

Structured light in Möbius-strip-like fiber resonators

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In this paper, we study the propagation of structured light in fiber resonators that in effect are generalizations of the Möbius strip. In particular, we address two examples of such systems: ring resonators based on strongly elliptical twisted two-mode fibers and on bielliptical fibers composed of two consecutive sections of elliptical fibers with different orientations of deformation ellipses. We study formation of whispering gallery modes (WGMs) in such systems. By analyzing orbital Stokes parameters of such WGMs we prove that those modes may form in certain cross-section optical vortices (OVs) with modulus unity topological charges. We demonstrate that the ability of resonators with effective topology of a Möbius strip to possess of vortex WGMs can be used for generation of OV frequency combs from Laguerre-Gaussian modes. We emphasize that to guarantee generation of vortex comb in a Möbius ring resonator parameters should be such as to warrant the existence of vortex WGMs in the resonator. We also study the effect of the spin-orbit interaction (SOI) on generation of OV frequency combs by Möbius-strip-like resonators on base of elliptical twisted fibers. We study the influence of the SOI and show that its presence does not prevent implementation of vortex WGMs in orthogonal polarizations and generation of OV combs in those polarizations.

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I. INTRODUCTION

Recent decades have been marked with a surge of interest in physical structures featuring topological properties of a Möbius strip—a single-surface geometric object. Although it is difficult to establish priority in fabrication of Möbius strips, it seems likely that among natural sciences the first to address such systems was chemistry [1]. In physics, the first demonstration of Möbius strip topology relates to a resonator cavity with embedded copper wire [2]. The idea of implementing Möbius topology by cavities has been used in recent researches [3,4]. Because of technical difficulties, the Möbius strip structure in the optical range has been demonstrated much later using modern technique of direct laser writing lithography [5]. Along with such efforts, there has been a steady interest in synthesis and creation of spatially localized Möbius-like objects [6], on the nanoscale included [7–10].

In general, studying Möbius-like systems has also stimulated progress in other fields of physics, in particular, quantum mechanics [11–13]. It has been shown that a Möbius ring of radiofrequency resonators features rotational symmetry typical to fermions, under which two round-trips are required to restore the initial configuration of the normal mode [14]. Such abstract approach to unveiling the topology of the Möbius strip helped to reveal its presence in a planar system of

coupled resonators [15]. Recently, it has been demonstrated construction of waveguide transmission line networks with embedded Möbius strip and Klein bottle topologies [16]. On the background of efforts to find physical devices featuring Möbius topology it was unpredicted the emergence of a trend to search for such topology in the field's patterns [17–23]. It has been stressed that in optical polarization patterns the presence of Möbius topology could be beneficial for fabrication of microstructures with a special topology. Such technologies might be critical for creating metamaterials with specific optical properties or molecular shape, selective membranes and substrates included. Recently, Möbius-type topology has been revealed in fractional angular momentum phase distribution in photonic-crystal microring resonators [24].

To date, it is well known that the unique properties of Möbius-strip-like systems are closely associated with the topological phase that arises while the system's depicting point evolves in a parametric space [4]. In optical examples of Möbius surfaces, such nondynamical phase usually coincides with the Pancharatnam-Berry phase, which arises because of changes in polarization state [3–6]. That phase equals quantitatively to the solid angle subtended by the trajectory of the state's depicting point on the Poincaré sphere (PS). Meanwhile, the physics of the topological phase could differ from such mechanism since that phase is relevant even for purely scalar fields [11,12]. In this case, it is more correct to refer to the topological Berry phase. Understanding this opens wider prospects in creation of Möbius optical systems, not limited by the frames of originally suggested geometrical example. In this way, it has recently been demonstrated that the Möbius topology can be present in a system without explicit properties of the Möbius strip, namely in a fiber ring laser [25]. The authors of that paper suggested forming the Möbius loop by

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a strongly anisotropic circular fiber. In their method, Möbius topology concerns only the behavior of material anisotropy axis. Rotating the ends of the fiber ring in the cut plane, the authors enabled controlling the Pancharatnam phase, a goal usually achieved by more elaborated techniques [3,4,6]. Later, generation of pulse trains with Möbius mode-locked fiber lasers has been demonstrated [26].

In spite of versatility of examples of optical Möbius strips, they are mostly limited to the study of the influence of topology on the state of polarization [27]. Meanwhile, it is well established that light may possess orbital degrees of freedom, which is characteristic to structured light beams [28]. Evolution of a system in a parametric space may affect such orbital degrees of freedom, resulting in a special Berry phase [29–31]. Upon such evolution, in a structured beam any its composing element that depends on the orbital number $l = \pm 1, \pm 2, \dots$ as $\exp(il\varphi)$ [such fields are also known as optical vortices (OVs) with topological charge (TC) l], where φ is the azimuthal angle, develops an additional topological factor of $\exp(i\Omega)$. Here Ω is the solid angle subtended by the trajectory of the system in some space of parameters. Recent papers have provided examples of using Möbius-like topology for generating OVs by means of a Möbius ring resonator (MRR) based on a strongly elliptical fiber and through designing a compound cavity [32,33]. In particular, it was shown [32] that a Möbius topology leads to emerging of vortex whispering gallery modes (WGMs) in MRRs, which fact facilitates generating OVs and vortex combs by such resonators. In addition, it was noted that generation of OVs is critically sensitive to insertion losses, which occur at the cut plane [25].

However, that paper employed scalar-approximation expressions for spun-fiber modes thus neglecting effects of the spin-orbit interaction (SOI). For a solitary straight twisted fiber one can reliably assess the effect of the SOI on the mode structure of such fiber and determine twist parameter, at which modes can be approximately treated as scalar-approximation circularly polarized fields implied in Ref. [32]. Regrettably, in resonator systems SOI effects may resonantly accumulate in this way making it unreliable to judge applicability of the scalar theory basing only on the expressions for vector modes of twisted fibers. This makes mandatory developing fully vectorial theory that enables studying SOI effects for twisted fibers in resonator configurations.

Somewhat earlier it has been shown that, actually, it is not imperative to fabricate just an object with a strict Möbius strip topology: it is sufficient to emulate it with a polarization-rotating element (etched corner) embedded into the structure of a rectangular waveguide ring resonator [34]. Introducing such element mixes eigenmodes of perfect waveguide and leads to emerging of a new mode structure. This role of Möbius topology has been singled out in Ref. [32]. However, no supporting example for that deduction has been provided, which makes it somewhat ill grounded.

In this connection, the main objective of the present paper is to study propagation of structured light in fiber resonators with topology resembling the one of the Möbius strip. We develop a fully vectorial theory of such resonator systems that allows for the combined action of the SOI and twisted geometry of the resonator. We study formation of WGMs in

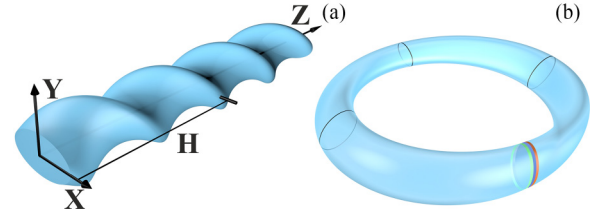


FIG. 1. (a) General view of an elliptical twisted fiber. The large axis of the cross-section ellipse at the output end makes an angle θ with the Z axis of the laboratory frame (X,Y,Z). (b) The Möbius ring resonator, in which the output end of the twisted fiber is rotated with respect to the input end through an angle $\theta = \pi$. Red ellipse shows the cut plane.

MRRs on base of strongly elliptical twisted two-mode fibers and show that at certain conditions such modes may form in a cross section an OV with modulus unity TC. In contrast to our previous approach [32], we study formation of such vortex state not in the cut plane of the MRR, but along the whole Möbius ring using to this end the formalism of the orbital PS. We demonstrate that such MRR being side coupled to the bus fiber can be used for generation of OVs and OV frequency combs. We show that to facilitate generation of vortex comb in a MRR the resonator parameters should be such as to warrant the existence of vortex WGMs in the resonator. We study the effect of the SOI on such generation and show that its presence does not totally prevent implementation of vortex WGMs in orthogonal polarizations and generation of OV combs in those polarizations. We also determine resonator parameters, at which the effect of the SOI can be disregarded.

In addition, we provide an example of how the effective Möbius strip topology may be emulated in a bielliptical ring resonator, that is, a resonator based on a two-mode elliptical fiber, in which a section of elliptical fiber with a different orientation of deformation ellipse is embedded. Specifically, we show that under certain conditions WGMs of such bielliptical resonator may form OVs in full analogy with genuine MRR. By studying trajectories of vortex WGMs of such resonator and the MRR on the orbital PS we emphasize the difference between those two types of resonators with Möbius-strip-like topology. We show that bielliptical fiber ring resonator emulates operation of MRR and can be used for generation of OVs and vortex combs when side coupled to a bus fiber.

II. VORTEX WGMs IN MÖBIUS ELLIPTICAL FIBER RESONATORS

In this section, we consider the first example of a system with a Möbius strip topology, which is produced from a strongly elliptical fiber by twisting it [see Fig. 1(a)] and then connecting its free ends. If the twist angle θ equals π , then one arrives at a MRR in its rigorous definition [see Fig. 1(b)]. Note that this is the only option available for strips. For elliptical fibers with small eccentricity it is possible to fabricate closed rings at any $\theta \neq \pi$ thus arriving at Möbius-like resonators, which in the literature also called Möbius resonators.

We start from a fully vectorial theory that allows for effects of the SOI. First, we address a straight twisted strongly elliptical fiber with a pitch of chiral lattice $H \equiv 2\pi/q$

[Fig. 1(a)]. Unlike in our previous paper on MRRs, we proceed from vector Bloch $l = 1$ modes written in the global Cartesian frame [34]:

$$\begin{aligned} |\psi_1\rangle &= F_1(r) \cos \Phi (\sin \xi e^{-iqz} \vec{c}_+ + \cos \xi e^{iqz} \vec{c}_-), \\ |\psi_2\rangle &= F_1(r) \cos \Phi (\cos \xi e^{-iqz} \vec{c}_+ - \sin \xi e^{iqz} \vec{c}_-), \\ |\psi_3\rangle &= F_1(r) \sin \Phi (\sin \xi e^{-iqz} \vec{c}_+ - \cos \xi e^{iqz} \vec{c}_-), \\ |\psi_4\rangle &= F_1(r) \sin \Phi (\cos \xi e^{-iqz} \vec{c}_+ + \sin \xi e^{iqz} \vec{c}_-), \end{aligned} \quad (1)$$

which are elliptically polarized in the local frame connected with the axes of deformation ellipse. Here F_1 is the radial function, $\Phi = \varphi - qz$, $\vec{c}_{\pm} = \text{col}(1, \pm i)$ are vectors of left (right) circular polarizations, (r, φ, z) are cylindrical polar coordinates and the basis of linear polarizations is implied. Specific form of phase Φ suggests that the areas of maximal intensity of the modes are aligned along the axes of deformation ellipse. Governing parameter ξ is defined as $\tan 2\xi = -|B_1|/4\tilde{\beta}_1 q$, $\pi/4 < \xi < \pi/2$, where $\tilde{\beta}_1$ is the scalar propagation constant of the circular fiber mode. Propagation constants of modes Eq. (1) are $\beta_i = \tilde{\beta}_1 + \Delta\beta_i$, where

$$\begin{aligned} \Delta\beta_{1,2} &\approx \frac{1}{2\tilde{\beta}_1} \left(D_1 \pm \frac{1}{2} \sqrt{16q^2 \tilde{\beta}_1^2 + B_1^2} \right), \\ \Delta\beta_{3,4} &\approx \frac{1}{2\tilde{\beta}_1} \left(-D_1 \pm \frac{1}{2} \sqrt{16q^2 \tilde{\beta}_1^2 + B_1^2} \right). \end{aligned} \quad (2)$$

In these expressions D_1 is induced by ellipticity and B_1 is the SOI constant [34]. Note that Eqs. (1) and (2) are valid only at $D_1 \gg B_1$, $q\tilde{\beta}_1$, that is, where the effect of form birefringence prevails over twist and the SOI ones.

Following the approach outlined in Refs. [25,32], at the first step, one has to study how the section of a fiber transforms arbitrary incoming field. We choose such field as $|\Psi_{in}\rangle = \sum_{k=1}^4 C_i \vec{e}_i$, where basis vectors are as follows:

$$\{\vec{e}_i\} = \{\cos \varphi \vec{c}_+, \sin \varphi \vec{c}_+, \cos \varphi \vec{c}_-, \sin \varphi \vec{c}_-\}. \quad (3)$$

After lengthy but straightforward calculations, one can obtain for the decomposition coefficients of the outgoing field $|\Psi_{out}\rangle = \sum_{k=1}^4 C'_i \vec{e}_i$ the following expression:

$$\vec{C}'(z) = \hat{M}(z) \vec{C}, \quad (4)$$

where elements of $\hat{M}$ are given in the appendix and decomposition coefficients form the vectors in the standard manner.

Consider now an elliptical fiber of length d , whose output end is rotated through an angle θ with respect to the input end. As is evident, in the resulting twisted fiber $q = \theta/d$. Connecting the free ends while no imparting any additional twist we obtain the model of the MRR described in Ref. [32]. In this situation, it is possible to define the mode of such a resonator cavity in the standard manner. If vector $\vec{C}$ of the incoming superposition is transformed in the Möbius ring as $\vec{C} \rightarrow \hat{M}(z=d, \theta) \vec{C}$, then $\vec{C}$ is the eigenvector provided $\hat{M} \vec{C} = \mu \vec{C}$. With necessity, eigenvalues of $\hat{M}$ should be unimodular: $\mu = \exp(i\Gamma)$. Then the WGM of the MRR is obtained where Γ is a multiple of 2π : $\Gamma = 2\pi n$, where $n = 0, \pm 1, \pm 2 \dots$

Generally, such eigenvectors correspond to the fields with nonuniform polarization and no connection can be established

with OV. However, in the limiting case, $D_1 \gg q\tilde{\beta}_1 \gg B_1$, where the modes Eq. (1) are uniformly polarized,

$$\begin{aligned} |\psi_1\rangle &= F_1(r) \cos \Phi \vec{c}_+, & |\psi_2\rangle &= F_1(r) \cos \Phi \vec{c}_-, \\ |\psi_3\rangle &= F_1(r) \sin \Phi \vec{c}_+, & |\psi_4\rangle &= F_1(r) \sin \Phi \vec{c}_-, \end{aligned} \quad (5)$$

and propagation constants are doubly degenerate:

$$\Delta\beta_1 = \Delta\beta_2 = D_1/2\tilde{\beta}_1 \equiv \delta\beta, \quad \Delta\beta_3 = \Delta\beta_4 = -\delta\beta, \quad (6)$$

WGMs can form OVs in certain cross sections of the closed fiber loop. Indeed, in this limiting case $\hat{M}$ turns into a Cartesian sum of identical matrices, each operating in the subspaces of right and left circular polarizations, $\hat{M} = \hat{\mu} \oplus \hat{\mu}$, where

$$\hat{\mu} = e^{i\tilde{\beta}_1 d} \begin{pmatrix} \cos \theta \exp(i\chi) & -\sin \theta \exp(-i\chi) \\ \sin \theta \exp(i\chi) & \cos \theta \exp(-i\chi) \end{pmatrix} \quad (7)$$

and $\chi = \delta\beta d$. Eigenvalues μ in this case read as $\mu_{1,2} = \exp(i\tilde{\beta}_1 d \pm iX)$, where $\cos X = \cos \theta \cos \chi$. Corresponding eigenvectors of $\hat{\mu}$ are as follows:

$$\vec{C}^{\pm} = \text{col}(\sin \theta e^{-i\chi}, \cos \theta e^{i\chi} - e^{\pm iX}) \equiv \text{col}(C_1^{\pm}, C_2^{\pm}).$$

This result has first been obtained within the frameworks of the scalar approximation theory in our previous paper on MRRs [32]. Propagating fields that correspond to such eigenvectors are as follows:

$$|\Psi_{1,2}\rangle = F_1(r) [C_1^{\pm} e^{iz\delta\beta} \cos \Phi + C_2^{\pm} e^{-iz\delta\beta} \sin \Phi] e^{i\tilde{\beta}_1 z} \vec{c}_{\pm}. \quad (8)$$

Note that in Ref. [32] the study of orbital angular momentum (OAM) of the field Eq. (8) was limited to the cut plane $z = 0$, which makes the results presented there less general. Analogous expressions for the second subspace are obtained through the following replacement in Eq. (8): $\vec{c}_+ \rightarrow \vec{c}_-$. The WGM mode is realized if

$$\tilde{\beta}_1 d \pm X = 2\pi n. \quad (9)$$

It is evident that the structured light field described by Eq. (8) does not maintain its structure upon propagation in the ring, which is quite natural since in elliptical fibers the incident fields generally undergo various mode conversions [35]. Such conversions are connected with the changes of OAM, so that WGMs do not possess of well-defined OAM. This circumstance has not been properly emphasized in Ref. [32], so that the following analysis is completely original. Despite the existing field conversion, it is possible to characterize such WGMs by the maximum value of OAM along the loop. To this end, it is convenient to study the evolution of the field Eq. (8) by plotting its trajectory on the orbital PS [36,37]. As can be inferred from that papers, a field $|\Phi\rangle \propto a \cos \varphi + b \sin \varphi$ can be depicted on such PS of unity radius by the point $(S_1, S_2, S_3)/S_0$, where

$$\begin{aligned} 2S_0 &= |a|^2 + |b|^2, & 2S_1 &= |a|^2 - |b|^2, \\ S_2 &= -\text{Re } ab^*, & S_3 &= -\text{Im } ab^*. \end{aligned} \quad (10)$$

The poles of such PS correspond to vortex states with well-defined OAM, so that the poles $(0, 0, \pm 1)$ correspond to OVs with TCs ± 1 , whereas Laguerre-Gaussian (LG) modes

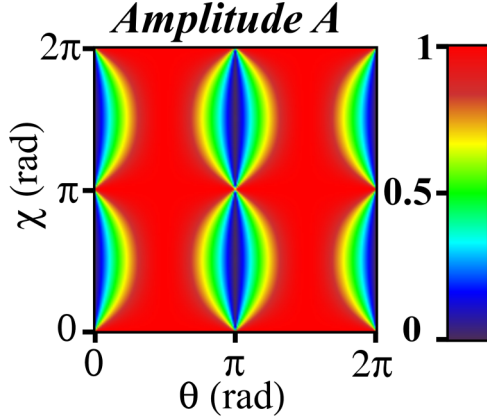


FIG. 2. Maximal value A of reduced third orbital Stokes parameter Eq. (12) vs θ and χ for eigenfields $|\Psi_{1,2}\rangle$ [Eq. (8)].

are located on the equator. For the field Eq. (8) one has

$$2S_0 = |C_1^\pm|^2 + |C_2^\pm|^2, \\ S_3 = -\text{Re}[C_1^\pm (C_2^\pm)^*] \sin 2z\delta\beta - \text{Im}[C_1^\pm (C_2^\pm)^*] \cos 2z\delta\beta. \quad (11)$$

Evidently, maximal value A of the ratio S_3/S_0 is

$$A = 2|C_1^\pm||C_2^\pm|(|C_1^\pm|^2 + |C_2^\pm|^2)^{-1} \quad (12)$$

and achieves its maximum at $|C_1^\pm| = |C_2^\pm|$. In such points OAM of the WGM is ± 1 , so that one can classify such specific mode as a vortex WGM. As can be easily shown using analytical expressions for $C_{1,2}^\pm$ coefficients, A is the same for both eigenfields $|\Psi_{1,2}\rangle$, so that it is sufficient to plot this function only for one of eigenfields Eq. (8). The corresponding plots of A is shown in Fig. 2. Examples of trajectories of vortex WGMs on the orbital PS will be given further in the text. As is seen from this figure, at $\theta = \pi$, that is where the genuine topology of the Möbius strip is implemented in the resonator, no vortex WGM appears in the system in study.

From expressions for $C_i^\pm$ one can readily establish that vortex WGMs are implemented either if $\theta = \pm\pi/2$ or if $\chi = \pi n$. This analytical result is also confirmed by Fig. 2. For the first root one has $\vec{C}^\pm = \text{col}(e^{-i\chi}, \pm i)$, which not necessarily corresponds to OV in the input cross section. Equation (9) in this case reduces to $\beta_1 d = (2n+1)\pi \pm \pi/2$. For the second root one has $\vec{C}^\pm = \text{col}(1, \pm i)$, which exactly corresponds to OV field in the cut pane. Equation (9) in this case reads as $\beta_1 d = 2\pi n \pm \theta$. Note that Ref. [32] provides more restricted conditions for implementation of the vortex WGM. This is primarily connected with the fact that the corresponding treatment in that paper was limited to the cut plane $z = 0$.

Finally, consider the influence of the SOI on implementation of WGMs. In all our examples the SOI corrections become comparable with the twist-induced ones at pitch $H \sim 30$ cm, so that for reasonably small loop lengths d it seems justified neglecting SOI. Nevertheless, for greater optical contrast Δ this pitch value may decrease so that it is helpful to address this point in more details. We start from the notion that in the presence of SOI $\hat{M}$ does not decompose into a Cartesian sum and WGM is not uniformly polarized. Indeed, in this case the eigenvector $\vec{C}$ of $\hat{M}(z = d, \theta)$ determine the

WGM's field in a cross section z through the components of vector $\vec{C}'(z)$ [Eq. (4)] as $\sum_i C'_i \vec{e}_i$. The latter field is inhomogeneously polarized and cannot form any OV of the known types. It might have seemed that in this situation the formation of vortex WGM is impossible. However, one can study the behavior of the WGM's field in orthogonal polarization components separately. For example, the field in the right circular polarization (RCP) is given by $C'_1(z) \cos \varphi + C'_2(z) \sin \varphi$; the field in the left circular polarization (LCP) is determined analogously through coefficients $C'_{3,4}(z)$. For such fields one can study evolution of partial orbital Stokes parameters $S_i^\pm$, where upper sign indicates polarization of the studied field, using the same definitions Eq. (10). Figure 3 shows reduced third Stokes parameters $S_3^\pm/S_0^\pm$ for orthogonal polarizations as functions of cross section's coordinate and wavelength. As is seen, this ratio can reach unity by modulus for both polarizations regardless of the SOI magnitude. Its slow dependence on wavelength guarantees that by adjusting λ one can make optical path $\beta_1 d$ such that a resonance condition on the WGM implementation is satisfied. Naturally, such condition in the presence of SOI may differ from the one given by Eq. (8); however, in the presented examples the increment of $\beta_1 d$ over the wavelength range taken in Fig. 3 is $\sim 3.5\pi$, so that one can select the desired wavelength from this diapason to meet the condition on the vortex WGM implementation.

One should also mention that the presence of the SOI does not lead to any noticeable redistribution of energy between polarizations with the increase of z . The role of SOI is reduced to forming the eigenvector $\vec{C}$ that determines initial distribution of energy between polarizations. Such slow dependence of energy in polarizations on z can be explained using basic facts on the influence of fiber perturbations on the evolution of polarization and orbital angular momenta in fibers. As is well established [35], OAM is mainly affected by form perturbations, whereas spin angular momentum connected with polarization state is rather changed by material anisotropy. The SOI can also change both these quantities, however, at a much smaller rate. In our MRR its ellipticity results in relatively fast changes of OAM (represented by third Stokes parameter). In the absence of material anisotropy, spin angular momentum (along with the beam's polarization state) varies slowly as compared to OAM. This results in observed approximate conservation of energy stored in polarization components. Note that for side-coupled MRRs such isolation breaks and the notion of vortex WGMs in orthogonally polarized components becomes less meaningful and instructive. We discuss this question in Sec. IV.

III. VORTEX WGMs IN BIELLIPTICAL FIBER RING RESONATORS

As has been suggested in Ref. [32] (although with no further argumentation), the main principle of operation of Möbius resonators based on elliptical twisted fibers lies in rotating cosine (sine) elliptical fiber mode so as to make it mismatched with itself at the cut plane. Such mismatching leads to excitation of the complementary sine (cosine) mode at the cut plane, which fact necessitates using the approach outlined in the previous section. One should note that if cross sections at the cut plane are completely matched, which

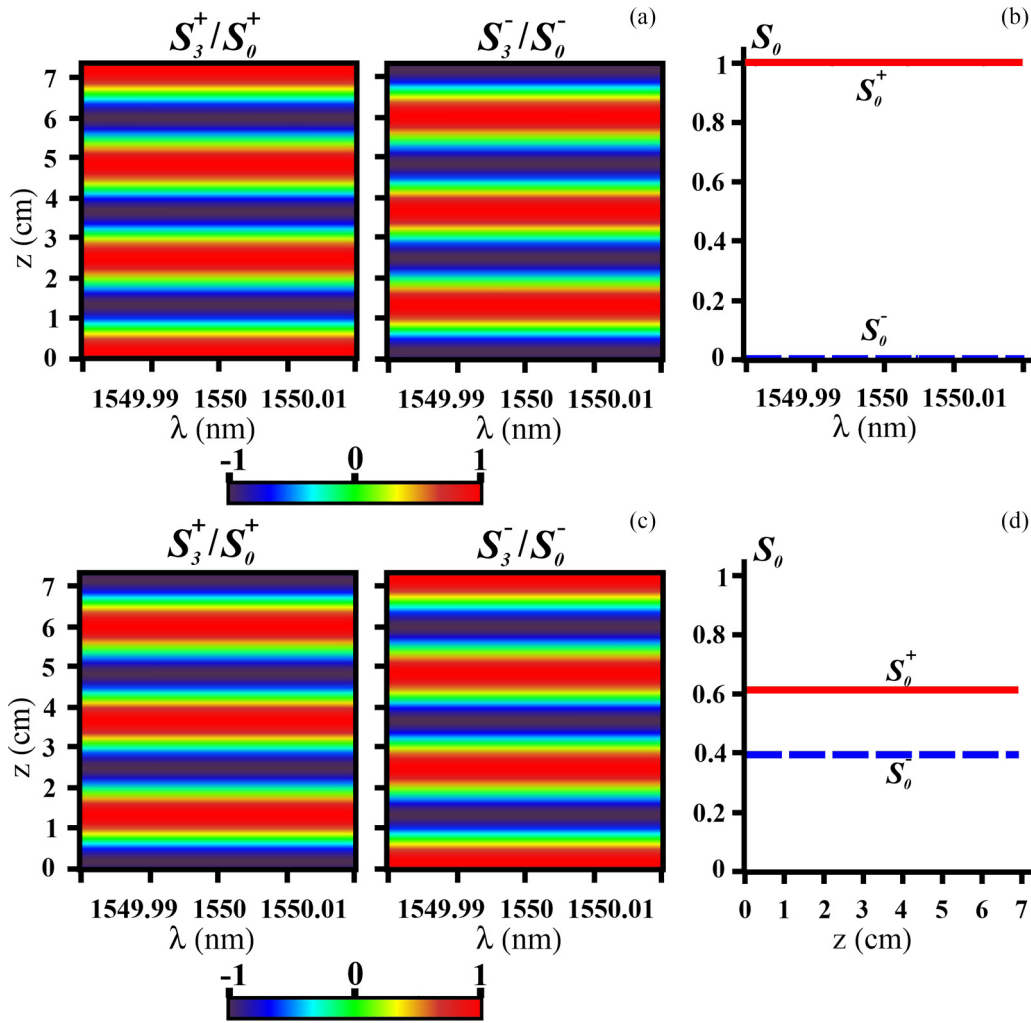


FIG. 3. Reduced third orbital Stokes parameters $S_3^{\pm}/S_0^{\pm}$ for eigenfields corresponding to eigenvectors of $\hat{M}(z=d, \theta)$ in the presence of SOI; z is the coordinate of cross section and λ is wavelength. [(a) and (b)] Twist corrections prevail over SOI ones: twist/SOI = 5; [(c) and (d)] SOI corrections prevail over twist ones: SOI/twist = 2. [(b) and (d)] Total power stored in RCP (red solid lines) and LCP (blue dashed lines) components vs coordinate of cross section.

takes place exactly where a literary Möbius strip topology is implemented, at $\theta = \pi$, then no such complementary orthogonal mode appears at such plane and no vortex WGM arises. However, for such adiabatically twisted elliptical fibers the SOI, although suppressed by ellipticity effects, may dominate over effects of twisting banning in this way the possibility of vortex WGMs realization. In such situation one can benefit from using in this scheme an antiadiabatic twisting that is abrupt changing the direction of the form anisotropy. To this end, in this section we consider a ring formed by an elliptical fiber of length d [see Fig. 4(a)], in which a section of another elliptical fiber of length s is inserted.

By using such a model we benefit from the ideas of Ref. [38], where it has been shown that Möbius strip topology can be emulated with a polarization-rotating element embedded into the structure of a rectangular waveguide ring resonator. We assume that all the fibers in question are two-mode ones. Let the principal axes of deformation ellipses make an angle α , as is shown in Fig. 4(b). It is known [35] that if effects of ellipticity prevail over the ones of the SOI, then $l = 1$ modes are given by four linearly (or circularly)

polarized odd and even LG modes. One should note that since in weakly guiding fibers the SOI is very small, the fiber ellipticity needs not be pronounced: In our examples, to ensure regime of prevailing ellipticity effects, it is sufficient to choose eccentricity greater than 0.4. Since in this approximation orthogonally polarized modes with opposite oddities are nearly

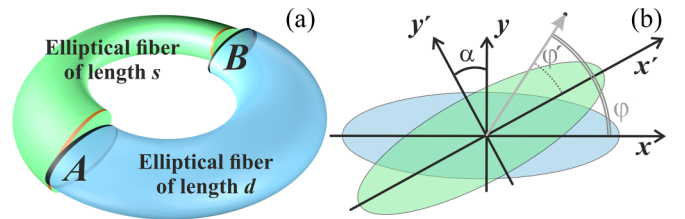


FIG. 4. (a) Scheme of a bielliptical fiber ring resonator formed of two identical elliptical fibers. (b) Mutual orientation of deformation ellipses in the cut planes A and B: Large axes of the cross-section ellipses make an angle α in such planes. The degree of ellipticity is strongly exaggerated.

degenerate, it is sufficient for our needs to use two of such modes with the same polarization and opposite oddity:

$$\vec{\psi}_1 = \vec{c}_+ F_1(r) \cos \varphi, \quad \vec{\psi}_3 = \vec{c}_+ F_1(r) \sin \varphi. \quad (13)$$

Note that instead of circularly polarized modes one can also choose here linearly polarized modes. The propagation constants of these modes are $\beta_{1,3} = \beta_1 \pm \delta\beta$, which complies with Eq. (6). For simplicity, we assume that both the loop and the inset are made of fibers with identical parameters.

To determine WGMs of such resonator consider round-trip evolution of the superposition $C_1 \vec{\psi}_1 + C_2 \vec{\psi}_3$ of eigenmodes Eq. (13) excited at the cross section A [see Fig. 4(a)]. It is convenient to describe such process through the transformation of the vector $\vec{C} = \text{col}(C_1, C_2)$. Then the section AB transforms this initial state as $\vec{C} \rightarrow \exp(i\tilde{\beta}_1 d) \hat{\Gamma}(\chi) \vec{C}$, where $\hat{\Gamma}(\chi) = \text{diag}(e^{i\chi}, e^{-i\chi})$ and diag denotes a diagonal matrix. To establish the action of section BA on the incoming field $a_1 \cos \varphi +$

$a_2 \sin \varphi$ one has first to recast this field in φ' coordinate measured in the rotated frame (x', y') : $a'_1 \cos \varphi' + a'_2 \sin \varphi'$, which is achieved via the following transformation of vector $\vec{a} = \text{col}(a_1, a_2)$: $\vec{a}' = \hat{R}(\alpha) \vec{a}$, where $\hat{R}(\alpha) = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$ is rotation matrix in the space spanned over $\cos \varphi'$, $\sin \varphi'$ orthogonal functions. At the second step, the field vector $\vec{a}'$ should be propagated into $\vec{a}' \rightarrow \exp(i\tilde{\beta}_1 s) \hat{\Gamma}(\psi) \vec{a}'$, where $\psi = s\delta\beta$. Returning to φ coordinate is accomplished through multiplying the outgoing vector by $\hat{R}(-\alpha)$. In this way, initial state vector $\vec{C}$ is transformed while round-trip evolution as

$$\vec{C} \rightarrow \exp(iL\tilde{\beta}_1) \hat{M}(\psi, \alpha) \hat{\Gamma}(\chi) \vec{C} \equiv \hat{\mathcal{M}} \vec{C}, \quad (14)$$

where $\hat{M}(\psi, \alpha) = \hat{R}(-\alpha) \hat{\Gamma}(\psi) \hat{R}(\alpha)$ and $L = d + s$. It is helpful to notice that elliptical inset affects the incoming vector $\vec{C}$ in full analogy with the action of anisotropic slab upon a linearly polarized field as assumed in the Jones formalism [39]. Explicitly,

$$\hat{M} \hat{\Gamma}(\chi) = \begin{pmatrix} (\cos \psi + i \sin \psi \cos 2\alpha) \exp(i\chi) & i \sin \psi \sin 2\alpha \exp(-i\chi) \\ i \sin \psi \sin 2\alpha \exp(i\chi) & (\cos \psi - i \sin \psi \cos 2\alpha) \exp(-i\chi) \end{pmatrix}. \quad (15)$$

The initially excited field is the mode of the bielliptical fiber ring resonator if $\vec{C}$ is the eigenvector of $\hat{\mathcal{M}}$: $\hat{\mathcal{M}} \vec{C} = \bar{\mu} \vec{C}$. Eigenvalues of $\hat{\mathcal{M}}$ are readily found and can be represented as

$$\bar{\mu}_{1,2} = \exp(i\tilde{\beta}_1 L \pm i\Psi), \quad (16)$$

where

$$\cos \Psi = \cos \psi \cos \chi - \sin \psi \sin \chi \cos 2\alpha. \quad (17)$$

Here $|\cos \Psi| \leq 1$, so that there are no spectral gaps. Eigenvectors of $\hat{\mathcal{M}}$ correspond to the following fields at the cut plane:

$$\begin{aligned} |\Psi_{1,2}\rangle &= F_1(r) \{ i e^{-i\chi} \sin \psi \sin 2\alpha \cos \varphi \\ &\quad + [e^{\pm i\Psi} - e^{i\chi} (\cos \psi + i \sin \psi \cos 2\alpha)] \sin \varphi \} \vec{c}_+. \end{aligned} \quad (18)$$

$$\frac{S_3}{S_0} = \sin \psi \sin 2\alpha \frac{\cos \psi \cos 2\chi - \cos(\chi \pm \Phi) - \sin 2\chi \sin \psi \cos 2\alpha}{1 - \cos \psi \cos(\pm \Phi - \chi) - \sin \psi \cos 2\alpha \sin(\pm \Phi - \chi)}. \quad (20)$$

Vortex eigenfield is realized at $z = 0$ only if $S_3/S_0 = \pm 1$. Plots of this ratio at rotation angle $\alpha = \pi/4$ are given in Fig. 5. At other values of rotation angle the picture somewhat changes; however, the regions where $S_3/S_0 = \pm 1$ are observed at any α . Satisfying the condition $S_3/S_0 = \pm 1$ at an arbitrary α needs adjusting a number of parameters, wavelength and ellipticity degree included. Furthermore, the implementation of the vortex WGM imposes additional restrictions on the system's parameters.

It is helpful to illustrate this complexity analytically in a special case $\alpha = \pi/4$, $\psi = \pi/2$. As follows from Eq. (20), in this case one has $S_3/S_0 = -\cos(\chi \pm \Psi)$. Simultaneously, from Eq. (17) it follows $\cos \Psi = 0$ or $\Psi = \pi/2 + \pi n$. Obviously, $S_3/S_0 = \pm 1$ at $\chi \pm \Psi = \pi m$ or $\chi = \pi/2 + \pi m$.

These fields form WGMs if the round-trip phase increment is a multiple of 2π , which is realized at the resonance lengths L_n :

$$\tilde{\beta}_1 L_n \pm \Psi = 2\pi n. \quad (19)$$

As in the previous example, the fields Eq. (18) do not maintain their composition upon evolution in the loop and in some cross sections can form OV's. Analogously, such fields should be called vortex eigenfields. Let us demonstrate that such bielliptical fiber ring resonators can possess of such fields. To simplify our task we study only the field's composition at $z = 0$. The ratio S_3/S_0 for the field Eq. (18) in this cross section is readily found to be

Finally, these conditions should be augmented with WGM implementation condition Eq. (19), so that one arrives at the following conditions of the vortex WGM implementation:

$$\begin{aligned} s\delta\beta &= \pi/2 + \pi n, \\ d\delta\beta &= \pi/2 + \pi k, \\ \tilde{\beta}_1 L &= \pi/2 + \pi m, \end{aligned} \quad (21)$$

where $n, k, m = 0, \pm 1, \pm 2, \dots$

To highlight the difference between vortex WGMs in elliptical fiber MRRs and bielliptical fiber ring resonators and to elucidate the meaning of conditions Eq. (21), it is helpful to plot the evolutions of the states excited at the cut planes of these resonators in terms of their trajectories on the orbital PS. Consider first the WGM's evolution in the bielliptical fiber

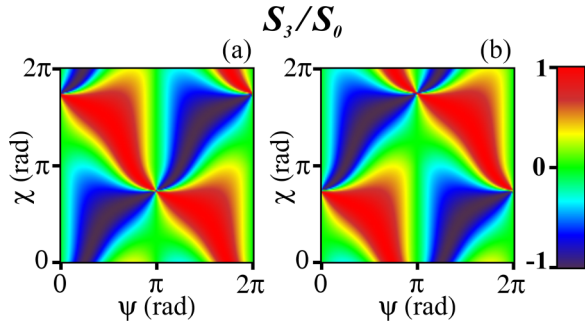


FIG. 5. Reduced third orbital Stokes parameter Eq. (20) vs ψ and χ for eigenfields $|\Psi_1\rangle$ (left) and $|\Psi_2\rangle$ (right) [Eq. (18)]. Rotation angle $\alpha = \pi/4$.

ring resonator. If $\alpha = \pi/4$ and $\psi = \delta\beta d = \pi/2$, then in the starting point A the field is the OV with TC +1. The evolution of such vortex state in the section AB is described on the PS by the curve $S_1 = 0$, $S_2 = -\sin 2\delta\beta z$, $S_3 = \cos 2\delta\beta z$ (note that $S_0 = 1$), where $0 \leq z \leq d$. This curve ends at the point where $z = d$, that is, $(0, 0, -1)$. Here we have allowed for the second condition in Eq. (21): $2\delta\beta d = \pi$.

Similarly, one can determine evolution in segment BA, $S_1 = -\sin 2\delta\beta z$, $S_2 = 0$, $S_3 = \cos 2\delta\beta z$, where $0 \leq z \leq s$. The end point of this curve is $(0, 0, 1)$, which is due to the first condition in Eq. (21): $2\delta\beta s = \pi$. Such end point is exactly the starting point for the total evolution curve. This path on the PS is shown in Figs. 6(a) and 6(b). This example shows that the propagating OV with TC 1 continuously passes over variety of states, including negatively charged OV. In this way, one can speak of a vortex WGM only in a narrow sense, as of a state, whose self-replicating trajectory crosses at least one of the poles on the PS. Evidently, such definition of a mode drastically differs from the one for circular fiber modes. In a wider context, if we understand a mode as the eigenvector of some eigenstate equation, then WGMs of multimode fiber MRRs are similar to circular fiber modes. Geometrical interpretation of WGM state evolution also helps to elucidate the role of the first two conditions [Eq. (21)]: They have to be fulfilled in order to make the trajectory closed. If such conditions are not satisfied, then the trajectory is not closed. An example of such curve is given in Figs. 6(c) and 6(d). One should note that depicting point in segments AB and BA evolves along circles, which is seen from Figs. 6(b) and 6(d).

In the elliptical fiber MRR vortex WGM is implemented either if $\theta = \pm\pi/2$ or if $\chi = \pi n$. In the first case for the initial state vector $\vec{C}^- = \text{col}(e^{-i\chi}, -i)$ the orbital Stokes parameters evolve as

$$S_1 = \sin \frac{\pi z}{d} \sin[\delta\beta(2z - d)], \quad S_2 = \cos \frac{\pi z}{d} \sin[\delta\beta(2z - d)], \\ S_3 = \cos[\delta\beta(2z - d)].$$

Note that here $S_0 = 1$. In addition, since $\delta\beta d = \chi$ the starting point is $(0, -\sin \chi, \cos \chi)$. This trajectory (see Fig. 7) is always closed and includes the north pole. In the second case for the initial state vector $\vec{C}^- = \text{col}(1, -i)$ the Stokes parameters read as

$$S_1 = \sin \frac{2\theta z}{d} \sin 2\delta\beta z, \quad S_2 = -\cos \frac{2\theta z}{d} \sin 2\delta\beta z, \\ S_3 = -\cos 2\delta\beta z.$$

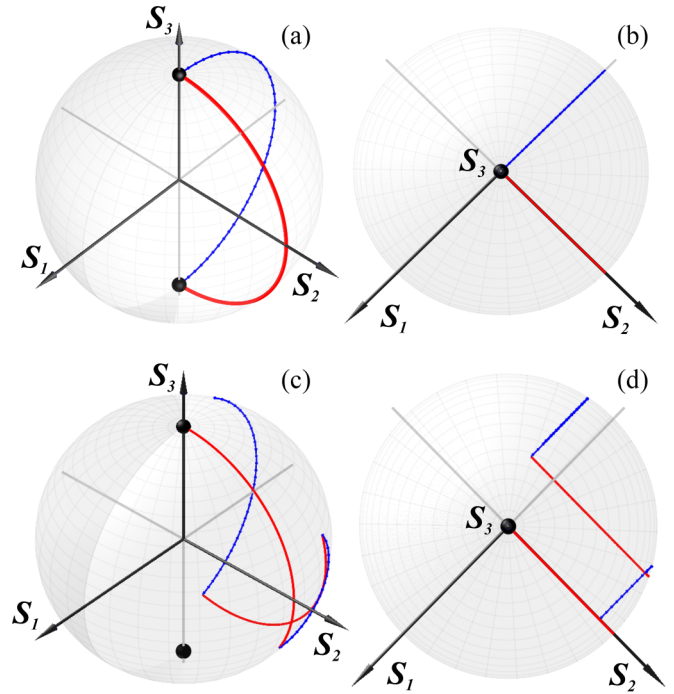


FIG. 6. Evolution of vortex eigenfields in a bielliptical ring resonator as mapped on the orbital PS; [(a) and (c)] axonometric projections and [(b) and (d)] top views. [(a) and (b)] WGM implementation condition Eq. (21) is satisfied, $d\delta\beta = \pi/2$. [(c) and (d)] Condition Eq. (21) is not satisfied, $d\delta\beta = \pi/3$; two round trips are shown. In both cases the starting point is the north pole, $s = d$. Red curves show evolution in the AB segment and blue curves show evolution in the BA segment. Note that in both segments evolution of the point that depicts the field's state takes place along circles.

This trajectory is also closed. In addition, it always includes both the north and the south poles. In this way, MRRs based on twisted elliptical fibers prove to be tolerable with respect to variations of the value of displacement angle θ .

As a concluding remark to this section, we emphasize that emulation of Möbius-like topology in bielliptical fiber ring resonators does not rely on any additional physical mechanism other than already presented in MRRs based on twisted elliptical fibers. In both these types of resonators a unique WGM that consists of simultaneously two eigenmodes of straight fibers (strongly elliptical and twisted elliptical, cor-

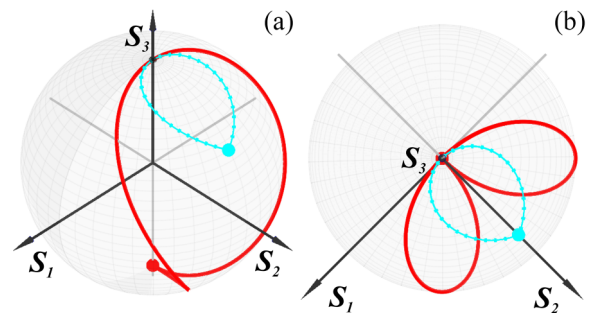


FIG. 7. Evolution of vortex eigenfields in elliptical fiber MRR as mapped on the orbital PS; (a) axonometric projection and (b) top view. Red curves, $d\delta\beta = \pi$; cyan curves, $d\delta\beta = 0.4\pi$.

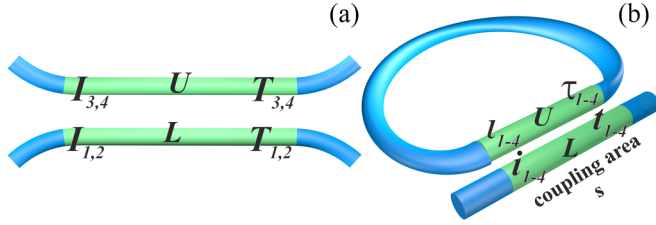


FIG. 8. The scheme of a MRR side coupled to the bus fiber. (a) Principal scheme of operation of the directional coupler on the incoming field vector $\vec{I} = \text{col}(I_1, I_2, I_3, I_4)$; it is shown correspondence between input and output coefficients and the coupler's ends. (b) Correspondence between input and output coefficients of the coupler and incoming and outgoing fields in the Möbius elliptical fiber ring as assumed in Eqs. (25) and (26).

respondingly) appears due to a mismatch of orientations of deformation ellipses in the cut plane. Eventually, just this property is responsible for the appearance of vortex WGMs and the ability of these systems to generate OV. All the differences between these resonators (including WGM behavior on the PS) seem irrelevant to their ability to generate OV. An antiadiabatic nature of twisting in bielliptical fiber ring resonators seems beneficial to their robustness with respect to SOI effects; however, this does not alter the mechanism of WGMs formation.

IV. GENERATION OF OV IN MÖBIUS FIBER RESONATORS

The presence of resonance WGMs in the considered systems suggests using them for generating OV. We assume the simplest scheme, in which MRR is side coupled to the bus fiber. Main elements of all such schemes are directional couplers [see Fig. 8(a)], which, in the scalar approximation, couple OV only with the same polarization. In addition, the strength of coupling does not depend on polarization's sign [40], so that the action of the directional coupler on the incoming field is the same for all polarizations. In particular, for the incoming field $\Psi_{in} = I_1|l, L\rangle + I_2|-l, L\rangle + I_3|l, U\rangle + I_4|-l, U\rangle$, where $|\pm l, L/U\rangle$ are OV of the same polarization, which sign is irrelevant, and of TCs $\pm l$ localized on the lower (upper) arms of the coupler, the action of the coupler can be represented by the following transformation of the incoming vector $\vec{I} = \text{col}(I_1, I_2, I_3, I_4)$:

$$\vec{T} = \exp(i\Lambda\tilde{\beta}_l) \begin{pmatrix} a & b & c & e \\ b & a & e & c \\ c & e & a & b \\ e & c & b & a \end{pmatrix} \vec{I} \equiv \hat{T}\vec{I}. \quad (22)$$

Here $\vec{T} = \text{col}(T_1, T_2, T_3, T_4)$ describes the outgoing field $\Psi_{out} = T_1|l, L\rangle + T_2|-l, L\rangle + T_3|l, U\rangle + T_4|-l, U\rangle$ and Λ is the length of the coupling area. Matrix elements are $a = \cos \kappa_1 \cos \kappa_2$, $b = -\sin \kappa_1 \sin \kappa_2$, $c = i \sin \kappa_1 \cos \kappa_2$, and $e = i \cos \kappa_1 \sin \kappa_2$, where κ_1 and κ_2 describe processes of tunneling of OV without and with the inversion of their TCs. The correspondence between input and output coefficients and the coupler's ends is shown in Fig. 8(a). Note that $\hat{T}$ is the same for both circular polarizations. Such formalism is convenient

for the description of OV generation in side-coupled MRRs and their analogs.

A. Generation of OV in Möbius elliptical fiber ring resonators

Let us consider first transmission properties of MRR based on an elliptical fiber, which is coupled to the bus fiber. It is convenient here to recast the transfer matrix $\hat{T}$ Eq. (22) using the basis of even $|EV\rangle$ and odd $|OD\rangle$ LG modes Eq. (3):

$$|EV, \pm\rangle = \cos \varphi \tilde{c}_{\pm}, \quad |OD, \pm\rangle = \sin \varphi \tilde{c}_{\pm}. \quad (23)$$

In this basis that matrix factorizes in the same manner for both polarizations and becomes

$$\hat{T} = e^{i\Lambda\tilde{\beta}_l} \begin{pmatrix} \cos \eta_1 & i \sin \eta_1 \\ i \sin \eta_1 & \cos \eta_1 \end{pmatrix} \oplus e^{i\Lambda\tilde{\beta}_l} \begin{pmatrix} \cos \eta_2 & i \sin \eta_2 \\ i \sin \eta_2 & \cos \eta_2 \end{pmatrix} \equiv \hat{\tau}_1 \oplus \hat{\tau}_2, \quad (24)$$

where $\eta_{1,2} = \kappa_1 \pm \kappa_2$. Such factorization conveys the fact that in the scalar approximation even and odd LG modes evolve in directional couplers independently of each other. Note that in the basis Eq. (23) decomposition of both the incoming and outgoing fields is assumed to be

$$\Psi_{\pm} = x_{1\pm}|EV, \pm, L\rangle + x_{2\pm}|OD, \pm, L\rangle + x_{3\pm}|EV, \pm, U\rangle + x_{4\pm}|OD, \pm, U\rangle$$

and is analogous for both polarizations.

Let the incoming and outgoing fields in the bus fiber be

$$|i\rangle = i_1|EV, +\rangle + i_2|OD, +\rangle + i_3|EV, -\rangle + i_4|OD, -\rangle, \\ |t\rangle = t_1|EV, +\rangle + t_2|OD, +\rangle + t_3|EV, -\rangle + t_4|OD, -\rangle. \quad (25)$$

In the same manner we represent the fields at the upper ends of the coupler as

$$|\iota\rangle = \iota_1|EV, +\rangle + \iota_2|OD, +\rangle + \iota_3|EV, -\rangle + \iota_4|OD, -\rangle, \\ |\tau\rangle = \tau_1|EV, +\rangle + \tau_2|OD, +\rangle + \tau_3|EV, -\rangle + \tau_4|OD, -\rangle. \quad (26)$$

The assignment of input and output coefficients to the coupler's ends is shown in Fig. 8(b). Then these decomposition coefficients are connected through the following relations:

$$\vec{\tau} = \hat{M}(d)\vec{\tau}, \quad \begin{pmatrix} t_1 \\ \tau_1 \end{pmatrix} = \hat{\tau}_1 \begin{pmatrix} i_1 \\ \tau_1 \end{pmatrix}, \quad \begin{pmatrix} t_2 \\ \tau_2 \end{pmatrix} = \hat{\tau}_2 \begin{pmatrix} i_2 \\ \tau_2 \end{pmatrix}, \\ \begin{pmatrix} t_3 \\ \tau_3 \end{pmatrix} = \hat{\tau}_1 \begin{pmatrix} i_3 \\ \tau_3 \end{pmatrix}, \quad \begin{pmatrix} t_4 \\ \tau_4 \end{pmatrix} = \hat{\tau}_2 \begin{pmatrix} i_4 \\ \tau_4 \end{pmatrix}, \quad (27)$$

where $\vec{\tau} = \text{col}(\tau_1, \tau_2, \tau_3, \tau_4)$, $\vec{\iota} = \text{col}(\iota_1, \iota_2, \iota_3, \iota_4)$. Here we treat the incoming field $\vec{i}$ as set in the experiment. The amplitudes $t_{1,3}$ allow one to calculate the amplitudes $T_{1,2}$ of outgoing RCP OV as $T_{1,2} = (t_1 \pm t_3)/\sqrt{2}$ and, consequently, their powers $P_{1,2} = |T_{1,2}|^2$. For outgoing LCP OV $|\pm l, -\rangle$ corresponding powers are $P_{3,4} = |t_2 \pm t_4|^2/2$. One should emphasize that the above given theory is fully vectorial and allows for the SOI in the Möbius ring, which has not been the subject of our previous work [32].

The presence of the SOI that couples polarization and orbital degrees of freedom may result in redistribution of energy between orthogonal polarizations, in this way making

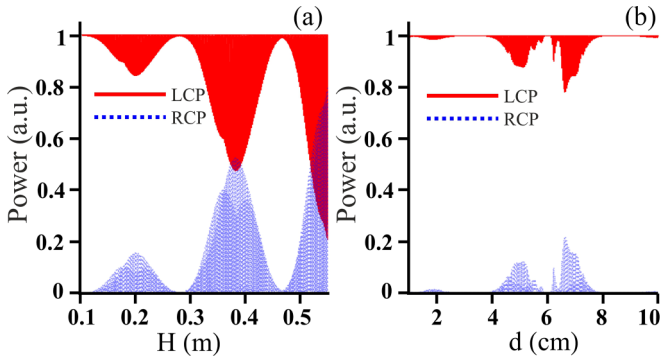


FIG. 9. Total power stored in LG modes with the left (red solid curves) and the right (blue dotted curves) circular polarization. Incoming field is even LCP LG mode. Parameters: $\Lambda = 2$ cm; (a) $\theta = \pi/4$, $d = \theta/q$ and (b) $H = 12$ cm.

the scalar approximation theory used in Ref. [32] inapplicable. To establish validity of such approximation we consider total powers of RCP and LCP components of the outgoing field if the incoming field in the bus fiber is even LCP LG mode. By fixing the total twist angle $\theta = \text{const}$ and varying the loop length d (or pitch H) one can scan the whole range of twist-induced term $q\tilde{\beta}_1$. Figure 9(a) shows that for small H (large q) redistribution of energy stored in the initial polarization (LCP) into the opposite one (RCP) is negligible. As H increases (and q decreases) the magnitudes of twist-induced $q\tilde{\beta}_1$ and SOI term B_1 become comparable and the scalar-approximation theory becomes less reliable. This is manifested through essential redistribution of the energy of the incident beam into the opposite polarization. In this way, to guarantee the validity of the scalar approximation one should use small-length loops (less than 2 cm in our example). Otherwise, generation of OV with the given polarization would be less effective.

In this connection, a question arises: Does the smallness of H warrant the absence of energy redistribution between orthogonal polarizations? To answer it, in Fig. 9(b) we have plotted powers of RCP and LCP components of the outgoing field versus loop length d for the previous example, in which the pitch value is preserved. Note that this can be achieved only at a cost of increasing θ . As is seen, starting from some loop length, energy starts to redistribute between polarizations. As d increases, the power curves behave similarly to Fig. 8(a). In this way, for such a complicated system effects of the SOI can be essential for transmission characteristics even though they are negligible in the structure of modes. Summarizing, the scalar approximation is reliable for relatively small values of pitches and loop length.

If the twisting effects prevail over the SOI, then MRR is suitable for generating OV frequency combs. That problem has gained topicality in connection with application of structured light in optical communications. Recently, it has been suggested generation of such frequency comb with OVs of different TCs on transmission peaks using integrated nonlinear microresonators [41]. Frequency combs with OVs of the same TC on all transmission peaks can be realized using parity-time-symmetric ring resonators [42] or add-drop filters on coupled ring optical waveguides (CROWs) [43,44]. Figure 10(a) demonstrates generation of vortex frequency

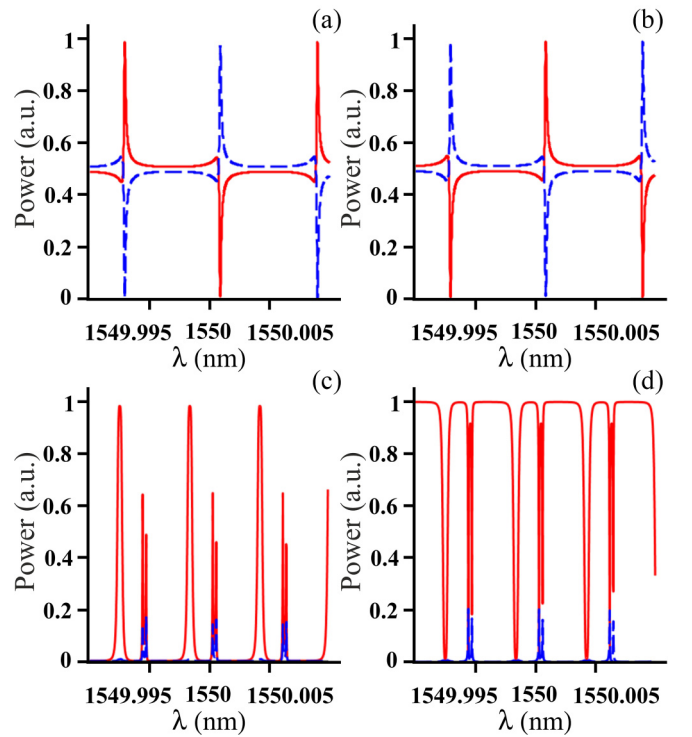


FIG. 10. Transmission of LG modes through MRR side coupled to a bus fiber in a strong twisting limit $q\tilde{\beta}_1 \gg B_1$. (a) Power stored in OVs with TCs $+1$ (red solid curves) and -1 (blue dashed curves) if the incident beam is LCP even LG mode. (b) Analogous curves for excitation with LCP odd LG mode. Parameters: $d = 7.35$ cm, $\theta = 0.59$ rad, and $\Lambda = 2.6$ cm. (c) RCP LG mode frequency comb generated from incident LCP even LG mode; red solid curve, power stored in the even LG mode; blue dashed curve, power of the odd LG mode. Parameters: $d = 11.25$ cm, $\theta = 0.59$ rad, and $\Lambda = 2.3$ cm; twisting effects are comparable with the SOI ones. (d) Transmission curve for incident LCP even LG mode; stop band filter implementation. Note that transmitted modes also have LCP.

comb from input LCP even LG mode. The distance between peaks of the generated comb is 2 GHz, whereas the width of the peaks is approximately 130 MHz. In the presented example $D_1 \gg q\tilde{\beta}_1 \gg B_1$ and MRR does not change polarization of the incoming field. If one inverts the oddity of the incoming LG mode, then in the transmitted field the curves for positively and negatively topologically charged OVs change places. This property is illustrated in Fig. 10(b).

If the twisting effects are comparable with the SOI ones ($D_1 \gg q\tilde{\beta}_1 \cong B_1$), then MRRs may exhibit some other features, which might be of interest for applications. As has been argued in Ref. [45], to increase capacity of information transmission channels that use higher-order modes one may not limit oneself with using OVs as the only information carriers. Signal multiplexing could be realized using even and odd LG modes. In this scenario, a property of MRRs to form LG frequency combs from the incoming LG mode could be useful. Figure 10(c) shows a frequency comb formed by the MRR from an incident LCP even LG mode. Note that the resulting comb is composed of RCP even LG modes, so that the comb's polarization is opposite to the one of the incident beam. Such polarization inversion takes place owing to the

presence of the SOI comparable with the twist effects. Finally, MRRs can operate as stopband filters for LG mode signals. Figure 10(d) shows an example of such transmission curve for incident LCP even LG mode. Note that unlike in Fig. 10(c), polarization of transmitted modes coincides with polarization of the incident field.

An important feature of resonator-based devices is robustness of their performances with respect to fabrication errors, external perturbations and the like. It is expected that the position of transmission curves should be sensitive to such deviations; however, what is to be verified is the robustness of general pattern of transmission curves. Consider first the influence of the SOI on frequency combs generation. In the majority of our examples we deal with the case where twist-induced corrections prevail over SOI ones (twist/SOI = 5). In this case, the transmission curve Fig. 11(a) almost coincides with the one given in Fig. 10(a). Simultaneously, in the LCP component a pronounced frequency comb is formed, however, with regrettably small power [see Fig. 11(b)]. For relatively large SOI (SOI/twist = 2) frequency comb with reasonably reduced power appears in one of polarizations, whereas in the other it is absent [see Figs. 11(c) and 11(d)]. Note that in this case the powers in polarizations are not conserved and, as is seen from Fig. 11(e), at certain wavelengths exhibit fast dynamics. Note that here we rely on the optimal displacement angle value of $\theta = \pi/2$. Meanwhile, this value was analytically estimated for no-SOI case, so that for large SOI such choice might not be quite reliable. In Fig. 11(f) we present the transmission curve for $\theta = 43^\circ$, where peaks for OV with TC -1 are largely suppressed. Summarizing, although the SOI deteriorates scalar-approximation transmission curves, they prove to be tolerant to the presence of spin-orbit coupling and even at large SOI constants one can achieve generation of vortex frequency combs by adjusting resonator's parameters.

Variations of ellipticity degree in the cross section may also affect transmission characteristics of MRRs. To assess the influence of such sort of imperfections we plotted transmission curves for three ellipticity constants: D_1 , $2D_1$, and $3D_1$, respectively. As is evident from Fig. 12(a), triple increase in ellipticity for the whole resonator results only in a 10% power loss in the generated comb with no detectable shift in its positioning. We believe that a random change of ellipticity degree would result in lesser effects. Another source of fabrication imperfection is connected with the presence of insertion losses at the cut plane. We assess the influence of losses integrally, that is, by the following renormalization of the propagation constant: $\tilde{\beta}_1 \rightarrow \tilde{\beta}_1 + i\gamma$, γ being attenuation coefficient. As follows from Fig. 12(b), at $\gamma = 0.1 \text{ m}^{-1}$, which for the given d roughly corresponds to 1% loss at the cut plane, frequency comb practically vanishes. This seems to be a rather low tolerance to insertion losses, especially allowing for the fact that we neglected losses in the coupler. In this situation some pumping might be in order to restore comb generation. Note that this is just the case for experimentally implemented MRRs on monomode fibers [25].

As a concluding remark, one should emphasize the importance of vortex WGMs implementation in the MRR for generation of the vortex frequency comb. This important point has been missing in our initial treatment of MRRs presented in Ref. [32]. The main thesis is that to guarantee

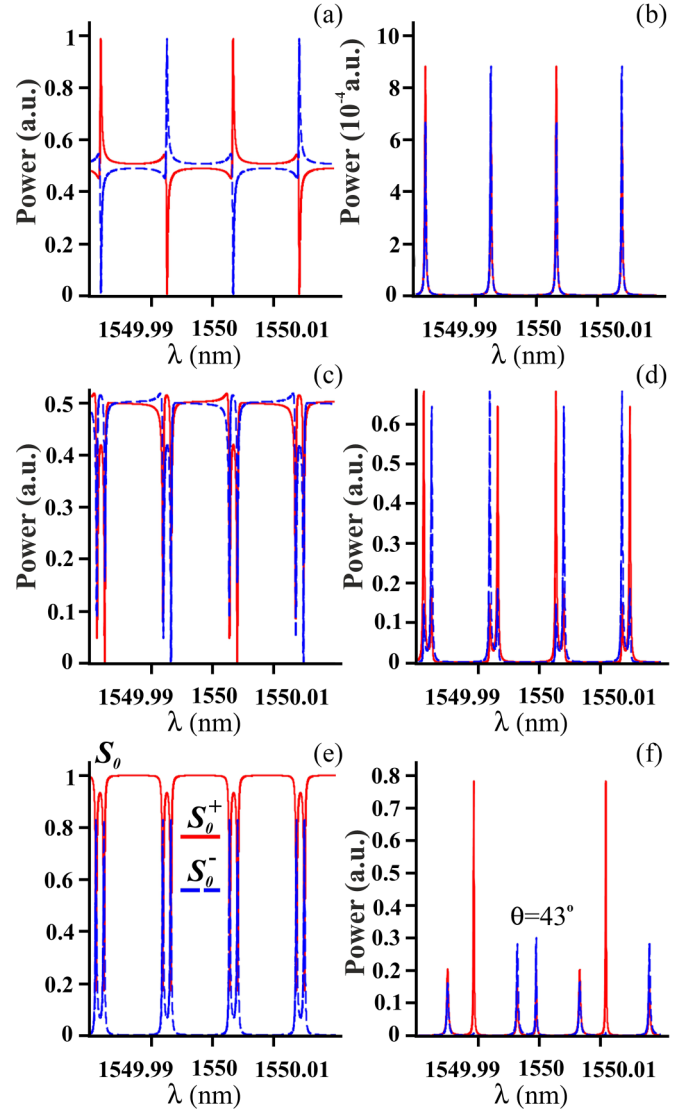


FIG. 11. Effect of SOI on generation of frequency combs in the MRR. [(a) and (b)] Twist corrections prevail over SOI ones: twist/SOI = 5. [(c)–(f)] SOI corrections prevail over twist ones: SOI/twist = 2. Everywhere except (e): Red solid curves, transmission of OVs with TCs +1; blue dashed curves, with TC -1 . [(a) and (c)] LCP component; [(b), (d), and (f)] RCP component. (e) Total powers stored in LCP outcoming OVs (red solid curve) and RCP outcoming OVs (blue solid curve). Parameters: $d = 7.35 \text{ cm}$, and $\Lambda = 2.59 \text{ cm}$; $\theta = \pi/2$ everywhere save (f), where $\theta = 43^\circ$.

generation of vortex comb in a MRR one has to choose resonator parameters such as to warrant in it the existence of vortex WGMs. Figure 13 shows the examples of transmission curves if the MRR is excited with LCP even LG mode. As is seen, transmitted powers for OVs with TCs ± 1 form vortex combs only if initial resonator parameters (θ , χ) are such that enable the existence of vortex WGM in the resonator. In those cases maximal value of reduced third orbital Stokes parameter (see also Fig. 2) should be close to unity. Note that the variation of the wavelength in the given range results in negligible displacements of the initial points on (θ , χ) plane: θ remains constant, whereas χ changes by approximately 10^{-4} rad .

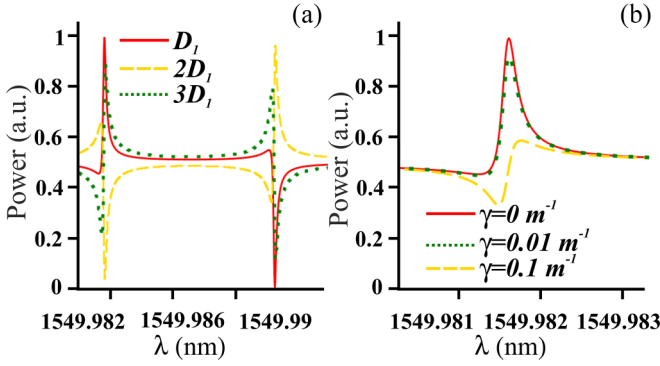


FIG. 12. (a) Effect of ellipticity degree on generation of frequency combs in the MRR. Transmission curves are shown for indicated values of ellipticity constants in units of some reference constant D_1 . (b) Effect of attenuation on generation of frequency combs in the MRR. Magnitudes of attenuation constant γ are shown in the legend.

B. Generation of OV's in bielliptical fiber ring resonators

In this section we study the scheme where a bielliptical fiber ring resonator is side coupled to the bus fiber in the same manner as shown in Fig. 8(b). Here we disregard the SOI. In this assumption, the evolution is factorized in polarizations so that the incoming and outgoing fields in the bus fiber have the same polarization:

$$|i\rangle = i_1|EV, +\rangle + i_2|OD, +\rangle, \quad |t\rangle = t_1|EV, +\rangle + t_2|OD, +\rangle. \quad (28)$$

Analogously, the fields at the upper ends of the coupler are

$$|\iota\rangle = \iota_1|EV, +\rangle + \iota_2|OD, +\rangle, \quad |\tau\rangle = \tau_1|EV, +\rangle + \tau_2|OD, +\rangle. \quad (29)$$

Correspondence between input and output coefficients and the coupler's ends is analogous to Fig. 8(b), with the only

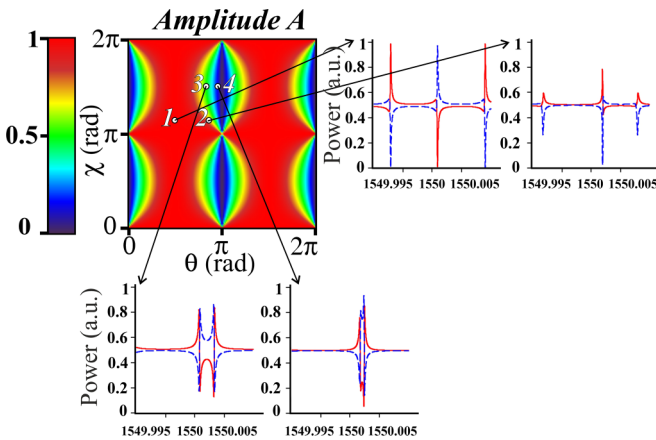


FIG. 13. Correlation between maximal values A of reduced third orbital Stokes parameter (Fig. 2) of a WGM and the quality of vortex frequency comb generated by a MRR. Power stored in OV's with TCs +1 and -1 are shown by red solid and blue dashed curves, respectively. The incident beam is the LCP even LG mode. Point 1: $d = 7.35 \text{ cm}$, $\theta = 1.54 \text{ rad}$; point 2: $d = 7.35 \text{ cm}$, $\theta = 2.71 \text{ rad}$; point 3: $d = 8.2 \text{ cm}$, $\theta = 2.615 \text{ rad}$; point 4: $d = 8.2 \text{ cm}$, $\theta = 3 \text{ rad}$.

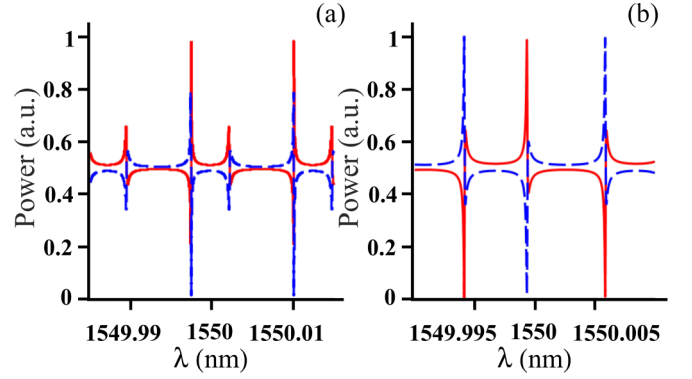


FIG. 14. Generation of vortex frequency combs from LG modes in a ring bielliptical resonator coupled to a bus fiber. Red solid curves, power stored in OV's with TCs +1; blue dashed curves, power of OV's with TCs -1. The incident beam, LCP even LG mode; $\alpha = \pi/4$, $\Lambda = 2.63 \text{ cm}$. (a) $d = s = 5 \text{ cm}$ and (b) $d = 10s$, $s = 1 \text{ cm}$.

correction that negatively polarized components are absent. One can easily establish the desired connections between the introduced decomposition coefficients:

$$\hat{\mathfrak{M}}\left(\begin{matrix} \tau_1 \\ \tau_2 \end{matrix}\right) = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \quad \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \hat{\tau}_1 \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}, \quad \begin{pmatrix} t_2 \\ t_1 \end{pmatrix} = \hat{\tau}_2 \begin{pmatrix} i_2 \\ i_1 \end{pmatrix}, \quad (30)$$

where $\hat{\tau}_i$ should be taken from Eq. (24). Powers $P_{1,2}$ of outgoing OV's with TCs ± 1 can be calculated analogously.

Figure 14 shows transmission curves for OV's if the incident beam is LCP even LG mode. These characteristics are similar to those presented in Fig. 10(a) and prove possibility to generate vortex frequency combs from LG modes. One should note that transmission curve is robust to variations of the displacement angle: For $\alpha = \pi/8$ the only difference of the transmission curve as compared to Fig. 14(a) is in its shifting by 85 MHz. The presence of vortex comb in the transmitted signal is also robust to variations of relative distances d and s . Indeed, Fig. 14(b) shows transmission curves for the case $d = 10s$, which basically coincide with the ones in Fig. 14(a). Such a degree of freedom allows for controlling the parameters of the generated vortex combs: For Fig. 14(a) distance between peaks is 1.58 GHz and peak width is 73.7 MHz, whereas for Fig. 14(b) distance between peaks is 1.46 GHz and peak width is 107.4 MHz. Generally, such parameters have the same order as for vortex combs generated with MRRs.

The last important point to address is robustness of vortex frequency comb generation to variations of resonator's parameters, such as Λ , s , d , and ellipticity parameter δ , which is critically important to experimental feasibility of vortex combs generation. Detailed analysis shows that while the positions of individual peaks are sensitive to variations of the mentioned parameters, the general pattern of the frequency comb maintains its form in a reasonable range of such parameters' deviation from their optimal values. So for a resonator with $d = 10 \text{ cm}$, $s = 1 \text{ cm}$, $\Lambda = 2.63 \text{ cm}$, $\alpha = 45^\circ$, and $\delta = 0.05$ allowable error is $|\Delta\alpha| \sim 5^\circ$, $|\Delta d| \sim 2 \text{ mm}$, $|\Delta\delta| \sim 0.015$, $|\Delta s| \sim 1 \text{ mm}$. We believe that this gives quite a reasonable manufacturing tolerance window for practical

realizations of such resonators capable of generating vortex combs. The most likely reason for the observed stability of frequency combs is that here we rather deal with two different physical systems: a closed Möbius ring and a side-coupled Möbius ring. For the latter system, formation of an ideal vortex after a cycle is not that critical as for the Möbius loop: Such a partially formed vortex would partially tunnel into the bus fiber reducing the effect of its incapability to form an ideal vortex WGM. The resulting vortex comb appears to be not ideal; however, such nonideality concerns only peak powers and not the distances between the peaks. In addition, presented for bielliptical resonators theory of vortex WGM implementation has a rather illustrative function since it studies field distribution only in the cut plane. Studying orbital Stokes parameters along the whole loop might have given less strict limitations as those presented by Eq. (21).

The suggested scheme of OV generation belongs to the class of all-fiber vortex generators. With regard to fiber-optics communications, this property favorably distinguishes this scheme from other numerous methods to create an OV [46]. We believe that its main application can be seen within such a long-standing trend in fiber OV applications [47]. In particular, the ability of Möbius-strip-like resonators to generate structured-light frequency combs could be useful for multiplexing techniques. In addition, the idea of Möbius resonators has a great, currently not realized, potential for generating dual frequency combs, which is the key element for dual-comb spectroscopy [48,49]. Creating vortex frequency combs by resonators with embedded Möbius-strip-like topology is advantageous as compared to earlier proposed fiber optical schemes. Such resonators are simpler to implement than both PT-symmetric resonators [42] and CROW-endowed add-drop filters [43]. The latter systems are incapable of generating OV combs from topologically neutral LG modes and require vortex feed. The most remarkable property of MRR is that it allows controlling the influence of the SOI by changing the pitch of the twisted Möbius ring. Indeed, in conventional fiber resonator systems the role of the SOI can be changed only by altering intrinsic fiber parameters, which implies choosing another fiber. On the contrary, MRRs enable one to control the effect of the SOI by merely varying the twist pitch, in this way transiting from scalar to fully vectorial cases. The latter possibility renders MRRs an additional degree of freedom in engineering their spectral responses.

V. CONCLUSION

In this paper, we have studied the propagation of structured light in fiber resonators with effective topology of the Möbius strip. In particular, we have addressed formation of whispering gallery modes in ring resonators on base of strongly elliptical twisted two-mode fibers and bielliptical fibers, that is, the fibers composed of two consecutive sections of elliptical fibers with differently oriented deformation ellipses. By studying orbital Stokes parameters of such WGMs we have shown that at certain conditions those modes may form

in certain cross-section OVs with modulus unity topological charges. We have demonstrated that ring resonators with effective Möbius strip topology being coupled to the bus fiber can be used for generation of OV frequency combs from LG modes. We have stressed that to guarantee generation of vortex comb in a MRR the resonator parameters should be such as to warrant the existence of vortex WGMs in the resonator. We have also studied the effect of the spin orbit interaction on generation of OV frequency combs in Möbius resonators on base of elliptical twisted fibers. We have demonstrated that the presence of the SOI does not prevent implementation of vortex WGMs in orthogonal polarizations and generation of OV combs in those polarizations.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available because they contain commercially sensitive information. The data are available from the authors upon reasonable request.

APPENDIX

Elements of matrix $\hat{M}$ [Eq. (4)]:

$$\begin{aligned}
 M_{11} &= \cos qz \exp(-iqz) [\sin^2 \gamma \exp(i\beta_1 z) + \cos^2 \gamma \exp(i\beta_2 z)] \\
 M_{12} &= -\sin qz \exp(-iqz) [\sin^2 \gamma \exp(i\beta_3 z) + \cos^2 \gamma \exp(i\beta_4 z)] \\
 M_{13} &= \frac{1}{2} \sin 2\gamma \cos qz \exp(-iqz) [\exp(i\beta_1 z) - \exp(i\beta_2 z)] \\
 M_{14} &= \frac{1}{2} \sin 2\gamma \sin qz \exp(-iqz) [\exp(i\beta_3 z) - \exp(i\beta_4 z)] \\
 M_{21} &= \sin qz \exp(-iqz) [\sin^2 \gamma \exp(i\beta_1 z) + \cos^2 \gamma \exp(i\beta_2 z)] \\
 M_{22} &= \cos qz \exp(-iqz) [\sin^2 \gamma \exp(i\beta_3 z) + \cos^2 \gamma \exp(i\beta_4 z)] \\
 M_{23} &= \frac{1}{2} \sin 2\gamma \sin qz \exp(-iqz) [\exp(i\beta_1 z) - \exp(i\beta_2 z)] \\
 M_{24} &= \frac{1}{2} \sin 2\gamma \cos qz \exp(-iqz) [\exp(i\beta_4 z) - \exp(i\beta_3 z)] \\
 M_{31} &= \frac{1}{2} \sin 2\gamma \cos qz \exp(iqz) [\exp(i\beta_1 z) - \exp(i\beta_2 z)] \\
 M_{32} &= \frac{1}{2} \sin 2\gamma \sin qz \exp(iqz) [\exp(i\beta_3 z) - \exp(i\beta_4 z)] \\
 M_{33} &= \cos qz \exp(iqz) [\cos^2 \gamma \exp(i\beta_1 z) + \sin^2 \gamma \exp(i\beta_2 z)] \\
 M_{34} &= -\sin qz \exp(iqz) [\cos^2 \gamma \exp(i\beta_3 z) + \sin^2 \gamma \exp(i\beta_4 z)] \\
 M_{41} &= \frac{1}{2} \sin 2\gamma \sin qz \exp(iqz) [\exp(i\beta_1 z) - \exp(i\beta_2 z)] \\
 M_{42} &= \frac{1}{2} \sin 2\gamma \cos qz \exp(iqz) [\exp(i\beta_4 z) - \exp(i\beta_3 z)] \\
 M_{43} &= \sin qz \exp(iqz) [\cos^2 \gamma \exp(i\beta_1 z) + \sin^2 \gamma \exp(i\beta_2 z)] \\
 M_{44} &= \cos qz \exp(iqz) [\cos^2 \gamma \exp(i\beta_3 z) + \sin^2 \gamma \exp(i\beta_4 z)].
 \end{aligned}$$

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