantum system, the time required to transport a given state $|\psi_0\rangle$ to a target state $|\psi_T\rangle$ via unitary dynamics generated by the Hamiltonian H_t is given by

$$T = \frac{\hbar l(\phi_t)|_0^T}{\langle\langle \Delta H_t^{\text{nc}} \rangle\rangle_T}, \quad (9)$$

when H_t^{nc} is computed using the basis $\{|a_i\rangle\}$ (where the initial state $|\psi_0\rangle$ is included in this basis), and writing $|\psi_t\rangle = \sum_{i=0}^{d-1} c_i |a_i\rangle$ we may define the length of the path in a real Hilbert space as $l(\phi_t)|_0^T = \int_0^T \sqrt{\langle\dot{\phi}_t|\dot{\phi}_t\rangle} dt$ traced out by the real vector $|\phi_t\rangle = \sum_{i=0}^{d-1} |c_i| |a_i\rangle$ and $|\dot{\phi}_t\rangle \equiv \frac{d}{dt} |\phi_t\rangle$.

Proof. Let us consider a d -dimensional quantum system which is described by pure states, the time evolution of which is governed by the Hamiltonian H_t . Using similar reasoning to

that used in Theorem 1, we may write

$$(\Delta H_t^{\text{nc}})^2 dt^2 = \frac{\hbar^2}{4} \sum_{i=0}^{d-1} \frac{\langle a_i | d\rho_t | a_i \rangle^2}{\langle a_i | \rho_t | a_i \rangle}. \quad (10)$$

Now, $\langle a_i | d\rho_t | a_i \rangle = d\langle a_i | \rho_t | a_i \rangle$ as $|a_i\rangle$ is time independent and hence

$$(\Delta H_t^{\text{nc}})^2 dt^2 = \frac{\hbar^2}{4} \sum_{i=0}^{d-1} \frac{(d|c_i|^2)^2}{|c_i|^2} = \hbar^2 \sum_{i=0}^{d-1} (d|c_i|)^2, \quad (11)$$

where $c_i = \langle a_i | \psi_t \rangle$. Integrating Eq. (11) yields

$$T = \frac{\hbar}{\langle\langle \Delta H_t^{\text{nc}} \rangle\rangle_T} \int_0^T \sqrt{\sum_{i=0}^{d-1} (d|c_i|)^2}, \quad (12)$$

where $\langle\langle \Delta H_t^{\text{nc}} \rangle\rangle_T$ is the time average of ΔH_t^{nc} .

Defining the vector $|\phi_t\rangle = \sum_{i=0}^{d-1} |c_i| |a_i\rangle$ in a real Hilbert space, its time derivative is given as $|\dot{\phi}_t\rangle = \sum_{i=0}^{d-1} \frac{d|c_i|}{dt} |a_i\rangle$. Hence, we have $\langle\dot{\phi}_t|\dot{\phi}_t\rangle = \sum_{i=0}^{d-1} \left(\frac{d|c_i|}{dt}\right)^2$. Now the length of the curve traced out by vector $|\phi_t\rangle$ in real Hilbert space during the evolution is given by

$$l(\phi_t)|_0^T = \int_0^T \sqrt{\langle\dot{\phi}_t|\dot{\phi}_t\rangle} dt = \int_0^T \sqrt{\sum_{i=0}^{d-1} \left(\frac{d|c_i|}{dt}\right)^2} dt. \quad (13)$$

Using Eqs. (13) and (12), we arrive at Eq. (9). ■

It is worth mentioning that $l(\phi_t)|_0^T$ is exactly equal to the Wootters length S_{PD} on the space of probability distributions (PD). S_{PD} is defined as the integral of the Wootters metric on the space of probability distributions, i.e., $S_{\text{PD}} = \int_\gamma dS_{\text{PD}}$, where $(dS_{\text{PD}})^2 = \frac{1}{4} \sum_{i=0}^{d-1} (dp_i)^2 / p_i = \frac{1}{4} F^{\text{cl}} dt^2$, $p_i = |c_i|^2$, γ is the curve traced out by a quantum system on the space of probability distributions when observed in some basis, and F^{cl} is just the classical Fisher information with respect to that basis $\{|a_i\rangle\}_{i=0}^{d-1}$ [18,70]. If the survival probability p_t is monotonically decreasing, then for two-dimensional quantum systems (i.e., $d = 2$), $l(\phi_t)|_0^T$ reduces to Θ_{0T} provided the path γ does not retrace itself. Therefore Theorem 1 may be considered a special case of Theorem 2.

Theorem 2 accurately determines the time required for arbitrary unitary dynamics of a d -dimensional system. However, calculating $l(\phi_t)|_0^T$ can be challenging compared to Θ_{0T} , which is why we derive another speed limit for both finite- and infinite-dimensional quantum systems. However, it turns out to be an inequality.

Theorem 3. For finite- or infinite-dimensional quantum systems, the time required to transport a given state $|\psi_0\rangle$ to a target state $|\psi_T\rangle$ via unitary dynamics generated by the Hamiltonian H_t is lower bounded by the following inequality:

$$T \geq T_{\text{QSL}}^{\text{IMT}} \equiv \frac{\hbar \Theta_{0T}}{\langle\langle \Delta H_t^{\text{nc}} \rangle\rangle_T} \geq \frac{\hbar \Theta_{0T}}{\langle\langle \Delta H_t \rangle\rangle_T} \equiv T^{\text{MT}} \quad (14)$$

where H_t^{nc} is determined using an orthogonal basis that includes the initial state and T^{MT} is the time-dependent generalized form of the MT bound. [We therefore call the left-hand inequality of Eq. (14) the improved MT bound for the quantum speed limit, $T_{\text{QSL}}^{\text{IMT}}$.]

Proof. Let us consider finite- or infinite-dimensional quantum systems, which are described using pure states. The time evolution of a given quantum system can be described by the Liouville–von Neumann equation, given as

$$\frac{d\rho_t}{dt} = -\frac{i}{\hbar}[H_t, \rho_t], \quad (15)$$

where H_t is the driving Hamiltonian of the quantum system, $\rho_t = |\psi_t\rangle\langle\psi_t|$, and $|\psi_t\rangle = U_t|\psi_0\rangle$ is the time-evolved initial state $|\psi_0\rangle$.

Taking the expectation of this equation in the initial state then yields

$$\frac{d}{dt}|\langle\psi_0|\psi_t\rangle|^2 = -\frac{i}{\hbar}\text{Tr}(\rho_0[H_t, \rho_t]) = \frac{i}{\hbar}\text{Tr}(\rho_t[H_t, \rho_0]). \quad (16)$$

As H_t can be expressed as $H_t = H_t^{\text{nc}} + H_t^{\text{cl}}$, and our orthonormal eigenbasis, $\{|\psi_k\rangle\}_{k=0}^{d-1}$, defining these terms involves the initial state, we see that H_t^{cl} commutes with the initial state, ρ_0 . Hence, with this basis choice, we have $[H_t, \rho_0] = [H_t^{\text{nc}}, \rho_0]$. Next, using Eq. (16) and the Robertson-Heisenberg uncertainty relation [71], $\frac{1}{2}|\langle[A, B]\rangle| \leq \Delta A \Delta B$, we obtain

$$\left| \frac{d}{dt}|\langle\psi_0|\psi_t\rangle|^2 \right| = \left| \frac{i}{\hbar}\text{Tr}(\rho_t[H_t^{\text{nc}}, \rho_0]) \right| \leq \frac{2}{\hbar} \Delta \rho_0 \Delta H_t^{\text{nc}}, \quad (17)$$

where $(\Delta \rho_0)^2 \equiv \text{Tr}(\rho_t \rho_0^2) - [\text{Tr}(\rho_t \rho_0)]^2 = p_t - p_t^2$ and $p_t = \text{Tr}(\rho_t \rho_0) = |\langle\psi_0|\psi_t\rangle|^2$. Consequently, Eq. (17) becomes

$$\Delta H_t^{\text{nc}} dt \geq \frac{\hbar}{2} \frac{|dp_t|}{\sqrt{p_t - p_t^2}}. \quad (18)$$

Integrating Eq. (18) and rearranging it now yields

$$\begin{aligned} T &\geq \frac{\hbar}{2\langle\langle\Delta H_t^{\text{nc}}\rangle\rangle_T} \int_0^T \frac{|dp_t|}{\sqrt{p_t - p_t^2}} \\ &\geq \frac{\hbar}{2\langle\langle\Delta H_t^{\text{nc}}\rangle\rangle_T} \left| \int_0^T \frac{dp_t}{\sqrt{p_t - p_t^2}} \right| \\ &= \frac{\hbar \Theta_{0T}}{\langle\langle\Delta H_t^{\text{nc}}\rangle\rangle_T} \equiv T_{\text{QSL}}^{\text{IMT}}, \end{aligned}$$

where we have used the triangle inequality for the integral to obtain the last inequality.

To finish off the proof, we need only show that $T_{\text{QSL}}^{\text{IMT}} \geq T^{\text{MT}}$, with the time-dependent version of T^{MT} given in the statement of the theorem. Recall that

$$(\Delta H_t)^2 = (\Delta H_t^{\text{nc}})^2 + (\Delta H_t^{\text{cl}})^2 \geq (\Delta H_t^{\text{nc}})^2, \quad (19)$$

hence $\langle\langle\Delta H_t^{\text{nc}}\rangle\rangle_T \leq \langle\langle\Delta H_t\rangle\rangle_T$ and the result follows. ■

This theorem yields an improvement over the usual MT bound by removing the unnecessary "classical" contribution to the uncertainty in the energy. It can be shown that Hamiltonians, H_{opt} , that saturate the MT bound will also saturate the improved MT bound. In this case, we have $T_{\text{QSL}}^{\text{IMT}} = T^{\text{MT}}$ because such Hamiltonians have a vanishing classical part (i.e., $H_{\text{opt}}^{\text{cl}} = 0$). Further, we may identify the quantity $\Delta H_t^{\text{nc}}/\hbar$ as the actual speed of evolution of a pure quantum system in the Hilbert space, improving on Ref. [5]'s claim that the speed of evolution is $\Delta H_t/\hbar$ (with distances, in each case, measured via the Fubini-Study metric [5]).

We note that while all optimal Hamiltonians for the standard MT bound, H_{opt} , are self-inverse, the converse is not true. In general, ΔH_t^{nc} turns out to be time dependent even for time-independent Hamiltonians. However, ΔH_t^{nc} is time independent when the Hamiltonian is optimal. We now show that all self-inverse Hamiltonians also saturate the improved MT bound.

Corollary 1. The bound derived in Theorem 3 saturates for the unitary dynamics generated by time-independent self-inverse Hamiltonians, i.e., for which $H^2 = I$ (where we set $\hbar = 1$ and make the Hamiltonian dimensionless).

See the Appendix for the detailed proof of the above corollary. While proving this corollary we made no assumptions about the dimension of the system or the initial state. Therefore, it equally holds for d -dimensional systems. It is important to mention the significance of self-inverse Hamiltonians because they are highly useful in both single- and multiqubit systems due to their ability to describe a wide range of physical interactions between qubits. Additionally, self-inverse Hamiltonians can be used to implement a variety of single-qubit and two-qubit quantum gates, including rotations, phase shifts, and the controlled-NOT gate. Therefore the bound (14) is highly useful for quantum computing. Moreover, the bound (14) can be saturated for a more general class of Hamiltonians, but finding such Hamiltonians is an extremely difficult task.

It is worth mentioning that the unitary dynamics for which the improved MT bound saturates, Eq. (14), may not be the fastest dynamics. To identify the fastest unitary dynamics, we need to estimate the evolution time and compare it with the time taken by other Hamiltonians to reach the target state from a given initial state. In short, we need to minimize the exact evolution time determined by exact QSLs over all possible Hamiltonians that connect these states, i.e., $T_{\text{opt}} = \min_H T$. Estimating the speed limits presented in this paper requires information about the time-evolved state and generator of the dynamics (i.e., the driving Hamiltonian), which is slightly more involved than calculating the MT bound or ML bound for a time-independent Hamiltonian, as the latter only requires knowledge of the generator of the dynamics. However, for time-dependent Hamiltonians, the MT bound requires knowledge of the time-evolved state which would still not allow us to improve the MT bound itself. Also, all other QSLs for any dynamics introduced to date require the same amount of information as our exact QSLs, but they fail to accurately estimate the exact evolution time. In this sense, our exact quantum speed limits outperform previously obtained results.

It is important to mention that Theorem 3 gives a tighter bound than the standard (original) MT bound. Theorem 3 aims to demonstrate that removing the classical part makes the original MT bound attainable for more general types of unitary dynamics, as shown in Corollary 1. We only refer to "exact speed limits" for bounds given in Theorems 1 and 2, as they are exact. The reason behind referring to Theorems 1 and 2 as exact is because they estimate the evolution time, not because they are derived from exact uncertainty relations. However, Theorem 1 is a special case of Theorem 2.

III. APPLICATIONS

Relation between exact speed limit, parameter estimation, and circuit complexity: If θ is an unknown parameter encoded by the unitary operator $U_\theta = e^{-\frac{iH\theta}{\hbar}}$ in the quantum state, then, without loss of generality, the bound obtained in Theorem 3 can be generalized as follows:

$$\theta \geq \frac{\hbar \Theta_{0\theta}}{\langle\langle \Delta H^{\text{nc}} \rangle\rangle_\theta} \geq \frac{\hbar \Theta_{0\theta}}{\langle\langle \Delta H \rangle\rangle_\theta} = \frac{\hbar \Theta_{0\theta}}{\Delta H}, \quad (20)$$

where the final step relies on the fact that $[H, U_\theta] = 0$.

Similarly, we can obtain a tighter quantum Cramér-Rao bound for unitary parameter estimation for pure probe states (see the Appendix):

$$\Delta\theta \geq \frac{1}{2n\Delta H^{\text{nc}}} \geq \frac{1}{2n\Delta H}, \quad (21)$$

where n is the number of measurements. Here, in the above equation the last inequality provides the original Cramér-Rao bound [18] and the first inequality can be considered as a tighter Cramér-Rao bound.

In quantum computation, one of the challenges is to find an efficient circuit for implementing a unitary operation $U_\theta = e^{-\frac{iH\theta}{\hbar}}$ (where θ is a known parameter), which can be used to solve computational problems like Grover's search algorithm or Shor's factoring algorithm. The minimum number of gates needed to implement a specific unitary transformation, taking the system from a reference state $|\psi_0\rangle$ to a target state $|\psi_\theta\rangle$, is known as its circuit complexity [72–74]. This is now widely used in many other areas including high-energy physics [73,75,76] and condensed-matter physics [77]. It has been found that the geodesic distance $d(I, U_\theta)$ (the shortest distance between the identity operator and the unitary operator of interest) serves as a good measure of circuit complexity. Therefore, the circuit complexity for a given reference state may be defined as [73,76,78,79]

$$\mathcal{C} \equiv d(I, U_\theta) = \arccos |\langle \psi | U_\theta | \psi \rangle| = \Theta_{0\theta}. \quad (22)$$

Interestingly, the circuit complexity reduces to the Hilbert-space angle. Thus, the following relation can be derived using the above bound of Eq. (20):

$$\mathcal{C} \leq \frac{\theta \langle\langle \Delta H^{\text{nc}} \rangle\rangle_\theta}{\hbar} \leq \frac{\theta \Delta H}{\hbar}. \quad (23)$$

The above inequality provides an upper bound on computational complexity. For an optimal Hamiltonian, H_{opt} , this bound saturates, implying that the circuit complexity may be precisely estimated as $\mathcal{C} = \omega \theta$, where ω is the coupling strength of H_{opt} .

IV. CONCLUSIONS

In this work, we have derived the exact speed limit for two-dimensional closed quantum systems as well as higher-dimensional quantum systems described by pure states. The exact quantum speed limits are the consequence of a novel exact uncertainty relation for pure states introduced by Hall [67–69]. Additionally, we have derived a quantum speed limit for both finite- and infinite-dimensional quantum systems and show that if the generator of dynamics is self-inverse the bound saturates. In support of our findings, we also provided

an example of a qubit system (see the Appendix). Moreover, we have identified the key reason why the MT bound does not saturate for most unitary processes: the generators (the driving Hamiltonians) contain some classical contribution. To obtain exact QSLs for most unitary processes, it is necessary to remove this "classicality" from the generators of the dynamics because it commutes with the state and does not contribute to its evolution. The QSLs derived here can be seen as an improved version of the MT bound because they depend on the variance of the nonclassical part of the generator rather than the variance of the generator itself. The exact QSLs are solely dependent on the variance of the nonclassical part of the generator since it is the only component responsible for the system's evolution. The nonclassical part of the generator strongly depends on the initial and time-evolved state of the given quantum system. Our exact quantum speed limits can accurately estimate the time required to perform any given task using quantum devices. Moreover, the exact speed limits can be thought of as an exact time-energy uncertainty relation.

Since the exact QSLs have been derived using fundamentally well-established concepts (noncommutativity of quantum-mechanical observables and uncertainty relations), therefore, we believe that they can be easily verified in future experiments. Our results will significantly enhance our understanding of the time-energy uncertainty relation in quantum physics. Moreover, we believe that these results will have numerous applications in the rapidly developing area of quantum technologies including quantum computation, quantum control, quantum communication, quantum thermal machines, and quantum energy storage devices. The ideas presented in this work open the door to finding exact speed limits for mixed states as well as for open quantum systems [80]. Moreover, this approach can be extended beyond the standard speed limits because it can be helpful to derive the exact or tight speed limits for other quantities such as operator flow, entanglement growth, information scrambling, and the growth of operator complexity.

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APPENDIX

1. Detailed proof of Theorem 1

Let us consider a two-dimensional quantum system which is described by pure states, the time evolution of which is governed by the Hamiltonian H_t . Let us take $A = \Pi_{\psi_0} = |\psi_0\rangle\langle\psi_0|$, $B = H_t$, and $\rho_t = |\psi_t\rangle\langle\psi_t|$ (where $|\psi_t\rangle = U_t|\psi_0\rangle$) in the exact uncertainty relation Eq. (2). Then, using the definition of $\delta_{H_t} \Pi_{\psi_0}$ defined in Eq. (3), we

obtain

$$(\delta_{H_t} \Pi_{\psi_0})^{-2} = \frac{\langle \psi_0 | \frac{i}{\hbar} [H_t, |\psi_t\rangle\langle\psi_t|] | \psi_0 \rangle^2}{\langle \psi_0 | \psi_t \rangle \langle \psi_t | \psi_0 \rangle} + \frac{\langle \psi_0^\perp | \frac{i}{\hbar} [H_t, |\psi_t\rangle\langle\psi_t|] | \psi_0^\perp \rangle^2}{\langle \psi_0^\perp | \psi_t \rangle \langle \psi_t | \psi_0^\perp \rangle}, \quad (\text{A1})$$

where $\{|\psi_0\rangle, |\psi_0^\perp\rangle\}$ is the eigenbasis of A .

Now, by using the Liouville–von Neumann equation $\frac{d\rho_t}{dt} = -\frac{i}{\hbar} [H_t, \rho_t]$ (where $\rho_t = |\psi_t\rangle\langle\psi_t|$), we obtain

$$\begin{aligned} (\delta_{H_t} \Pi_{\psi_0})^{-2} &= \frac{\langle \psi_0 | \frac{d\rho_t}{dt} | \psi_0 \rangle^2}{|\langle \psi_0 | \psi_t \rangle|^2} + \frac{\langle \psi_0^\perp | \frac{d\rho_t}{dt} | \psi_0^\perp \rangle^2}{|\langle \psi_0^\perp | \psi_t \rangle|^2} \\ &= \frac{\left(\frac{d}{dt} |\langle \psi_0 | \psi_t \rangle|^2\right)^2}{|\langle \psi_0 | \psi_t \rangle|^2} + \frac{\left(\frac{d}{dt} |\langle \psi_0^\perp | \psi_t \rangle|^2\right)^2}{|\langle \psi_0^\perp | \psi_t \rangle|^2} \\ &= \frac{\left(\frac{d}{dt} |\langle \psi_0 | \psi_t \rangle|^2\right)^2}{|\langle \psi_0 | \psi_t \rangle|^2} + \frac{\left(\frac{d}{dt} |\langle \psi_0 | \psi_t \rangle|^2\right)^2}{1 - |\langle \psi_0 | \psi_t \rangle|^2}, \\ &= \left(\frac{d}{dt} |\langle \psi_0 | \psi_t \rangle|^2\right)^2 \\ &\quad \times \left[\frac{1}{|\langle \psi_0 | \psi_t \rangle|^2 - |\langle \psi_0 | \psi_t \rangle|^4} \right], \end{aligned} \quad (\text{A2})$$

where we have used the fact that $\langle \psi_0 | \frac{d\rho_t}{dt} | \psi_0 \rangle = \frac{d}{dt} |\langle \psi_0 | \psi_t \rangle|^2$ and $|\langle \psi_0^\perp | \psi_t \rangle|^2 = 1 - |\langle \psi_0 | \psi_t \rangle|^2$.

Next, utilizing the exact uncertainty relation [Eq. (2)], we obtain

$$\begin{aligned} \Delta H_t^{\text{nc}} &= \frac{\hbar}{2} (\delta_{H_t} \Pi_{\psi_0})^{-1} \\ &= \pm \frac{\hbar}{2} \frac{\frac{d}{dt} |\langle \psi_0 | \psi_t \rangle|^2}{\sqrt{|\langle \psi_0 | \psi_t \rangle|^2 - |\langle \psi_0 | \psi_t \rangle|^4}}. \end{aligned} \quad (\text{A3})$$

Since ΔH_t^{nc} is always positive in Eq. (A3), therefore we assume that $p_t = |\langle \psi_0 | \psi_t \rangle|^2$ is monotonically decreasing and hence

$$\Delta H_t^{\text{nc}} = -\frac{\hbar}{2} \frac{\frac{d}{dt} p_t}{\sqrt{p_t - p_t^2}}. \quad (\text{A4})$$

After integrating both sides of Eq. (A4) with respect to time, we obtain

$$T \left(\frac{1}{T} \int_0^T \Delta H_t^{\text{nc}} dt \right) = -\frac{\hbar}{2} \int_0^T \frac{\frac{d}{dt} p_t}{\sqrt{p_t - p_t^2}} dt. \quad (\text{A5})$$

Now defining $\langle \Delta H_t^{\text{nc}} \rangle_T \equiv \frac{1}{T} \int_0^T \Delta H_t^{\text{nc}} dt$ as the time-average uncertainty of the nonclassical part of the driving Hamiltonian in state $|\psi_t\rangle$, we obtain

$$\begin{aligned} T &= -\frac{\hbar}{2 \langle \Delta H_t^{\text{nc}} \rangle_T} 2[\arcsin(\sqrt{p_t})]_0^T \\ &= -\frac{\hbar}{\langle \Delta H_t^{\text{nc}} \rangle_T} \left[\arcsin(|\langle \psi_0 | \psi_T \rangle|) - \frac{\pi}{2} \right] \\ &= \frac{\hbar}{\langle \Delta H_t^{\text{nc}} \rangle_T} \arccos(|\langle \psi_0 | \psi_T \rangle|) \\ &= \frac{\hbar \Theta_{0T}}{\langle \Delta H_t^{\text{nc}} \rangle_T}. \end{aligned} \quad (\text{A6})$$

Example 1. Consider a quantum system consisting of a single qubit with an initial state of $|0\rangle$, the evolution of which is governed by a nonoptimal Hamiltonian $H = \hbar \hat{n} \cdot \vec{\sigma}$, where $\hat{n} = (n_x, n_y, n_z)$ is a unit vector. To determine the quantum speed limit time $T_{\text{QSL}}^{\text{IMT}}$ required to reach the final state $|\psi_T\rangle = -in_z|0\rangle + (n_y - in_x)|1\rangle$ from the given initial state $|\psi_0\rangle$, we require the following quantities: $|\psi_t\rangle = (\cos t - in_z \sin t)|0\rangle + (n_y - in_x) \sin t|1\rangle$, $\Theta_{0T} = \arccos(\sqrt{n_z})$ and $\langle \Delta H_t^{\text{nc}} \rangle_T = \frac{2}{\pi} \arccos(\sqrt{n_z})$. After putting these quantities into Eq. (14), we find that $T_{\text{QSL}}^{\text{IMT}} = \frac{\pi}{2}$, which is equal to the actual evolution time. Hence, the bound (14) is tight and attainable for the given unitary evolution. As we have assumed a general time-independent Hamiltonian for the single qubit in this example, we can conclude that the bound given by Eq. (14) is tight for all unitary processes generated by time-independent Hamiltonians in such systems. This example can also be used to verify the claim of Theorem 1.

Example 2. Consider a two-qubit system with an initial state $|\psi_0\rangle = |00\rangle$, the time evolution of which is governed by a nonoptimal Hamiltonian $H = \sigma_x \otimes I + I \otimes \sigma_x$ (we have taken $\hbar = 1$). To determine the exact time required to reach the final state $|11\rangle$ from the given initial state $|00\rangle$, we need to calculate ΔH_t^{nc} and $l(\phi_t)|_0^T$ (as defined in Theorem 2). The time-evolved state of the quantum system at any time t under the unitary $U_t = e^{-iHt} = (\cos t I - i\sigma_x \sin t) \otimes (\cos t I - i\sigma_x \sin t)$ is given by

$$|\psi_t\rangle = \cos^2 t |00\rangle - i \sin t \cos t (|01\rangle + |10\rangle) - \sin^2 t |11\rangle. \quad (\text{A7})$$

The basis of the two-qubit Hilbert space is given by $|a_0\rangle = |00\rangle$, $|a_1\rangle = |01\rangle$, $|a_2\rangle = |10\rangle$, and $|a_3\rangle = |11\rangle$. It is now straightforward to verify that the classical part of the driving Hamiltonian is zero, resulting in $\Delta H = \Delta H^{\text{nc}} = \sqrt{2}$. The length traced out by the vector $|\phi_t\rangle$ in the real Hilbert space, connecting the initial state $|00\rangle$ and final state $|11\rangle$, is given by

$$l(\phi_t)|_0^T = \int_0^T \sqrt{\sum_{i=0}^3 \left(\frac{d|c_i|}{dt} \right)^2} dt = \frac{\pi}{\sqrt{2}}. \quad (\text{A8})$$

We have used $c_1 = \cos^2 t$, $c_2 = c_3 = i \sin t \cos t$, and $c_4 = \sin^2 t$ to determine $l(\phi_t)|_0^T$. The exact time required to reach the target state $|11\rangle$ from the initial state $|00\rangle$ via the given driving Hamiltonian is

$$T = \frac{l(\phi_t)|_0^T}{\Delta H^{\text{nc}}} = \frac{\pi}{2}.$$

2. Detailed proof of Corollary 1

To show that the bound in Eq. (9) is saturated for self-inverse driving Hamiltonians, we first need to calculate $\langle \Delta H_t^{\text{nc}} \rangle_T = \frac{1}{T} \int_0^T \Delta H^{\text{nc}} dt$. Recall

$$\begin{aligned} (\Delta H^{\text{nc}})^2 &= (\Delta H)^2 - (\Delta H^{\text{cl}})^2 \\ &= \langle \psi_t | H^2 | \psi_t \rangle - \langle \psi_t | (H^{\text{cl}})^2 | \psi_t \rangle, \end{aligned} \quad (\text{A9})$$

where to arrive at the second expression, we use the fact that $\langle \psi_t | H^{\text{cl}} | \psi_t \rangle = \langle \psi_t | H | \psi_t \rangle$ [67]. From the definition of H^{cl} in

Eq. (4) where the eigenbasis is $\{|\psi_k\rangle\}_{k=0}^{d-1}$ we may write

$$\langle\psi_t|(H^{\text{cl}})^2|\psi_t\rangle = \frac{1}{4} \sum_{k=0}^{d-1} \frac{(\langle\psi_k|H|\psi_t\rangle\langle\psi_t|\psi_k\rangle + \text{c.c.})^2}{|\langle\psi_k|\psi_t\rangle|^2}, \quad (\text{A10})$$

where c.c. denotes the complex conjugate. Therefore, Eq. (A9) reduces to

$$(\Delta H^{\text{nc}})^2 = \langle\psi_t|H^2|\psi_t\rangle - \sum_{k=0}^{d-1} \frac{(\langle\psi_k|H|\psi_t\rangle\langle\psi_t|\psi_k\rangle + \text{c.c.})^2}{4|\langle\psi_k|\psi_t\rangle|^2}. \quad (\text{A11})$$

Now, for a self-inverse Hamiltonian, i.e., one where $H^2 = I$, the unitary evolution operator may be expressed as $U_t = e^{-itH} = \cos(t)I - i\sin(t)H$. Here we have taken $\hbar = 1$ and H is dimensionless.

Substituting $|\psi_t\rangle = U_t|\psi_0\rangle$ into Eq. (A11) and after simplification we obtain

$$\Delta H^{\text{nc}} = \frac{\sqrt{1 - \langle\psi_0|H|\psi_0\rangle^2} \cos t}{\sqrt{1 - (1 - \langle\psi_0|H|\psi_0\rangle^2) \sin^2 t}}. \quad (\text{A12})$$

Integrating this expression we find

$$\begin{aligned} \langle\langle\Delta H^{\text{nc}}\rangle\rangle_T &= \frac{1}{T} \int_0^T \Delta H^{\text{nc}} dt \\ &= \frac{1}{T} \arcsin[\sqrt{1 - \langle\psi_0|H|\psi_0\rangle^2} \sin T] \\ &= \frac{1}{T} \arccos[\sqrt{\cos^2 T + \langle\psi_0|H|\psi_0\rangle^2 \sin^2 T}] \\ &= \frac{1}{T} \arccos(|\langle\psi_0|\psi_T\rangle|) \equiv \frac{1}{T} \Theta_{0T}, \end{aligned} \quad (\text{A13})$$

where the identity $\arcsin(x) = \arccos(\sqrt{1-x^2})$ has been used. Substituting Eq. (A13) into the definition of $T_{\text{QSL}}^{\text{IMT}}$ into Eq. (14), we obtain $T_{\text{QSL}}^{\text{IMT}} = T$ and we see that the bound is saturated.

3. Tighter quantum Cramér-Rao bound

The quantum Cramér-Rao bound represents a fundamental limit on the precision with which an unknown parameter θ can be estimated using a quantum probe [18]. For an observable A , the precision can be characterized by the error-propagation formula

$$(\Delta\theta)^2 = \frac{(\Delta A)^2}{n(|\partial_\theta\langle A\rangle|)^2}, \quad (\text{A14})$$

where ΔA denotes the variance of A in state ρ_θ and n is the number of independent measurements.

To establish a tighter lower bound on $\Delta\theta$, we first consider the rate of change of the expectation value of the observable A with respect to the parameter θ . For a unitary parameter encoded by the Hamiltonian H , the probe state after parameter encoding is described as $\rho_\theta = U_\theta \rho_0 U_\theta^\dagger$ with $U_\theta = e^{-iH\theta}$ (taking $\hbar = 1$ and ρ_0 as a pure state). The rate of change of the expectation value of the observable A can be

expressed as

$$\frac{d\langle A\rangle}{d\theta} = \text{tr}\left(\frac{d\rho_\theta}{d\theta}A\right) + \text{tr}\left(\rho_\theta \frac{dA}{d\theta}\right), \quad (\text{A15})$$

where $\langle A\rangle = \text{tr}(\rho_\theta A)$.

Given the unitary evolution, this expression simplifies (in the Heisenberg picture) to

$$\frac{d\langle A\rangle}{d\theta} = i \text{tr}([\rho_0, H]A_\theta) = i \text{tr}([\rho_0, H^{\text{nc}}]A_\theta), \quad (\text{A16})$$

where $A_\theta = U_\theta^\dagger A U_\theta$. The second equality in the above equation was obtained using the fact that H^{cl} commutes with the initial state (see the proof of Theorem 3). Applying the Robertson uncertainty relation [71] to the expression for the rate of change of the expectation value of the observable A , we obtain

$$\left|\frac{d\langle A\rangle}{d\theta}\right| = |\text{tr}(i[\rho_0, H^{\text{nc}}]A_\theta)| \leq 2\Delta A_\theta \Delta H^{\text{nc}}, \quad (\text{A17})$$

where ΔX represents the variance of the observable X in state ρ_0 . Substituting this bound into the error-propagation formula, we obtain a tighter lower bound on $\Delta\theta$:

$$\Delta\theta \geq \frac{1}{2n\Delta H^{\text{nc}}} \geq \frac{1}{2n\Delta H}. \quad (\text{A18})$$

The first inequality in the above relation provides a tighter bound on the precision of the unknown parameter θ . The second inequality represents the original quantum Cramér-Rao bound [18]. Since the original Cramér-Rao bound is only attainable for optimal probe states and the improved bound (the first inequality in the above equation, i.e., we refer to it as a tighter Cramér-Rao bound) can be achieved for more general probe states. It is important to note that the above bound is only applicable to the unitary encoding of parameters in pure states.

4. Tighter speed limits on observables

If we consider time as a parameter in Eq. (A17), we obtain a tighter bound on the rate of change of observables in the unitary dynamics of quantum system with pure states, i.e.,

$$\left|\frac{d\langle A\rangle}{dt}\right| = \frac{1}{\hbar} |\text{tr}(i[\rho_0, H_t^{\text{nc}}]A_t)| \leq \frac{2}{\hbar} \Delta A_t \Delta H_t^{\text{nc}}. \quad (\text{A19})$$

We consider observable A to be positive and normalized, i.e., $\tilde{A} = A/\|A\|_{\text{op}}$, where the operator norm $\|A\|_{\text{op}}$ is the largest eigenvalue of observable A . The eigenvalue of $\tilde{A} = A/\|A\|_{\text{op}}$ lies between 0 and 1. Now, we replace A by $\tilde{A} = A/\|A\|_{\text{op}}$ in Eq. (A19), and then after integrating Eq. (A19) we obtain the following bound on evolution time:

$$T \geq \frac{\hbar}{2\langle\langle H_t^{\text{nc}}\rangle\rangle_T} \left| \arcsin\left\langle \frac{A_T}{\|A_0\|_{\text{op}}} \right\rangle - \arcsin\left\langle \frac{A_0}{\|A_0\|_{\text{op}}} \right\rangle \right|. \quad (\text{A20})$$

To obtain the above equation, we utilize the fact that since $\|\tilde{A}_t\|_{\text{op}} \leq 1$ we have $\langle\tilde{A}_t^2\rangle \leq \langle\tilde{A}_t\rangle$, and we also use the inequality $|\int f(t) dt| \leq \int |f(t)| dt$. It is important to note that the above speed limits on observables [Eqs. (A19) and (A20)]

are tighter than bounds discussed in Refs. [34–36] for unitary evolution of pure states. It is worthwhile mentioning that in the case of a closed quantum battery described by a pure state, the

above bounds [Eqs. (A19) and (A20)] provide a tighter bound on charging power and charging time, respectively, which is discussed in Ref. [34].

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