

Quasinormal modes in dispersive photonic time crystals

Calvin M. Hooper^{✉,*}, Ian R. Hooper[✉], and Simon A. R. Horsley[✉]*School of Physics and Astronomy, University of Exeter, Stocker Road, Exeter EX4 4QL, United Kingdom* (Received 20 October 2025; revised 2 March 2026; accepted 26 March 2026; published 29 April 2026)

Quasinormal modes characterize the transient response of static optical cavities. Here we introduce the notion of a Floquet quasinormal mode to describe transient responses in photonic time crystals. Contrasting their static counterparts, exceptional points associated with symmetry transitions are an inherent feature, as modes spontaneously and nonperturbatively lock their phase to the oscillations of the material. We further investigate the limiting behavior of the Floquet quasinormal modes in large cavities. Distinct nonperturbative behavior arises in time-modulated systems as increasingly large time-crystal cavities come closer to achieving the maximum gain predicted from a bulk wave number band gap.

DOI: [10.1103/m5ww-n7ws](https://doi.org/10.1103/m5ww-n7ws)

I. INTRODUCTION

Noether's theorem [1] links energy conservation to the translational invariance of a system in time. This applies as much to particle physics as to classical wave propagation. Yet when a wave-supporting medium is explicitly time dependent, time translation symmetry is broken. The propagation characteristics of the wave then become sensitive to the time delay relative to the material modulation, a sensitivity that is connected to the pumping of energy into or out of the system [2–13]. We should remember, however, that such energy must always be supplied externally to the medium: while the wave inside a material appears to receive free energy, the laboratory still pays a power bill.

Perhaps the clearest example of this unusual behavior occurs in photonic time crystals [2,8,10,12,14–16]. What makes these periodically driven media so striking, aside from their analytic simplicity in the Fourier domain, is the exponential growth in time they provide to waves. Indeed, as long as the wave frequency is close to commensurate with the material's time variations, pumping of energy to or from the wave will compound exponentially in time [2,7–10,12,15–17]. Whether energy is pumped into or out of an incident wave is determined entirely by the wave's relative phase with respect to the oscillation of the medium [18–21].

Analysis of periodically modulated media has largely followed via an analogy with spatial crystals [12,14,16,17]. For instance, the preceding discussion of exponential gain in periodically driven media can be understood directly in terms of the theory of band gaps. In spatial crystals, the discrete rather than continuous translational symmetry replaces momentum as a conserved quantity with the quasimomentum, which is

periodic in reciprocal space. By direct analogy, frequency is no longer conserved in time crystals, replaced with the periodic quasifrequency.

Similarly, while spatial crystals possess a *frequency* band gap within which the wave number becomes complex, temporal crystals possess a *wave number* band gap within which the frequency becomes complex (corresponding to the aforementioned exponential growth or decay of fields as a function of time). Furthermore, in both cases, the maximum growth and decay rates within a spatial or temporal crystal band gap may be found analogously, by substituting in quasimomenta or quasifrequencies with an increasing range of imaginary parts until the corresponding frequency or wave number ceases to be real. In each case, this occurs at an exceptional point as the real frequencies or wave numbers within the band gap break into complex conjugate pairs for complex quasimomenta or quasifrequencies with too much growth or decay to be supported by the spatial or temporal crystal.

Perhaps more important than the similarities between space and time crystals are their differences. For instance, the preservation of frequency by spatial crystals allows second quantization to proceed in a straightforward manner. By contrast, time crystals inherently couple positive and negative wave frequencies, thus mixing the associated creation and annihilation operators. This mixing allows photons to be extracted not only from thermal fluctuations [22,23], but also from the vacuum state [23–26].

As soon as we consider wave propagation in the time domain, we must face the linked complexities of dispersion and dissipation [27]. Both effects modify the outcome of proposed experiments: dispersion naturally limits the compression of pulses by analog black holes, as predicted by Horsley *et al.* [24,25], while intrinsic loss, inherent to certain methods of time modulation [28,29], can overwhelm any anticipated wave amplification, closing wave number band gaps [18,30]. Conversely, coupling between both wave and material degrees of freedom can open qualitatively new band gaps, not constrained to the center or edge of the Floquet-Brillouin zone [31]. Previously, we investigated the amplification of waves in a dispersive, dissipative slab through calculating the

*Contact author: ch1122@exeter.ac.uk

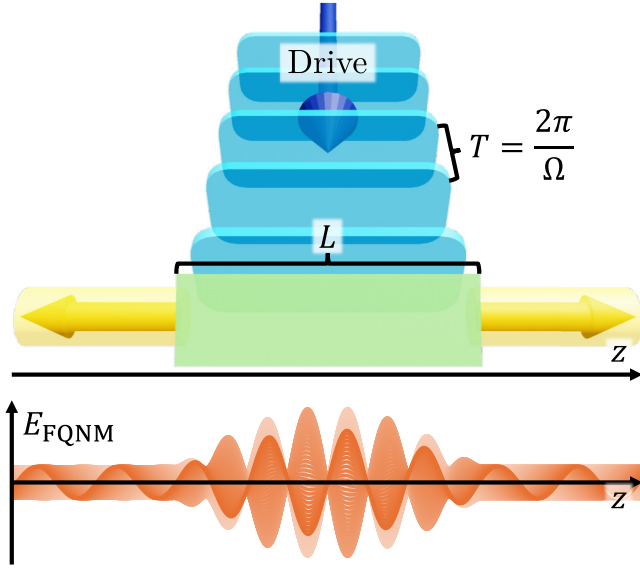


FIG. 1. A schematic of the (Floquet) quasnormal mode problem: A photonic time-crystal cavity may be realized by changing the material properties of a cavity of length L , through a pumping with periodicity T , and corresponding angular frequency $\Omega = 2\pi/T$. We then consider how any fields initially within the cavity propagate over time, whether amplified by the driving, or decaying as they leak out. This behavior may be characterised by considering representative "modes" of the cavity (see Sec. I A) which evolve by only a constant phase after every time period T . One such mode is shown, with its history plotted with steadily fading lines. Since the external fields contain information about the past state of the cavity, they typically grow exponentially for lossy cavities. However, in time-varying systems, gain is possible within the cavity, and so they may also decay in space or maintain a constant amplitude (as pictured).

continuous wave transmission operator [18]. In this paper we study the transient behavior of the same system (see Fig. 1) by extending the quasnormal mode concept to periodically driven materials (Floquet quasnormal modes, FQNMs).

A. Quasnormal modes

The modes of a static system are those fields which maintain their spatial profile, while varying in time as the exponential of a complex frequency, with a real part determining the rate of oscillations, and an imaginary part corresponding to the inverse decay time of the mode. The origin of such decay is twofold: intrinsic losses, where energy from the wave is lost to heat in a material, and radiative losses, where waves leak from a region of space into their surroundings. Of these mechanisms, the exponentially growing spatial profiles associated with radiative loss have historically proven problematic, with such modes dubbed quasnormal modes (QNMs [32–34]). They are useful because a discrete set of such modes can be employed to understand the otherwise complex dynamics of open systems.

An additional complexity associated with this setting is the potential for temporal modulations to impart net gain on a wave, resulting in potentially exponentially increasing fields. Of course, conservation of energy limits how much power may be pumped into a wave, and so nonlinear effects

thus inevitably prevent this exponential growth from continuing indefinitely. This limitation may be considered from two perspectives, depending on the application for which the quasnormal modes are intended. When long-time, or steady-state dynamics are important, such as when considering second quantization of electromagnetic waves in the presence of a material [26] by the use of quasnormal modes [35,36], one must constrain all QNMs (and henceforth FQNMs) to decay, confining their frequencies to the lower half plane [37]. The alternative perspective is to consider these modes as a convenient tool for calculating the time evolution of a system, in which case, no such constraint need be applied for sufficiently short times that any nonlinearities may be neglected.

Of course, such analogies are only useful if the notion of a QNM may be extended to the time-varying case. And it is not immediately obvious that this should be the case: in time-varying media, the wave frequency is not conserved. One might thus question whether the concept of a mode is naturally applicable. It seems it is: as discussed in [27], propagation in the material can still be described using a linear operator, which has associated eigenfunctions. These eigenfunctions contain a spread of frequencies that are interconverted such that their combined spectrum is unmodified after a single modulation period. Given the difficulty of understanding wave propagation in dispersive, time-varying materials, might it be useful to thus extend the QNM concept to time-varying media (FQNMs)?

Here we present such an extension, quantifying the trajectories traced in the complex plane by the FQNM quasifrequencies under continuous variations of a cavity. These modes no longer conserve complex frequency, but rather the more general complex Floquet quasifrequency (defined as the ladder of frequency components equivalent modulo the driving frequency). We identify a number of regions where nonperturbative behavior is in fact fundamental to understanding the behavior of FQNMs. We will demonstrate that, in contrast to previous work [38], this nonperturbative behavior is very robust, not requiring any artificially applied symmetry.

This nonperturbative behavior is not merely a numerical challenge: exceptional points arising in the QNMs of a static cavity have already been found to have direct implications on emission from such cavities [39,40].

B. Existing work in time-varying media

The results of this work also provide an underpinning to the other recent works in this area [41,42], as all use essentially the same techniques to characterize the modes of time-varying media, while considering different aspects of the same problem.

For instance, [41] presents a practical perturbation theory for the calculation of FQNM quasifrequencies. However, this paper demonstrates that many phenomena found in time-varying media cannot be arrived at by perturbing from static media, and thus our results must be applied to find an appropriate starting point for perturbative analyses. Similarly, the results of [42] on the topic of scattering applies Keldysh's Theorem [43] to find completeness relations between isolated modes, while this work applies Fredholm theory [44] to identify where this assumption of isolation breaks down.

This work is also directly relevant to experiments: as the practical applications and limitations of time-varying media have become better understood, focus has shifted from the slow processes initially studied [45] towards rapid modulation of material properties. Indeed, sufficiently fast processes for the realization of time crystals has now been achieved in regimes from microwaves [29,46] to THz [20,28] and near-infrared [11,19], or outside of electromagnetism, in acoustics [47–50] or water waves [51,52]. Furthermore, while achieving sufficiently rapid modulation and low loss for the realization of wave number band gaps at optical frequencies has thus far proven difficult, this work presents an alternative setting in which the symmetry transition central to such band gaps [18] may be realized without the requirement of net gain. This is not to say, however, that net gain is unrealistic: it is certainly possible in both acoustics [48,50] and microwaves [46,53–55]. In such a context, (F)QNM determine whether, and if so how quickly, a system will saturate.

C. Outline

The structure of this paper is then as follows: In Sec. II we apply the operator formalism of Horsley *et al.* [27] to show that the FQNM problem can be written in a particularly compact and intuitive form. We then consider a number of general properties regarding the trajectories of FQNM quasifrequencies in the complex plane (our justification for the continuity and discreteness of these trajectories is provided in Appendix B). In Sec. III we prove how, in contrast to static media, exceptional points are a ubiquitous feature in the trajectories of FQNM under continuous variations of the system parameters. We show that these exceptional points are a necessary consequence of the symmetry presented in [18] (a link noted in quantum mechanics [56], and later investigated more generally [57]). In Sec. IV we consider the global distribution of FQNM in the complex plane through the limit of large slab lengths. We demonstrate that the introduction of time variations has a radical effect on this limit, again due to the symmetry described in [18]. Indeed, these limits directly describe how the gain (and loss) present in the finite time-crystal cavities of experiment eventually approaches the theoretical predictions for an infinite bulk medium. Finally, in Sec. V we verify our results in a particular example, characterizing the origin of the transmission poles observed in [18], in terms of their associated modes.

II. QUASINORMAL MODES IN A TIME-VARYING MATERIAL

In static media, an efficient method for computing the time-evolution of waves in a cavity is to study its QNMs. These modes are solutions to the undriven wave equation, assuming time harmonic evolution with frequency ω . For a perfectly closed, dissipation-free system these frequencies are purely real valued and correspond to the usual normal mode frequencies. Meanwhile for a general system that is both open (outgoing boundary conditions) and dissipative, the quasinormal mode frequencies ω are complex, where $\text{Im}[\omega] < 0$ (or $\text{Im}[\omega] > 0$) corresponds to loss (or gain). These QNMs provide a framework for analyzing the linear

response of a cavity, including both short-time transients and the long-time steady state.¹

Due to the assumption of harmonic time evolution, the QNM eigenvalue problem requires a system with time translational symmetry, an assumption that fails for time-varying media. But under the constraint of periodicity in time, the Floquet quasifrequency ω_0 [59] takes its place. Hereafter T denotes the period of modulation of the material parameters, with $\Omega = \frac{2\pi}{T}$ its angular frequency. The harmonic ansatz for FQNM is then given by

$$\Psi(\mathbf{r}, t; \omega_0) = e^{-i\omega_0 t} \bar{\Psi}(\mathbf{r}, t; \omega_0), \quad (1)$$

where ω_0 is the Bloch frequency and Ψ is a vector containing both the electric and magnetic fields, as well as the material response (e.g., the electric polarization and its time derivative). $\bar{\Psi}$ differs from Ψ as it is periodic with period T , i.e., $\bar{\Psi}(\mathbf{r}, t; \omega_0) = \bar{\Psi}(\mathbf{r}, t + T; \omega_0)$. Equation (1) thus corresponds to fields that are restored up to a scalar multiple of $e^{-i\omega_0 T}$ after a single period has elapsed. Note that there is the usual nonuniqueness of ω_0 in the definition (1), where we may shift its value by any multiple of Ω , keeping the function $\bar{\Psi}$ periodic: $e^{-i\omega_0 t} \bar{\Psi}(\mathbf{r}, t; \omega_0) = e^{-i(\omega_0 + m\Omega)t} [e^{ims\Omega t} \bar{\Psi}(\mathbf{r}, t; \omega_0)]$.

In Appendix A we describe how such an FQNM eigenvalue problem may be set up in general. However, for the contents of this paper, we will direct our attention towards wave propagation at normal incidence (along the z axis), propagating through a temporally periodic dielectric slab. Here the transverse electric field E obeys a generalization of the one-dimensional wave equation,

$$\frac{\partial^2 E}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \{[1 + \Pi_L(z)\chi(\partial_t, t)]E\}, \quad (2)$$

where the function Π_L is zero everywhere except for inside the dielectric

$$\Pi_L(z) = \begin{cases} 1 & |z| < \frac{L}{2} \\ 0 & |z| \geq \frac{L}{2} \end{cases}. \quad (3)$$

The quantity $\chi(\partial_t, t)$ appearing in Eq. (2) is the operator representing the dielectric susceptibility of the dispersive, time-varying medium.

Although our findings are general, we will—where necessary—consider the same example as in [18], and assume the time-varying Drude susceptibility

$$\chi(\partial_t, t) = [1 + \eta \cos(\Omega t)] \frac{\omega_{\text{pl},0}^2}{\partial_t^2 + \gamma \partial_t}, \quad (4)$$

with the parameters $\eta = 0.2$, $\gamma = 0.01 \Omega$, $\omega_{\text{pl},0} = 0.3 \Omega$, where $\Omega = 1 \text{ rad s}^{-1}$, and with cavity lengths L measured in units of c/Ω .

In addition, note the particular ordering chosen for the ∂_t and t operators in (4), which corresponds to a particular microscopic model of the material dynamics, with direct physical implications [60]. This ordering is chosen as an example

¹To state this for *any* initial condition of the cavity requires the notion of quasinormal mode completeness, which has not been rigorously proven true in general [58].

which allows for the amplification of incident waves [18], something which is not generally true [29].

We now follow [27], calculating the effect of wave propagation on the frequency spectrum of the electric field. Following Eq. (1), individual solutions for E are given by

$$E(z, t; \omega_0) = e^{-i\omega_0 t} \sum_{m=-\infty}^{\infty} \tilde{E}_m(z) e^{-im\Omega t}. \quad (5)$$

An important symmetry in this representation was presented in [18], which we now generalize to the complex quasifrequencies of QNMs.

Time-domain fields are real valued and are hence symmetric under $\mathcal{C}$ symmetry, where $\mathcal{C}$ is the complex conjugate operator $\mathcal{C}a = a^*\mathcal{C}$. However, the Floquet ansatz (1) immediately breaks this symmetry for any ω_0 where $e^{-i\omega_0 T}$ is not real valued. To obtain a real time-domain field in this case, one must thus add together modes (5) with Floquet quasifrequencies ω_0 and $-\omega_0^*$.

Yet the modes given in Eq. (5) do not always break complex conjugation symmetry. If, for example, $\text{Re}\{\omega_0\} = 0$, the functions $E(z, t; \omega_0)$ are real valued when the Fourier components obey $\tilde{E}_m = \mathcal{R}\tilde{E}_m\mathcal{R}$, where $\mathcal{R}$ reverses the Fourier spectrum of a wave, i.e., $\mathcal{R}\tilde{E}_m\mathcal{R} = \tilde{E}_{-m}$. If this is the case we say the mode is $\mathcal{RC}$ symmetric, a term introduced in Ref. [18], and which indicates a standing wave dependence of the wave in time, analogous to the standing wave solutions at the edge of the Brillouin zone in a spatially periodic medium.

Similar to wave solutions in a spatially periodic medium, as we can translate the Bloch frequency ω_0 by any multiple of Ω , this $\mathcal{RC}$ symmetry can hold about any axis parallel to the imaginary axis where the real part of ω_0 is an integer multiple of $\Omega/2$. As a result, for these special choices of ω_0 the solution (1) can be chosen to correspond to a real time-domain field, while for other values of the Bloch frequency, two different solutions must always be combined. Henceforth, we denote with A_m the axis corresponding to frequencies with a real part $\frac{m\Omega}{2}$.

We now treat the Fourier components $\tilde{E}_m$ defined in Eq. (5) as the components of a single infinite vector $|\tilde{E}\rangle$. The latter is then governed by Eq. (2), with operators substituted by their frequency domain counterparts

$$\begin{aligned} E &\mapsto |\tilde{E}\rangle, \\ \partial_t &\mapsto -i\hat{\omega}, \\ \cos(q\Omega t + \phi) &\mapsto \hat{\Delta}_q(\phi), \end{aligned} \quad (6)$$

where

$$\begin{aligned} \hat{\omega} &= \text{diag}(\cdots \quad \omega_0 - \Omega \quad \omega_0 \quad \omega_0 + \Omega \quad \cdots), \\ (\hat{\Delta}_q(\phi))_{ij} &= \frac{1}{2} \begin{cases} e^{i\phi} & i - j = -q \\ e^{-i\phi} & i - j = q \\ 0 & \text{otherwise} \end{cases}, \end{aligned} \quad (7)$$

for $q \geq 1$. Note that we include the index "q" in Eq. (6) so that, e.g., Eq. (4) may be extended to an arbitrary time dependence of the plasma frequency, written as a Fourier sum. Furthermore, when applying the substitution rules of Eq. (6) to Eq. (4), it immediately follows that the components of the resulting operator are all meromorphic. In any practical

system exhibiting exponentially bounded gain, this results in an operator which is (up to some finite imaginary offset) analytic in the upper half plane, allowing the Kramers-Kronig relations to be applied similarly to the static case. The general (non-time-crystalline) case is considered in detail in [61].

Combining Eqs. (2) and (6) leads to an operator-valued version of the Helmholtz equation,

$$\frac{\partial |\tilde{E}\rangle}{\partial z^2} + \frac{\hat{\omega}^2}{c^2} \left[1 - \Pi_L(z)(1 + \eta\hat{\Delta}_1(0)) \frac{\omega_{\text{pl},0}^2}{\hat{\omega}^2 + i\gamma\hat{\omega}} \right] |\tilde{E}\rangle = 0. \quad (8)$$

As in these definitions, throughout the rest of this paper we will often leave the dependence of any Floquet operator on ω_0 as implicit for the sake of brevity.

At this point, we note that any time-domain operator which maps a real valued physical field to another real valued field must preserve $\mathcal{RC}$ symmetry. Thus, for any such operator $\mathcal{O}(\omega_0)$, we require $\mathcal{RC}\mathcal{O}(\omega_0)\mathcal{R}\mathcal{C} = \mathcal{O}(-\omega_0^*)$, which, for the special case of ω_0 lying on any symmetry axis A_m reduces to

$$\mathcal{R}_m \mathcal{C} \mathcal{O}(\omega_0) \mathcal{R}_m \mathcal{C} = \mathcal{O}(\omega_0), \quad (9)$$

where $\mathcal{R}_m$ denotes a reflection of the components of $\tilde{E}$ about the axis A_m . This symmetry can be easily verified for the example operators, $-i\hat{\omega}$ and $\hat{\Delta}_q(\phi)$ defined in Eq. (7). This symmetry of operators under transformations involving a pair of operators is closely related to the theory of $\mathcal{PT}$ symmetry [56,57]. In the context of $\hat{K}^2$, this symmetry is directly related to the formation of wave number band gaps [18] and associated exceptional points [62].

Solving Eq. (2) proceeds as usual (as described in [27]), albeit with care regarding operator ordering. We substitute the operator-valued generalization of a sum of traveling waves, $|\tilde{E}\rangle = e^{i\hat{K}z}|\tilde{E}_+\rangle + e^{-i\hat{K}z}|\tilde{E}_-\rangle$ for the region within the slab $|z| < \frac{L}{2}$, and the corresponding outgoing waves, $|\tilde{E}\rangle = e^{i\frac{\hat{\omega}}{c}(\pm z - \frac{L}{2})}|\tilde{E}_{\text{out},\pm}\rangle$, in the region outside $|z| > \frac{L}{2}$. Requiring continuity of these expression at the slab boundary, solutions to Eq. (8) reduce to the requirement that the outgoing field vector, $\tilde{E}_{\text{out}}$ is in the nullspace of $\hat{Q}$,

$$\hat{Q}|\tilde{E}_{\text{out},+}\rangle = 0, \quad (10)$$

where the operator, $\hat{Q}$ (which equals the inverse of the transmission operator $\hat{t}$ given in Ref. [18]) is given by

$$\hat{Q} = \left(\frac{1 + \hat{n}}{2} e^{-i\hat{K}L} \frac{1 + \hat{n}}{2} - \frac{1 - \hat{n}}{2} e^{i\hat{K}L} \frac{1 - \hat{n}}{2} \right) \hat{n}^{-1}, \quad (11)$$

where we have introduced the following operator definitions of the frequency domain refractive index, wave number squared, and susceptibility,

$$\begin{aligned} \hat{n} &= \left(\frac{\hat{\omega}}{c} \right)^{-1} \hat{K}, \\ \hat{K}^2 &= \frac{\hat{\omega}^2}{c^2} (1 + \hat{\chi}), \\ \hat{\chi} &= -(1 + \eta\hat{\Delta}_1(0)) \frac{\omega_{\text{pl}}^2}{\hat{\omega}^2 + i\gamma\hat{\omega}}. \end{aligned} \quad (12)$$

Equation (10) is solved by varying the complex value of the Bloch frequency ω_0 until there is at least one zero eigenvalue

of $\hat{Q}$. The associated vector $|\tilde{E}\rangle$ then tells us the Fourier spectrum of the FQNM of our time-varying slab. In Fig. 2 we present the simplest possible example using this formalism, generalizing the QNMs [Fig. 2(a)] of a static system to their FQNM equivalents [Fig. 2(b)] without adding any time variation. In this case, the modes of the static system are copied into a series of replicas separated by gaps of the modulation frequency. This is analogous to the copies of the dispersion relation of a homogeneous medium obtained through assuming spatial periodicity, with the separation given by the reciprocal lattice vector. Each replica is a different representation of the same physical field, with offsets of $m\Omega$ in the complex plane precisely balanced by discrete shifts in the Floquet spectrum. In addition to the Floquet symmetry demonstrated in this figure, the predicted symmetry axes A_m are also immediately observed, and clearly remain as slab thickness is varied.

Aside from edge cases where a material response drops to zero at certain frequencies [18], $\hat{Q}$ will generally be infinite-dimensional. Fortunately (see Appendix C), the modes of an infinite system, within a given region of the complex plane, may be approximated by the modes of a sequence of finite-dimensional approximations. Further details on our numerical procedures may be found in Appendix D. For intuitive purposes, in this paper it suffices to note that the dimension of approximation chosen corresponds directly to the number of replicas present for any given mode.

There are a few subtleties within the definitions (12). First, we might worry that the infinite number of choices of operator square root, $\hat{K} = [\hat{K}^2]^{1/2}$ each leads to a different prediction when applying Eq. (10). However, close examination shows that $\hat{Q}$ is a holomorphic function of the operator $\hat{K}^2$, and thus independent of this choice of square root. Similarly, divergences of $\hat{n}^{-1}$ cancel when considering the full expression for $\hat{Q}$.

By contrast, the behavior of the operator $\hat{Q}$ when $\hat{K}^2$ diverges, e.g., at a complex resonant frequency, is more problematic. In the usual QNM case, this corresponds to an essential singularity, with an infinite number of modes appearing at nearby complex frequencies. In Appendix B we apply Fredholm theory [44] to demonstrate that this idea generalises to the case of FQNMs in dispersive one-dimensional slabs.

For future reference, we define the set S_R as FQNM quasifrequencies of the system when with coupling is removed between the waves and material. In Appendix B, we demonstrate that, away from these points all FQNMs have a finite spacing, and move continuously under perturbation. Thus, for our remaining analyses, we focus on understanding FQNMs in the remainder of the complex plane (which we term the set S_0).

III. DEGENERACIES AND EXCEPTIONAL POINTS

In a static system, $\mathcal{RC}$ symmetry implies whenever a mode crosses a symmetry axis A_m , it must collide with another such mode [see Fig. 2(c)]. This close proximity ensures that such modes are greatly affected by the frequency coupling introduced by time modulation.

However, the effect of this coupling on mode trajectories is heavily constrained by symmetry, resulting in three distinct

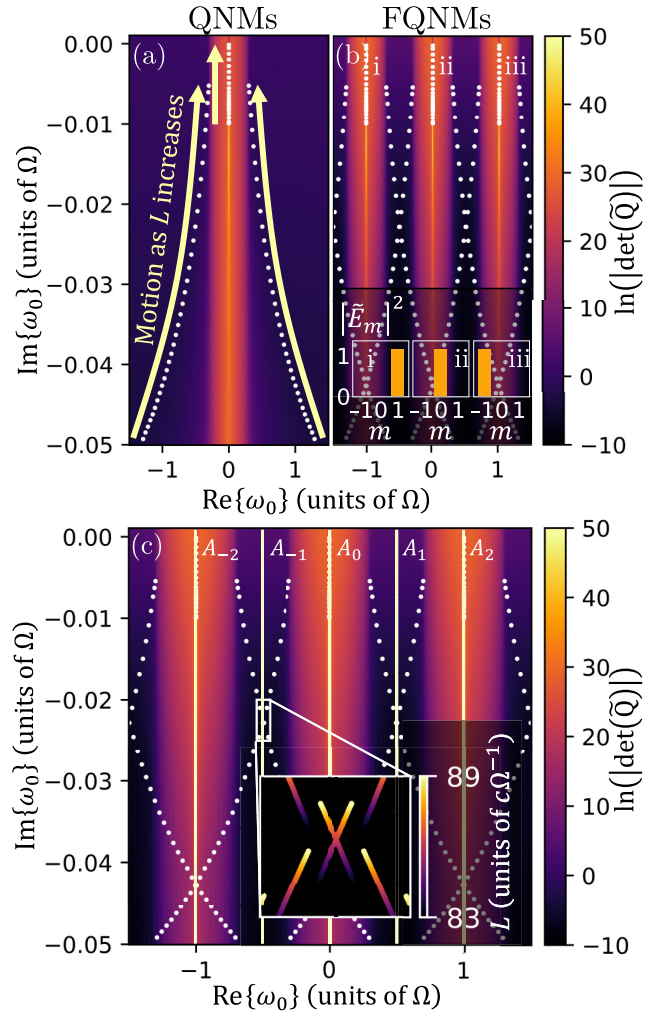


FIG. 2. The QNMs and FQNMs of a static Drude metal slab: The parameters are those given in the main text, with $\eta = 0$, setting the time modulation to zero, and a slab length of $L = 85 \text{ } c\Omega^{-1}$ (aside from the inset of panel c). In all panels we show a color plot of the indicator function $\ln(|\det(\hat{Q})|)$, the divergence of this quantity indicating a nonempty kernel, and thus a solution to (10). Numerically determined roots of $\det(\hat{Q}) = 0$ are plotted with white dots. Here $\hat{Q}$ approximates the operator $\hat{Q}$ derived in the main text, but truncated to a 5×5 matrix. (a) The QNMs of a static Drude metal. The band of large $|\det(\hat{Q})|$ around $\text{Re}(\omega_0) = 0$ corresponds to a region of suppressed transmission. The modes within this region are thus mostly confined to the material, and only weakly couple to electromagnetic waves. (b) The FQNMs of a static Drude metal. These can be immediately identified as shifted copies of the QNMs of panel (a), shifted in frequency by integer multiples of Ω . Denoted with i–iii are three such copies, with their associated eigenvectors plotted componentwise in the inset. Note that the frequency shifts of each eigenvector precisely cancel the shifted quasifrequency ω_0 , such that each replica FQNM corresponds to the same time-domain field. (c) The crossing of FQNM quasifrequencies about symmetry axes (labeled with A_m). Inset: FQNMs crossing a symmetry axis. The FQNM quasifrequencies are plotted as colored points denoting slab lengths between $L = 83 \text{ } c\Omega^{-1}$ and $L = 89 \text{ } c\Omega^{-1}$. By definition of the symmetry axis, as a given mode crosses an $\mathcal{RC}$ axis A_m , it must collide with its $\mathcal{RC}$ -symmetric pair. In static media, these modes aren't coupled, and so no additional behavior arises at these crossings, a fact expected to change for finite modulations.

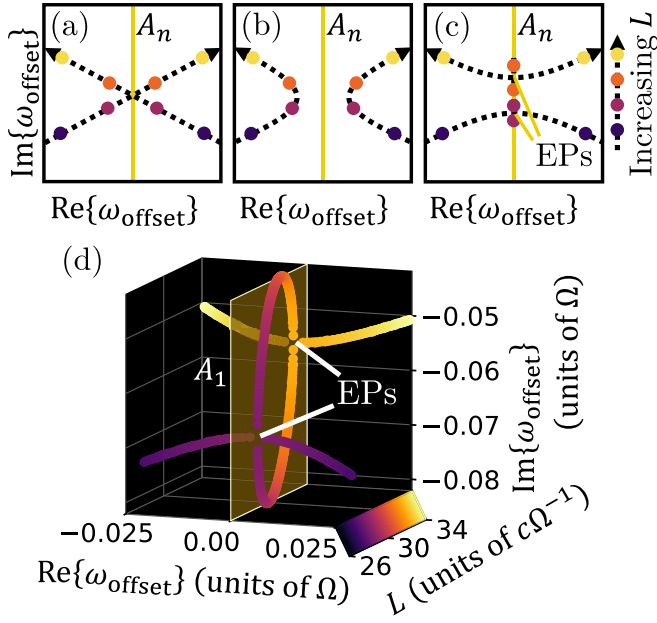


FIG. 3. $\mathcal{RC}$ -symmetry (un)breaking of FQNM in a photonic time crystal: In all plots, we consider frequency offsets ω_{offset} relative to an $\mathcal{RC}$ -symmetry axis A_m (specified as A_1 in panel d). The exceptional points (labeled "EPs") in panels (c) and (d) represent symmetry transitions only observed in time-varying media. In panels (a)–(c) we plot the possible qualitative behaviors allowed by symmetry for FQNM approaching a symmetry axis A_m . (a) If the modes are not coupled by time variations in the system, they will pass one another unperturbed. This is most likely for static media (see Fig. 2). Away from the crossing point, the behavior of later modes can be expected to line up with this example. (b) As is commonly expected from Hermitian systems, a pair of modes may experience avoided crossing. (c) Our system obeys $\mathcal{RC}$ symmetry rather than Hermitian symmetry, which possesses the same capacity for spontaneous symmetry transitions as the well-studied $\mathcal{PT}$ symmetry [56,57]. Thus, a pair of modes may be pulled together by coupling, colliding in an exceptional point where they transition from possessing $\mathcal{RC}$ symmetry only as a pair to each individually preserving it, with each lying directly on the symmetry axis A_m . (d) An example of (c) in the (sampled) FQNM trajectories of a Drude metal [parameters given below Eq. (4)] about the A_1 symmetry axis. These trajectories are plotted as a function of length between $L = 26 \text{ c}\Omega^{-1}$ and $L = 34 \text{ c}\Omega^{-1}$. Although these occur only for precise values of L , continuity of mode frequencies guarantees that such values of L must occur during a symmetry transition.

behaviors, represented schematically their respective panels of Fig. 3.

In the first case we have an *unperturbed crossing*. Modes cross the symmetry axis A_m with a smooth path, their coupling begin zero on A_m . Since this requires the coupling to be exactly zero, this is the least likely to be observed in a given setup. Second we have *avoided crossing*. Interaction between the two modes prevents their collision entirely, and neither touches A_m . Finally we have *exceptional point crossing*. Two modes attract as they approach A_m , rapidly colliding in an exceptional point, before rapidly splitting again into a pair lying along A_m . This process is then undone to allow the pair to leave A_m .

Out of these possibilities, of particular interest is Fig. 3(c), as the rapid collision of two modes at an exceptional point is a nonperturbative phenomenon which is commonly observed (due to $\mathcal{RC}$ symmetry) in the behavior of FQNM [see Fig. 3(d)]. This behavior is closely analogous to the $\mathcal{PT}$ symmetry of [56], with exceptional points occurring when the modes transition from being symmetric as a pair about A_m , to being individually symmetric and lying on A_m .

This symmetry transition also has direct physical consequences. Modes away from a symmetry axis A_m , represent fields where the oscillation frequency does not have a fixed phase relationship with the modulation of the material parameters. As a result, the exponential decay rate for these modes remains the same regardless of any relative offset between oscillations in a mode and modulation of the slab.² By contrast, modes lying precisely on an axis A_m possess a Floquet quasifrequency with a real part of $m\Omega/2$. As a result, following Eq. (1), the form of the corresponding fields in the time domain are preserved after each period of oscillation, up to a sign. Physically, this corresponds to energy being stored in a consistent place—whether in the field or the material—throughout each modulation period, and thus having energy consistently pumped or removed. This phase sensitivity is a key feature of time crystals and has been discussed by various authors [18–20,30].

We have thus demonstrated that the phase sensitivity is not a feature solely of bulk time crystals, but that it also arises within finite time crystals. In addition, by viewing this through the language of symmetry transitions, we can see that this is in fact a robust feature of time-varying media.

IV. LIMITING STRUCTURE OF MODES IN LARGE SLABS

The above has considered how time modulation affects the motion of relatively few modes in the complex plane. However, taking the limit of large slabs allows us to connect propagation in the bulk of a time-varying medium and the FQNM of a finite slab, which are more relevant to typical experiments. In particular, we demonstrate that the complex frequency associated with the maximum gain in a wave number band gap is realized in finite slabs as a limit point in the set of FQNM as slab length is increased to infinity.

Figure 4 shows a simple example for the case of a static medium, where many modes approach a single limiting complex frequency as $L \rightarrow \infty$. In static media, FQNM frequencies may be obtained directly from the equivalent QNM condition. Replacing operators with scalars, Eq. (10) reduces to

$$(1 - [r(\omega)e^{ik(\omega)L}]^2) = 0, \quad (13)$$

where $k(\omega)$ and $r(\omega) = \frac{1-n(\omega)}{1+n(\omega)}$ are the frequency-dependent wave number and reflectivity of a static system with refractive index $n(\omega)$.

²Over shorter timescales, especially for frequencies which are very nearly commensurate, some initial amplification or decay may occur, before eventually the mode shifts out of phase with the driving field. For further discussion, see [63].

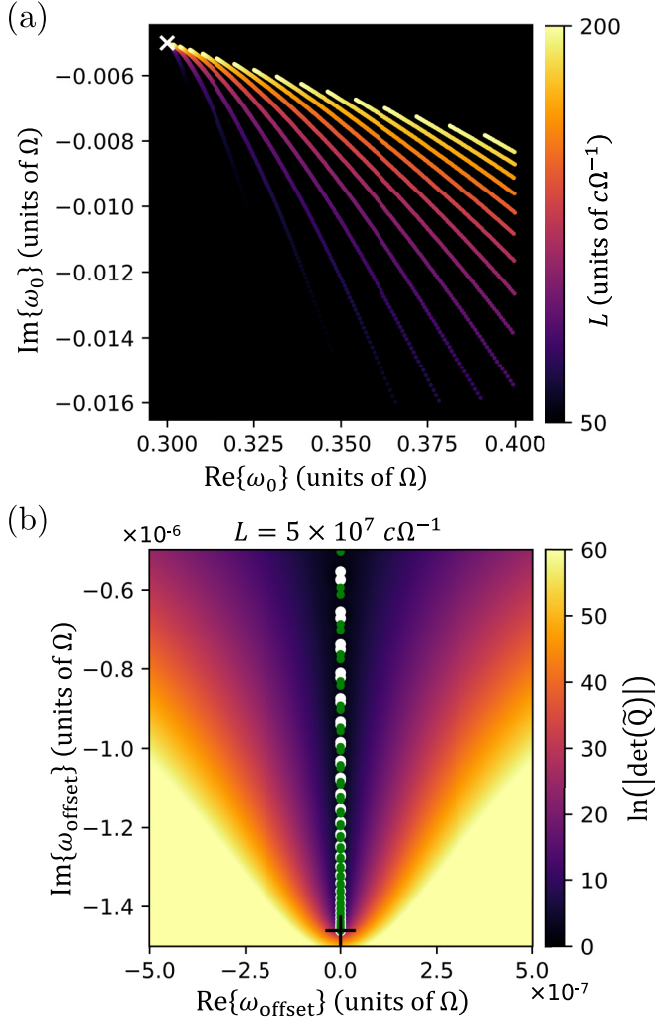


FIG. 4. QNMs and FQNMs of static and time-varying slabs approach limit points as slab length increases: (a) Plotted as points: sampled trajectories of QNMs for slab lengths increasing from $L = 50 c\Omega^{-1}$ to $L = 200 c\Omega^{-1}$. The white cross corresponds to a zero wave number point of the Drude model (discussed further in the main text), steadily approached by the QNMs. (b) For illustrative purposes, in this example, rather than a Drude model, we consider the 4×4 truncation $\tilde{Q}$ of the FQNM operator $\tilde{Q}$ for $\hat{\chi} = 1 + 0.2\hat{\Delta}_1(0)$. We plot the FQNM indicator function as a colormap, with FQNM quasifrequencies highlighted by white dots, and their approximations (see Sec. IV A 2) plotted with smaller green dots. All frequencies are plotted relative to the $\mathcal{RC}$ -symmetry axis A_1 with $\omega_{\text{offset}} = \omega_0 - \frac{\Omega}{2}$. For a very thick slab ($L = 5 \times 10^7 c\Omega^{-1}$), modes clustered around this exceptional limit point, associated with the weak third wave number band gap in the dispersion relation of $\hat{K}^2(\omega_0)$.

The limiting frequency evident in Fig. 4(a) corresponds to the point $\omega_{n=0}$ where the permittivity vanishes: the uniform mode of the slab remains at a fixed frequency, and other increasingly long wavelength modes tending towards this complex frequency with increasing L . In the limit of an infinitely long slab, this ever more closely spaced set of QNMs will all satisfy the bulk dispersion relation, $k = \omega\sqrt{\epsilon(\omega)}/c$, parameterized by some complex value of k .

To see this clustering of QNM frequencies directly, we take frequencies surrounding the zero index point, $k^2(\omega_{n=0}) = 0$, writing the frequency as $\omega = (1 + \delta)\omega_{n=0}$ and $k(\omega) \approx \mathcal{K}\sqrt{\delta}$, for $|\delta| \ll 1$. Note that, as $L \rightarrow \infty$, $k(\omega)L$ varies significantly faster than $r(\omega)$, so the latter can be well approximated as unity. More specifically, our approximation is to take the $\delta \rightarrow 0$ limit, although allowing L to increase sufficiently that $k(\omega)L$ does not decay to 0.

Under these conditions, the solutions to (13) can be approximated as

$$\omega \approx \omega_{n=0} \left(1 + \left(\frac{m\pi}{\mathcal{K}L} \right)^2 \right) \forall m \in \mathbb{Z}, \quad (14)$$

where the quadratic dependence on m is responsible for the clustering of modes around the zero index frequency identified in Fig. 4(a). Taking the $L \rightarrow \infty$ limit and replacing m with the continuous variable $k = m\pi/L$ this becomes

$$\omega \rightarrow \omega_{n=0} + \frac{c^2 k^2}{\epsilon'(\omega_S) \omega_{n=0}^2}, \quad (15)$$

where $\mathcal{K} = \sqrt{\epsilon'(\omega_S)} \omega_{n=0}^{3/2}/c$. Equation (15) is just the bulk dispersion relation for a frequency close to the zero index point, parameterized by a real wave vector k .

A. FQNM limits of a time-varying slab

1. The single-mode case

How does this behavior of the QNM spectrum carry over to the case of a time-varying slab? To answer this we first consider the analog of a zero index point, a complex Floquet frequency, $\omega_{n=0}$ where a single eigenvalue of $\hat{K}^2(\omega_{n=0})$ vanishes. In the case of time-varying media this point represents the analog of the uniform slab mode: a wave within the slab that has fixed zero wave vector and is composed of a combination of frequencies with the same relative amplitude after a single period of modulation. As we shall show, for such time-varying systems there are also a set of FQNMs with different Floquet frequencies that, with increasing L , approach the point of zero index, where $\hat{K}^2(\omega_{n=0}) = 0$.

To find the complex frequency ω_0 of one of these modes we use Eqs. (10) and (11), premultiplying $\tilde{Q}$ to write the FQNM condition as

$$[1 - (\hat{r}e^{i\hat{K}L})^2]|\tilde{E}_{\text{int}}\rangle = 0, \quad (16)$$

where $|\tilde{E}_{\text{int}}\rangle = (\hat{n}^{-1} - 1)|\tilde{E}_{\text{out},+}\rangle$, and $\hat{r} = (1 - \hat{n})(1 + \hat{n})^{-1}$ as the time-varying analog to the static reflection operator.

Choosing the root $\hat{K} \approx [\hat{K}^2]^{1/2}$ such that eigenvalues of $e^{i\hat{K}L}$ decay with increasing L eliminates all other modes from (16) except those which—as in Eq. (15)—correspond to a real eigenvalue of $\hat{K}$. Here we assume that there is only one such eigenvalue (with corresponding eigenvector $|0\rangle$) for each Floquet frequency ω_0 , which thus turns the propagation operator into a projector, $e^{i\hat{K}L} \approx e^{ik_0(\omega_0)L}|0\rangle\langle 0|$.

Using the same limit as before for our approximation, where $\delta \rightarrow 0$ unless multiplied by the very large L , the eigenvector $|0\rangle$ becomes essentially constant, with $\hat{r}|0\rangle \approx |0\rangle$. Thus, being now equivalent to a 1×1 matrix

problem, Eq. (16) reduces to the scalar form

$$(1 - e^{2ik_0(\omega_0)L})|\tilde{E}_{\text{int}}\rangle \approx 0, \quad (17)$$

where the eigenvector is $|\tilde{E}_{\text{int}}\rangle = |0\rangle$, and k_0 is the relevant real eigenvalue of $\hat{K}$.

Equation (17) is equivalent to our earlier condition (13) for static media. Thus, provided only a single eigenvalue of the $\hat{K}$ operator is real valued as a function of the Floquet frequency ω_0 , we will have the same cluster of FQNM, approaching the zero index point with increasing slab length.

Since these limit points are qualitatively identical to those of the previous section, we term both “static” limit points.

2. The two-mode case

The gain associated with the wave number band gap of a time crystal arises from complex quasifrequencies ω_0 which despite their complex nature are associated with a real wave number. Mathematically, this corresponds to investigating when $\hat{K}^2(\omega_0)$ possesses real eigenvalues. Fortunately, the $\mathcal{RC}$ symmetry noted in Sec. II for ω_0 lying on a symmetry axis is directly analogous [18] to the $\mathcal{PT}$ symmetry investigated for non-Hermitian Hamiltonians in quantum mechanics [64], where the question of when an operator possesses real eigenvalues is well studied in terms of symmetric and symmetry-broken phases.

In our case, whenever ω_0 lies on a symmetry axis A_m , $\hat{K}^2(\omega_0)$ may fall into either of these phases—a symmetric phase with real eigenvalues, or a symmetry-broken phase where eigenvalues are found in complex-conjugate pairs—separated by an exceptional point where the operator cannot be diagonalized. As such, the wave number band gap in a time crystal corresponds precisely to this symmetric phase, with the maximum gain possible determined by the maximum $\text{Im}\{\omega_0\}$ in the symmetric phase, and thus by the position of the exceptional point marking the end of such a phase.

Such exceptional points, naturally requiring the coupling between two frequencies, are inherently time-varying in nature. And, as we will demonstrate, form limit points precisely analogous to those of the previous section.

To demonstrate this, we return to our methods of the previous sections: consider frequencies $\omega_0 = (1 + \delta)\omega_{\text{EP}}$ around an exceptional point ω_{EP} where a pair of eigenvalues of $\hat{K}^2$ transition from their symmetric (real) phase to their symmetry-broken (complex conjugate pair) phase. Characteristic of an exceptional point is that this collision does not occur linearly in δ , but with a square root dependence—a dependence inherited by the eigenvalues of $\hat{K} = [\hat{K}^2]^{1/2}$. This square root collision can be written explicitly by considering writing the eigenvalues $k_{\pm}(\omega_0)$ in terms of their average $\bar{k}$ and half-splitting Δk , such that $k_{\pm}(\omega_0) = \bar{k}(\omega_0) \pm \Delta k(\omega_0)$, which, around the exceptional point (to first order in $\sqrt{\delta}$), reduces to

$$k_{\pm}(\omega_0) \approx \bar{k}(\omega_{\text{EP}}) \pm \Delta K \sqrt{\delta}. \quad (18)$$

Of course, any eigenvectors associated with the symmetric phase cannot decay in magnitude under evolution by $e^{i\hat{K}L}$. Thus, following our approach from the previous section, assume that all other modes of $\hat{K}^2$ contain at least a small amount of loss (although this assumption is actually

unnecessary) such that, for large slab lengths L , $e^{i\hat{K}L}$ projects to 0 all but two modes.

It would be convenient to follow our prior analyses by finding a diagonalization for $\hat{K}^2$ which holds to constant order in $\sqrt{\delta}$, then representing $e^{i\hat{K}L}$ using this basis. However, $\delta = 0$ corresponds, by definition, to an exceptional point of $\hat{K}^2$ where diagonalization is impossible. Instead, we apply the Schur decomposition to at least write $\hat{K}^2$ in upper diagonal form.

We thus consider only a single eigenvector $\hat{K}^2|+\rangle = k_+^2|+\rangle$, letting $\langle +| = (|+\rangle)^\dagger$, normalized as for a standard orthonormal basis, before introducing the second vector $|-\rangle$ to span the remaining space. However, in contrast to diagonalization, $|-\rangle$ is not an eigenvector of $\hat{K}^2$, but instead defined by its orthonormality to $|+\rangle$, with $\langle -| = (|-\rangle)^\dagger$, $\langle -|-\rangle = 1$ and $\langle +|-\rangle = 0$.

In this basis, $\hat{K}^2$ is then written as

$$\hat{K}^2 = k_+^2|+\rangle\langle +| + \kappa^2|+\rangle\langle -| + k_-^2|-\rangle\langle -|, \quad (19)$$

with $\kappa^2 = \langle +|\hat{K}^2|-\rangle$ as a dimensionful quantity encoding the extent to which $|-\rangle$ fails to be an eigenvalue of $\hat{K}^2$.

Sylvester’s formula for functions of 2×2 matrices then allows us to explicitly find $e^{i\hat{K}L}$ as we approach the exceptional point:

$$e^{i\hat{K}L} = e^{i\bar{k}L} \left(e^{i\Delta kL} \frac{\hat{K}^2 - k_-^2}{k_+^2 - k_-^2} - e^{-i\Delta kL} \frac{\hat{K}^2 - k_+^2}{k_+^2 - k_-^2} \right). \quad (20)$$

However, about the exceptional point, $k_+^2 - k_-^2 = 4\bar{k}\Delta k$ tends to zero as $\sqrt{\delta}$. To assess the resulting divergence, the projectors in Sylvester’s formula may be written in terms of the $|\pm\rangle$ basis:

$$\frac{\hat{K}^2 - k_{\pm}^2}{k_+^2 - k_-^2} = \mp |\mp\rangle\langle \mp| + \frac{\kappa^2}{4\bar{k}\Delta k} |\pm\rangle\langle -|. \quad (21)$$

For fixed L , this divergence is canceled by the fact that $e^{i\Delta kL} \rightarrow 1$. However, for arbitrarily large L , and small δ , this cancellation does not occur, and as $\delta \rightarrow 0$ the exponential is dominated by

$$e^{i\hat{K}L} \approx e^{i\bar{k}L} \frac{i\kappa^2}{2\bar{k}\Delta k} \sin(\Delta kL) |+\rangle\langle -|. \quad (22)$$

Thus, in this limit, Eq. (16) tends towards the form

$$-\left(\hat{r} e^{i\bar{k}L} \frac{i\kappa^2}{2\bar{k}\Delta k} \sin(\Delta kL) |+\rangle\langle -| \right)^2 |\tilde{E}_{\text{int}}\rangle \approx 0. \quad (23)$$

By letting $|\tilde{E}_{\text{int}}\rangle = \hat{r}|+\rangle$, and dividing out constants in L , this then reduces to

$$(1 - e^{2i\Delta k(\omega_0)L})^2 |\tilde{E}_{\text{int}}\rangle \approx 0, \quad (24)$$

which, in precise analogy with both Eqs. (13) and (17), has double roots at

$$\omega \approx \omega_{\text{EP}} \left(1 + \left(\frac{m\pi}{\Delta K L} \right)^2 \right) \forall m \in \mathbb{Z}. \quad (25)$$

The effect of higher order corrections in δ is then simply to split these pairs of roots by a small amount.

Given that the origin of these limit points is qualitatively distinct from the “static” limit points described above—arising only due to coupling between different frequencies in a time-varying medium—we henceforth refer to these modes as “exceptional” limit points.

Note that this divergence associated with the defective point $\delta \rightarrow 0$ actually makes our earlier assumption that $e^{i\hat{K}L}$ becomes a 2×2 projector unnecessary. The defective point naturally dominates as long as the root $\hat{K} = [\hat{K}^2]^{1/2}$ is chosen we stated: so that $e^{i\hat{K}L}$ is never exponentially growing in L . In Fig. 4 we demonstrate the accuracy of our analysis by approximating the FQNM of a large slab by the roots of $\det\{\sin[\hat{K}(\omega_0)]L\} = 0$. This example confirms our analysis even in the case that $e^{i\hat{K}L}$ does not project down onto a 2×2 subspace.

This analysis demonstrates the clustering of modes around the exceptional point frequencies ω_{EP} associated with the maximum and minimum loss in the wave number band gaps of a system. The simple approximation presented also bears direct relevance to experiments, directly relating the thickness of a slab to how well it realizes the gain predicted by the wave number band gap of its bulk.

V. MODES OF A TIME-VARYING DRUDE SLAB

In this section we demonstrate the accuracy of our analyses by returning to our Drude model example in the case of a very thick slab. We note that our results identify the origin of the diverging transmission coefficients we noted previously in [18]. These poles arise whenever an FQNM crosses the real frequency axis. Indeed, such crosses are guaranteed by the wave number band gap in this material, which places an exceptional limit point above the real axis, directly at the point of maximum gain of the bulk system.

In more generality, to assess the validity of our analyses throughout this paper, we note the following predictions made over the preceding sections, as applied to a slab of increasing length:

- (1) The trajectories followed by any given FQNM will be largely smooth.
- (2) An exception to statement 1 occurs whenever two modes collide along a symmetry axis A_m , resulting in a pair of exceptional points when the modes enter and leave A_m .
- (3) Modes will tend towards both static and exceptional limit points, as defined in the previous section, associated respectively with refractive index zeros and the (complex frequency) limits of wave number band gaps.

In particular, we focus on frequencies around the $\mathcal{RC}$ -symmetry axis A_1 . We thus consider the frequency variable $\omega_{\text{offset}} = \omega_0 - \frac{\Omega}{2}$ and plot these FQNM in Fig. 5.

Statements 1 and 2 are confirmed by trajectories such as Fig. 5(i), which progresses smoothly aside from at A_1 , where two modes collide and then split in a pair of exceptional points, similar to those seen in Fig. 3(d). However, aside from collisions on A_1 , all trajectories remain continuous.

Meanwhile, statement 3 is backed up most obviously by Fig. 5(ii), as most of the modes pictured approach a static limit point. However, this is not true of every mode [see Fig. 5(iii)]. In addition, two pairs of modes collide in an exceptional

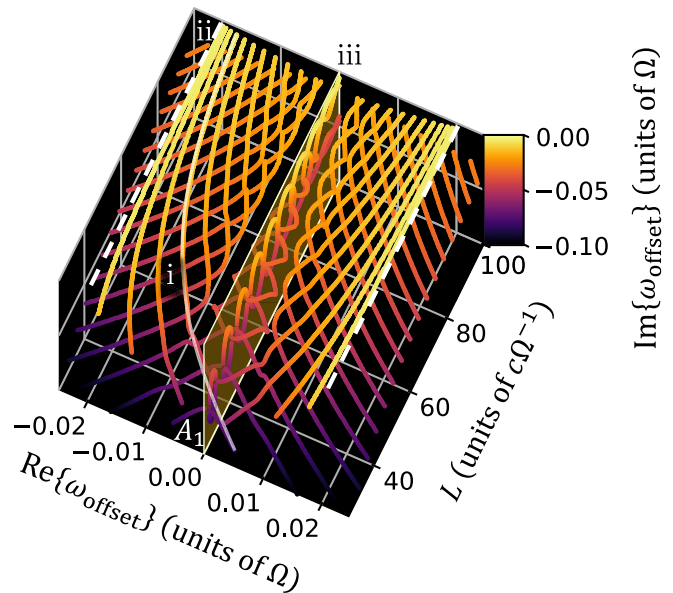


FIG. 5. FQNM in a Drude slab for a large range of increasing lengths: Sampled FQNM trajectories are plotted as points colored by the imaginary part of their quasifrequency. For readability, only modes for a 2×2 truncation of $\hat{\chi}$ are plotted. The $\mathcal{RC}$ -symmetry axis A_1 is shown as a translucent yellow plane, with a light yellow outline. (i) An otherwise smooth FQNM trajectory (qualitatively presented with a transparent white line) in a static system is broken in time-varying media by $\mathcal{RC}$ -symmetry transitions about the A_1 axis. (ii) Most FQNM trajectories have limiting behavior similar to static systems, approaching refractive index zeros (white dashed lines) as slab length increases. (iii) However, for sufficiently thick slabs, pairs of modes become permanently confined to A_1 and isolated from the static limit points (ii). These modes are associated closely with the bulk wave number band gap, and responsible for the diverging transmission coefficients of the slabs presented in [18]. Noting their isolation, one might predict that they tend towards an exceptional limit point, which is validated when considering much larger slabs, as in Fig. 6.

point, and remain stuck to the symmetry axis A_1 . This pairing up is precisely that expected of modes tending towards an exceptional limit point.

Indeed, an exceptional point of $\hat{K}^2$ can be found, associated with $\mathcal{RC}$ -symmetry breaking, at $\omega^* \sim 0.0072\Omega i + \frac{\Omega}{2}$, predicting this as a limit point at the maximum gain available in a bulk wave number band gap. In Fig. 6 we demonstrate that this analysis is indeed correct: as the length of the Drude slab increases, ever more modes become confined to A_1 , oscillating in pairs while tending towards ω^* .

In Fig. 6 this prediction is confirmed: as the length of the Drude slab increases, ever more modes become confined to A_1 , tending towards an exceptional limit point associated with $\mathcal{RC}$ symmetry breaking in $\hat{K}^2$. As discussed in the previous section, this exceptional point corresponds precisely to the maximum gain possible for a real wave number excitation in a bulk medium.

Furthermore, since the modes associated with the onset of gain are trapped to the axis A_1 by $\mathcal{RC}$ symmetry, it follows that no sequence of smooth perturbations through real

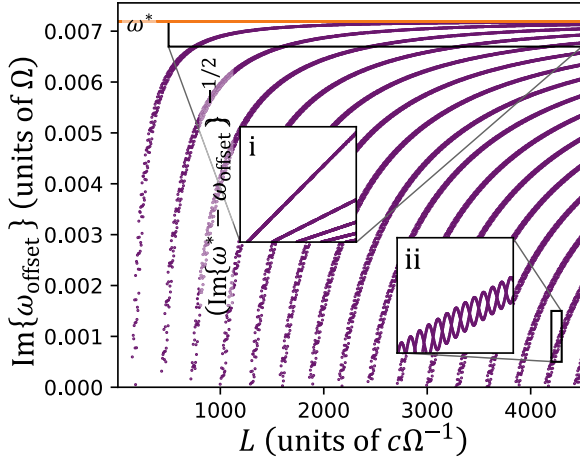


FIG. 6. An exceptional limit point due to $\mathcal{RC}$ symmetry in a Drude metal slab: The sampled FQNM trajectories of a Drude metal slab converge to an exceptional limit point at ω^* as its length increases from $L = 10 c\Omega^{-1}$ to $L = 4500 c\Omega^{-1}$. Eventually additional modes join the pairs shown in Fig. 5(iii) in becoming confined to the A_1 axis (pictured here), and over this large range of slab lengths, these converge slowly towards the limit point ω^* , associated with the maximum gain in the bulk wave number band gap. (i) To verify the asymptotics of Eq. (25), we plot $(\text{Im}\{\omega^* - \omega_{\text{offset}}\})^{1/2}$. The linear results thus show good agreement with the theory of Sec. IV A 2. (ii) We demonstrate that the apparent noise in the first half of the figure is really just a sampling artefact as pairs of orthogonal modes oscillate very rapidly.

physical systems (which must preserve $\mathcal{RC}$ symmetry) could have predicted the onset of this gain.

VI. SUMMARY AND CONCLUSIONS

Over the course of this paper, we have introduced the notion of Floquet quasnormal modes (FQNMs), solutions to the wave equation in a periodically modulated cavity with outgoing boundary conditions with a definite Floquet quasifrequency. These FQNMs allow for a transient analysis of photonic time-crystal slabs, extending the driven response we investigated previously [18]. We note that not only are collisions between FQNMs guaranteed by symmetry, but that these result in exceptional points where pairs of FQNMs undergo a symmetry transition, becoming temporarily phase locked to the modulation of the slab.

By directing our focus towards understanding how the FQNMs of a system vary under continuous changes to said system, we uncover a number of general properties of their trajectories in the complex plane. Indeed, we show that while these trajectories are typically smooth, FQNMs regularly collide along particular symmetry axes, forming exceptional points which interrupt this smoothness. Furthermore, by considering the limit points of these trajectories, we show that such collisions are essentially inevitable. In particular, we demonstrate that such nonperturbative collisions are actually required to understand the modes which realize gain in media with a wave number band gap.

This nonperturbative approach contrasts other recent work in this area [41,42] by focusing primarily on

nonperturbative features of the FQNM distribution. This focus allows us to identify the pitfalls to be encountered by future practical applications of perturbative approaches. In particular, due to the nonperturbative nature of collisions, we anticipate that any future perturbation theories aiming to provide convergent results must carefully select an initial model from which to perturb. In particular, our results suggest that direct perturbative analyses will typically not converge unless beginning from time-varying, rather than static, cavities.

ACKNOWLEDGMENTS

C.M.H. acknowledges financial support from the Engineering and Physical Sciences Research Council (EPSRC) of the United Kingdom via the Exeter University Physics DTP. I.R.H. and S.A.R.H. acknowledge financial support from the EPSRC via the META4D Programme Grant (No. EP/Y015673/1). S.A.R.H. thanks the Royal Society and TATA for financial support (Grant No. RPG-2016-186).

DATA AVAILABILITY

The data that support the findings of this article are openly available [65].

APPENDIX A: FQNMS AS A (NON-)LINEAR EIGENVALUE PROBLEM

It is worth briefly justifying the sense in which the FQNM problem as derived from Eq. (2) (with outgoing boundary conditions) is related to an eigenvalue problem. To this end, we consider the problem

$$\left(\partial_t - \begin{pmatrix} L_{WW} & L_{WM} \\ L_{MW} & L_{MM} \end{pmatrix}\right) \begin{pmatrix} \psi_W \\ \psi_M \end{pmatrix} = 0, \quad (\text{A1})$$

where ψ_W and ψ_M are real fields associated with the wave and material, respectively. Throughout this section, explicit dependence on space and time will be taken as implicit. On their own, these fields would evolve with $\partial_t \psi_Q = L_{QQ} \psi_Q$ for $Q \in \{W, M\}$, where L_{WW} is assumed to act on the space of outgoing waves ψ_W . For our problem, these fields are coupled by the operators L_{WM} and L_{MW} , respectively describing coupling from the material to the wave, and vice versa. Throughout this work, we will frequently assume that we are dealing with light interacting with some dispersive material, such that L_{WW} contains no time variation, with any time variations contained either in the evolution of the material field, or in the coupling to or from it. More compactly, this can be written as

$$(\partial_t - L)\Psi = 0, \quad (\text{A2})$$

where

$$\Psi = \begin{pmatrix} \psi_W \\ \psi_M \end{pmatrix} \quad (\text{A3})$$

and

$$L = \begin{pmatrix} L_{WW} & L_{WM} \\ L_{MW} & L_{MM} \end{pmatrix}. \quad (\text{A4})$$

We briefly note that, since Ψ consists entirely of real fields, $L^* = L$. Finally, we introduce the periodicity constraint of time crystals, that $L(t + \frac{2\pi}{\Omega}) = L(t)$, at which stage it also

becomes convenient to introduce the Floquet ansatz for solutions:

$$\Psi(t; \omega_0) = e^{-i\omega_0 t} \bar{\Psi}(t; \omega_0), \quad (\text{A5})$$

where $\bar{\Psi}(t + \frac{2\pi}{\Omega}; \omega_0) = \bar{\Psi}(t; \omega_0)$. Substituting this ansatz into Eq. (A2) gives

$$(\partial_t - i\omega_0 - L)\bar{\Psi}(\omega_0) = 0. \quad (\text{A6})$$

However, the periodicity of L also implies that it can be expressed as the Fourier series,

$$L(t) = \sum_{n=-\infty}^{\infty} e^{-in\Omega t} L_n, \quad (\text{A7})$$

where $[\partial_t, L_n] = 0$, and $L_n = L_{-n}^*$. With this, we then apply the method of [27] to convert this problem to the frequency domain with the mappings $\bar{\Psi}(t; \omega_0) \mapsto \tilde{\Psi}(\omega; \omega_0)$, $\partial_t \mapsto -i\omega$, and $t \mapsto -i\partial_\omega$, giving

$$\tilde{L} = \sum_{n=-\infty}^{\infty} e^{-n\Omega \partial_\omega} \tilde{L}_n, \quad (\text{A8})$$

where $[\tilde{L}_n, \omega] = 0$, and $e^{-n\Omega \partial_\omega}$ are powers of the discrete frequency translation operator,

$$e^{-\Omega \partial_\omega} \tilde{\Psi}(\omega; \omega_0) = \tilde{\Psi}(\omega - \Omega; \omega_0). \quad (\text{A9})$$

We are then left with the following linear eigenvalue problem in ω_0 :

$$[-i(\omega + \omega_0) - \tilde{L}] \tilde{\Psi}(\omega; \omega_0) = 0. \quad (\text{A10})$$

Practically, however, most QNM calculations do not take close account of material degrees of freedom, instead working with wave equations similar in form to Eq. (2), where the material degrees of freedom have been eliminated and replaced with some general susceptibility-like operator encoding the material response. To this end, we reexpand Eq. (A10) in terms of its wave and material components:

$$\left(-i(\omega + \omega_0) - \begin{pmatrix} \tilde{L}_{WW} & \tilde{L}_{WM} \\ \tilde{L}_{MW} & \tilde{L}_{MM} \end{pmatrix} \right) \begin{pmatrix} \tilde{\Psi}_W(\omega; \omega_0) \\ \tilde{\Psi}_M(\omega; \omega_0) \end{pmatrix} = 0. \quad (\text{A11})$$

Considering the lower equation, we find

$$\tilde{\Psi}_M(\omega; \omega_0) = (-i(\omega + \omega_0) - \tilde{L}_{MM})^{-1} \tilde{L}_{MW} \tilde{\Psi}_W(\omega; \omega_0). \quad (\text{A12})$$

Thus, under the condition that we are away from any resonance of the decoupled material field (i.e., $[-i(\omega + \omega_0) - \tilde{L}_{MM}]^{-1}$ is bounded for the ω_0 under consideration), it follows that the material fields can indeed be successfully eliminated, resulting in the nonlinear eigenvalue problem,

$$\begin{aligned} & (-i(\omega + \omega_0) - \tilde{L}_{WW}) \tilde{\Psi}_W(\omega; \omega_0) \\ & = \tilde{L}_{WM} (-i(\omega + \omega_0) - \tilde{L}_{MM})^{-1} \tilde{L}_{MW} \tilde{\Psi}_W(\omega; \omega_0), \end{aligned} \quad (\text{A13})$$

defined away from ω_0 associated with material resonance. Here the left-hand side corresponds to the free space wave equation, while the right-hand side corresponds to the material response normally encoded in the susceptibility.

APPENDIX B: DISCRETENESS AND CONTINUITY OF FQNM TRAJECTORIES

The behavior of the eigenvalues of finite-dimensional linear operators is well studied. As an $n \times n$ operator M is continuously varied, its n eigenvalues (counted with algebraic multiplicity) vary continuously [66–68]. Furthermore, as its eigenvalues form a finite set, they are necessarily discrete (i.e., lacking in accumulation points). These properties all follow almost immediately from noting that the eigenvalues λ are solutions to the equation $\det(\lambda - M) = 0$. Unfortunately, for infinite-dimensional operators such as those considered in Appendix A, the determinant is defined only under particular conditions [69], and the properties above may not necessarily be assumed.

This situation is improved significantly by Fredholm theory [44,70], which generalizes a number of these properties from finite-dimensional to infinite-dimensional operators. In particular, one considers the nonlinear eigenvalue problem $[1 - L(\lambda)]\Psi = 0$ in regions of λ such that $L(\lambda)$ is holomorphic, noting that eigenvalues λ correspond to poles in the generalized resolvent operator $[1 - L(\lambda)]^{-1}$. However, the majority of results from Fredholm theory, including generalizations of the continuity and discreteness properties mentioned above, apply only when $L(\lambda)$ is a compact operator [70].

The precise definition of compact operators is technical, and fortunately unnecessary for our purposes. Instead, we consider the subset of compact operators known as Hilbert-Schmidt operators [44], noting the sufficient condition that an operator is Hilbert-Schmidt if it can be represented by an integral transform with a square-integrable kernel [71]. The final property of use to us will be the fact that the composition of a Hilbert-Schmidt operator with a bounded operator remains Hilbert-Schmidt.

Fortunately, Eq. (A13) is relatively simple to convert into Fredholm form, taking on the form

$$(1 - \tilde{G}_{WW}(\omega_0) \tilde{L}_{WM} \tilde{G}_{MM}(\omega_0) \tilde{L}_{MW}) \tilde{\Psi}_W(\omega_0) = 0, \quad (\text{B1})$$

where $\tilde{G}_{QQ}(\omega_0) = [-i(\omega - \omega_0) - \tilde{L}_{QQ}]^{-1}$ for $Q \in \{W, M\}$, with $\tilde{G}_{WW}(\omega_0)$ corresponding to the Green's function free space wave equation with Floquet boundary conditions in time and outgoing boundary conditions in space. Meanwhile, $\tilde{G}_{MM}(\omega_0)$ corresponds to a material response with Floquet boundary conditions decoupled from any wave propagation. This form of Eq. (A13) then holds away from resonances of either the wave or the material.

To this end, consider a generalization of the 1 + 1-dimensional wave equation considered in the main text:

$$(c^{-2} \partial_z^2 - \partial_t^2) E = \partial_t^2 \chi E, \quad (\text{B2})$$

where E obeys the Floquet ansatz [see Eq. (1)] over a single period T with outgoing spatial boundary conditions. χ meanwhile is an operator encoding the polarization response of the system to an electric field (with the same Floquet boundary conditions), through the relation

$$P = \epsilon_0 \chi E. \quad (\text{B3})$$

We will not, however, consider χ to be an entirely arbitrary operator. Indeed, when polarization responses are due to

the motion of massive charges (a reasonable assumption for physical systems), then we further require that $\|\partial_t^2 P\|/\|E\|$ is bounded (where $\|\cdot\|$ denotes the L^2 norm), and thus that $\partial_t^2 \chi$ is a bounded operator (note that this is a distinct condition from possessing a bounded integral kernel). To draw analogy to Fredholm theory, we may then write Eq. (B2) in the form

$$[1 - (c^{-2}\partial_z^2 - \partial_t^2)^{-1}\partial_t^2\chi]E = 0. \quad (\text{B4})$$

Since $\partial_t^2 \chi$ is a bounded operator, results from Fredholm theory will apply under the condition that $(c^{-2}\partial_z^2 - \partial_t^2)^{-1}$ is compact. Fortunately, this inverse, with the boundary conditions specified earlier, may be found explicitly as

$$\begin{aligned} & [(c^{-2}\partial_z^2 - \partial_t^2)^{-1}\psi_S](z, t) \\ &= \int_0^T \int_{-\frac{w}{2}}^{\frac{w}{2}} G(z - z', t - t'; \omega_0) \psi_S(z', t') dz' dt', \end{aligned} \quad (\text{B5})$$

in terms of the Green's function

$$\begin{aligned} G(z, t; \omega_0) &= -\frac{1}{2c} \sum_{n=0}^{\lceil \frac{w}{cT} \rceil} \Theta\{c(t + nT) - |z|\} e^{i\omega_0 T} \\ &\quad + \frac{e^{i(\lceil \frac{w}{cT} \rceil + 1)\omega_0 T}}{1 - e^{i\omega_0 T}}, \end{aligned} \quad (\text{B6})$$

which holds away from $e^{i\omega_0 T} = 1$ and $\omega_0 = \infty$, corresponding directly to the spectrum of the wave equation, and where w is the length of a contiguous region containing the entire cavity.

Since $G(z, t; \omega_0)$ is square integrable, it immediately follows that $(c^{-2}\partial_z^2 - \partial_t^2)^{-1}$ is a Hilbert-Schmidt operator, and thus that $(c^{-2}\partial_z^2 - \partial_t^2)^{-1}\partial_t^2\chi$ is compact (once again, away from wave and material resonances). Thus, the following theorems may be immediately applied:

(1) The Analytic Fredholm Theorem [44], which proves that poles in ω_0 of $[1 - L(\omega_0)]^{-1}$ are isolated for closed regions of the complex plane where $L(\omega_0)$ is compact. The only FQNM accumulation points occur at wave or material resonances.

(2) Steinberg's Theorem [72] then proves that the poles in ω_0 of $[1 - L(\omega_0; \lambda)]^{-1}$ trace out continuous trajectories under the continuous variation of a real parameter λ , under continuity in λ , and otherwise identical conditions to the Analytic Fredholm Theorem.

APPENDIX C: APPROXIMATIONS OF FQNMS

A central assumption in this work, and that of others papers in this area [18,41,42], is that the response of time crystals may be studied by considering coupling between only a finite number of Fourier modes, rather than the infinite number actually present in the system. Fortunately, in the previous Appendix we demonstrated that our FQNM problem could be well described through compact operators, which are fortunately precisely those operators which are well approximated (in a norm sense) by sequences of finite-dimensional operators.

It is known in general that even very small perturbations of an operator (in a norm sense) may result in discontinuous changes to its spectrum, an effect which is mitigated when a root is isolated [73]. While this continuity has been studied for

linear eigenvalue problems, we briefly present a derivation for Fredholm problems restricted to compact operators.

As in the previous Appendix, we direct our attention to poles of an operator of the form

$$R(\omega_0; L) = [1 - L(\omega_0)]^{-1}, \quad (\text{C1})$$

for ω_0 in an open region $O \subset \mathbb{C}$, where O is chosen such that $L(\omega_0)$ is compact and holomorphic over the closure of O (denoted $\tilde{O}$).

By statement 1 in the previous Appendix, it then follows that all poles ω_0 in O are isolated. Furthermore, since O is an open set, for any pole $\tilde{\omega}_0$, there exists η_0 such that all loops of the form $\Gamma_\eta = \{\omega_0 \in \mathbb{C} | |\omega_0 - \tilde{\omega}_0| = \eta\}$, for all $0 < \eta < \eta_0$ lie within $\mathbb{C}$. Furthermore, since $\tilde{\omega}_0$ is an isolated pole, η_0 may be chosen such that the only pole enclosed by Γ_η is $\tilde{\omega}_0$. Then consider a general perturbation $\Delta(\omega_0)$, similarly assumed to be holomorphic and compact over $\tilde{O}$, and construct

$$R(\omega_0; L + \xi \Delta) = \{1 - [L(\omega_0) + \xi \Delta(\omega_0)]\}^{-1}, \quad (\text{C2})$$

for $\xi \in [0, 1]$.

Meanwhile, statement 2 ensures that, for any pole to leave the loop Γ_η , there must have been some ξ_0 such that the pole intersected with Γ_η , and thus such that $R(\omega_0; \xi_0)$ diverged for some $\omega_0 \in \Gamma_\eta$. To prove continuity, we demonstrate that for any $0 < \eta < \eta_0$, we can always pick a ball of $\Omega(\omega_0)$ in an appropriate metric space such that $R(\omega_0; L + \xi \Delta)$ remains finite for $\omega_0 \in \Gamma_\eta$ for all $\xi \in [0, 1]$. To this end, note that $R(\omega_0; L)$ is finite on Γ_η , and thus,

$$R(\omega_0; L + \xi \Delta) = [1 - \xi R(\omega_0; L) \Delta(\omega_0)]^{-1} R(\omega_0; L). \quad (\text{C3})$$

A sufficient condition for this to be finite on Γ_η for all ξ is then that

$$\max_{\omega_0 \in \Gamma_\eta} \{\|R(\omega_0; L)\| \|\Delta(\omega_0)\|\} < 1. \quad (\text{C4})$$

By upper bounding this quantity further, we arrive at the following, stricter, sufficient condition:

$$\max_{\omega_0 \in O} \|\Delta(\omega_0)\| < \left(\max_{\omega_0 \in \Gamma_\eta} \|R(\omega_0; L)\| \right)^{-1}. \quad (\text{C5})$$

Since $\max_{\omega_0 \in \Gamma_\eta} \|R_0(\omega_0)\|$ is finite for any η , the bound on $\max_{\omega_0 \in O} \|\Delta(\omega_0)\|$ is nonzero for any loop size η .

Thus, the poles in ω_0 of $R(\omega_0; L)$, defined over the region $\omega_0 \in O$ are continuous under variations in L , with the metric $\rho_O(L_1, L_2) = \max_{\omega_0 \in O} \|L_1(\omega_0) - L_2(\omega_0)\|$.

Finally, recall that, by assumption, $L(\omega_0)$ was compact over $\tilde{O}$. Therefore, for any ω_0 , $L(\omega_0)$ is well approximated for any ω_0 by a sequence of holomorphic, finite-dimensional approximations $L_n(\omega_0)$, such that $\lim_{n \rightarrow \infty} \|L_n(\omega_0) - L(\omega_0)\| = 0$. Furthermore, since $\tilde{O}$ is a closed, simply connected subset of $\mathbb{C}$, it is also compact, and thus this convergence implies convergence in the metric $\lim_{n \rightarrow \infty} [\rho_O(L_n, L)] = 0$.

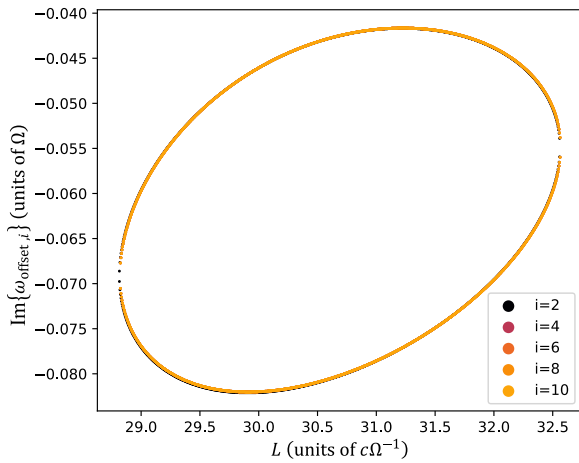


FIG. 7. Convergence of FQNM with frequency coupling truncated to finite dimensions about an exceptional point: The FQNM frequencies within the plane A_1 in Fig. 3, plotted as a function of a cavity length L , for a range of i -dimensional approximations (where i is chosen to be even to preserve $\mathcal{RC}$ -symmetry about A_1). The trajectories of the two modes pictured both begin and end with a pair of exceptional points where FQNM quasifrequencies are particularly sensitive to perturbations. However, despite this worst-case scenario, convergence is still incredibly rapid. Furthermore, the same qualitative behavior is predicted by all orders of approximation.

By the results of the previous section, our FQNM problem can be written equivalently as a study of these poles, and

thus, the poles of L_n within O converge to those of L , as $n \rightarrow \infty$.

APPENDIX D: NUMERICAL IMPLEMENTATION

In Appendix C we demonstrated that the modes of the true infinite-dimensional problem are converged to by the modes of finite-dimensional truncations. Thus, our routine to find FQNM quasifrequencies is split into two steps:

(1) Find all FQNM quasifrequencies of the truncated problem (represented by $\tilde{Q}_i$), where i is the dimension of the approximation, within a given region $R \subset \mathbb{C}$.

(2) Increase i until all of the modes in R have converged.

Step 1 is straightforward. Since $\det[\tilde{Q}_i(\omega_0)]$ is holomorphic, by applying the maximum modulus principle, one finds that $I_i(\omega_0) = |\det[\tilde{Q}_i(\omega_0)]|$ has local minima only at the zeros of $I_i(\omega_0)$, and thus at the approximate FQNM of the system. We then simply sample $\ln[I_i(\omega_0)]$ over R and find any local minima. More sophisticated root-finding algorithms could certainly be used, but this suffices for the purposes of this paper. The one detail with which we take special care is to ensure that our truncations $\tilde{Q}_i$ all preserve any relevant symmetries we are investigating, which typically corresponds to considering truncations of either even or odd order, depending on the symmetry axis A_n we aim to faithfully represent.

In step 2, we apply this method to I_i , until its roots have converged to within a desired tolerance (see Fig. 7). Since our results focus on large, nonperturbative features, we find very small truncations typically suffice.

- [1] E. Noether, Invariante Variationsprobleme, *Nachr. d. Ges. d. Wiss. zu Göttingen, Math.-Phys. Kl.* **1918**, 235 (1918).
- [2] E. Galiffi, R. Tirole, S. Yin, H. Li, S. Vezzoli, P. A. Huidobro, M. G. Silveirinha, R. Sapienza, A. Alù, and J. B. Pendry, Photonics of time-varying media, *Adv. Photon.* **4**, 014002 (2022).
- [3] C. Caloz and Z.-L. Deck-Léger, Spacetime metamaterials—part II: Theory and applications, *IEEE Trans. Antennas Propag.* **68**, 1583 (2020).
- [4] A. Cartella, T. F. Nova, M. Fechner, R. Merlin, and A. Cavalleri, Parametric amplification of optical phonons, *Proc. Natl. Acad. Sci. USA* **115**, 12148 (2018).
- [5] G. Castaldi, C. Rizza, N. Engheta, and V. Galdi, Multiple actions of time-resolved short-pulsed metamaterials, *Appl. Phys. Lett.* **122**, 021701 (2023).
- [6] X. Zhu, H.-W. Wu, Y. Zhuo, Z. Liu, and J. Li, Effective medium for time-varying frequency-dispersive acoustic metamaterials, *Phys. Rev. B* **108**, 104303 (2023).
- [7] J. C. Serra, E. Galiffi, P. A. Huidobro, J. B. Pendry, and M. G. Silveirinha, Particle-hole instabilities in photonic time-varying systems, *Opt. Mater. Express* **14**, 1459 (2024).
- [8] J. B. Pendry, Air conditioning for photons, *Opt. Mater. Express* **14**, 407 (2024).
- [9] Z. Dong, X. Chen, and L. Yuan, Extremely narrow band in moiré photonic time crystal, *Phys. Rev. Lett.* **135**, 033803 (2025).
- [10] E. Lustig, Y. Sharabi, and M. Segev, Topological aspects of photonic time crystals, *Optica* **5**, 1390 (2018).
- [11] A. C. Harwood, S. Vezzoli, T. V. Raziman, C. Hooper, R. Tirole, F. Wu, S. A. Maier, J. B. Pendry, S. A. R. Horsley, and R. Sapienza, Space-time optical diffraction from synthetic motion, *Nat. Commun.* **16**, 5147 (2025).
- [12] M. M. Asgari, P. Garg, X. Wang, M. S. Mirmoosa, C. Rockstuhl, and V. Asadchy, Theory and applications of photonic time crystals: A tutorial, *Adv. Opt. Photon.* **16**, 958 (2024).
- [13] D. L. Sounas and A. Alù, Non-reciprocal photonics based on time modulation, *Nat. Photon.* **11**, 774 (2017).
- [14] E. Lustig, O. Segal, S. Saha, C. Fruhling, V. M. Shalaev, A. Boltasseva, and M. Segev, Photonic time-crystals—Fundamental concepts, *Opt. Express* **31**, 9165 (2023).
- [15] H. He, S. Zhang, J. Qi, F. Bo, and H. Li, Faraday rotation in nonreciprocal photonic time-crystals, *Appl. Phys. Lett.* **122**, 051703 (2023).
- [16] J. G. Gaxiola-Luna and P. Halevi, Growing fields in a temporal photonic (time) crystal with a square profile of the permittivity $\epsilon(t)$, *Appl. Phys. Lett.* **122**, 011702 (2023).
- [17] G. Trainiti, Y. Xia, J. Marconi, G. Cazzulani, A. Erturk, and M. Ruzzene, Time-periodic stiffness modulation in elastic metamaterials for selective wave filtering: Theory and experiment, *Phys. Rev. Lett.* **122**, 124301 (2019).
- [18] C. M. Hooper, J. R. Capers, I. R. Hooper, and S. A. R. Horsley, Symmetry-protected lossless modes in dispersive time-varying media, *Phys. Rev. A* **111**, 033507 (2025).

- [19] E. Galiffi, A. C. Harwood, S. Vezzoli, R. Tirole, A. Alù, and R. Sapienza, Optical coherent perfect absorption and amplification in a time-varying medium, *Nat. Photon.* **20**, 163 (2026).
- [20] E. Hendry, C. M. Hooper, W. P. Wardley, and S. A. R. Horsley, Effects due to generation of negative frequencies during temporal diffraction, [arXiv:2507.03491](#).
- [21] X. Wang, M. S. Mirmoosa, V. S. Asadchy, C. Rockstuhl, S. Fan, and S. A. Tretyakov, Metasurface-based realization of photonic time crystals, *Sci. Adv.* **9**, eadg7541 (2023).
- [22] A. Vertiz-Conde, I. Liberal, and J. E. Vázquez-Lozano, Dispersion effects in thermal emission from temporal metamaterials: High-frequency cutoffs, *Opt. Lett.* **50**, 1097 (2025).
- [23] J. E. Vázquez-Lozano and I. Liberal, Incandescent temporal metamaterials, *Nat. Commun.* **14**, 4606 (2023).
- [24] S. A. R. Horsley and J. B. Pendry, Quantum electrodynamics of time-varying gratings, *Proc. Natl. Acad. Sci. USA* **120**, e2302652120 (2023).
- [25] S. A. R. Horsley and J. B. Pendry, Traveling wave amplification in stationary gratings, *Phys. Rev. Lett.* **133**, 156903 (2024).
- [26] J. Park, K. Lee, R.-Y. Zhang, H.-C. Park, J.-W. Ryu, G. Y. Cho, M. Y. Lee, Z. Zhang, N. Park, W. Jeon, J. Shin, C. T. Chan, and B. Min, Spontaneous emission decay and excitation in photonic time crystals, *Phys. Rev. Lett.* **135**, 133801 (2025).
- [27] S. A. R. Horsley, E. Galiffi, and Y.-T. Wang, Eigenpulses of dispersive time-varying media, *Phys. Rev. Lett.* **130**, 203803 (2023).
- [28] A. Tomadin, S. M. Hornett, H. I. Wang, E. M. Alexeev, A. Candini, C. Coletti, D. Turchinovich, M. Kläui, M. Bonn, F. H. L. Koppens, E. Hendry, M. Polini, and K.-J. Tielrooij, The ultrafast dynamics and conductivity of photoexcited graphene at different Fermi energies, *Sci. Adv.* **4**, eaar5313 (2018).
- [29] H. Moussa, G. Xu, S. Yin, E. Galiffi, Y. Ra'di, and A. Alù, Observation of temporal reflection and broadband frequency translation at photonic time interfaces, *Nat. Phys.* **19**, 863 (2023).
- [30] X. Wang, P. Garg, M. S. Mirmoosa, A. G. Lamprianidis, C. Rockstuhl, and V. S. Asadchy, Expanding momentum bandgaps in photonic time crystals through resonances, *Nat. Photon.* **19**, 149 (2025).
- [31] M. G. Ozlu, V. Mkhitarian, C. B. Fruhling, A. Boltasseva, and V. M. Shalae, Floquet engineering of polaritonic amplification in dispersive photonic time crystals, *Phys. Rev. Res.* **7**, 023214 (2025).
- [32] H. Benisty, J.-J. Greffet, and P. Lalanne, Basics of resonators and cavities, in *Introduction to Nanophotonics* (Oxford University Press, Oxford, 2022), pp. 159–196.
- [33] J. Ren, S. Franke, and S. Hughes, Quasinormal mode theories and applications in classical and quantum nanophotonics, in *Advances in Near-Field Optics*, edited by R. Gordon (Springer International Publishing, Cham, 2023), pp. 87–135.
- [34] P. T. Kristensen and S. Hughes, Modes and mode volumes of leaky optical cavities and plasmonic nanoresonators, *ACS Photon.* **1**, 2 (2014).
- [35] S. Franke, S. Hughes, M. K. Dezfouli, P. T. Kristensen, K. Busch, A. Knorr, and M. Richter, Quantization of quasinormal modes for open cavities and plasmonic cavity quantum electrodynamics, *Phys. Rev. Lett.* **122**, 213901 (2019).
- [36] S. Franke, J. Ren, and S. Hughes, Quantized quasinormal-mode theory of coupled lossy and amplifying resonators, *Phys. Rev. A* **105**, 023702 (2022).
- [37] B. VanDrunen, J. Ren, S. Franke, and S. Hughes, Gain-compensated metal cavity modes and a million-fold improvement of Purcell factors, *Optica Quantum* **2**, 85 (2024).
- [38] Z. Zhang, F. Zhang, Z. Xu, Y. Hu, H. Bao, and H. Shen, Realizing exceptional points by Floquet dissipative couplings in thermal atoms, *Phys. Rev. Lett.* **133**, 133601 (2024).
- [39] J. Ren, S. Franke, and S. Hughes, Quasinormal modes, local density of states, and classical Purcell factors for coupled loss-gain resonators, *Phys. Rev. X* **11**, 041020 (2021).
- [40] J. Ren, S. Franke, and S. Hughes, Connecting classical and quantum mode theories for coupled lossy cavity resonators using quasinormal modes, *ACS Photon.* **9**, 138 (2022).
- [41] A. C. Valero, S. Gladyshev, D. Globosits, S. Rotter, E. A. Muljarov, and T. Weiss, Resonant states of structured photonic time crystals, [arXiv:2506.01472](#).
- [42] B. Vial and R. V. Craster, Quasinormal modes of Floquet media slabs, *Phys. Rev. B* **112**, 214201 (2025).
- [43] M. V. Keldysh, On the completeness of the eigenfunctions of some classes of non-selfadjoint linear operators, *Russ. Math. Surv.* **26**, 15 (1971).
- [44] M. Renardy and R. C. Rogers, Operator theory, in *An Introduction to Partial Differential Equations*, edited by J. E. Marsden, L. Sirovich, and S. S. Antman (Springer, New York, 2004), p. 228.
- [45] L. Fan, J. Wang, L. T. Varghese, H. Shen, B. Niu, Y. Xuan, A. M. Weiner, and M. Qi, An all-silicon passive optical diode, *Science* **335**, 447 (2012).
- [46] I. R. Hooper, D. B. Phillips, and S. A. R. Horsley, Harnessing the frequency eigenchannels of ultrafast time-varying media, [arXiv:2508.12753](#).
- [47] X. Wen, X. Zhu, H. W. Wu, and J. Li, Realizing spatiotemporal effective media for acoustic metamaterials, *Phys. Rev. B* **104**, L060304 (2021).
- [48] X. Wen, X. Zhu, A. Fan, W. Y. Tam, J. Zhu, H. W. Wu, F. Lemoult, M. Fink, and J. Li, Unidirectional amplification with acoustic non-Hermitian space-time varying metamaterial, *Commun. Phys.* **5**, 18 (2022).
- [49] M. Malléjac and R. Fleury, Experimental realization of an active time-modulated airborne acoustic circulator at audible frequencies, *J. Sound Vib.* **618**, 119246 (2025).
- [50] W. C. Wong, G. Chaplain, and J. Li, Eigenmode-guided amplification via spatiotemporal active acoustic metamaterials, [arXiv:2506.17687v2](#).
- [51] M. Koukouraki, P. Petitjeans, A. Maurel, and V. Pagneux, Characterization of temporal aiming for water waves with an anisotropic metabathymetry, *Phys. Rev. B* **112**, 214301 (2025).
- [52] V. Bacot, M. Labousse, A. Eddi, M. Fink, and E. Fort, Time reversal and holography with spacetime transformations, *Nat. Phys.* **12**, 972 (2016).
- [53] A. L. Cullen, A travelling-wave parametric amplifier, *Nature (London)* **181**, 332 (1958).
- [54] P. K. Tien, Parametric amplification and frequency mixing in propagating circuits, *J. Appl. Phys.* **29**, 1347 (1958).
- [55] F. Hyde, Varactor-diode parametric amplifiers, *Electron. Power* **10**, 197 (1964).
- [56] C. M. Bender and S. Boettcher, Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry, *Phys. Rev. Lett.* **80**, 5243 (1998).
- [57] W. D. Heiss, Exceptional points of non-Hermitian operators, *J. Phys. A: Math. Gen.* **37**, 2455 (2004).

- [58] C. Sauvan, T. Wu, R. Zarouf, E. A. Muljarov, and P. Lalanne, Normalization, orthogonality, and completeness of quasinormal modes of open systems: The case of electromagnetism, *Opt. Express* **30**, 6846 (2022).
- [59] M. Lyubarov, Y. Lumer, A. Dikopoltsev, E. Lustig, Y. Sharabi, and M. Segev, Amplified emission and lasing in photonic time crystals, *Science* **377**, 425 (2022).
- [60] B. Dixon, C. M. Hooper, I. R. Hooper, and S. A. R. Horsley, Scattering in time-varying Drude-Lorentz models, *J. Opt.* **28**, 035101 (2026).
- [61] D. M. Solís and N. Engheta, Functional analysis of the polarization response in linear time-varying media: A generalization of the Kramers-Kronig relations, *Phys. Rev. B* **103**, 144303 (2021).
- [62] Y. Sun, S. Fan, and G. Hu, Formulation of dispersive and dissipative time-varying media as a Floquet matrix eigenproblem, *Phys. Rev. Lett.* **135**, 156903 (2025).
- [63] I. Kiorpelidis, F. K. Diakonov, G. Theocharis, and V. Pagneux, Transient amplification in stable Floquet media, *Phys. Rev. B* **110**, 134315 (2024).
- [64] Q.-h. Wang, 2×2 PT-symmetric matrices and their applications, *Philos. Trans. R. Soc. A* **371**, 20120045 (2013).
- [65] C. M. Hooper, I. R. Hooper, and S. A. R. Horsley, Code repository (2026), doi:[10.17605/OSF.IO/3V5EM](https://doi.org/10.17605/OSF.IO/3V5EM).
- [66] L. Qi, Some simple estimates for singular values of a matrix, *Linear Algebra Appl.* **56**, 105 (1984).
- [67] M. Carlsson and O. Rubin, A Hilbert-space variant of Geršgorin's circle theorem, *Math. Nachr.* **297**, 3095 (2024).
- [68] A. M. Ostrowski, *Solution of Equations and Systems of Equations: Pure and Applied Mathematics: A Series of Monographs and Textbooks*, edited by P. A. Smith and S. Eilenberg (Academic Press, New York, 1966), Vol. 9.
- [69] C. Boyd and N. Snigireva, On the analyticity of the Fredholm determinant, *Monatshefte Math.* **190**, 675 (2019).
- [70] G. W. Hanson and A. B. Yakovlev, Introductory linear operator theory, in *Operator Theory for Electromagnetics: An Introduction* (Springer, New York, 2002), p. 129.
- [71] J. Zweck, Y. Latushkin, and E. Gallo, A regularity condition under which integral operators with operator-valued kernels are trace class, *Bol. Soc. Mat. Mex.* **31**, 38 (2025).
- [72] S. Steinberg, Meromorphic families of compact operators, *Arch. Ration. Mech. Anal.* **31**, 372 (1968).
- [73] M. Ahues, A. Largillier, and B. Limaye, *Spectral Computations for Bounded Operators*, Applied Mathematics (Chapman & Hall/CRC, Philadelphia, 2001).